axioms

Article

Sharp Coefficient Bounds for a Subclass of Bounded Turning Functions with a Cardioid Domain

Lei Shi [1], Hari Mohan Srivastava [2,3,4,5], Nak Eun Cho [6] and Muhammad Arif [7,*]

1. School of Mathematics and Statistics, Anyang Normal University, Anyang 455002, China; shimath@aynu.edu.cn
2. Department of Mathematics and Statistics, University of Victoria, Victoria, BC V8W 3R4, Canada; harimsri@math.uvic.ca
3. Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
4. Department of Mathematics and Informatics, Azerbaijan University, 71 Jeyhun Hajibeyli Street, Baku AZ1007, Azerbaijan
5. Section of Mathematics, International Telematic University Uninettuno, 00186 Rome, Italy
6. Department of Applied Mathematics, Pukyong National University, Busan 48513, Republic of Korea; necho@pknu.ac.kr
7. Department of Mathematics, Abdul Wali khan University Mardan, Mardan 23200, Pakistan

* Correspondence: marifmaths@awkum.edu.pk

Abstract: In the present paper, we give a new simple proof on the sharp bounds of coefficient functionals related to the Carathéodory functions and make a correction on the extremal functions. The result is further used to investigate some initial coefficient bounds on a subclass of bounded turning functions $\mathcal{R}_{\wp}$ associated with a cardioid domain. For functions in this class, we calculate the bounds of the Fekete–Szegö-type inequality and the second- and third-order Hankel determinants. All the results are proved to be sharp.

Keywords: univalent function; cardioid domain; coefficient bounds; Hankel determinant

MSC: 30C45; 30C80

Citation: Shi, L.; Srivastava, H.M.; Cho, N.E.; Arif, M. Sharp Coefficient Bounds for a Subclass of Bounded Turning Functions with a Cardioid Domain. *Axioms* **2023**, *12*, 775. https://doi.org/10.3390/axioms12080775

Academic Editor: Miodrag Mateljevic

Received: 1 July 2023
Revised: 5 August 2023
Accepted: 8 August 2023
Published: 10 August 2023

1. Introduction and Definitions

Let $\mathcal{H}(\mathbb{D})$ represent the family of functions which are analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{A}$ denote the subfamily of $\mathcal{H}(\mathbb{D})$ consisting of functions in the form of

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}. \tag{1}$$

Suppose that $\mathcal{P}$ indicates the class of the class of all functions p that are analytic in $\mathbb{D}$ with $\Re(p(z)) > 0$ and

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

If $p \in \mathcal{P}$, it is a Carathéodory function. Assume that the set $\mathcal{S} \subset \mathcal{A}$ contains all univalent functions in $\mathbb{D}$. Using the Koebe theorem, it is known that for each univalent function $f \in \mathcal{S}$, there exist an inverse function f^{-1} defined at least on a disc of radius $1/4$ with the Taylor's series of the form

$$f^{-1}(w) := w + \sum_{n=2}^{\infty} B_n w^n, \quad |w| < \frac{1}{4}. \tag{3}$$

For two functions $F_1, F_2 \in \mathcal{H}(\mathbb{D})$, we say F_1 is subordinate to F_2, written by $F_1 \prec F_2$, if there exists a function u which is analytic in $\mathbb{D}$ with $u(0) = 0$ and $|u(z)| < 1$, such that

$F_1(z) = F_2(u(z)), z \in \mathbb{D}$. The function u is called a Schwarz function. In the case of F_2 being univalent in $\mathbb{D}$, then we have the relation

$$F_1(z) \prec F_2(z) \quad (z \in \mathbb{D}) \quad \Longleftrightarrow \quad F_1(0) = F_2(0) \quad \text{and} \quad F_1(\mathbb{D}) \subset F_2(\mathbb{D}).$$

In geometric function theory, the most basic and important subfamilies of the set $\mathcal{S}$ are the family $\mathcal{S}^*$ of starlike functions, the family $\mathcal{C}$ of convex functions and the family $\mathcal{R}$ of bounded turning functions. The interested readers are referred to [1], (Chapter II) . In 1994, Ma and Minda [2] introduced a class of analytic univalent functions $\varphi(z)$, which maps $\mathbb{D}$ onto the starlike domain with respect to $\varphi(0) = 1$ in the right half plane and is symmetric about the real axis. The Ma and Minda classes of $\mathcal{C}(\varphi)$, $\mathcal{S}^*(\varphi)$ and $\mathcal{R}(\varphi)$ are characterized, respectively, as

$$\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec \varphi(z) \right\},$$
$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$
$$\mathcal{R}(\varphi) := \{ f \in \mathcal{A} : f'(z) \prec \varphi(z) \},$$

see [3,4]. By considering different image domains $\varphi(\mathbb{D})$, various classes $\mathcal{C}(\varphi)$, $\mathcal{S}^*(\varphi)$ and $\mathcal{R}(\varphi)$ of univalent functions were considered in recent years. For example, setting $\varphi(z) = \sqrt{1+z}$, we obtain the class $\mathcal{S}^*_L = \mathcal{S}^*(\sqrt{1+z})$, which represents the collection of functions in the class $\mathcal{A}$ that $\frac{zf'(z)}{f(z)}$ lies in the domain bounded by the lemniscate of Bernoulli $\left|w^2 - 1\right| = 1$, see [5]. Choosing $\tilde{\varphi} = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$, $\mathcal{S}^*_p = \mathcal{S}^*(\tilde{\varphi})$ is the class of parabolic starlike functions. For functions $f \in \mathcal{S}^*_p$, its image of $\frac{zf'(z)}{f(z)}$ under $\mathbb{D}$ is the parabolic domain given by $\{w \in \mathbb{C} : \Re(w) > |w-1|\}$, see [6]. The class $\mathcal{S}^*_c = \mathcal{S}^*\left(1 + \frac{4}{3}z + \frac{2}{3}z^2\right)$ is a collection of starlike functions $f \in \mathcal{A}$ where $\frac{zf'(z)}{f(z)}$ lies in the domain bounded by the cardiod $\Omega_c = \left\{u + iv : \left(9u^2 + 9v^2 - 18u + 5\right)^2 - 16\left(9u^2 + 9v^2 - 6u + 1\right) = 0\right\}$; for further reading we refer to [7]. In [8], Wani and Swaminathan investigated the class $\mathcal{S}^*_{Ne} = \mathcal{S}^*\left(1 + z - \frac{1}{3}z^3\right)$, consisting of functions associated with a nephroid domain. For other related works, see, for instance, [9–11]. Recently, S. Sivaprasad Kumar et al. [12] introduced and studied a class of starlike functions defined by

$$\mathcal{S}^*_{\mathcal{C}} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + ze^z =: \wp(z), \quad z \in \mathbb{D} \right\}, \tag{4}$$

where $\wp(z)$ maps the unit disk onto a cardioid domain.

Motivated by the above works, we now consider a subfamily $\mathcal{R}_\wp$ of bounded turning functions defined by

$$\mathcal{R}_\wp := \{ f \in \mathcal{A} : f'(z) \prec 1 + ze^z, \quad z \in \mathbb{D} \}. \tag{5}$$

For given parameters $q,\ n \in \mathbb{N} = \{1, 2, \cdots\}$, the Hankel determinant $H_{q,n}(f)$ was defined by Pommerenke [13,14] for a function $f \in \mathcal{S}$ of the form (1) as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad a_1 = 1. \tag{6}$$

The upper bounds of $\left|H_{q,n}(f)\right|$ have been investigated for different subclasses of univalent functions. By applying Schwarz Lemma [15,16], Selin Aydinoğlua and Bülent Nafi Örnek [17] determined the sharp bounds of Hankel determinant $\mathcal{H}_{2,1}(f) = a_3 - a_2^2$ for the class $\mathcal{M}_\alpha$, defined by the condition $f \in \mathcal{A}$ and $\left|\left(\frac{z}{f(z)}\right)^2 f'(z) - \alpha\right| < 1$, where $\alpha \in \mathbb{C}$. Of note, the Hankel determinant $\mathcal{H}_{2,1}(f)$ is also known as Fekete–Szegö inequality. The absolute sharp bounds of the functional $H_{2,2}(f) = a_2a_4 - a_3^2$ were found in [18,19] for each of the sets $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{R}$. The Hankel determinant of order three is given as

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = -a_5a_2^2 + 2a_2a_3a_4 - a_3^3 + a_5a_3 - a_4^2. \tag{7}$$

The estimation of the determinant $|H_{3,1}(f)|$ seems a little harder compared to the bound of $|H_{2,2}(f)|$, see [20–22]. In 2010, Babalola [23] obtained the upper bound of $|H_{3,1}(f)|$ for the families of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$. Later on, many authors obtained non-sharp bounds on $|H_{3,1}(f)|$ for different subfamilies of univalent functions, see, for example, [24–26]. The sharp bound of the third Hankel determinant for convex functions $\mathcal{C}$ was obtained in [27]. For $f \in \mathcal{S}^*$, the upper bound of $|H_{3,1}(f)|$ was finally proved to be $\frac{4}{9}$ by Kowalczyk et al. [28]. For the bounded turning functions $\mathcal{R}$, the sharp upper bound of third Hankel determinant was calculated to be $\frac{1}{4}$ in [29]. For some subclasses of convex functions, starlike functions and bounded turning functions, some sharp bounds of third Hankel determinant were also obtained in [30–33].

In the current article, our main goal is to calculate the sharp bounds on some initial coefficients for the class $\mathcal{R}_\wp$ of bounded turning functions linked with a cardioid domain. We also obtain the Fekete–Szegö inequality, and the sharp bounds of the second- and third-order Hankel determinants for this class. In proof of our results, we give a new simple proof of an estimation for the Carathéodory function and correct an error on the extremal function in Lemma 2.1 of [34].

2. A Set of Lemmas

The key to the proof of our results is the following lemmas.

Lemma 1 ([35])**.** *Let $p \in \mathcal{P}$ be given by (2). Then, we have*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{8}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\delta, \tag{9}$$

$$\begin{aligned} 8c_4 = {} & c_1^4 + \left(4 - c_1^2\right)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - c_1^2\right)\left(1 - |x|^2\right) \\ & \cdot \left[c_1(x-1)\delta + \overline{x}\delta^2 - \left(1 - |\delta|^2\right)\rho\right] \end{aligned} \tag{10}$$

for some complex numbers x, δ and ρ, such that $|x| \le 1$, $|\delta| \le 1$ and $|\rho| \le 1$.

Lemma 2 ([36])**.** *If $p \in \mathcal{P}$ has the form (2), then*

$$|c_n| \le 2 \ \text{for} \ \ n \ge 1. \tag{11}$$

Lemma 3 ([37])**.** *For any complex number μ and $p \in \mathcal{P}$,*

$$|c_{n+k} - \mu c_n c_k| \le 2\max\{1, |2\mu - 1|\}. \tag{12}$$

Lemma 4 ([38]). *Let $\omega(z) = \sum_{n=1}^{\infty} w_k z^k$ be a Schwarz function. Then, for real numbers μ and ν, we have the following sharp estimate given by*

$$\Psi(\omega) = \left| w_3 + \mu w_1 w_2 + \nu w_1^3 \right| \leq \Phi(\mu, \nu), \tag{13}$$

where $\Phi(\mu,\nu)$ is defined by

$$\Phi(\mu,\nu) = \begin{cases} 1, & (\mu,\nu) \in \mathbb{D}_1 \bigcup \mathbb{D}_2 \bigcup \{(2,1)\}, \\ |\nu|, & (\mu,\nu) \in \bigcup\limits_{k=3}^{7} \mathbb{D}_k, \\ \frac{2}{3}(|\mu|+1)\sqrt{\frac{|\mu|+1}{3(|\mu|+1+\nu)}}, & (\mu,\nu) \in \mathbb{D}_8 \bigcup \mathbb{D}_9, \\ \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\sqrt{\frac{\mu^2-4}{3(\nu-1)}}, & (\mu,\nu) \in \mathbb{D}_{10} \bigcup \mathbb{D}_{11} \setminus \{(2,1)\}, \\ \frac{2}{3}(|\nu|-1)\sqrt{\frac{|\mu|-1}{3(|\mu|-1-\nu)}}, & (\mu,\nu) \in \mathbb{D}_{12}, \end{cases} \tag{14}$$

and

$$\begin{aligned}
\mathbb{D}_1 &= \left\{(\mu,\nu) : |\mu| \leq \frac{1}{2},\ -1 \leq \nu \leq 1\right\}, \\
\mathbb{D}_2 &= \left\{(\mu,\nu) : \frac{1}{2} \leq |\mu| \leq 2,\ \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leq \nu \leq 1\right\}, \\
\mathbb{D}_3 &= \left\{(\mu,\nu) : |\mu| \leq \frac{1}{2},\ \nu \leq -1\right\}, \\
\mathbb{D}_4 &= \left\{(\mu,\nu) : |\mu| \geq \frac{1}{2},\ \nu \leq -\frac{2}{3}(|\mu|+1)\right\}, \\
\mathbb{D}_5 &= \{(\mu,\nu) : |\mu| \leq 2,\ \nu \geq 1\}, \\
\mathbb{D}_6 &= \left\{(\mu,\nu) : 2 \leq |\mu| \leq 4,\ \nu \geq \frac{1}{12}\left(\mu^2+8\right)\right\}, \\
\mathbb{D}_7 &= \left\{(\mu,\nu) : |\mu| \geq 4,\ \nu \geq \frac{2}{3}(|\mu|-1)\right\}, \\
\mathbb{D}_8 &= \left\{(\mu,\nu) : \frac{1}{2} \leq |\mu| \leq 2,\ -\frac{2}{3}(|\mu|+1) \leq \nu \leq \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1)\right\}, \\
\mathbb{D}_9 &= \left\{(\mu,\nu) : |\mu| \geq 2,\ -\frac{2}{3}(|\mu|+1) \leq \nu \leq \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}\right\}, \\
\mathbb{D}_{10} &= \left\{(\mu,\nu) : 2 \leq |\mu| \leq 4,\ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \leq \nu \leq \frac{1}{12}\left(\mu^2+8\right)\right\}, \\
\mathbb{D}_{11} &= \left\{(\mu,\nu) : |\mu| \geq 4,\ \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4} \leq \nu \leq \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4}\right\}, \\
\mathbb{D}_{12} &= \left\{(\mu,\nu) : |\mu| \geq 4,\ \frac{2|\mu|(|\mu|-1)}{\mu^2-2|\mu|+4} \leq \nu \leq \frac{2}{3}(|\mu|-1)\right\}.
\end{aligned}$$

The following Lemma was obtained by Virendra Kumar et al. [34] in 2019. As the authors point out, it is of independent interest as well. Unfortunately, there are some minor mistakes on the extremal function. Next, we will give a new more simple proof of this result using Lemma 4.

Lemma 5. *Let $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \in \mathcal{P}$. Then, for any real number σ,*

$$\left|\sigma p_3 - p_1^3\right| \le \begin{cases} 2|\sigma - 4|, & \sigma < \frac{4}{3}, \\ 2\sigma\sqrt{\frac{\sigma}{\sigma-1}}, & \sigma \ge \frac{4}{3}. \end{cases}$$

The above estimate is sharp.

Proof. Let $p \in \mathcal{P}$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots, \quad z \in \mathbb{D}.$$

Suppose that $\omega(z) = \frac{p(z)-1}{p(z)+1}$. Clearly, $\omega(0) = 0$. Since $p(z)$ lies in the right half plane and $\frac{z-1}{z+1}$ maps the right half plane to the unit disk, we know $|\omega(z)| < 1$. Thus, ω is a Schwarz function. Assume that

$$\omega(z) = w_1 z + w_2 z^2 + w_3 z^3 + w_4 z^4 + \cdots, \quad z \in \mathbb{D}.$$

From the fact that

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + 2w_1 z + \left(2w_1^2 + 2w_2\right)z^2 + \left(2w_1^3 + 4w_2 w_1 + 2w_3\right)z^3 + \cdots,$$

we have $p_1 = 2w_1$, $p_2 = 2\left(w_2 + w_1^2\right)$ and $p_3 = 2\left(w_3 + 2w_2 w_1 + w_1^3\right)$. It follows that

$$\sigma p_3 - p_1^3 = 2\sigma w_3 + 4\sigma w_1 w_2 + (2\sigma - 8)w_1^3.$$

As $\sigma = 0$ the proof is trivial, we assume that $\sigma \neq 0$ in the following. Then, we obtain

$$\left|\sigma p_3 - p_1^3\right| = 2|\sigma|\left|w_3 + 2w_1 w_2 + \frac{\sigma - 4}{\sigma} w_1^3\right|. \tag{15}$$

Let $\mu = 2$, $\nu = \frac{\sigma-4}{\sigma}$. Clearly, (μ, ν) is only possible to lie in the disk $\mathbb{D}_4$, $\mathbb{D}_5$, $\mathbb{D}_6$, $\mathbb{D}_8$ and $\mathbb{D}_9$, which can be further specified as

$$\begin{aligned} \mathbb{D}_4 &= \left\{(\mu,\nu) : |\mu| \ge \frac{1}{2},\ \nu \le -2\right\}, \\ \mathbb{D}_5 &= \{(\mu,\nu) : |\mu| \le 2,\ \nu \ge 1\}, \\ \mathbb{D}_6 &= \{(\mu,\nu) : 2 \le |\mu| \le 4,\ \nu \ge 1\}, \\ \mathbb{D}_8 &= \left\{(\mu,\nu) : \frac{1}{2} \le |\mu| \le 2,\ -2 \le \nu \le 1\right\}, \\ \mathbb{D}_9 &= \{(\mu,\nu) : |\mu| \ge 2,\ -2 \le \nu \le 1\}. \end{aligned}$$

For $\sigma < 0$, it is observed that $\nu > 1$ and $(\mu,\nu) \in \mathbb{D}_5 \bigcup \mathbb{D}_6$. Thus, we have

$$\left|w_3 + 2w_1 w_2 + \frac{\sigma - 4}{\sigma} w_1^3\right| \le |\nu| = \frac{\sigma - 4}{\sigma}. \tag{16}$$

For $0 < \sigma < \frac{4}{3}$, we see $\nu < -2$ and $(\mu,\nu) \in \mathbb{D}_4$. Thus,

$$\left|w_3 + 2w_1 w_2 + \frac{\sigma - 4}{\sigma} w_1^3\right| \le |\nu| = -\frac{\sigma - 4}{\sigma}. \tag{17}$$

For $\sigma \geq \frac{4}{3}$, we know $-2 \leq \nu < 1$ and $(\mu, \nu) \in \mathbb{D}_8 \bigcup \mathbb{D}_9$. Then we deduce that

$$\left| w_3 + 2w_1 w_2 + \frac{\sigma - 4}{\sigma} w_1^3 \right| \leq \frac{2}{3}(|\mu| + 1)\sqrt{\frac{|\mu| + 1}{3(|\mu| + 1 + \nu)}} = \sqrt{\frac{\sigma}{\sigma - 1}}. \tag{18}$$

Combining (15)–(18), the result of Lemma 5 follows. □

Remark 1. *In [34], the authors gave an extremal function f given by*

$$f(z) = \frac{1 - z^2}{1 - 2\sqrt{\frac{\sigma}{\sigma-1}}z + z^2}, \quad \sigma > \frac{4}{3}.$$

Let $q = \sqrt{\frac{\sigma}{\sigma-1}}$. It is seen that

$$f(z) = 1 + 2qz + 2\left(2q^2 - 1\right)z^2 + 2q\left(4q^2 - 3\right)z^3 + 2\left(8q^4 - 8q^2 + 1\right)z^4 + \cdots, \quad z \in \mathbb{D}.$$

We know $f \notin \mathcal{P}$ because $c_1 = 2q > 2$. Hence, the extremal function is not correct, since it is not a Carathéodory function. Indeed, the extremal function $\widehat{f}$ for $\sigma > \frac{4}{3}$ can be defined by taking

$$\hat{f}(z) = 1 + \hat{p}_1 z + \hat{p}_2 z^2 + \hat{p}_3 z^3 + \cdots, \quad z \in \mathbb{D}, \tag{19}$$

where $\hat{p}_1 = \sqrt{\frac{\sigma}{\sigma-1}}$, $\hat{p}_2 = -\frac{\sigma-2}{\sigma-1}$ and $\hat{p}_3 = -\frac{2\sigma-3}{\sigma}\sqrt{\left(\frac{\sigma}{\sigma-1}\right)^3}$.

3. Coefficient Bounds for the Family $\mathcal{R}_\wp$

We begin this section by finding the bounds on some initial coefficients for functions in the class $\mathcal{R}_\wp$.

Theorem 1. *If $f \in \mathcal{R}_\wp$ has the series representation of the form (1), then*

$$|a_2| \leq \frac{1}{2},$$
$$|a_3| \leq \frac{1}{3},$$
$$|a_4| \leq \frac{\sqrt{14}}{14} \approx 0.2673.$$

These bounds are best possible.

Proof. Let $f \in \mathcal{R}_\wp$. Then (5) can be written by Schwarz function as

$$f'(z) = 1 + \omega(z)e^{\omega(z)}, \quad z \in \mathbb{D}. \tag{20}$$

Assuming that

$$\omega(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots, \quad z \in \mathbb{D}, \tag{21}$$

and

$$p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots, \quad z \in \mathbb{D}. \tag{22}$$

It is seen that $p \in \mathcal{P}$ and

$$\omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots}, \quad z \in \mathbb{D}. \tag{23}$$

From (1), we obtain

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 + \cdots. \tag{24}$$

By simplifications and using the series expansion of (23), we obtain

$$\begin{aligned} 1 + \omega(z)e^{\omega(z)} = 1 &+ \frac{1}{2}c_1z + \frac{1}{2}c_2z^2 + \left(-\frac{1}{16}c_1^3 + \frac{1}{2}c_3\right)z^3 \\ &+ \left(\frac{1}{24}c_1^4 - \frac{3}{16}c_1^2c_2 + \frac{1}{2}c_4\right)z^4 + \cdots. \end{aligned} \tag{25}$$

Comparing (24) and (25), we have

$$a_2 = \frac{1}{4}c_1, \tag{26}$$

$$a_3 = \frac{1}{6}c_2, \tag{27}$$

$$a_4 = \frac{1}{8}\left(-\frac{1}{8}c_1^3 + c_3\right), \tag{28}$$

$$a_5 = \frac{1}{10}\left(\frac{1}{12}c_1^4 - \frac{3}{8}c_1^2c_2 + c_4\right). \tag{29}$$

For a_2 and a_3, implementing Lemma 2, we obtain $|a_2| \leq \frac{1}{2}$ and $|a_3| \leq \frac{1}{3}$. For a_4, an application of Lemma 5 leads us to $|a_4| \leq \frac{1}{4}\sqrt{\frac{8}{7}} = \frac{\sqrt{14}}{14}$. The equality of $|a_2|$ and $|a_3|$ are achieved by the functions f_1 and f_2 given, respectively, by

$$f_1(z) = \int_0^z (1 + te^t)dt = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{8}z^4 + \frac{1}{30}z^5 + \cdots, \quad z \in \mathbb{D}, \tag{30}$$

$$f_2(z) = \int_0^z \left(1 + t^2e^{t^2}\right)dt = z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{14}z^7 + \frac{1}{54}z^9 + \cdots, \quad z \in \mathbb{D}. \tag{31}$$

The equality on the bounds of $|a_4|$ is obtained by f_3 defined by

$$f_3(z) = \int_0^z \left(1 + \omega(t)e^{\omega(t)}\right)dt, \quad z \in \mathbb{D}, \tag{32}$$

where $\omega(z) = \frac{p(z)-1}{p(z)+1}$ and

$$p(z) = 1 + \frac{\sqrt{56}}{7}z - \frac{6}{7}z^2 - \frac{26\sqrt{14}}{49}z^3 + \cdots, \quad z \in \mathbb{D}. \tag{33}$$

It is verified that

$$f_3(z) = z + \frac{\sqrt{14}}{14}z^2 - \frac{1}{7}z^3 - \frac{\sqrt{14}}{14}z^4 + \cdots, \quad z \in \mathbb{D}. \tag{34}$$

The proof of Theorem 1 is thus completed. □

Theorem 2. *If f is of the form (1) belonging to $\mathcal{R}_\wp$, then*

$$\left|a_3 - \gamma a_2^2\right| \leq \max\left\{\frac{1}{3}, \left|\frac{3\gamma - 4}{12}\right|\right\}, \quad \gamma \in \mathbb{C}.$$

This inequality is sharp.

Proof. Employing (26) and (27), we may write

$$\left|a_3-\gamma a_2^2\right|=\frac{1}{6}\left|c_2-\frac{3}{8}\gamma c_1^2\right|.$$

An application of (12) leads us to

$$\left|a_3-\gamma a_2^2\right|\leq \max\left\{\frac{1}{3},\left|\frac{3\gamma-4}{12}\right|\right\}.$$

This result is sharp for the functions f_1 and f_2 given by (30) and (31). □

Theorem 3. *Let $f\in\mathcal{R}_\wp$. Then*

$$|a_2a_3-a_4|\leq\frac{1}{4}.$$

This inequality is sharp with the extremal function f_4 given by

$$f_4(z)=\int_0^z\left(1+t^3e^{t^3}\right)dt=z+\frac{1}{4}z^4+\frac{1}{7}z^7+\frac{1}{20}z^{10}+\frac{1}{78}z^{13}+\cdots,\quad z\in\mathbb{D}. \tag{35}$$

Proof. Using (26)–(28), we have

$$|a_2a_3-a_4|=\frac{1}{8}\left|c_3-\frac{1}{3}c_1c_2-\frac{1}{8}c_1^3\right|. \tag{36}$$

From (21) and (22), it is noted that

$$c_1=2w_1, \tag{37}$$
$$c_2=2\left(w_2+w_1^2\right), \tag{38}$$
$$c_3=2\left(w_3+2w_1w_2+w_1^3\right). \tag{39}$$

Hence, we obtain

$$|a_2a_3-a_4|=\frac{1}{4}\left|w_3+\frac{4}{3}w_1w_2-\frac{1}{6}w_1^3\right|. \tag{40}$$

Taking $\mu=\frac{4}{3}$ and $\nu=-\frac{1}{6}$, we know $(\mu,\nu)\in\mathbb{D}_2$. Using Lemma 4, we easily obtain

$$|a_2a_3-a_4|\leq\frac{1}{4}.$$

Clearly, the bound is sharp with the extremal function given by (35). □

Theorem 4. *If $f\in\mathcal{R}_\wp$, then*

$$|H_{2,2}(f)|=\left|a_2a_4-a_3^2\right|\leq\frac{1}{9}.$$

The inequality is sharp with the extremal function given by (31).

Proof. From (26)–(28), we have

$$H_{2,2}(f)=-\frac{1}{256}c_1^4+\frac{1}{32}c_1c_3-\frac{1}{36}c_2^2.$$

Let $f\in\mathcal{R}_\wp$ and $f_\theta(z)=e^{-i\theta}f(e^{i\theta}z)$, $\theta\in\mathbb{R}$. We have $|H_{2,2}(f_\theta)|=|H_{2,2}(f)|$ for all $\theta\in\mathbb{R}$. Hence, when estimating the upper bounds of $|H_{2,2}(f)|$, we may assume a_2 of f to be real, and thus $c_1 := c \in [0,2]$. Using (8) and (9) to express c_2 and c_3 in terms of $c_1 = c$, we obtain

$$\begin{aligned}|H_{2,2}(f)| = \bigg| &-\frac{7}{2304}c^4 + \frac{1}{576}c^2\left(4-c^2\right)x - \frac{1}{1152}\left(4-c^2\right)\left(c^2+32\right)x^2 \\ &+\frac{1}{64}c\left(4-c^2\right)\left(1-|x|^2\right)\delta\bigg|.\end{aligned}$$

With the aid of the triangle inequality, replacing $|\delta| \le 1,\ |x| = t \le 1$ and taking $c \in [0,2]$, we obtain

$$\begin{aligned}|H_{2,2}(f)| \le &\frac{7}{2304}c^4 + \frac{1}{576}c^2\left(4-c^2\right)t + \frac{1}{1152}\left(4-c^2\right)\left(c^2+32\right)t^2 \\ &+ \frac{1}{64}c\left(4-c^2\right)\left(1-t^2\right) =: K(c,t).\end{aligned}$$

It is noted that

$$\frac{\partial K}{\partial t} = \frac{1}{576}c^2(4-c^2) + \frac{1}{576}\left(4-c^2\right)\left(c^2-18c+32\right)t \ge 0$$

for $t \in [0,1]$, thus $K(c,t) \le K(c,1)$. Putting $t = 1$ gives

$$|H_{2,2}(f)| \le \frac{7}{2304}c^4 + \frac{1}{576}c^2\left(4-c^2\right) + \frac{1}{1152}\left(4-c^2\right)\left(c^2+32\right) =: \chi(c).$$

Since $\chi(c) = \frac{1}{2304}\left(c^4 - 40c^2 + 256\right)$ and $\chi'(c) \le 0$ on $[0,2]$, we know χ is decreasing for $c \in [0,2]$ and

$$|H_{2,2}(f)| \le \chi(0) = \frac{1}{9}.$$

The equality is obtained by the extremal function defined by (31). This completes the proof of Theorem 4. □

Theorem 5. *If $f \in \mathcal{R}_\wp$ has the form (1), then*

$$|H_{3,1}(f)| \le \frac{1}{16}.$$

This inequality is sharp with the extremal function f_4 given by (35).

Proof. From the definition, $H_{3,1}(f)$ can be written as

$$H_{3,1}(f) = 2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5. \tag{41}$$

Let $c_1 = c$. By putting (26)–(29) into (41), we obtain

$$\begin{aligned}H_{3,1}(f) = \frac{1}{552{,}960}\Big(&-423c^6 + 1344c^4c_2 + 2160c^3c_3 - 3456c^2c_2^2 - 3456c^2c_4 \\ &+5760cc_2c_3 - 2560c_2^3 + 9216c_2c_4 - 8640c_3^2\Big).\end{aligned} \tag{42}$$

Let $f \in \mathcal{R}_\wp$ and $f_\theta = e^{-i\theta}f\left(e^{i\theta}z\right),\ \theta \in \mathbb{R}$. Note that $|H_{3,1}(f_\theta)| = |H_{3,1}(f)|$ for all $\theta \in \mathbb{R}$, we may also assume that $c \in [0,2]$. Suppose that $b = 4 - c^2$. Using (8)–(10), we obtain

$$
\begin{aligned}
H_{3,1}(f) = \frac{1}{552{,}960}\Big\{&-71c^6 + 2304b^2x^3 - 320b^3x^3 + 576c^2bx^2 + 144c^4bx^3 - 612c^4bx^2 \\
&+72c^4bx + 36c^2b^2x^4 - 288c^2b^2x^3 - 816c^2b^2x^2 - 2160b^2\Big(1-|x|^2\Big)^2\delta^2 \\
&+576c^2b\Big(1-|x|^2\Big)\Big(1-|\delta|^2\Big)\rho - 576c^3bx\Big(1-|x|^2\Big)\delta - 576c^2b\overline{x}\Big(1-|x|^2\Big)\delta^2 \\
&+936c^3b\Big(1-|x|^2\Big)\delta - 144cb^2x^2\Big(1-|x|^2\Big)\delta - 2304b^2|x|^2\Big(1-|x|^2\Big)\delta^2 \\
&-576cb^2x\Big(1-|x|^2\Big)\delta + 2304b^2x\Big(1-|x|^2\Big)\Big(1-|\delta|^2\Big)\rho\Big\},
\end{aligned}
$$

where $\rho, x, \delta \in \overline{\mathbb{D}} := \{z : |z| \le 1\}$. Observing that $H_{3,1}(f)$ can be written as

$$
H_{3,1}(f) = \frac{1}{552{,}960}\Big[d_1(c,x) + d_2(c,x)\delta + d_3(c,x)\delta^2 + \Phi(c,x,\delta)\rho\Big],
$$

with

$$
\begin{aligned}
d_1(c,x) &= -71c^6 + \Big(4-c^2\Big)\Big[\Big(4-c^2\Big)\Big(1024x^3 + 32c^2x^3 + 36c^2x^4 - 816c^2x^2\Big) \\
&\quad +576c^2x^2 - 612c^4x^2 + 144c^4x^3 + 72c^4x\Big], \\
d_2(c,x) &= 72\Big(4-c^2\Big)\Big(1-|x|^2\Big)\Big[\Big(4-c^2\Big)\Big(-2cx^2\Big) - 32cx + 13c^3\Big], \\
d_3(c,x) &= 144\Big(4-c^2\Big)\Big(1-|x|^2\Big)\Big[\Big(4-c^2\Big)\Big(-|x|^2 - 15\Big) - 4c^2\overline{x}\Big], \\
\Phi(c,x,\delta) &= 576\Big(4-c^2\Big)\Big(1-|x|^2\Big)\Big(1-|\delta|^2\Big)\Big[c^2 + 4x\Big(4-c^2\Big)\Big].
\end{aligned}
$$

Taking $|x| = t,\ |\delta| = y$ and utilizing the fact $|\rho| \le 1$, we obtain

$$
\begin{aligned}
|H_{3,1}(f)| &\le \frac{1}{552{,}960}\Big[|d_1(c,x)| + |d_2(c,x)|y + |d_3(c,x)|y^2 + |\Phi(c,x,\delta)|\Big]. \\
&\le \frac{1}{552{,}960}[\Gamma(c,t,y)],
\end{aligned} \tag{43}
$$

where

$$
\Gamma(c,t,y) = h_1(c,t) + h_2(c,t)y + h_3(c,t)y^2 + h_4(c,t)\Big(1-y^2\Big),
$$

with

$$
\begin{aligned}
h_1(c,t) &= 71c^6 + \Big(4-c^2\Big)\Big[\Big(4-c^2\Big)\Big(1024t^3 + 32c^2t^3 + 36c^2t^4 + 816c^2t^2\Big) \\
&\quad +576c^2t^2 + 144c^4t^3 + 612c^4t^2 + 72c^4t\Big], \\
h_2(c,t) &= 72\Big(4-c^2\Big)\Big(1-t^2\Big)\Big[\Big(4-c^2\Big)\Big(2ct^2\Big) + 13c^3 + 32ct\Big], \\
h_3(c,t) &= 144\Big(4-c^2\Big)\Big(1-t^2\Big)\Big[\Big(4-c^2\Big)\Big(t^2+15\Big) + 4c^2t\Big], \\
h_4(c,t) &= 576\Big(4-c^2\Big)\Big(1-t^2\Big)\Big[c^2 + 4t\Big(4-c^2\Big)\Big].
\end{aligned}
$$

Now, we have to maximize Γ in the closed cuboid $\Theta := [0,2] \times [0,1] \times [0,1]$. It is not hard to see that $\Gamma(0,0,1) = 34{,}560$. Thus, we have $\max_{(c,t,y)\in\Theta}\{\Gamma(c,t,y)\} \ge 34{,}560$. We aim to prove that the maximum values of Γ with $(c,t,y) \in \Theta$ is simply equal to 34,560. For this, we first show that the maximum value of Γ is obtained on the face $y = 1$ of Θ.

On the face $t = 1$, it reduces to $\Gamma(c,1,y) = r_1(c) = 127c^6 - 3312c^4 + 8256c^2 + 16{,}384$. Then,

$$
\frac{\partial r_1}{\partial c} = 6c\Big(127c^4 - 2208c^2 + 2752c\Big).
$$

Putting $\frac{\partial r_1}{\partial c} = 0$, we obtain the only critical point $\hat{c}_0 = \sqrt{\frac{1104-136\sqrt{47}}{127}} \approx 1.1625$ for $c \in (0,2)$. Therefore, $\max r_1(c) \approx 21{,}805.95$ with the maximum value attained on $c = \hat{c}_0$. Thus, we assume that $t < 1$. Furthermore, for the points on the face $c = 2$, $\Gamma(2,t,y) \equiv 4544$ for all $(t,y) \in [0,1] \times [0,1]$. Hence, we further assume that $c < 2$.

Let $(c,t,y) \in [0,2) \times [0,1) \times [0,1]$. By differentiating Γ partially with respect to y, we obtain

$$\frac{\partial \Gamma}{\partial y} = h_2(c,t) + 2[h_3(c,t) - h_4(c,t)]y.$$

Obviously, we have

$$\left.\frac{\partial H}{\partial y}\right|_{y=0} = h_2(c,t) \geq 0.$$

Let

$$\left.\frac{\partial H}{\partial y}\right|_{y=1} = h_2(c,t) + 2[h_3(c,t) - h_4(c,t)] =: \zeta_1(c,t). \tag{44}$$

It is noted that

$$\zeta_1(c,t) = 72(4-c^2)(1-t^2)\zeta_2(c,t),$$

where

$$\zeta_2(c,t) = (4-c^2)(2ct^2 + 4t^2 + 60 - 64t) + 13c^3 + 32ct + 16c^2t - 16c^2.$$

Clearly, we have

$$\zeta_2(c,t) \geq (4-c^2)(4t^2 + 60 - 64t) + 13c^3 + 16c^2t - 16c^2 =: \eta(c,t).$$

Suppose that $\eta(c,t) = \eta_0 + \eta_1 t + \eta_2 t^2$, where $\eta_0 = 240 - 76c^2 + 13c^3$, $\eta_1 = 80c^2 - 256$ and $\eta_2 = 16 - 4c^2$. Taking η as a polynomial of degree 2 with respect to t, we know $\eta_2 > 0$ and the symmetric axis t_0 is defined as

$$t_0 = -\frac{\eta_1}{2\eta_2} = \frac{2(16-5c^2)}{4-c^2}.$$

Let $\tilde{c}_0 = \frac{4}{\sqrt{5}}$. For $c \in [\tilde{c}_0, 2)$, it is observed that $t_0 \leq 0$. Then, the minimum value of η is achieved on $t = 0$. We thus have

$$\eta(c,t) \geq \eta(c,0) = \eta_0 \geq 40 > 0, \quad c \in [\tilde{c}_0, 2). \tag{45}$$

Let $\bar{c}_0 = \frac{2\sqrt{7}}{3}$. It is seen that $t_0 \geq 1$ for $c \in [0, \bar{c}_0]$. It follows that

$$\eta(c,t) \geq \eta(c,1) = \eta_0 + \eta_1 + \eta_2 = 13c^3 \geq 0, \quad c \in [0, \bar{c}_0]. \tag{46}$$

Assume that $c \in (\bar{c}_0, \tilde{c}_0)$. Then $t_0 \in (0,1)$. Hence, the minimum value of η is obtained on $t = t_0$. This leads to

$$\eta(c,t) \geq \eta(c,t_0) = \eta_0 - \frac{\eta_1^2}{4\eta_2} = \frac{\iota(c)}{4-c^2},$$

where

$$\iota(c) = -13c^5 - 324c^4 + 52c^3 + 2016c^2 - 3136, \quad c \in (\bar{c}_0, \tilde{c}_0).$$

It is calculated that ι achieves its minimum value of about 56.9731 on $c = \tilde{c}_0$, thus we know

$$\eta(c,t) > 0, \quad c \in (\bar{c}_0, \tilde{c}_0). \tag{47}$$

Combining (45)–(47), we have $\eta(c,t) \geq 0$ on $[0,2) \times [0,1)$, which leads to $\zeta_1(c,t) \geq 0$ for all $(c,t) \in [0,2) \times [0,1)$. Therefore, we have $\frac{\partial \Gamma}{\partial y}\Big|_{y=1} \geq 0$. As $\frac{\partial \Gamma}{\partial y}$ is a linear continuous function with respect to y, we have

$$\frac{\partial \Gamma}{\partial y} \geq \min\left\{ \frac{\partial \Gamma}{\partial y}\Big|_{y=0}, \frac{\partial \Gamma}{\partial y}\Big|_{y=1} \right\} \geq 0, \quad y \in [0,1].$$

Hence, $\Gamma(c,t,y) \leq \Gamma(c,t,1)$ for all $(c,t,y) \in [0,2) \times [0,1) \times [0,1]$. Based on the above discussions, it reduces to find the global maximum value of Γ on the face $y = 1$ of Θ. On the face $y = 1$, we have

$$\begin{aligned}\Gamma(c,t,1) = 71c^6 &+ \left(4-c^2\right)^2\left[36(c^2-4c-4)t^4 + 32(c^2+32)t^3 + 48(17c^2+3c-42)t^2 + 2160\right] \\ &+ (4-c^2)\left[144c(c^3-4c-16)t^3 + 36c^2(17c^2-26c+16)t^2 + 72c(c^3+8c+32)t + 936c^3\right] \\ &=: \Lambda(c,t).\end{aligned}$$

By observing that $c^2 - 4c - 4 \leq 0$ and $c^3 - 4c - 16 \leq 0$ for $c \in [0,2)$, we have

$$\begin{aligned}\Lambda(c,t) \leq 71c^6 &+ \left(4-c^2\right)^2\left[32(c^2+32)t^3 + 48(17c^2+3c-42)t^2 + 2160\right] \\ &+ (4-c^2)\left[36c^2(17c^2-26c+16)t^2 + 72c(c^3+8c+32)t + 936c^3\right] \\ &=: Q(c,t).\end{aligned}$$

Furthermore, using $17c^2 - 26c + 16 \geq 0$, $t^3 \leq t^2 \leq t$ leads to

$$\begin{aligned}Q(c,t) \leq 71c^6 &+ \left(4-c^2\right)^2\left[32(c^2+32)t^2 + 48(17c^2+3c-42)t^2 + 2160\right] \\ &+ (4-c^2)\left[36c^2(17c^2-26c+16)t + 72c(c^3+8c+32)t + 936c^3\right] \\ &= 4(4-c^2)R(c,t) + 71c^6 + 2160(4-c^2)^2 + 936(4-c^2)c^3 \\ &=: W(c,t),\end{aligned}$$

where

$$R(c,t) = 4(4-c^2)(53c^2+9c-62)t^2 + 9c(19c^3-26c^2+32c+64)t.$$

Clearly, if $c \geq 1$, we have $53c^2 + 9c - 62 \geq 0$ and $19c^3 - 26c^2 + 32c + 64 \geq 0$, which leads to

$$R(c,t) \leq R(c,1), \quad c \in [1,2).$$

Then, we obtain

$$W(c,t) \leq 4(4-c^2)R(c,1) + 71c^6 + 2160(4-c^2)^2 + 936(4-c^2)c^3 =: \varrho_1(c), \quad c \in [1,2).$$

In virtue of $\varrho_1(c) = 235c^6 + 144c^5 - 4032c^4 - 3456c^3 + 8832c^2 + 11{,}520c + 18{,}688$ obtaining its maximum value of about 32,192.46 on $c \approx 1.1053$ for $c \in [1,2)$, we have $\Lambda(c,t) < 34{,}560$ on $[1,2) \times [0,1)$. Suppose that $c \in [0,1)$ and $m(c) = 19c^3 - 26c^2 + 32c + 64$. It is noted that $m'(c) = 54c^2 - 52c + 32 \geq 0$ for $c \in [0,1)$. Thus, we have $m(c) \in [64,89)$. Since $0 < 4 - c^2 \leq 4$ and $c^2 \leq c$, it is not hard to see that

$$R(c,t) \leq 992(c-1)t^2 + 801ct =: V(c,t).$$

Let $V(c,t) = v_1 t + v_2 t^2$, where $v_1 = 801c$ and $v_2 = 992(c-1)$. Obviously, we have $v_2 < 0$. Considering V as a polynomial of degree 2 with respect to t, we obtain the symmetric axis $\bar{t}_0$ defined by

$$\bar{t}_0 = -\frac{v_1}{2v_2} = \frac{801c}{1984(1-c)}. \tag{48}$$

For $c > \dot{c}_0 = \frac{1984}{2785} \approx 0.7124$, we have $\bar{t}_0 > 1$. Then, the maximum value of V is attained on $t = 1$, which implies that $V(c,t) \leq V(c,1) = 1793c - 992$. Then,

$$W(c,t) \leq 4(4-c^2)V(c,1) + 71c^6 + 2160(4-c^2)^2 + 936(4-c^2)c^3 =: \varrho_2(c), \quad c \in [\dot{c}_0, 1).$$

It is calculated that

$$\varrho_2(c) = 71c^6 - 936c^5 + 2160c^4 - 3428c^3 - 13{,}312c^2 + 28{,}688c + 18{,}688, \quad c \in [\dot{c}_0, 1),$$

which obtains its maximum value of about 32,127.89 on $c \approx 0.8966$. Hence, we obtain

$$\Gamma(c,t) < 34560, \quad (c,t) \in [\dot{c}_0, 1) \times [0,1).$$

For $c \in [0, \dot{c}_0)$, we have $t_0 \in [0,1)$. Then, we obtain

$$V(c,t) \leq -\frac{v_1^2}{4v_2} = \frac{801^2}{3968} \cdot \frac{c^2}{1-c} \leq \frac{162c^2}{1-c} \leq 162c^2,$$

which yields to

$$W(c,t) \leq 648(4-c^2)c^2 + 71c^6 + 2160(4-c^2)^2 + 936(4-c^2)c^3 =: \varrho_3(c), \quad c \in [0, \dot{c}_0).$$

In light of

$$\varrho_3(c) = 71c^6 - 936c^5 + 1512c^4 + 3744c^3 - 14{,}688c^2 + 34{,}560, \quad c \in [0, \dot{c}_0),$$

it is not hard to see that ϱ_3 achieves its maximum value 34,560 on $c = 0$. Therefore, we conclude that

$$\Lambda(c,t) \leq 34{,}560, \quad (c,t) \in [0,2) \times [0,1).$$

From the above cases, we obtain

$$\Gamma(c,t,y) \leq 34{,}560 \quad \text{on } [0,2] \times [0,1] \times [0,1].$$

Using (43), it follows that

$$|H_{3,1}(f)| \leq \frac{1}{552{,}960}[\Gamma(c,t,y)] \leq \frac{1}{16} = 0.0625.$$

The proof of Theorem 5 is thus completed. □

Author Contributions: The idea was proposed by L.S. and improved by H.M.S., L.S., N.E.C. and M.A. wrote and completed the calculations. H.M.S. and M.A. checked all the results. All authors read and approved the final manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their gratitude for the referees' valuable suggestions which truly improved the present work.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Duren, P. *Univalent Functions*; Grundlehren der Mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo; Springer: Berlin/Heidelberg, Germany, 1983.
2. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; pp. 157–169.
3. Cho, N.E.; Kwon, O.S.; Lecko, A.; Sim, Y.J. Sharp estimates of generalized zalcman functional of early coefficients for Ma-Minda type functions. *Filomat* **2018**, *32*, 6267–6280. [CrossRef]
4. Sharma, P.; Raina, R.K.; Sokół, J. Certain Ma-Minda type classes of analytic functions associated with the crescent-shaped region. *Anal. Math. Phys.* **2019**, *9*, 1887–1903. [CrossRef]
5. Madaan, V.; Kumar, A.; Ravichandran V. Starlikeness associated with lemniscate of Bernoulli. *Filomat* **2019**, *33*, 1937–1955. [CrossRef]
6. Ali, R.M.; Ravichandran, V. Uniformly convex and uniformly starlike functions. *Ramanujan Math. Newsl.* **2011**, *355*, 16–30.
7. Sharma, K.; Jain, N.K.; Ravichandran, V. Starlike functions associated with a cardioid. *Afr. Mat.* **2016**, *27*, 923–939. [CrossRef]
8. Wani, L.A.; Swaminathan, A. Starlike and convex functions associated with a nephroid domain. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 79–104. [CrossRef]
9. Mendiratta, R.; Nagpal, S.; Ravichandran, V. On a subclass of strongly starlike functions associated with exponential function. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 365–386. [CrossRef]
10. Goel, P.; Sivaprasad Kumar, S. Certain class of starlike functions associated with modified sigmoid function. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 957–991. [CrossRef]
11. Gandhi, S. Radius estimates for three leaf function and convex combination of starlike functions. In *Mathematical Analysis 1: Approximation Theory ICRAPAM*; Deo, N., Gupta, V., Acu, A., Agrawal, P., Eds.; Springer: Singapore, 2018; Volume 306.
12. Sivaprasad Kumar, S.; Kamaljeet, G. A cardioid domain and starlike functions. *Anal. Math. Phys.* **2021** *11*, 54. [CrossRef]
13. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *J. Lond. Math. Soc.* **1966**, *1*, 111–122. [CrossRef]
14. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* **1967**, *14*, 108–112. [CrossRef]
15. Mateljević, M. Schwarz lemma and Kobayashi metrics for harmonic and holomorphic functions. *J. Math. Anal. Appl.* **2018**, *464*, 78–100. [CrossRef]
16. Mateljević, M.; Mutavdžic, N.; Örnek, B.N. Note on some classes of holomorphic functions related to Jack's and Schwarz's lemma. *Appl. Anal. Discret. Math.* **2022**, *16*, 111–131. [CrossRef]
17. Aydinoğlua, S.; Örnek, B.N. Estimates concerned with Hankel determinant for (α) class. *Filomat* **2022**, *36*, 3679–3688. [CrossRef]
18. Janteng, A.; Halim, S.A.; Darus, M. Coefficient inequality for a function whose derivative has a positive real part. *J. Inequalities Pure Appl. Math.* **2006**, *7*, 1–5.
19. Janteng, A.; Halim, S.A.; Darus, M. Hankel determinant for starlike and convex functions. *Int. J. Math.* **2007**, *1*, 619–625.
20. Cho, N.E.; Kowalczyk, B.; Kwon, O.S.; Lecko, A.; Sim, Y.J. The bounds of some determinants for starlike functions of order alpha. *Bull. Malays. Math. Sci. Soc.* **2018**, *41*, 523–535. [CrossRef]
21. Shi, L.; Srivastava, H.M.; Arif, M.; Hussain, S.; Khan, H. An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function. *Symmetry* **2019**, *11*, 598. [CrossRef]
22. Srivastava, H.M.; Khan, B.; Khan, N.; Tahir, M.; Ahmad, S.; Khan, N. Upper bound of the third Hankel determinant for a subclass of q-starlike functions associated with the q-exponential function. *Bull. Sci. Math.* **2021**, *167*, 102942. [CrossRef]
23. Babalola, K.O. On $H_3(1)$ Hankel determinant for some classes of univalent functions. *Inequal. Theory Appl.* **2010**, *6*, 1–7.
24. Kwon, O.S.; Lecko, A.; Sim, Y.J. The bound of the Hankel determinant of the third kind for starlike functions. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 767–780. [CrossRef]
25. Altınkaya, Ş.; Yalçın, S. Third Hankel determinant for Bazilevič functions. *Adv. Math.* **2016**, *5* , 91–96.
26. Zaprawa, P.; Obradović, M.; Tuneski, N. Third Hankel determinant for univalent starlike functions. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. *Ser. A Mat.* **2021**, *115*, 1–6.
27. Kowalczyk, B.; Lecko, A.; Sim, Y.J. The sharp bound of the Hankel determinant of the third kind for convex functions. *Bull. Aust. Math. Soc.* **2018**, *97*, 435–445. [CrossRef]
28. Kowalczyk, B.; Lecko, A.; Thomas, D.K. The sharp bound of the third Hankel determinant for starlike functions. *Forum Math.* **2022**, *34*, 1249–1254. [CrossRef]
29. Kowalczyk, B.; Lecko, A. The sharp bound of the third Hankel determinant for functions of bounded turning. *Bol. Soc. Mat. Mex.* **2021**, *27*, 69. [CrossRef]
30. Lecko, A.; Sim, Y.J. Śmiarowska B. The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2. *Complex Anal. Oper. Theory* **2019**, *13*, 2231–2238. [CrossRef]
31. Riaz, A.; Raza, M.; Thomas, D.K. Hankel determinants for starlike and convex functions associated with sigmoid functions. *Forum Math.* **2022**, *34*, 137–156. [CrossRef]
32. Shi, L.; Arif, M.; Ullah, K.; Alreshidi, N.; Shutaywi, M. On sharp estimate of third Hankel determinant for a subclass of starlike functions. *Fractal Fract.* **2022**, *6*, 437. [CrossRef]

33. Wang, Z.G.; Raza, M.; Arif, M.; Ahmad, K. On the third and fourth Hankel determinants for a subclass of analytic functions. *Bulletin of the Malaysian Mathematical Sciences Society* **2022**, *45*, 323 –359. [CrossRef]
34. Kumar, V.; Cho, N.E.; Ravichandran, V.; Srivastava, H.M. A sharp coefficient bounds for starlike functions associated with the Bell numbers. *Math. Slovaca* **2019**, *69*, 1053–1064. [CrossRef]
35. Kwon, O.S.; Lecko, A.; Sim, Y.J. On the fourth coefficient of functions in the Carathéodory class. *Comput. Methods Funct. Theory* **2018**, *18*, 307–314. [CrossRef]
36. Pommerenke, C. *Univalent Function*; Vanderhoeck & Ruprecht: G öttingen, Germany, 1975.
37. Libera, R.J.; Złotkiewicz, E.J. Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, II. *Proc. Am. Math. Soc.* 1984, *92*, 58–60.
38. Prokhorov, D.V.; Szynal, J. Inverse coefficients for (α, β)-convex functions. *Ann. Univ. Mariae Curie-Sklodowska (Sect. A)* **1981**, *35*, 125–143.