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Weighted Pseudo- θ -Almost Periodic Sequence and Finite-Time Guaranteed Cost Control for Discrete-Space and Discrete-Time Stochastic Genetic Regulatory Networks with Time Delays

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Abstract: This paper considers the dual hybrid effects of discrete-time stochastic genetic regulatory networks and discrete-space stochastic genetic regulatory networks in difference formats of exponential Euler difference and second-order central finite difference. The existence of a unique-weight pseudo- θ -almost periodic sequence solution for discrete-time and discrete-space stochastic genetic regulatory networks on the basis of discrete constant variation formulation is discussed, as well as the theory of semi-flow and metric dynamical systems. Furthermore, a finite-time guaranteed cost controller is constructed to reach global exponential stability of these discrete networks via establishing a framework of drive, response, and error networks. The results indicate that spatial diffusions of non-negative dense coefficients have no influence on the global existence of the unique weighted pseudo- θ -almost periodic sequence solution of the networks. The present study is a basic work in the consideration of discrete spatial diffusion in stochastic genetic regulatory networks and serves as a foundation for further study.

Keywords: discrete spatial diffusion; discrete time; stochastic; weighted pseudo- θ -almost periodicity; finite-time guaranteed cost controller; finite difference method

MSC: 35B15



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1. Introduction

Genetic regulatory networks (GRNs) have been widely recognized due to their possible usages [1]. GRNs are actually a complex dynamical system that describes the regulatory mechanisms of DNA, mRNA, and protein interactions in biological systems at the molecular level [2,3]. The analysis of genetic regulatory networks is not only an important way to understand and grasp the operation mechanisms of the activity of cellular life [4], but also has promising applications in the fields of disease genetic prediction and drug target screening [5–8]. For this reason, it is necessary and valuable to propose suitable mathematical models to represent expression mechanisms and signal transduction pathways. Currently, GRNs are generally modeled by Boolean models, Bayesian models, and differential equation models. Two of the most widely used models are Boolean models and differential equation models [9–11]. In particular, differential equations describe the concentration changes in proteins and mRNA [12,13]. This model has received more attention because of its higher accuracy and its ability to accurately describe the nonlinear dynamic behaviour of biological systems.

In general, the majority of models utilised to characterize GRNs in the currently available literature suppose that the concentrations of mRNAs and proteins are spatially

homogeneous at all times. However, this assumption has some limitations, for example the diffusion phenomenon should be considered for the case of non-uniform distributions of gene product concentrations [14]. Therefore, the issue of kinetic analysis of GRNs with reaction–diffusion effects is worth investigating. Moreover, time delay is often inevitable due to the finite processing time of interactions among agents and may influence system performance. As of recently, a great deal of results on GRNs with time delays can be found in the literature, see, e.g., Han et al. [14], who established an asymptotic stability criterion for reaction–diffusion delayed GRNs under Dirichlet and Neumann boundary conditions, respectively, insightfully recognizing that diffusion–reaction information can reduce the conservation of the system. Robust state estimation of delayed genetic regulatory networks with reaction–diffusion terms and uncertainty terms under the Dirichlet boundary condition is considered by Zou et al. in [15]. Xie et al. [16] discuss the stability of genetic regulatory networks, centralised spatial diffusion, and discrete and infinite distribution delays.

During the processes of both computational simulation and analysis, engineers often use discrete-time continuous models to evaluate their structural behaviour. The signals received and operated in digital networks are dependent on discrete-time rather than continuous-time. Therefore, discrete-time GRNs have been studied by many authors. For example, Xue et al. [17] investigate the problems of state boundary description and reachable set estimation for discrete-time delayed genetic regulatory networks with bounded perturbations. Liu et al. [18] study the problem of exponential stability analysis of discrete genetic regulatory networks with time-varying discrete-time delays and unbounded distributed time delays. Yue et al. [19] investigate the dynamics of discrete-time genetic models and obtain conditions for the existence and stability of fixed points. It is shown that the discrete-time genetic network undergoes fold bifurcation, flip bifurcation, and Neimark–Sacker bifurcation, illuminating the richer dynamical properties of the discrete-time genetic model than the original continuous-time model. It is worth noting that most of the results on GRNs only concern discrete-time GRNs [17–22], while the results on spatial discrete GRNs have not received sufficient attention in existing studies, probably owing to the partial ineffectiveness of traditional methods in space–time continuous networks, such as the Lyapunov–Krasovskii general functions in discrete-space and -time networks, and the difficulty of computing the difference. To date, there are several reports referring to space–time discrete models [23–25]; nevertheless, the models of stochastic space–time discrete GRNs have not been deeply addressed.

It is well known that stochastic uncertainty is inevitable in various dynamical systems, with reference to its ability to alter the mechanical properties of genetic regulatory networks in practical applications. Therefore, the dynamic behaviour of delayed stochastic genetic regulatory networks has been extensively studied, see the literature [26–29]. For example, Xu et al. [26] investigate the input state stability problem of stochastic gene regulation networks with multiple time delays, and give sufficient conditions for the mean square exponential input state stability of the system using the Lyapunov generalization, Ito’s formula, and Dynkin’s formula. Wang [29] investigates the dual effects of discrete space and discrete time in stochastic genetic regulatory networks by means of exponential Eulerian differences and central finite differences. In addition, finite-time guaranteed cost control is a very effective method in the engineering field due to its many advantages in practical applications, see references [30–37]. The advantages of finite-time guaranteed cost controllers are listed below: (1) Stability. A finite-time guaranteed cost controller is a feedback controller that adjusts the system to remain stable when it is subject to external disturbances or internal changes. (2) Reliability. It can adjust the control strategy adaptively depending on the state of the system, so as to increase the reliability of the system. In summary, the finite-time cost-preserving controller is an advanced control method for

genetic regulatory networks with many advantages that can assist the system to be more stable, reliable, and robust, and optimise the performance index of the system.

On the other hand, the global exponential stability and almost periodic nature of GRNs are significant and necessary dynamical behaviours that have been extensively researched by many authors in the last two decades, see the literature [18,38–42]. Particularly in stochastic models, the notion of θ -almost periodicity was first introduced in the paper [43] on the basis of semi-flow and metric dynamical system theories, and the existence of θ -almost periodicity for several continuous-time stochastic models was investigated [44,45]. First, pseudo-almost periodicity was introduced in the early 1990s by Zhang [46] as a natural extension of classical probability periodicity. Since then, pseudo-approximate periodic solutions of differential equations have attracted a lot of attention. In the literature [47], Diagana extended pseudo-almost periodicity to weighted pseudo-almost periodicity and reported a number of excellent contributions on weighted pseudo-almost periodicity, see references [48–50]. However, the study of the θ -almost periodicity of stochastic discrete-time GRNs, not to mention weighted pseudo- θ -almost periodicity, influenced by spatial diffusion, has not been addressed in depth so far.

Based on the above motivation, the main purpose of this paper is to establish discrete-time stochastic genetic regulatory networks (SGRNs) for discrete-space diffusion using exponential Euler difference and central finite difference methods. On this ground, a discrete constant variation formula for discrete SGRNs is derived. On the basis of the discrete constant variation formula, the weighted pseudo- θ -almost periodicity of discrete SGRNs with discrete spatial diffusion is investigated by combining the theory of semi-fluid dynamical systems and metric dynamical systems. In the end, a finite-time guaranteed cost controller for this type of SGRN is designed by the construction of a drive, response, and error network framework. The main studies and innovations of this paper are briefly summarised in turn as follows.

- (1) Discrete-time and discrete-space SGRNs are newly introduced, which extends the studied models in reports [18,40].
- (2) The weighted pseudo- θ -almost periodicity of this class of SGRNs is considered for the first time, which complements the works on the almost periodicity of GRNs in references [12,38].
- (3) Finite-time cost-preserving controllers are designed for this class of SGRNs.

Plan of this paper: In Section 2, a formula for discrete-time and discrete-space SGRNs is given and the concept of weighted pseudo- θ -almost periodicity is presented. Section 3 discusses the global existence of unique weighted pseudo- θ -almost periodic sequence solutions for discrete-time and discrete-space SGRNs on the basis of the theory of semifluid and metric dynamical systems, the discrete constant variation formula, and the fixed-point theorem. Furthermore, in Section 4, finite-time cost-preserving controllers are designed by constructing a framework of drive, response, and error networks for discrete-time and discrete-space SGRNs. Section 5 gives numerical examples of discrete-time and discrete-space SGRNs achieving weight pseudo- θ -almost periodicity, finite-time guaranteed cost control, and global exponential stability. The conclusions and main points of the paper are given in Section 6.

Symbols: $\mathbb{R}^n$ denotes the space of n -dimensional real vectors; $\mathbb{Z}$ is the field of integral numbers; $\mathbb{N}_0 = \{0, 1, 2, \dots\}$; $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$; $\mathbb{N}_a^b = \{a, a + 1, \dots, b\}$ for any $a, b \in \mathbb{Z}$; $I_J = I \cap J$, $\forall I, J \subseteq \mathbb{R}$. Let $\xi_1 = (1, 0, \dots, 0)^T$, $\xi_2 = (0, 1, \dots, 0)^T, \dots, \xi_m = (0, 0, \dots, 1)^T$.

Define $N_p \in \mathbb{N}$ for $p \in \mathbb{N}_1^m$, $\bar{U}_v := \{v = (v_1, \dots, v_m)^T : (v_p, p) \in (\mathbb{N}_1^{N_p-1}, \mathbb{N}_1^m)\}$, $\partial \bar{U}_v := \bar{U}_v \setminus \bar{U}_v$, $\bar{U}_v = \{v = (v_1, \dots, v_m)^T : (v_p, p) \in (\mathbb{N}_0^{N_p}, \mathbb{N}_1^m)\}$.

For any function $f : \bar{U}_v \times \mathbb{Z}$ to $\mathbb{R}^n$, we denote as $f := f_k^{(v)} = (f_{1,k}^{(v)}, \dots, f_{n,k}^{(v)})^T$, where $(v, k) \in \bar{U}_v \times \mathbb{Z}$. Sometimes, $f = (f_1, \dots, f_n)^T$ is used for simplicity.

2. Problem Formulation

In this section, firstly, discrete-time and discrete-space SGRNs are presented, which can be considered as discrete formats of continuous-time SGRNs with reaction diffusion. Secondly, the constant variation formula of the discrete network is obtained by dividing the discrete network into two discrete sub-networks based on the theory of difference equations. In the next step, important inequalities are given, such as the Minkowski inequality in Lemma 2. Finally, the definition of weighted pseudo- θ -almost periodicity is presented.

2.1. Space–Time Discrete Stochastic GRNs

This article considers the following space–time discrete stochastic genetic regulatory networks (GRNs) in the Euler form of

$$\begin{cases} \mathbf{m}_{i,k+1}^{(v)} = e^{-a_{i,k}h} \mathbf{m}_{i,k}^{(v)} + \frac{1 - e^{-a_{i,k}h}}{a_{i,k}} \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,k}^{(v)} \right. \\ \left. + \sum_{j=1}^m b_{ij,k} f_j(\mathbf{p}_{j,k-\sigma_{j,k}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,k} g_j(\mathbf{p}_{j,k-\mu_{j,k}}^{(v)}) w_{1j,k} + I_{i,k} \right], \\ \mathbf{p}_{i,k+1}^{(v)} = e^{-c_{i,k}h} \mathbf{p}_{i,k}^{(v)} + \frac{1 - e^{-c_{i,k}h}}{c_{i,k}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{\hbar_q}^2 \mathbf{p}_{i,k}^{(v)} + d_{i,k} \mathbf{m}_{i,k}^{(v)} + \sum_{j=1}^m \omega_{ij,k} \eta_j(\mathbf{m}_{j,k-\nu_{j,k}}^{(v)}) w_{2j,k} \right] \end{cases} \quad (1)$$

for $(v, k) \in \mathbb{U}_v \times \mathbb{Z}$ and $i = 1, 2, \dots, m$; $\mathbf{m}_i$ and $\mathbf{p}_i$ denote the concentrations of the i th mRNA and i th protein, respectively;

$$\tilde{a}_i = a_i - 2 \sum_{q=1}^n \frac{\Theta_{iq}}{\hbar^2}, \quad \tilde{c}_i = c_i - 2 \sum_{q=1}^n \frac{\Pi_{iq}}{\hbar^2},$$

$\tilde{a}_i > 0$ and $\tilde{c}_i > 0$ are the decay rates of the i th mRNA and i th protein, respectively; Θ_{iq} and Π_{iq} represent the transmission diffusion matrixes, where $\tilde{\Delta}_{\hbar_q}^2$ means the discrete-space operator denoted by

$$\tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,\cdot}^{(\cdot)} := \frac{\mathbf{m}_{i,\cdot}^{(v+\zeta_q)} + \mathbf{m}_{i,\cdot}^{(v-\zeta_q)}}{\hbar^2}, \quad \tilde{\Delta}_{\hbar_q}^2 \mathbf{p}_{i,\cdot}^{(\cdot)} := \frac{\mathbf{p}_{i,\cdot}^{(v+\zeta_q)} + \mathbf{p}_{i,\cdot}^{(v-\zeta_q)}}{\hbar^2}, \quad q \in \mathbb{N}_1^n;$$

$\hbar$ and h denote the length of the space and time steps in order; γ_{ij} and ω_{ij} stand for noise intensities; $d_i > 0$ is the translation rate; $I_i = \sum_{j \in \mathbf{I}_i} w_{ij}$, $w_{ij} \geq 0$ is bounded and $\mathbf{I}_i$ is the set of all the j which is a repressor of gene i ; $b_{ij} = w_{ij}$ if transcription factor j is an activator of gene i , $b_{ij} = 0$ if there is no link from node j to i , and $b_{ij} = -w_{ij}$ if transcription factor j is a repressor of gene i ; f_j , g_j , and η_j are Hill functions; $w_{1j,k} := \frac{1}{\hbar} [\mathbf{w}_{1j}(kh + h) - \mathbf{w}_{1j}(kh)]$, $w_{2j,k} := \frac{1}{\hbar} [\mathbf{w}_{2j}(kh + h) - \mathbf{w}_{2j}(kh)]$, and $i, j = 1, 2, \dots, m$; $\mathbf{w}_{11}, \dots, \mathbf{w}_{1m}, \mathbf{w}_{21}, \dots, \mathbf{w}_{2m}$ are scalar mutually independent two-sided standard Brown motions on complete probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbf{P})$ with filtration

$$\mathcal{F}_k = \sigma\{(w_{11,q}, \dots, w_{1m,q}, w_{21,q}, \dots, w_{2m,q}) : q \in (-\infty, k)_{\mathbb{Z}}\}, \forall k \in \mathbb{Z}.$$

The Dirichlet boundary conditions of GRNs Equation (1) are described as

$$\mathbf{m}_{i,k}^{(v)} \Big|_{v \in \partial \mathbb{U}_v} = 0 = \mathbf{p}_{i,k}^{(v)} \Big|_{v \in \partial \mathbb{U}_v}, \quad \forall k \in \mathbb{Z}. \quad (2)$$

Herein, $\mathbb{U}_v$ can be regarded as a discrete form of the rectangle area $\mathbb{U}$ in $\mathbb{R}^m$, which is described by

$$\mathbb{U} = \left\{ x = (x_1, x_2, \dots, x_m)^T \in \mathbb{R}^m : 0 < x_p < L_p := \hbar N_p, p \in \mathbb{N}_1^m \right\}.$$

Let $\mathbf{m}_{i,k}^{(v)} = M_i(v\hbar, kh)$ and $\mathbf{p}_{i,k}^{(v)} = P_i(v\hbar, kh)$ for $(v, k) \in \mathcal{U}_v \times \mathbb{Z}$. Then, GRNs Equation (1) is a full discretization scheme of the following stochastic GRNs with reaction diffusions

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} M_i(x, t) = \sum_{q=1}^n \frac{\partial}{\partial x_q} \left[\Theta_{iq} \frac{\partial M_i(x, t)}{\partial x_q} \right] - \tilde{a}_i(t) M_i(x, t) \\ \quad + \sum_{j=1}^m b_{ij}(t) f_j(P_j(x, t - \sigma_j(t))) + I_i(t) + \sum_{j=1}^m \gamma_{ij}(x, t) g_j(P_j(x, t - \mu_j(t))) \frac{d}{dt} \mathbb{W}_{1j}(t), \\ \frac{\partial}{\partial t} P_i(x, t) = \sum_{q=1}^n \frac{\partial}{\partial x_q} \left[\Pi_{iq} \frac{\partial P_i(x, t)}{\partial x_q} \right] - \tilde{c}_i(t) P_i(x, t) \\ \quad + d_i(x, t) M_i(x, t) + \sum_{j=1}^m \omega_{ij}(x, t) \eta_j(M_j(x, t - \nu_j(t))) \frac{d}{dt} \mathbb{W}_{2j}(t), \\ M_i(x, t) \Big|_{x \in \partial \mathcal{U}} = 0 = P_i(x, t) \Big|_{x \in \partial \mathcal{U}}, \end{array} \right. \quad (3)$$

where $x = (x_1, \dots, x_n)^T \in \mathcal{U} \subseteq \mathbb{R}^n$ refers to a space variable.

The discrete techniques in SGRNs Equation (1) are therefore Eulerian difference (ED) for Brownian motion, exponential Eulerian difference (EED) for time variables, and central finite difference (CFD) for spatial variables, respectively. For more information on ED, EED, and CFD, please see the literature [51–55].

Remark 1. By using Euler differences, reports [18,40] considered discrete-time GRNs without spatial diffusions. In this article, SGRNs Equation (1) extends the models in reports [18,40].

Lemma 1. GRNs Equation (1) can be expressed as

$$\left\{ \begin{array}{l} \mathbf{m}_{i,k}^{(v)} = \prod_{s=k_0}^{k-1} e^{-a_{i,s}h} \mathbf{m}_{i,k_0}^{(v)} + \sum_{v=k_0}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{i,s}h} (1 - e^{-a_{i,v}h})}{a_{i,v}} \\ \quad \times \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,v}^{(v)} + \sum_{j=1}^m b_{ij,v} f_j(\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,v} g_j(\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}) w_{1j,v} + I_{i,v} \right], \\ \mathbf{p}_{i,k}^{(v)} = \prod_{s=k_0}^{k-1} e^{-c_{i,s}h} \mathbf{p}_{i,k_0}^{(v)} + \sum_{v=k_0}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-c_{i,s}h} (1 - e^{-c_{i,v}h})}{c_{i,v}} \\ \quad \times \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{\hbar_q}^2 \mathbf{p}_{i,v}^{(v)} + d_{i,v} \mathbf{m}_{i,v}^{(v)} + \sum_{j=1}^m \omega_{ij,v} \eta_j(\mathbf{m}_{j,v-\nu_{j,v}}^{(v)}) w_{2j,v} \right], \end{array} \right. \quad (4)$$

where $(v, k) \in \mathcal{U}_v \times [k_0, \infty)_{\mathbb{Z}}$ with some initial point $k_0 \in \mathbb{Z}$, $i = 1, 2, \dots, m$. Moreover, it holds that

$$\mathbf{m}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v} = 0 = \mathbf{p}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v}, \quad \forall k \in [k_0, \infty)_{\mathbb{Z}}, i = 1, 2, \dots, m.$$

Lemma 2 ([56] (Minkowski inequality)). If $X, Y \in L^2(\Omega, \mathbb{R})$, then

$$\left(\mathbb{E}|X + Y|^2 \right)^{\frac{1}{2}} \leq \left(\mathbb{E}|X|^2 \right)^{\frac{1}{2}} + \left(\mathbb{E}|Y|^2 \right)^{\frac{1}{2}}.$$

Lemma 3 ([56] (Hölder inequality)). Let $a_k, b_k : \mathbb{Z} \rightarrow \mathbb{R}$. Then,

$$\left| \sum_k a_k b_k \right|^2 \leq \sum_k |a_k|^2 \sum_k |b_k|^2.$$

Lemma 4. $\mathbb{E}|w_{1j,k}|^2 = \mathbb{E}|w_{2j,k}|^2 = \frac{1}{h}$ for $j = 1, 2, \dots, n$.

Proof. By the definition of $w_{1j,k}$ and Ito formula, it is derived that

$$\mathbf{E}|w_{1j,k}|^2 = \frac{1}{h^2} \mathbf{E} \left(\int_{kh}^{kh+h} d\mathbb{W}_{1j}(s) \right)^2 = \frac{1}{h^2} \mathbf{E} \int_{kh}^{kh+h} ds = \frac{1}{h}, \quad \forall j = 1, 2, \dots, n.$$

This completes the proof. $\square$

2.2. Weighted Pseudo-Almost Periodicity

In the following, assume that $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a norm linear space, and $L^p(\Omega, \mathbb{R}^n)$ denotes the set of all p th integrable $\mathbb{R}^n$ -valued random variables with the norm

$$\|u\|_p = \max_{1 \leq i \leq n} [\mathbf{E}|u_i|^p]^{1/p}, \quad \forall u = (u_1, \dots, u_n)^T \in L^p(\Omega, \mathbb{R}^n), p > 0,$$

in which $\mathbf{E}$ denotes the expectation operator with respect to probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Definition 1. Let $X \in \mathbb{X}$ and $\epsilon > 0$ be arbitrary. If $v = v_{\epsilon}$ and $\tau \in [a, a + v_{\epsilon}]_{\mathbb{Z}}$ for any $a \in \mathbb{Z}$, ensuring that

$$\|X_{k+\tau} - X_k\|_{\mathbb{X}} < \epsilon, \quad \forall k \in \mathbb{Z},$$

then $\{X_k\}$ is an almost periodic sequence. Herein, τ is called an ϵ -almost period of X . $AP(\mathbb{Z}, \mathbb{X})$ denotes the set of the whole almost periodic sequences.

Let $\mathbb{U}$ be the set of all weight sequences $\alpha : \mathbb{Z} \rightarrow (0, +\infty)$ satisfying

$$\frac{\alpha_{k+s}}{\alpha_k} \leq \bar{\alpha}, \quad \forall k \in \mathbb{Z}, s \in [0, \sigma_0]_{\mathbb{Z}};$$

$$\mu_k(\alpha) := \sum_{s=-k}^k \alpha_s \rightarrow +\infty, \quad \frac{1}{\mu_k(\alpha)} \sum_{s=-k-\sigma_0}^{-k} \alpha_s \rightarrow 0, \quad \text{as } k \rightarrow +\infty,$$

where $\sigma_0 = \max_{1 \leq j \leq m} \sup_{k \in \mathbb{Z}} \{\sigma_{j,k}, \mu_{j,k}, v_{j,k}\}$.

Define $\mathbb{B}(\mathbb{Z}, \mathbb{X})$ as the set of all bounded sequences from $\mathbb{Z}$ to $\mathbb{X}$ and

$$PAP_0^{\mu}(\mathbb{Z}, \mathbb{X}, \alpha) := \left\{ X \in \mathbb{B}(\mathbb{Z}, \mathbb{X}) : \lim_{k \rightarrow +\infty} \frac{1}{\mu_k(\alpha)} \sum_{s=-k}^k \alpha_s \|X_s\|_{\mathbb{X}} = 0 \right\}.$$

When $\mathbb{X} = L^2(\Omega, \mathbb{R}^n)$ or $\mathbb{R}^n$, we use $PAP_0^{\mu}(\mathbb{Z}, \mathbb{R}^n, \alpha)$ to denote $PAP_0^{\mu}(\mathbb{Z}, \mathbb{X}, \alpha)$.

Definition 2. Sequence $X \in \mathbb{B}(\mathbb{Z}, \mathbb{X})$ is said to be a weighted pseudo-almost periodic sequence or an α -pseudo-almost periodic sequence in the case $X = Y + Z$, where $Y \in AP(\mathbb{Z}, \mathbb{X})$, $Z \in PAP_0^{\mu}(\mathbb{Z}, \mathbb{X}, \alpha)$, and $\alpha \in \mathbb{U}$. The space of all α -pseudo-almost periodic sequences is represented by $PAP^{\mu}(\mathbb{Z}, \mathbb{X}, \alpha)$.

Supposing that $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ is a metric dynamical system, see the pioneering work in [57]. It holds that $\theta_k : \Omega \rightarrow \Omega$ is $\mathcal{F}$ -measurable, $\mathbf{P}(\theta_k^{-1}(A)) = \mathbf{P}(A)$ for any $A \in \mathcal{F}$, and $\theta_{s+k} = \theta_s \circ \theta_k, \forall s, k \in \mathbb{Z}$.

Definition 3. The translation to a sequential process X_k is defined as

$$\mathcal{L}_{\tau} X_k(\omega) := X_{k+\tau}(\theta_{-\tau}\omega), \quad \forall \omega \in \Omega, s, k, \tau \in \mathbb{Z}.$$

Definition 4. If $X_k \in \mathbb{X}, \forall k \in \mathbb{Z}$, then X is said to be θ -almost periodic in the case that for each $\epsilon > 0$ we can find at least one positive integer $\nu = \nu(\epsilon)$ and it has a constant $\tau = \tau(\epsilon) \in (a, a + \nu)_{\mathbb{Z}}$ for arbitrary $a \in \mathbb{Z}$, ensuring that

$$\|\mathcal{L}_\tau X_k - X_k\|_{\mathbb{X}} \leq \epsilon, \quad \forall k \in \mathbb{Z}.$$

Herein, τ is called an ϵ - θ -almost period of X . The space of all θ -almost periodic sequences is represented by $AP^\theta(\mathbb{Z}, \mathbb{X})$. If $\mathcal{L}_\tau X = X$ with $\tau \in \mathbb{Z}$, then X is said to be θ -periodic.

When $\mathbb{X} = L^p(\Omega, \mathbb{R}^n)$ with $p > 0$, then X is said to be θ -almost periodic in p -mean. If $p = 2$, the elements in $PAP^\theta(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n))$ are called a mean square θ -almost periodic sequence. Hereby, we use a simplified symbol $PAP^\theta(\mathbb{Z}, \mathbb{R}^n)$ to denote $PAP^\theta(\mathbb{Z}, L^2(\Omega, \mathbb{R}^n))$.

Definition 5. Sequence $X : \mathbb{Z} \rightarrow \mathbb{X}$ is said to be a weighted pseudo- θ -almost periodic sequence or α -pseudo- θ -almost periodic sequence in the case $X = Y + Z$, where $Y \in AP^\theta(\mathbb{Z}, \mathbb{X})$, $Z \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$, and $\alpha \in \mathbb{U}$. The space of all α -pseudo- θ -almost periodic sequences is represented by $PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{X}, \alpha)$. If Y is θ -periodic, then X is said to be a weighted pseudo- θ -periodic sequence or an α -pseudo- θ -periodic sequence.

When $\mathbb{X} = L^p(\Omega, \mathbb{R}^n)$ with $p > 0$, then X is said to be a weighted pseudo- θ -almost periodic sequence or an α -pseudo- θ -almost periodic sequence in p -mean. If $p = 2$, the elements in $PAP^{\theta, \mu}(\mathbb{Z}, L^p(\Omega, \mathbb{R}^n), \alpha)$ are called a weighted pseudo mean square θ -almost periodic sequence or an α -pseudo mean square θ -almost periodic sequence. Hereby, a simplified symbol $PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{R}^n, \alpha) := PAP^{\theta, \mu}(\mathbb{Z}, L^2(\Omega, \mathbb{R}^n), \alpha)$.

3. Mean Square α -Pseudo- θ -Almost Periodic Sequence

This section focuses on weighted pseudo- θ -almost periodic sequence solutions in the mean square sense of SGRNs Equation (1) based on stochastic calculus theory, the constant variation formula, and the Banach contraction mapping principle.

For any $\mathbf{u} = (\mathbf{m}, \mathbf{p})^T \in PAP^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$ with $\mathbf{m} = (\mathbf{m}_1, \dots, \mathbf{m}_m)^T$ and $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_m)^T$, define $\Gamma : PAP^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha) \rightarrow \mathbb{R}^{2m}$ by

$$\Gamma \mathbf{u} = \left((\Phi \mathbf{u})_1, \dots, (\Phi \mathbf{u})_m, (\Psi \mathbf{u})_1, \dots, (\Psi \mathbf{u})_m \right)^T,$$

where

$$\begin{cases} (\Phi \mathbf{u})_{i,k}^{(v)} = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{i,s}h}(1 - e^{-a_{i,v}h})}{a_{i,v}} \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{h_q}^2 \mathbf{m}_{i,v}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m b_{ij,v} f_j(\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,v} g_j(\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}) w_{1j,v} + I_{i,v} \right], \\ (\Psi \mathbf{u})_{i,k}^{(v)} = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-c_{i,s}h}(1 - e^{-c_{i,v}h})}{c_{i,v}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{h_q}^2 \mathbf{p}_{i,v}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m \omega_{ij,v} \eta_j(\mathbf{m}_{j,v-\nu_{j,v}}^{(v)}) w_{2j,v} + d_{i,v} \mathbf{m}_{i,v}^{(v)} \right], \quad \forall (v, k) \in \mathcal{U}_v \times \mathbb{Z}; \end{cases} \quad (5)$$

$$(\Phi \mathbf{u})_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v} = 0 = (\Psi \mathbf{u})_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v}, \quad \forall k \in \mathbb{Z}, i = 1, 2, \dots, m.$$

For $\forall \mathbf{u} = (\mathbf{m}, \mathbf{p}) \in PAP^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$, define the norm as follows:

$$\|\mathbf{u}\|_\infty = \sup_{(v,k) \in \mathcal{U}_v \times \mathbb{Z}} \max_{1 \leq i \leq m} \left\{ \|\mathbf{m}_{i,k}^{(v)}\|_2, \|\mathbf{p}_{i,k}^{(v)}\|_2 \right\},$$

in which $\|\mathbf{m}_{i,k}^{(v)}\|_2 = \left[\mathbf{E}(\mathbf{m}_{i,k}^{(v)})^2 \right]^{\frac{1}{2}}$ and $\|\mathbf{p}_{i,k}^{(v)}\|_2 = \left[\mathbf{E}(\mathbf{p}_{i,k}^{(v)})^2 \right]^{\frac{1}{2}}$ for all $(v, k) \in \mathcal{U}_v \times \mathbb{Z}$, $i = 1, 2, \dots, m$.

Define

$$\underline{a}_i := \inf_{k \in \mathbb{Z}} a_{i,k}, \quad \underline{c}_i := \inf_{k \in \mathbb{Z}} c_{i,k}, \quad \bar{d}_i := \sup_{k \in \mathbb{Z}} |d_{i,k}|,$$

$$\bar{I}_i := \sup_{k \in \mathbb{Z}} |I_{i,k}|, \quad \bar{b}_{ij} := \sup_{k \in \mathbb{Z}} |b_{ij,k}|, \quad \bar{\gamma}_{ij} := \sup_{k \in \mathbb{Z}} |\gamma_{ij,k}|, \quad \bar{\omega}_{ij} := \sup_{k \in \mathbb{Z}} |\omega_{ij,k}|,$$

where $i, j = 1, 2, \dots, m$.

In the later discussion of this paper, the following assumptions are necessary:

(g₁) a_i and c_i are $\mathbb{R}$ -valued almost periodic sequences; σ_j , μ_j , and ν_j are $\mathbb{Z}_0$ -valued almost periodic sequences; b_{ij} , γ_{ij} , ω_{ij} , I_i , and d_i are $\mathbb{R}$ -valued α -pseudo-almost periodic sequences.

(g₂) $f_j(0) = g_j(0) = \eta_j(0) = 0$ and there exist positive numbers L_j^f , L_j^g and L_j^η such that

$$|f_j(u) - f_j(v)| \leq L_j^f |u - v|, \quad |g_j(u) - g_j(v)| \leq L_j^g |u - v|, \quad |\eta_j(u) - \eta_j(v)| \leq L_j^\eta |u - v|$$

for any $u, v \in \mathbb{R}$, $j = 1, 2, \dots, m$.

(g₃) $\min_{1 \leq i \leq m} \{a_i, c_i\} > 0$.

3.1. α -Pseudo- θ -Almost Periodicity of Operator Γ

Define a coordinate function $\mathbf{w}_{pj,k}(\omega) := \mathbf{w}_{pj}(kh, \omega) := \omega_{pj,k}$ and $\theta = (\theta_k)_{k \in \mathbb{Z}}$, which is the dynamical system on $(\Omega, \mathcal{F}, \mathbf{P})$, as

$$\theta_k \omega(s) = \left(\omega_{11,k+s} - \omega_{11,k}, \dots, \omega_{1m,k+s} - \omega_{1m,k}, \omega_{21,k+s} - \omega_{21,k}, \dots, \omega_{2m,k+s} - \omega_{2m,k} \right)^T,$$

where $\omega = (\omega_{11}, \dots, \omega_{1m}, \omega_{21}, \dots, \omega_{2m})^T \in \Omega$, $k, s \in \mathbb{Z}$, $p = 1, 2$, $j = 1, 2, \dots, m$.

For any $k, \tau \in \mathbb{Z}$ and $\omega \in \Omega$, it holds that

$$\mathbf{w}_{pj,k+\tau}(\theta_{-\tau} \omega) = \mathbf{w}_{pj,k}(\omega) - \mathbf{w}_{pj,-\tau}(\omega), \quad p = 1, 2, j = 1, 2, \dots, m. \quad (6)$$

Lemma 5. Let $\sigma : \mathbb{Z} \rightarrow [0, \sigma_0]_{\mathbb{Z}}$ with $\sigma_0 > 0$ and $\Delta \sigma < 1$. If $\mathbf{x} \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$, then $\mathbf{x}_{k-\sigma_k} \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$, $\forall k \in \mathbb{Z}$.

Proof. By the definition of $PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$, we obtain

$$\begin{aligned} \frac{1}{\mu_k(\alpha)} \sum_{s=-k}^k \alpha_s \|\mathbf{x}_{s-\sigma_s}\|_{\mathbb{X}} &\leq \frac{\bar{\alpha}}{\mu_k(\alpha)} \sum_{q=-k-\sigma_k}^{k-\sigma_k} \alpha_q \|\mathbf{x}_q\|_{\mathbb{X}} \\ &\leq \frac{\bar{\alpha}}{\mu_k(\alpha)} \sum_{q=-k}^k \alpha_q \|\mathbf{x}_q\|_{\mathbb{X}} + \frac{\bar{\alpha}}{\mu_k(\alpha)} \sum_{q=-k-\sigma_0}^{-k} \alpha_q \|\mathbf{x}_q\|_{\mathbb{X}} \rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$. This completes the proof. $\square$

Corollary 1. If $\mathbf{x} \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$, then $\mathbf{x}_{k-1} \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$ for each $k \in \mathbb{Z}$.

Lemma 6. Let $\sigma : \mathbb{Z} \rightarrow [0, \sigma_0]_{\mathbb{Z}}$ be an almost periodic sequence, which satisfies the conditions in Lemma 5. If $\mathbf{x} \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{X}, \alpha)$, then $\mathbf{x}_{k-\sigma_k} \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{X}, \alpha)$, $\forall k \in \mathbb{Z}$.

Proof. Owing to $\mathbf{x} \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{X}, \alpha)$, then $\mathbf{x} = \hat{\mathbf{x}} + \check{\mathbf{x}}$, where $\hat{\mathbf{x}} \in AP^\theta(\mathbb{Z}, \mathbb{R})$ and $\check{\mathbf{x}} \in PAP_0^\mu(\mathbb{Z}, \mathbb{R}, \alpha)$. From Lemma 5, $\check{\mathbf{x}}_{k-\sigma_k} \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$, $\forall k \in \mathbb{Z}$. It suffices to prove $\hat{\mathbf{x}}_{k-\sigma_k} \in AP^\theta(\mathbb{Z}, \mathbb{X})$, $\forall k \in \mathbb{Z}$.

Let $\tau \in \mathbb{Z}$ be an ϵ -almost period of σ and $\hat{\mathbf{x}}, \epsilon \in (0, 1)$. Noting that $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$, so

$$|\sigma_{k+\tau} - \sigma_k| = 0 < \epsilon, \quad \forall k \in \mathbb{Z},$$

which derives

$$\|\mathcal{L}_\tau \hat{\mathbf{x}}_{k-\sigma_k} - \hat{\mathbf{x}}_{k-\sigma_k}\|_{\mathbb{X}} \leq \|\hat{\mathbf{x}}_{k+\tau-\sigma_{k+\tau}} - \hat{\mathbf{x}}_{k-\sigma_{k+\tau}}\|_{\mathbb{X}} + \|\hat{\mathbf{x}}_{k-\sigma_{k+\tau}} - \hat{\mathbf{x}}_{k-\sigma_k}\|_{\mathbb{X}} < \epsilon, \quad \forall k \in \mathbb{Z}.$$

Then, $\hat{\mathbf{x}}_{k-\sigma_k} \in AP^\theta(\mathbb{Z}, \mathbb{X}), \forall k \in \mathbb{Z}$. This completes the proof. $\square$

Lemma 7. If $b \in PAP^\mu(\mathbb{Z}, \mathbb{R}, \alpha)$, $\mathbf{x} \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{X}, \alpha)$ is bounded, $f(0) = 0$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ meets the Lipschitz condition with Lipschitz constant $L_f > 0$, then $bf(\mathbf{x}) \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{X}, \alpha)$.

Proof. Under the assumptions in Lemma 7, there exist $\hat{b} \in AP(\mathbb{Z}, \mathbb{R}), \check{b} \in PAP_0^\mu(\mathbb{Z}, \mathbb{R}, \alpha), \hat{\mathbf{x}} \in AP^\theta(\mathbb{Z}, \mathbb{X})$, and $\check{\mathbf{x}} \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$ such that

$$b = \hat{b} + \check{b}, \quad \mathbf{x} = \hat{\mathbf{x}} + \check{\mathbf{x}}.$$

For any $\tau \in \mathbb{Z}$,

$$\begin{aligned} \|\mathcal{L}_\tau \hat{b}_k f(\hat{\mathbf{x}}_k) - \hat{b}_k f(\hat{\mathbf{x}}_k)\|_{\mathbb{X}} &= \|\hat{b}_{k+\tau} f(\mathcal{L}_\tau \hat{\mathbf{x}}_k) - \hat{b}_k f(\hat{\mathbf{x}}_k)\|_{\mathbb{X}} \\ &\leq \|\hat{b}_{k+\tau} - \hat{b}_k\|_{L_f} \|\hat{\mathbf{x}}_{k+\tau}(\theta_{-\tau} \omega)\|_{\mathbb{X}} + |\hat{b}_k| L_f \|\mathcal{L}_\tau \hat{\mathbf{x}}_k - \hat{\mathbf{x}}_k\|_{\mathbb{X}}, \quad \forall k \in \mathbb{Z}, \end{aligned}$$

which implies $\hat{b}f(\hat{\mathbf{x}}) \in AP^\theta(\mathbb{Z}, \mathbb{X})$. Meanwhile,

$$\|bf(\mathbf{x}) - \hat{b}f(\hat{\mathbf{x}})\|_{\mathbb{X}} \leq |\check{b}| L_f \|\mathbf{x}\|_2 + |\hat{b}| L_f \|\check{\mathbf{x}}\|_{\mathbb{X}},$$

which induces $bf(\mathbf{x}) - \hat{b}f(\hat{\mathbf{x}}) \in PAP_0^\mu(\mathbb{Z}, \mathbb{X}, \alpha)$. This completes the proof. $\square$

Lemma 8. If $a \in AP(\mathbb{Z}, \mathbb{R})$ with $\underline{a} = \inf_{k \in \mathbb{Z}} a_k > 0$, $\mathbf{x} \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{R}, \alpha)$ is bounded and $\mathbf{x}_k$ is $\mathcal{F}_k$ -adaptive for each $k \in \mathbb{Z}$, then

$$\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \mathbf{x}_v \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{R}, \alpha), \quad \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \mathbf{x}_v w_{pj,v} \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{R}, \alpha), \quad \forall k \in \mathbb{Z},$$

where $p = 1, 2, j = 1, 2, \dots, m$.

Proof. Similar to Lemma 7, there exist $\hat{\mathbf{x}} \in AP^\theta(\mathbb{Z}, \mathbb{R})$ and $\check{\mathbf{x}} \in PAP_0^\mu(\mathbb{Z}, \mathbb{R}, \alpha)$ such that $\mathbf{x} = \hat{\mathbf{x}} + \check{\mathbf{x}}$.

Let $\bar{a} = \sup_{k \in \mathbb{Z}} a_k, \tau \in \mathbb{Z}$ be an ϵ -almost period of a and $\hat{\mathbf{x}}$,

$$\hat{\mathcal{I}}_{pj,k} := \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \hat{\mathbf{x}}_v w_{pj,v}, \quad \check{\mathcal{I}}_{pj,k} := \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \check{\mathbf{x}}_v w_{pj,v}, \quad \forall k \in \mathbb{Z},$$

where $p = 1, 2, j = 1, 2, \dots, m$. By using Equation (6) and the Minkowski and Hölder inequalities in Lemmas 2 and 3, we have

$$\begin{aligned} \|\mathcal{L}_\tau \hat{\mathcal{I}}_{pj,k} - \hat{\mathcal{I}}_{pj,k}\|_2 &= \left\{ \mathbf{E} \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \left(e^{a_s h} (e^{-a_{s+\tau} h} - e^{-a_s h}) \hat{\mathbf{x}}_{v+\tau} + (\hat{\mathbf{x}}_{v+\tau} - \hat{\mathbf{x}}_v) \right) w_{pj,v} \right]^2 \right\}^{\frac{1}{2}} \\ &\leq \left\{ \sum_{v=-\infty}^{k-1} e^{-\underline{a}(k-v-1)h} \sum_{v=-\infty}^{k-1} e^{-\underline{a}(k-v-1)h} \mathbf{E} \left[\left(e^{(\bar{a}-\underline{a})h} \epsilon h |\hat{\mathbf{x}}_{v+\tau}| + |\hat{\mathbf{x}}_{v+\tau} - \hat{\mathbf{x}}_v| \right) w_{pj,v} \right]^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1}{1-e^{-\underline{a}h}} \left(e^{(\bar{a}-\underline{a})h} h \sup_{k \in \mathbb{Z}} \|\hat{\mathbf{x}}_k\|_2 + 1 \right) h^{-\frac{1}{2}} \epsilon, \quad \forall k \in \mathbb{Z}, \end{aligned}$$

which implies $\hat{\mathcal{I}}_{pj} \in AP^\theta(\mathbb{Z}, \mathbb{R}), p = 1, 2, j = 1, 2, \dots, m$.

On the other hand, similar to the before derivation, we attain

$$\begin{aligned}\|\check{I}_{pj,k}\|_2 &= \left\{ \mathbf{E} \left[\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \check{\mathbf{x}}_v w_{pj,v} \right]^2 \right\}^{\frac{1}{2}} \\ &\leq \left[\sum_{v=-\infty}^{k-1} e^{-a(k-v-1)h} \sum_{v=-\infty}^{k-1} e^{-a(k-v-1)h} \mathbf{E}(\check{\mathbf{x}}_v^2 w_{pj,v}^2) \right]^{\frac{1}{2}} \\ &\leq \left[\frac{1}{1-e^{-ah}} \sum_{v=-\infty}^{k-1} \frac{1}{h} e^{-a(k-v-1)h} \|\check{\mathbf{x}}_v\|_2^2 \right]^{\frac{1}{2}}, \quad \forall k \in \mathbb{Z},\end{aligned}$$

which implies

$$\begin{aligned}\lim_{k \rightarrow +\infty} \frac{1}{\mu_k(\alpha)} \sum_{s=-k}^k \alpha_s \|\check{I}_{pj,s}\|_2 &\leq \lim_{k \rightarrow +\infty} \frac{1}{\mu_k(\alpha)} \left[\sum_{s=-k}^k \alpha_s \sum_{s=-k}^k \alpha_s \|\check{I}_{pj,s}\|_2^2 \right]^{\frac{1}{2}} \\ &\leq \lim_{k \rightarrow +\infty} \left[\frac{1}{\mu_k(\alpha)(1-e^{-ah})} \sum_{s=-k}^k \alpha_s \sum_{v=-\infty}^{s-1} \frac{1}{h} e^{-a(s-v-1)h} \|\check{\mathbf{x}}_v\|_2^2 \right]^{\frac{1}{2}} \\ &\leq \left[\frac{\sup_{k \in \mathbb{Z}} \|\check{\mathbf{x}}_k\|_2}{1-e^{-ah}} \sum_{q=0}^{\infty} e^{-aqh} \frac{1}{h} \lim_{k \rightarrow +\infty} \frac{1}{\mu_k(\alpha)} \sum_{s=-k}^k \alpha_s \|\check{\mathbf{x}}_{s-q-1}\|_2^2 \right]^{\frac{1}{2}} \quad (q = s - v - 1) \\ &= 0, \quad p = 1, 2, j = 1, 2, \dots, m.\end{aligned}$$

In the above computations, Corollary 1 and the principle of uniform convergence are employed. Thus, $\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \check{\mathbf{x}}_v w_{pj,v} \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{R}, \alpha)$, $\forall k \in \mathbb{Z}$. Furthermore, $\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} e^{-a_s h} \check{\mathbf{x}}_v \in PAP^{\theta, \mu}(\mathbb{Z}, \mathbb{R}, \alpha)$ can be similarly addressed, and $\forall k \in \mathbb{Z}$. This completes the proof. $\square$

Together with Lemmas 5–8, we derive the following:

Theorem 1. *Supposing that (g₁)–(g₃) hold. Then, Γ maps $PAP^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$ to $PAP^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$.*

3.2. Weighted Pseudo-Almost Periodic Sequence Solution to GRNs Equation (1)

Define

$$PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha) = \left\{ \mathbf{u} \in PAP^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha) : \|\mathbf{u} - \varphi\|_{\infty} \leq \frac{\varsigma \varphi_0}{1 - \varsigma} \right\},$$

where

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_m, 0, \dots, 0)^T, \quad \varphi_{i,k}^{(v)} = \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{i,s}h}(1 - e^{-a_{i,v}h})}{a_{i,v}} I_{i,v}$$

for all $(v, k) \in \mathcal{U}_v \times \mathbb{Z}$, $i = 1, 2, \dots, m$. From the definition of φ , we derive

$$\|\varphi\|_{\infty} = \max_{1 \leq i \leq m} \sup_{(v,k) \in \mathcal{U}_v \times \mathbb{Z}} \|\varphi_{i,k}^{(v)}\|_2 = \max_{1 \leq i \leq m} \sup_{(v,k) \in \mathcal{U}_v \times \mathbb{Z}} \sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{i,s}h}(1 - e^{-a_{i,v}h})}{a_{i,v}} I_{i,v} \leq \max_{1 \leq i \leq m} \frac{\bar{I}_i}{\underline{a}_i} = \varphi_0,$$

which induces

$$\|\mathbf{u}\|_{\infty} \leq \|\mathbf{u} - \varphi\|_{\infty} + \|\varphi\|_{\infty} \leq \frac{\varsigma \varphi_0}{1 - \varsigma} + \varphi_0 = \frac{\varphi_0}{1 - \varsigma}, \quad \forall \mathbf{u} \in PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha).$$

Theorem 2. *Let (g₁)–(g₃) be valid. GRNs Equation (1) possesses a unique weighted pseudo- or α -pseudo-almost periodic sequence solution if the following condition holds.*

(g₄) $\varsigma = \max\{\varsigma_1, \varsigma_2\} < 1$, where

$$\varsigma_1 = \max_{1 \leq i \leq m} \frac{1}{a_i} \left[\sum_{q=1}^n \frac{2|\Theta_{iq}|}{\hbar^2} + \sum_{j=1}^m \bar{b}_{ij} L_j^f + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g h^{-\frac{1}{2}} \right],$$

$$\varsigma_2 = \max_{1 \leq i \leq m} \frac{1}{c_i} \left[\sum_{q=1}^n \frac{2|\Pi_{iq}|}{\hbar^2} + \sum_{j=1}^m \bar{\omega}_{ij} L_j^\eta h^{-\frac{1}{2}} + \bar{d}_i \right].$$

Proof. Let us prove that the operator Γ is self-mapping from $PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$ to $PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$. Supposing that $\mathbf{u} = (\mathbf{m}, \mathbf{p})^T = (\mathbf{m}_1, \dots, \mathbf{m}_m, \mathbf{p}_1, \dots, \mathbf{p}_m)^T \in PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$. In view of Equation (5) and by utilizing the Minkowski and Hölder inequalities in Lemmas 2 and 3, we have

$$\begin{aligned} \left\| (\Phi \mathbf{u})_{i,k}^{(v)} - \varphi_{i,k}^{(v)} \right\|_2 &= \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{i,s}h} (1 - e^{-a_{i,v}h})}{a_{i,v}} \left[\sum_{q=1}^n |\Theta_{iq}| \tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,v}^{(v)} \right. \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^m b_{ij,v} f_j(\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,v} g_j(\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}) w_{1j,v} \right] \right)^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1-e^{-a_i h}}{a_i} \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left[\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,v}^{(v)}| \right. \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)}| + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}| |w_{1j,v}| \right] \right)^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1-e^{-a_i h}}{a_i} \left\{ \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \right. \\ &\quad \left. \times \mathbf{E} \left[\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,v}^{(v)}| + \sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)}| + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}| |w_{1j,v}| \right]^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{1-e^{-a_i h}} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left\{ \left(\mathbf{E} \left[\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,v}^{(v)}| \right. \right. \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)}| + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}| |w_{1j,v}| \right]^2 \right)^{\frac{1}{2}} \right\}^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{1-e^{-a_i h}} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left\{ \left(\mathbf{E} \left(\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{\hbar_q}^2 \mathbf{m}_{i,v}^{(v)}| \right)^2 \right)^{\frac{1}{2}} \right. \right. \\ &\quad \left. \left. + \left(\mathbf{E} \left(\sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)}| \right)^2 \right)^{\frac{1}{2}} + \left(\mathbf{E} \left(\sum_{j=1}^m \bar{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}| |w_{1j,v}| \right)^2 \right)^{\frac{1}{2}} \right\}^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{1-e^{-a_i h}} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left\{ \sum_{q=1}^n |\Theta_{iq}| \frac{2}{\hbar} \|\mathbf{u}\|_\infty \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f \|\mathbf{u}\|_\infty + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g \frac{1}{\sqrt{\hbar}} \|\mathbf{u}\|_\infty \right\}^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{1-e^{-a_i h}} \frac{1}{1-e^{-a_i h}} \left\{ \sum_{q=1}^n |\Theta_{iq}| \frac{2}{\hbar} \|\mathbf{u}\|_\infty \right. \right. \\ &\quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f \|\mathbf{u}\|_\infty + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g \frac{1}{\sqrt{\hbar}} \|\mathbf{u}\|_\infty \right\}^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{1}{a_i} \left(\sum_{q=1}^n |\Theta_{iq}| \frac{2}{\hbar} + \sum_{j=1}^m \bar{b}_{ij} L_j^f + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g \frac{1}{\sqrt{\hbar}} \right) \|\mathbf{u}\|_\infty \\ &\leq \frac{\varsigma \varphi_0}{1-\varsigma}, \quad i = 1, 2, \dots, m, \end{aligned} \tag{7}$$

as well as

$$\begin{aligned}
\|(\Psi \mathbf{u})_{i,k}^{(v)} - 0\|_2 &= \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{h_q}^2 \mathbf{p}_{i,v}^{(v)} \right. \right. \right. \\
&\quad \left. \left. + d_{i,v} \mathbf{m}_{i,v}^{(v)} + \sum_{j=1}^m \tilde{\omega}_{ij,v} \eta_j(\mathbf{m}_{j,v-v_{j,v}}^{(v)}) w_{2j,v} \right] \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \left[\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{h_q}^2 \mathbf{p}_{i,v}^{(v)}| \right. \right. \right. \\
&\quad \left. \left. + \bar{d}_i |\mathbf{m}_{i,v}^{(v)}| + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right] \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \right. \\
&\quad \times \mathbf{E} \left[\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{h_q}^2 \mathbf{p}_{i,v}^{(v)}| + \bar{d}_i |\mathbf{m}_{i,v}^{(v)}| + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right]^2 \Bigg\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{1-e^{-c_i h}} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \left\{ \left(\mathbf{E} \left[\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{h_q}^2 \mathbf{p}_{i,v}^{(v)}| \right. \right. \right. \right. \\
&\quad \left. \left. + \bar{d}_i |\mathbf{m}_{i,v}^{(v)}| + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right] \right)^2 \right\}^{\frac{1}{2}} \Bigg\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{1-e^{-c_i h}} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \left\{ \left(\mathbf{E} \left(\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{h_q}^2 \mathbf{p}_{i,v}^{(v)}| \right)^2 \right)^{\frac{1}{2}} \right. \right. \\
&\quad \left. \left. + \left(\mathbf{E} \left(\bar{d}_i |\mathbf{m}_{i,v}^{(v)}| \right)^2 \right)^{\frac{1}{2}} + \left(\mathbf{E} \left(\sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right)^2 \right)^{\frac{1}{2}} \right\}^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{1-e^{-c_i h}} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \right. \\
&\quad \times \left\{ \sum_{q=1}^n |\Pi_{iq}| \frac{2}{h} \|\mathbf{u}\|_\infty + \bar{d}_i \|\mathbf{u}\|_\infty + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta \frac{1}{\sqrt{h}} \|\mathbf{u}\|_\infty \right\}^2 \Bigg\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{1-e^{-c_i h}} \frac{1}{1-e^{-c_i h}} \right. \\
&\quad \times \left\{ \sum_{q=1}^n |\Pi_{iq}| \frac{2}{h} \|\mathbf{u}\|_\infty + \bar{d}_i \|\mathbf{u}\|_\infty + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta \frac{1}{\sqrt{h}} \|\mathbf{u}\|_\infty \right\}^2 \Bigg\}^{\frac{1}{2}} \\
&\leq \frac{1}{c_i} \left(\sum_{q=1}^n |\Pi_{iq}| \frac{2}{h} + \bar{d}_i + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta \frac{1}{\sqrt{h}} \right) \|\mathbf{u}\|_\infty \\
&\leq \frac{\varsigma \varphi_0}{1-\varsigma}, \quad i = 1, 2, \dots, m.
\end{aligned} \tag{8}$$

In the calculations of the stochastic terms of Equations (7) and (8), Lemma 4 has been employed.

Together with Equations (7) and (8), $\|\Gamma \mathbf{u} - \varphi\|_\infty \leq \frac{\varsigma \varphi_0}{1-\varsigma}$ and $\Gamma \mathbf{u}$ is well defined in space $(PAP_b^{\theta,\mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha), \|\cdot\|_\infty)$ for any $\mathbf{u} \in PAP_b^{\theta,\mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$.

In the end, the property of contraction to the operator Γ in space $PAP_b^{\theta,\mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$ will be demonstrated. Let $\mathbf{u} = (\mathbf{m}_1, \dots, \mathbf{m}_m, \mathbf{p}_1, \dots, \mathbf{p}_m)^T$ and $\tilde{\mathbf{u}} = (\tilde{\mathbf{m}}_1, \dots, \tilde{\mathbf{m}}_m, \tilde{\mathbf{p}}_1, \dots, \tilde{\mathbf{p}}_m)^T$ belong to space $PAP_b^{\theta,\mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$, it follows that

$$\begin{aligned}
& \left\| (\Phi \mathbf{u})_{i,k}^{(v)} - (\Phi \tilde{\mathbf{u}})_{i,k}^{(v)} \right\|_2 = \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-a_{i,s}h}(1-e^{-a_{i,v}h})}{a_{i,v}} \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{h,q}^2 (\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)}) \right. \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^m b_{ij,v} (f_j(\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)}) - f_j(\tilde{\mathbf{p}}_{j,v-\sigma_{j,v}}^{(v)})) \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^m \gamma_{ij,v} (g_j(\mathbf{p}_{j,v-\mu_{j,v}}^{(v)}) - g_j(\tilde{\mathbf{p}}_{j,v-\mu_{j,v}}^{(v)})) w_{1j,v} \right] \right)^2 \right\}^{\frac{1}{2}} \\
& \leq \frac{1-e^{-a_i h}}{a_i} \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left[\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{h,q}^2 (\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)})| \right. \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\sigma_{j,v}}^{(v)}| + \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\mu_{j,v}}^{(v)}| w_{1j,v} \right] \right)^2 \right\}^{\frac{1}{2}} \\
& \leq \frac{1-e^{-a_i h}}{a_i} \left\{ \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \mathbf{E} \left[\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{h,q}^2 (\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)})| \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\sigma_{j,v}}^{(v)}| + \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\mu_{j,v}}^{(v)}| w_{1j,v} \right] \right)^2 \right\}^{\frac{1}{2}} \\
& \leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{1-e^{-a_i h}} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left\{ \left(\mathbf{E} \left[\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{h,q}^2 (\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)})| \right. \right. \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\sigma_{j,v}}^{(v)}| + \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\mu_{j,v}}^{(v)}| w_{1j,v} \right] \right)^2 \right\}^{\frac{1}{2}} \right\}^{\frac{1}{2}} \quad (9) \\
& \leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{1-e^{-a_i h}} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left\{ \left(\mathbf{E} \left(\sum_{q=1}^n |\Theta_{iq}| |\tilde{\Delta}_{h,q}^2 (\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)})| \right)^2 \right)^{\frac{1}{2}} \right. \right. \\
& \quad \left. \left. + \left(\mathbf{E} \left(\sum_{j=1}^m \bar{b}_{ij} L_j^f |\mathbf{p}_{j,v-\sigma_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\sigma_{j,v}}^{(v)}| \right)^2 \right)^{\frac{1}{2}} \right. \right. \\
& \quad \left. \left. + \left(\mathbf{E} \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g |\mathbf{p}_{j,v-\mu_{j,v}}^{(v)} - \tilde{\mathbf{p}}_{j,v-\mu_{j,v}}^{(v)}| w_{1j,v} \right)^2 \right\}^{\frac{1}{2}} \right\}^{\frac{1}{2}} \\
& \leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{1-e^{-a_i h}} \sum_{v=-\infty}^{k-1} e^{-a_i(k-v-1)h} \left\{ \sum_{q=1}^n |\Theta_{iq}| \frac{2}{h} \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty} \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty} + \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g \frac{1}{\sqrt{h}} \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty} \right\}^2 \right\}^{\frac{1}{2}} \\
& \leq \frac{1-e^{-a_i h}}{a_i} \left\{ \frac{1}{(1-e^{-a_i h})^2} \left\{ \sum_{q=1}^n |\Theta_{iq}| \frac{2}{h} \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty} \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^m \bar{b}_{ij} L_j^f \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty} + \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g \frac{1}{\sqrt{h}} \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty} \right\}^2 \right\}^{\frac{1}{2}} \\
& \leq \frac{1}{a_i} \left(\sum_{q=1}^n |\Theta_{iq}| \frac{2}{h} + \sum_{j=1}^m \bar{b}_{ij} L_j^f + \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g \frac{1}{\sqrt{h}} \right) \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty} \\
& \leq \varsigma \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\infty}, \quad i = 1, 2, \dots, m,
\end{aligned}$$

as well as

$$\begin{aligned}
\|(\Psi \mathbf{u})_{i,k}^{(v)} - (\Psi \tilde{\mathbf{u}})_{i,k}^{(v)}\|_2 &= \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} \prod_{s=v+1}^{k-1} \frac{e^{-c_{i,s}h}(1-e^{-c_{i,v}h})}{c_{i,v}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{\hbar_q}^2 (\mathbf{p}_{i,v}^{(v)} - \tilde{\mathbf{p}}_{i,v}^{(v)}) \right. \right. \right. \\
&\quad \left. \left. + d_{i,v}(\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)}) + \sum_{j=1}^m \tilde{\omega}_{ij} \eta_j (\mathbf{m}_{j,v-v_{j,v}}^{(v)} - \tilde{\mathbf{m}}_{j,v-v_{j,v}}^{(v)}) w_{2j,v} \right] \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \mathbf{E} \left(\sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \left[\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{\hbar_q}^2 (\mathbf{p}_{i,v}^{(v)} - \tilde{\mathbf{p}}_{i,v}^{(v)})| \right. \right. \right. \\
&\quad \left. \left. + \tilde{d}_i |\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)}| + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)} - \tilde{\mathbf{m}}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right] \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \mathbf{E} \left[\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{\hbar_q}^2 (\mathbf{p}_{i,v}^{(v)} - \tilde{\mathbf{p}}_{i,v}^{(v)})| \right. \right. \\
&\quad \left. \left. + \tilde{d}_i |\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)}| + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)} - \tilde{\mathbf{m}}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right] \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{1-e^{-c_i h}} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \left\{ \left(\mathbf{E} \left[\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{\hbar_q}^2 (\mathbf{p}_{i,v}^{(v)} - \tilde{\mathbf{p}}_{i,v}^{(v)})| \right. \right. \right. \right. \\
&\quad \left. \left. + \tilde{d}_i |\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)}| + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)} - \tilde{\mathbf{m}}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right] \right)^2 \right\}^{\frac{1}{2}} \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{1-e^{-c_i h}} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \left\{ \left(\mathbf{E} \left(\sum_{q=1}^n |\Pi_{iq}| |\tilde{\Delta}_{\hbar_q}^2 (\mathbf{p}_{i,v}^{(v)} - \tilde{\mathbf{p}}_{i,v}^{(v)})| \right)^2 \right)^{\frac{1}{2}} \right. \right. \\
&\quad \left. \left. + \left(\mathbf{E} \left(\tilde{d}_i |\mathbf{m}_{i,v}^{(v)} - \tilde{\mathbf{m}}_{i,v}^{(v)}| \right)^2 \right)^{\frac{1}{2}} + \left(\mathbf{E} \left(\sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta |\mathbf{m}_{j,v-v_{j,v}}^{(v)} - \tilde{\mathbf{m}}_{j,v-v_{j,v}}^{(v)}| |w_{2j,v}| \right)^2 \right)^{\frac{1}{2}} \right\}^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{1-e^{-c_i h}} \sum_{v=-\infty}^{k-1} e^{-c_i(k-v-1)h} \left\{ \sum_{q=1}^n |\Pi_{iq}| \frac{2}{\hbar} \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty \right. \right. \\
&\quad \left. \left. + \tilde{d}_i \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta \frac{1}{\sqrt{\hbar}} \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty \right\}^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1-e^{-c_i h}}{c_i} \left\{ \frac{1}{(1-e^{-c_i h})^2} \left\{ \sum_{q=1}^n |\Pi_{iq}| \frac{2}{\hbar} \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty \right. \right. \\
&\quad \left. \left. + \tilde{d}_i \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta \frac{1}{\sqrt{\hbar}} \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty \right\}^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{1}{c_i} \left(\sum_{q=1}^n |\Pi_{iq}| \frac{2}{\hbar} + \tilde{d}_i + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta \frac{1}{\sqrt{\hbar}} \right) \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty \\
&\leq \varsigma \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty, \quad i = 1, 2, \dots, m.
\end{aligned}
\tag{10}$$

The inequalities in Equations (9) and (10) exhibit $\|\Gamma \mathbf{u} - \Gamma \tilde{\mathbf{u}}\|_\infty \leq \varsigma \|\mathbf{u} - \tilde{\mathbf{u}}\|_\infty, \forall \mathbf{u}, \tilde{\mathbf{u}} \in PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$. In line with assumption $(\mathbf{g}_1)$, the operator Γ is a contraction mapping. Consequently, Γ possess a unique fixed point $\hat{\mathbf{u}} = (\hat{\mathbf{m}}, \hat{\mathbf{p}})^T \in PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$, i.e., $\Gamma \hat{\mathbf{u}} = \hat{\mathbf{u}}$. Hence, $\hat{\mathbf{u}}$ is a unique weighted pseudo-almost periodic sequence to GRNs Equation (1). This completes the proof. $\square$

Remark 2. Articles [12,38] studied the existence of a unique (weighted pseudo) almost periodic solution of continuous-time GRNs without spatial diffusions. However, this paper not only regards the spatial diffusions, but also studies the corollary responding to multi-variable discrete GRNs. So Theorem 2 complements the works of [12,38].

4. Finite-Time Guaranteed Cost Controls in Exponential Form

In this section, finite-time guaranteed cost controllers for SGRNs Equation (1) are designed based on the drive network, response network, and error network. The global exponential stability of SGRNs Equation (1) in the mean square sense is also discussed.

4.1. The Frame of Controlling GRNs

Let $\hat{\mathbf{u}} = (\hat{\mathbf{m}}, \hat{\mathbf{p}})^T \in PAP_b^{\theta, \mu}(\mathcal{U}_v \times \mathbb{Z}, \mathbb{R}^{2m}, \alpha)$ be the unique weighted pseudo-almost periodic sequence to GRNs Equation (1), where $\hat{\mathbf{m}} = (\hat{\mathbf{m}}_1, \dots, \hat{\mathbf{m}}_m)^T$ and $\hat{\mathbf{p}} = (\hat{\mathbf{p}}_1, \dots, \hat{\mathbf{p}}_m)^T$. That is,

$$\begin{cases} \hat{\mathbf{m}}_{i,k+1}^{(v)} = e^{-a_{i,k}h} \hat{\mathbf{m}}_{i,k}^{(v)} + \frac{1 - e^{-a_{i,k}h}}{a_{i,k}} \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{h_q}^2 \hat{\mathbf{m}}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m b_{ij,k} f_j(\hat{\mathbf{p}}_{j,k-\sigma_{j,k}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,k} g_j(\hat{\mathbf{p}}_{j,k-\mu_{j,k}}^{(v)}) w_{1j,k} + I_{i,k} \right], \\ \hat{\mathbf{p}}_{i,k+1}^{(v)} = e^{-c_{i,k}h} \hat{\mathbf{p}}_{i,k}^{(v)} + \frac{1 - e^{-c_{i,k}h}}{c_{i,k}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{h_q}^2 \hat{\mathbf{p}}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m \omega_{ij,k} \eta_j(\hat{\mathbf{m}}_{j,k-\nu_{j,k}}^{(v)}) w_{2j,k} + d_{i,k} \hat{\mathbf{m}}_{i,k}^{(v)} \right], \quad \forall (v, k) \in \mathcal{U}_v \times \mathbb{Z}_0, \end{cases} \quad (11)$$

where $i = 1, 2, \dots, m$. The initial and boundary values of GRNs Equation (11) can be described as

$$\hat{\mathbf{m}}_{i,s}^{(\cdot)} = \hat{\phi}_{i,s}^{(\cdot)}, \quad \hat{\mathbf{p}}_{i,s}^{(\cdot)} = \hat{\phi}_{i,s}^{(\cdot)}, \quad \forall s \in [-\sigma_0, 0]_{\mathbb{Z}}; \quad \hat{\mathbf{m}}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v} = 0 = \hat{\mathbf{p}}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v}, \quad \forall k \in \mathbb{Z}_0,$$

where $i = 1, 2, \dots, m$.

A controlling network is constructed as below:

$$\begin{cases} \mathbf{m}_{i,k+1}^{(v)} = e^{-a_{i,k}h} \mathbf{m}_{i,k}^{(v)} + \frac{1 - e^{-a_{i,k}h}}{a_{i,k}} \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{h_q}^2 \mathbf{m}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m b_{ij,k} f_j(\mathbf{p}_{j,k-\sigma_{j,k}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,k} g_j(\mathbf{p}_{j,k-\mu_{j,k}}^{(v)}) w_{1j,k} + I_{i,k} \right] + \rho_{i,k}^{(v)}, \\ \mathbf{p}_{i,k+1}^{(v)} = e^{-c_{i,k}h} \mathbf{p}_{i,k}^{(v)} + \frac{1 - e^{-c_{i,k}h}}{c_{i,k}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{h_q}^2 \mathbf{p}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m \omega_{ij,k} \eta_j(\mathbf{m}_{j,k-\nu_{j,k}}^{(v)}) w_{2j,k} + d_{i,k} \mathbf{m}_{i,k}^{(v)} \right] + \varrho_{i,k}^{(v)}, \quad (v, k) \in \mathcal{U}_v \times \mathbb{Z}_0, \end{cases} \quad (12)$$

where $i = 1, 2, \dots, m$. The initial and boundary values of GRNs Equation (12) are given by

$$\mathbf{m}_{i,s}^{(\cdot)} = \phi_{i,s}^{(\cdot)}, \quad \mathbf{p}_{i,s}^{(\cdot)} = \phi_{i,s}^{(\cdot)}, \quad \forall s \in [-\sigma_0, 0]_{\mathbb{Z}}; \quad \mathbf{m}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v} = 0 = \mathbf{p}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v}, \quad \forall k \in \mathbb{Z}_0,$$

where $i = 1, 2, \dots, m$.

Let $\mathbf{e}_i = \mathbf{m}_i - \hat{\mathbf{m}}_i$ and $\mathbf{w}_i = \mathbf{p}_i - \hat{\mathbf{p}}_i$, $i = 1, 2, \dots, m$. Together with GRNs Equations (12) and (11), it yields

$$\begin{cases} \mathbf{e}_{i,k+1}^{(v)} = e^{-a_{i,k}h} \mathbf{e}_{i,k}^{(v)} + \frac{1 - e^{-a_{i,k}h}}{a_{i,k}} \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{e}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m b_{ij,k} \tilde{f}_j(\mathbf{w}_{j,k-\sigma_{j,k}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,k} \tilde{g}_j(\mathbf{w}_{j,k-\mu_{j,k}}^{(v)}) w_{1j,k} \right] + \rho_{i,k}^{(v)}, \\ \mathbf{w}_{i,k+1}^{(v)} = e^{-c_{i,k}h} \mathbf{w}_{i,k}^{(v)} + \frac{1 - e^{-c_{i,k}h}}{c_{i,k}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{w}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m \omega_{ij,k} \tilde{\eta}_j(\mathbf{e}_{j,k-\nu_{j,k}}^{(v)}) w_{2j,k} + d_{i,k} \mathbf{e}_{i,k}^{(v)} \right] + \varrho_{i,k}^{(v)}, \end{cases} \quad (13)$$

where

$$\tilde{f}_j(\mathbf{w}_j) = f_j(\mathbf{p}_j) - f_j(\hat{\mathbf{p}}_j), \quad \tilde{g}_j(\mathbf{w}_j) = g_j(\mathbf{p}_j) - g_j(\hat{\mathbf{p}}_j), \quad \tilde{\eta}_j(\mathbf{e}_j) = g_j(\mathbf{m}_j) - g_j(\hat{\mathbf{m}}_j),$$

in which $(v, k) \in \mathcal{U}_v \times \mathbb{Z}_0$, $i, j = 1, 2, \dots, m$.

The state feedback controller is designed:

$$\rho_{i,k}^{(\cdot)} = \kappa_i \mathbf{e}_{i,k}^{(\cdot)}, \quad \varrho_{i,k}^{(\cdot)} = \varkappa_i \mathbf{w}_{i,k}^{(\cdot)}, \quad \forall k \in \mathbb{Z}_0, \quad (14)$$

where κ_i and $\varkappa_i$ denote the controller gains to be determined later, $i = 1, 2, \dots, m$.

Substituting controller Equation (14) into the error network Equation (13) leads to

$$\begin{cases} \mathbf{e}_{i,k+1}^{(v)} = (e^{-a_{i,k}h} + \kappa_i) \mathbf{e}_{i,k}^{(v)} + \frac{1 - e^{-a_{i,k}h}}{a_{i,k}} \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{e}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m b_{ij,k} \tilde{f}_j(\mathbf{w}_{j,k-\sigma_{j,k}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,k} \tilde{g}_j(\mathbf{w}_{j,k-\mu_{j,k}}^{(v)}) w_{1j,k} \right], \\ \mathbf{w}_{i,k+1}^{(v)} = (e^{-c_{i,k}h} + \varkappa_i) \mathbf{w}_{i,k}^{(v)} + \frac{1 - e^{-c_{i,k}h}}{c_{i,k}} \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{w}_{i,k}^{(v)} \right. \\ \quad \left. + \sum_{j=1}^m \omega_{ij,k} \tilde{\eta}_j(\mathbf{e}_{j,k-\nu_{j,k}}^{(v)}) w_{2j,k} + d_{i,k} \mathbf{e}_{i,k}^{(v)} \right], \quad (v, k) \in \mathcal{U}_v \times \mathbb{Z}_0, \end{cases} \quad (15)$$

where $i = 1, 2, \dots, m$.

Similar to the derivation of Formula (4), we achieve

$$\begin{cases} \mathbf{e}_{i,k}^{(v)} = \prod_{s=0}^{k-1} (e^{-a_{i,s}h} + \kappa_i) \mathbf{e}_{i,0}^{(v)} + \sum_{v=0}^{k-1} \prod_{s=v+1}^{k-1} \frac{(e^{-a_{i,s}h} + \kappa_i)(1 - e^{-a_{i,v}h})}{a_{i,v}} \\ \quad \times \left[\sum_{q=1}^n \Theta_{iq} \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{e}_{i,v}^{(v)} + \sum_{j=1}^m b_{ij,v} \tilde{f}_j(\mathbf{w}_{j,v-\sigma_{j,v}}^{(v)}) + \sum_{j=1}^m \gamma_{ij,v} \tilde{g}_j(\mathbf{w}_{j,v-\mu_{j,v}}^{(v)}) w_{1j,v} \right], \\ \mathbf{w}_{i,k}^{(v)} = \prod_{s=0}^{k-1} (e^{-c_{i,s}h} + \varkappa_i) \mathbf{w}_{i,0}^{(v)} + \sum_{v=0}^{k-1} \prod_{s=v+1}^{k-1} \frac{(e^{-c_{i,s}h} + \varkappa_i)(1 - e^{-c_{i,v}h})}{c_{i,v}} \\ \quad \times \left[\sum_{q=1}^n \Pi_{iq} \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{w}_{i,v}^{(v)} + d_{i,v} \mathbf{e}_{i,v}^{(v)} + \sum_{j=1}^m \omega_{ij,v} \tilde{\eta}_j(\mathbf{e}_{j,v-\nu_{j,v}}^{(v)}) w_{2j,v} \right], \quad (v, k) \in \mathcal{U}_v \times \mathbb{Z}_0, \end{cases} \quad (16)$$

where $i = 1, 2, \dots, m$. Moreover, it holds that

$$\mathbf{e}_{i,s}^{(\cdot)} = \varphi_{i,s}^{(\cdot)} - \hat{\varphi}_{i,s}^{(\cdot)}, \quad \mathbf{w}_{i,s}^{(\cdot)} = \varphi_{i,s}^{(\cdot)} - \hat{\varphi}_{i,s}^{(\cdot)}, \quad \forall s \in [-\sigma_0, 0] \mathbb{Z}; \quad \mathbf{e}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v} = 0 = \mathbf{w}_{i,k}^{(v)} \Big|_{v \in \partial \mathcal{U}_v}, \quad \forall k \in \mathbb{Z}_0,$$

where $i = 1, 2, \dots, m$.

Definition 6. State feedback controller Equation (14) finite-time stabilises GRNs Equation (12) with a finite-time exponential convergent form in case the error networks Equation (15) achieves finite-time exponential stability, i.e., for any $\epsilon \in (0, 1)$ there exists $\delta > 0$, $\mu > 0$ and integer $K > 0$, ensuring that

$$\varphi_0 := \max_{1 \leq i \leq m} \max_{(v,s) \in \mathcal{U}_v \times [-\sigma_0, 0]_{\mathbb{Z}}} \left\{ \left\| \mathbf{e}_{i,s}^{(v)} \right\|_2, \left\| \mathbf{w}_{i,s}^{(v)} \right\|_2 \right\} < \delta$$

implies that

$$\max_{1 \leq i \leq m} \max_{v \in \mathcal{U}_v} \left\{ \left\| \mathbf{e}_{i,k}^{(v)} \right\|_2, \left\| \mathbf{w}_{i,k}^{(v)} \right\|_2 \right\} \leq \epsilon e^{-\mu k}, \quad \forall k \in [0, K]_{\mathbb{Z}}. \quad (17)$$

Herein, K is called the settling time.

Define a performance index J_c^K associated with the error networks Equation (15) by

$$J_c^K = \mathbf{E} \sum_{k=0}^K \max_{v \in \mathcal{U}_v} \mathbf{U}_k^{(v)T} F \mathbf{U}_k^{(v)},$$

where

$$\mathbf{U} = \text{col}(\mathbf{e}, \rho, \mathbf{w}, \varrho), \quad F = \text{diag}(P_1, Q_1, P_2, Q_2),$$

$$\mathbf{e} = \text{col}(\mathbf{e}_1, \dots, \mathbf{e}_m), \quad \mathbf{w} = \text{col}(\mathbf{w}_1, \dots, \mathbf{w}_m),$$

$$\rho = \text{col}(\rho_1, \dots, \rho_m), \quad \varrho = \text{col}(\varrho_1, \dots, \varrho_m),$$

$$P_i = P_i^T > 0, Q_i = Q_i^T > 0, i = 1, 2.$$

Definition 7. State feedback controller Equation (14) is said to be a finite-time guaranteed cost controller to GRNs Equation (12) in case it finite-time stabilises GRNs Equation (12) with an exponential convergent form and meets

$$J_c^K \leq \lambda,$$

where $\lambda > 0$ is a constant.

4.2. Design of Finite-Time Guaranteed Cost Controllers

From the first equation of the error networks Equation (16), we obtain

$$\begin{aligned} \left\| \mathbf{e}_{i,k}^{(v)} \right\|_2 &= \left[\mathbf{E} |\mathbf{e}_{i,k}^{(v)}|^2 \right]^{\frac{1}{2}} \leq (e^{-a_i h} + \kappa_i)^k \left\| \mathbf{e}_{i,0}^{(v)} \right\|_2 + \sum_{v=0}^{k-1} (e^{-a_i h} + \kappa_i)^{k-v-1} \frac{(1-e^{-a_i h})}{a_i} \\ &\quad \times \left[\sum_{q=1}^n |\Theta_{iq}| \left\| \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{e}_{i,v}^{(v)} \right\|_2 + \sum_{j=1}^m \tilde{b}_{ij} L_j^f \left\| \mathbf{w}_{j,v-\sigma_{j,v}}^{(v)} \right\|_2 + \sum_{j=1}^m \tilde{\gamma}_{ij} L_j^g \left\| \mathbf{w}_{j,v-\mu_{j,v}}^{(v)} \right\|_2 h^{-\frac{1}{2}} \right], \end{aligned} \quad (18)$$

where $v \in \mathcal{U}_v, i = 1, 2, \dots, m$. Similarly,

$$\begin{aligned} \left\| \mathbf{w}_{i,k}^{(v)} \right\|_2 &= \left[\mathbf{E} |\mathbf{w}_{i,k}^{(v)}|^2 \right]^{\frac{1}{2}} \leq (e^{-c_i h} + \varkappa_i)^k \left\| \mathbf{w}_{i,0}^{(v)} \right\|_2 + \sum_{v=0}^{k-1} (e^{-c_i h} + \varkappa_i)^{k-v-1} \frac{(1-e^{-c_i h})}{c_i} \\ &\quad \times \left[\sum_{q=1}^n |\Pi_{iq}| \left\| \tilde{\Delta}_{\tilde{h}_q}^2 \mathbf{w}_{i,v}^{(v)} \right\|_2 + \tilde{d}_i \left\| \mathbf{e}_{i,v}^{(v)} \right\|_2 + \sum_{j=1}^m \tilde{\omega}_{ij} L_j^\eta \left\| \mathbf{e}_{j,v-\nu_{j,v}}^{(v)} \right\|_2 h^{-\frac{1}{2}} \right], \end{aligned} \quad (19)$$

where $v \in \mathcal{U}_v, i = 1, 2, \dots, m$.

Equations (18) and (19) are equal to

$$\begin{aligned} \max_{v \in \mathcal{U}_v} \left\| \mathbf{e}_{i,k}^{(v)} \right\|_2 &\leq (e^{-a_i h} + \kappa_i)^k \max_{v \in \mathcal{U}_v} \left\| \mathbf{e}_{i,0}^{(v)} \right\|_2 + \sum_{v=0}^{k-1} (e^{-a_i h} + \kappa_i)^{k-v-1} \frac{(1-e^{-a_i h})}{a_i} \\ &\times \left[\sum_{q=1}^n |\Theta_{iq}| \max_{v \in \mathcal{U}_v} \left\| \tilde{\Delta}_{h_q}^2 \mathbf{e}_{i,v}^{(v)} \right\|_2 + \sum_{j=1}^m \bar{b}_{ij} L_j^f \max_{v \in \mathcal{U}_v} \left\| \mathbf{w}_{j,v-\sigma_{j,v}}^{(v)} \right\|_2 \right. \\ &\left. + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g \max_{v \in \mathcal{U}_v} \left\| \mathbf{w}_{j,v-\mu_{j,v}}^{(v)} \right\|_2 h^{-\frac{1}{2}} \right], \end{aligned} \quad (20)$$

and

$$\begin{aligned} \max_{v \in \mathcal{U}_v} \left\| \mathbf{w}_{i,k}^{(v)} \right\|_2 &\leq (e^{-c_i h} + \varkappa_i)^k \max_{v \in \mathcal{U}_v} \left\| \mathbf{w}_{i,0}^{(v)} \right\|_2 + \sum_{v=0}^{k-1} (e^{-c_i h} + \varkappa_i)^{k-v-1} \frac{(1-e^{-c_i h})}{c_i} \\ &\times \left[\sum_{q=1}^n |\Pi_{iq}| \max_{v \in \mathcal{U}_v} \left\| \tilde{\Delta}_{h_q}^2 \mathbf{w}_{i,v}^{(v)} \right\|_2 + \bar{d}_i \max_{v \in \mathcal{U}_v} \left\| \mathbf{e}_{i,v}^{(v)} \right\|_2 \right. \\ &\left. + \sum_{j=1}^m \bar{\omega}_{ij} L_j^\eta \max_{v \in \mathcal{U}_v} \left\| \mathbf{e}_{j,v-\nu_{j,v}}^{(v)} \right\|_2 h^{-\frac{1}{2}} \right], \quad i = 1, 2, \dots, m. \end{aligned} \quad (21)$$

Theorem 3. If (g₂) and the following assumptions are fulfilled,

(g₅) The control gains $\kappa_i = e^{-\hat{a}_i h} - e^{-a_i h}$ and $\varkappa_i = e^{-\hat{c}_i h} - e^{-c_i h}$, where $\hat{a}_i$ and $\hat{c}_i$ are positive constants, $i = 1, 2, \dots, m$.

(g₆) It holds that $1 - e^{-\underline{\alpha} h} < \max\{v_1, v_2\} < \frac{1 - e^{-\underline{\alpha} h}}{1 - e^{-\bar{\alpha} h}}$, where $\underline{\alpha} = \min_{1 \leq i \leq m} \{\hat{a}_i, \hat{c}_i\}$ and $\bar{\alpha} = \max_{1 \leq i \leq m} \{\hat{a}_i, \hat{c}_i\}$,

$$\begin{aligned} v_1 &= \max_{1 \leq i \leq m} \frac{(1 - e^{-a_i h})}{a_i} \left[\sum_{q=1}^n \frac{2|\Theta_{iq}|}{h^2} + \sum_{j=1}^m \bar{b}_{ij} L_j^f + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g h^{-\frac{1}{2}} \right], \\ v_2 &= \max_{1 \leq i \leq m} \frac{(1 - e^{-c_i h})}{c_i} \left[\sum_{q=1}^n \frac{2|\Pi_{iq}|}{h^2} + \bar{d}_i + \sum_{j=1}^m \bar{\omega}_{ij} L_j^\eta h^{-\frac{1}{2}} \right]. \end{aligned}$$

then state feedback controller Equation (14) is a finite-time guaranteed cost controller for GRNs Equation (12) with the settling time K satisfying

$$K < -\frac{1}{\bar{\alpha} h} \ln \left(1 - \frac{1 - e^{-\underline{\alpha} h}}{\max\{v_1, v_2\}} \right).$$

Proof. In accordance with (g₆), for any $\epsilon > 0$, we can select $\delta > 0$ and $0 < \mu < \underline{\alpha}$ to be small enough, causing

$$\begin{aligned} \frac{\delta}{\epsilon} + \max_{1 \leq i \leq m} \frac{1 - e^{-(\hat{a}_i - \mu)Kh}}{1 - e^{-(\hat{a}_i - \mu)h}} \frac{(1 - e^{-a_i h})e^{\mu(\sigma_0+1)h}}{a_i} \left[\sum_{q=1}^n \frac{2|\Theta_{iq}|}{h^2} + \sum_{j=1}^m \bar{b}_{ij} L_j^f + \sum_{j=1}^m \bar{\gamma}_{ij} L_j^g h^{-\frac{1}{2}} \right] &< 1, \\ \frac{\delta}{\epsilon} + \max_{1 \leq i \leq m} \frac{1 - e^{-(\hat{c}_i - \mu)Kh}}{1 - e^{-(\hat{c}_i - \mu)h}} \frac{(1 - e^{-c_i h})e^{\mu(\sigma_0+1)h}}{c_i} \left[\sum_{q=1}^n \frac{2|\Pi_{iq}|}{h^2} + \bar{d}_i + \sum_{j=1}^m \bar{\omega}_{ij} L_j^\eta h^{-\frac{1}{2}} \right] &< 1. \end{aligned}$$

A method of reduction to absurdity will be adapted here, supposing that Equation (17) holds. If not, then one of the following two cases must be valid.

(a) There exist $k_0 \in (0, T]_{\mathbb{Z}}$ and $i_0 \in \{1, 2, \dots, m\}$ such that

$$\max_{1 \leq i \leq m} \max_{v \in \mathcal{U}_v} \left\{ \left\| \mathbf{e}_{i,k}^{(v)} \right\|_2, \left\| \mathbf{w}_{i,k}^{(v)} \right\|_2 \right\} \leq \epsilon e^{-\mu k h}, \quad \forall k \in [0, k_0]_{\mathbb{Z}}; \quad \max_{v \in \mathcal{U}_v} \left\| \mathbf{e}_{i_0, k_0}^{(v)} \right\|_2 > \epsilon e^{-\mu k_0 h}.$$

(b) There exist $k_1 \in (0, T]_{\mathbb{Z}}$ and $i_1 \in \{1, 2, \dots, m\}$ ensuring

$$\max_{1 \leq i \leq m} \max_{v \in \mathcal{U}_v} \left\{ \left\| \mathbf{e}_{i,k}^{(v)} \right\|_2, \left\| \mathbf{w}_{i,k}^{(v)} \right\|_2 \right\} \leq \epsilon e^{-\mu k h}, \quad \forall k \in [0, k_1]_{\mathbb{Z}}; \quad \max_{v \in \mathcal{U}_v} \left\| \mathbf{w}_{i_1, k_1}^{(v)} \right\|_2 > \epsilon e^{-\mu k_1 h}.$$

If (a) holds, from Equation (20) and (g5) we obtain

$$\begin{aligned} \max_{v \in \mathcal{U}_v} \left\| \mathbf{e}_{i_0, k_0}^{(v)} \right\|_2 &\leq (e^{-a_{i_0} h} + \kappa_{i_0})^{k_0} \delta + \sum_{v=0}^{k_0-1} (e^{-a_{i_0} h} + \kappa_{i_0})^{k_0-v-1} \frac{(1-e^{-a_{i_0} h})}{a_{i_0}} \\ &\quad \times \left[\sum_{q=1}^n \frac{2|\Theta_{i_0 q}|}{h^2} + \sum_{j=1}^m \bar{b}_{i_0 j} L_j^f + \sum_{j=1}^m \bar{\gamma}_{i_0 j} L_j^g h^{-\frac{1}{2}} \right] e^{\mu \sigma_0 h} \epsilon e^{-\mu v h} \\ &\leq e^{-\hat{a}_{i_0} k_0 h} \delta + \sum_{v=0}^{k_0-1} e^{-\hat{a}_{i_0} h (k_0 - v - 1)} \frac{(1-e^{-a_{i_0} h})}{a_{i_0}} \\ &\quad \times \left[\sum_{q=1}^n \frac{2|\Theta_{i_0 q}|}{h^2} + \sum_{j=1}^m \bar{b}_{i_0 j} L_j^f + \sum_{j=1}^m \bar{\gamma}_{i_0 j} L_j^g h^{-\frac{1}{2}} \right] e^{\mu \sigma_0 h} \epsilon e^{-\mu v h} \\ &= e^{-\hat{a}_{i_0} k_0 h} \delta + \sum_{v=0}^{k_0-1} e^{-(\hat{a}_{i_0} - \mu) h (k_0 - v - 1)} \frac{(1-e^{-a_{i_0} h})}{a_{i_0}} e^{\mu (\sigma_0 + 1) h} \\ &\quad \times \left[\sum_{q=1}^n \frac{2|\Theta_{i_0 q}|}{h^2} + \sum_{j=1}^m \bar{b}_{i_0 j} L_j^f + \sum_{j=1}^m \bar{\gamma}_{i_0 j} L_j^g h^{-\frac{1}{2}} \right] \epsilon e^{-\mu k_0 h} \\ &\leq \left(\frac{\delta}{\epsilon} e^{-(\hat{a}_{i_0} - \mu) k_0 h} + \frac{1-e^{-(\hat{a}_{i_0} - \mu) k_0 h}}{1-e^{-(\hat{a}_{i_0} - \mu) h}} \frac{(1-e^{-a_{i_0} h}) e^{\mu (\sigma_0 + 1) h}}{a_{i_0}} \right. \\ &\quad \times \left. \left[\sum_{q=1}^n \frac{2|\Theta_{i_0 q}|}{h^2} + \sum_{j=1}^m \bar{b}_{i_0 j} L_j^f + \sum_{j=1}^m \bar{\gamma}_{i_0 j} L_j^g h^{-\frac{1}{2}} \right] \right) \epsilon e^{-\mu k_0 h} \\ &\leq \left\{ \frac{\delta}{\epsilon} + \frac{1-e^{-(\hat{a}_{i_0} - \mu) K h}}{1-e^{-(\hat{a}_{i_0} - \mu) h}} \frac{(1-e^{-a_{i_0} h}) e^{\mu (\sigma_0 + 1) h}}{a_{i_0}} \right. \\ &\quad \times \left. \left[\sum_{q=1}^n \frac{2|\Theta_{i_0 q}|}{h^2} + \sum_{j=1}^m \bar{b}_{i_0 j} L_j^f + \sum_{j=1}^m \bar{\gamma}_{i_0 j} L_j^g h^{-\frac{1}{2}} \right] \right\} \epsilon e^{-\mu k_0 h} \\ &\leq \epsilon e^{-\mu k_0 h}, \end{aligned}$$

which contradicts fact (a).

On the other hand, if (b) holds, from Equation (21), we can likewise compute

$$\begin{aligned} \max_{v \in \mathcal{U}_v} \left\| \mathbf{w}_{i_1, k_1}^{(v)} \right\|_2 &\leq (e^{-\zeta_{i_1} h} + \varkappa_{i_1})^{k_1} \delta + \sum_{v=0}^{k_1-1} (e^{-\zeta_{i_1} h} + \varkappa_{i_1})^{k_1-v-1} \frac{(1-e^{-\zeta_{i_1} h})}{\zeta_{i_1}} \\ &\quad \times \left[\sum_{q=1}^n \frac{2|\Pi_{i_1 q}|}{h^2} + \bar{d}_{i_1} + \sum_{j=1}^m \bar{\omega}_{i_1 j} L_j^\eta h^{-\frac{1}{2}} \right] e^{\mu \sigma_0 h} \epsilon e^{-\mu v h} \\ &\leq e^{-\hat{\zeta}_{i_1} k_1 h} \delta + \sum_{v=0}^{k_1-1} e^{-\hat{\zeta}_{i_1} h (k_1 - v - 1)} \frac{(1-e^{-\zeta_{i_1} h})}{\zeta_{i_1}} \\ &\quad \times \left[\sum_{q=1}^n \frac{2|\Pi_{i_1 q}|}{h^2} + \bar{d}_{i_1} + \sum_{j=1}^m \bar{\omega}_{i_1 j} L_j^\eta h^{-\frac{1}{2}} \right] e^{\mu \sigma_0 h} \epsilon e^{-\mu v h} \\ &= e^{-\hat{\zeta}_{i_1} k_1 h} \delta + \sum_{v=0}^{k_1-1} e^{-(\hat{\zeta}_{i_1} - \mu) h (k_1 - v - 1)} \frac{(1-e^{-\zeta_{i_1} h})}{\zeta_{i_1}} e^{\mu (\sigma_0 + 1) h} \\ &\quad \times \left[\sum_{q=1}^n \frac{2|\Pi_{i_1 q}|}{h^2} + \bar{d}_{i_1} + \sum_{j=1}^m \bar{\omega}_{i_1 j} L_j^\eta h^{-\frac{1}{2}} \right] \epsilon e^{-\mu k_1 h} \\ &\leq \left(\frac{\delta}{\epsilon} e^{-(\hat{\zeta}_{i_1} - \mu) k_1 h} + \frac{1-e^{-(\hat{\zeta}_{i_1} - \mu) k_1 h}}{1-e^{-(\hat{\zeta}_{i_1} - \mu) h}} \frac{(1-e^{-\zeta_{i_1} h}) e^{\mu (\sigma_0 + 1) h}}{\zeta_{i_1}} \right. \\ &\quad \times \left. \left[\sum_{q=1}^n \frac{2|\Pi_{i_1 q}|}{h^2} + \bar{d}_{i_1} + \sum_{j=1}^m \bar{\omega}_{i_1 j} L_j^\eta h^{-\frac{1}{2}} \right] \right) \epsilon e^{-\mu k_1 h} \\ &\leq \left\{ \frac{\delta}{\epsilon} + \frac{1-e^{-(\hat{\zeta}_{i_1} - \mu) K h}}{1-e^{-(\hat{\zeta}_{i_1} - \mu) h}} \frac{(1-e^{-\zeta_{i_1} h}) e^{\mu (\sigma_0 + 1) h}}{\zeta_{i_1}} \right. \\ &\quad \times \left. \left[\sum_{q=1}^n \frac{2|\Pi_{i_1 q}|}{h^2} + \bar{d}_{i_1} + \sum_{j=1}^m \bar{\omega}_{i_1 j} L_j^\eta h^{-\frac{1}{2}} \right] \right\} \epsilon e^{-\mu k_1 h} \\ &\leq \epsilon e^{-\mu k_1 h}, \end{aligned}$$

which contradicts fact (b). As a consequence, state feedback controller Equation (14) with the control gains in (g₅) stabilises GRNs Equation (12) in finite time.

In light of Definition 7, the finite-time guaranteed cost control will be displayed as follows. It holds that

$$\begin{aligned} \mathbf{U}^T \mathbf{U} &= \begin{pmatrix} \mathbf{e} & \rho & \mathbf{w} & \varrho \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \rho \\ \mathbf{w} \\ \varrho \end{pmatrix} \\ &= \sum_{i=1}^m (\mathbf{e}_i^2 + \rho_i^2 + \mathbf{w}_i^2 + \varrho_i^2) \\ &= \sum_{i=1}^m [(1 + \kappa_i^2) \mathbf{e}_i^2 + (1 + \varkappa_i^2) \mathbf{w}_i^2] \\ &\leq \sum_{i=1}^m (2 + \kappa_i^2 + \varkappa_i^2) \max\{\mathbf{e}_i^2, \mathbf{w}_i^2\}, \end{aligned}$$

which induces

$$J_c^K = \mathbf{E} \sum_{k=0}^K \max_{v \in \mathcal{U}_v} \mathbf{U}_k^{(v)T} \mathbf{U}_k^{(v)} \leq \vartheta \sum_{k=0}^K \max_{1 \leq i \leq m} \max_{v \in \mathcal{U}_v} \{\|\mathbf{e}_{i,k}^{(v)}\|_2, \|\mathbf{w}_{i,k}^{(v)}\|_2\} \leq \vartheta \sum_{k=0}^K \epsilon e^{-\mu k h} \leq \lambda,$$

where $\vartheta = \sum_{i=1}^m (2 + \kappa_i^2 + \varkappa_i^2)$ and $\lambda = \frac{\vartheta}{1 - e^{-\mu h}}$. Therefore, state feedback controller Equation (14) is a finite-time guaranteed cost controller for GRNs Equation (12). This completes the proof. $\square$

If $\kappa_i = 0 = \varkappa_i$ in feedback controller Equation (14), then $\rho_{i,k}^{(v)}$ and $\varrho_{i,k}^{(v)}$ are vanished from GRNs Equation (12), $\forall (v, k) \in \mathcal{U}_v \times \mathbb{N}_0$ and $i = 1, 2, \dots, m$. Based upon the proof of Theorem 3, we can easily obtain

Corollary 2. Let assumptions (g₂)–(g₄) hold. Then, GRNs Equation (1) is finite-time exponentially stable in a mean-square sense. Further, if (g₁) holds, then GRNs Equation (1) admits a unique weighted pseudo almost periodic sequence solution, which is finite-time exponentially stable in a mean-square sense.

5. Example

This section gives an experimental example to verify the feasibility of the main results for discrete-space and -time stochastic GRNs, which have been addressed in the previous sections of this article.

Considering the following discrete-time stochastic GRNs with discrete spatial diffusions

$$\left\{ \begin{aligned} \begin{pmatrix} \mathbf{m}_{1,k+1}^{(v)} \\ \mathbf{m}_{2,k+1}^{(v)} \end{pmatrix} &= \begin{pmatrix} e^{-9h} & 0 \\ 0 & e^{-10h} \end{pmatrix} \begin{pmatrix} \mathbf{m}_{1,k}^{(v)} \\ \mathbf{m}_{2,k}^{(v)} \end{pmatrix} + \begin{pmatrix} \frac{1-e^{-9h}}{9} & 0 \\ 0 & \frac{1-e^{-10h}}{10} \end{pmatrix} \left[\begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix} \tilde{\Delta}_h^2 \begin{pmatrix} \mathbf{m}_{1,k}^{(v)} \\ \mathbf{m}_{2,k}^{(v)} \end{pmatrix} \right. \\ &\quad + \begin{pmatrix} 1.2 \cos(k\pi + \frac{\pi}{4}) + e^{-|k|} & 0.5 \\ 0.3 & 1.8 \sin(k\pi + \frac{\pi}{4}) + e^{-|k|} \end{pmatrix} \begin{pmatrix} f_1(\mathbf{p}_{1,k-2}^{(v)}) \\ f_2(\mathbf{p}_{2,k-1}^{(v)}) \end{pmatrix} \\ &\quad \left. + \begin{pmatrix} 0.2 & 0.1 \\ 0 & 0.15 \end{pmatrix} \begin{pmatrix} g_1(\mathbf{p}_{1,k-1}^{(v)}) w_{11,k} \\ g_2(\mathbf{p}_{2,k-1}^{(v)}) w_{12,k} \end{pmatrix} + \begin{pmatrix} 1 + 0.2 \cos(k\pi + \frac{\pi}{4}) \\ 0.5 + 0.5 \sin(k\pi + \frac{\pi}{4}) \end{pmatrix} \right], \\ \begin{pmatrix} \mathbf{p}_{1,k+1}^{(v)} \\ \mathbf{p}_{2,k+1}^{(v)} \end{pmatrix} &= \begin{pmatrix} e^{-12h} & 0 \\ 0 & e^{-15h} \end{pmatrix} \begin{pmatrix} \mathbf{p}_{1,k}^{(v)} \\ \mathbf{p}_{2,k}^{(v)} \end{pmatrix} + \begin{pmatrix} \frac{1-e^{-12h}}{12} & 0 \\ 0 & \frac{1-e^{-15h}}{15} \end{pmatrix} \left[\begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix} \tilde{\Delta}_h^2 \begin{pmatrix} \mathbf{p}_{1,k}^{(v)} \\ \mathbf{p}_{2,k}^{(v)} \end{pmatrix} \right. \\ &\quad \left. + 0.1 \begin{pmatrix} |\mathbf{m}_{1,k}^{(v)}| \\ |\mathbf{m}_{2,k}^{(v)}| \end{pmatrix} + \begin{pmatrix} \cos(k\pi + \frac{\pi}{3}) + e^{-|k|} & 0.1 \\ 0 & \sin(k\pi + \frac{\pi}{5}) + e^{-|k|} \end{pmatrix} \begin{pmatrix} \eta_1(\mathbf{m}_{1,k-1}^{(v)}) w_{21,k} \\ \eta_2(\mathbf{m}_{2,k-2}^{(v)}) w_{22,k} \end{pmatrix} \right], \end{aligned} \right. \quad (22)$$

where $(v, k) \in (0, 10) \times \mathbb{Z}_0$,

$$\mathbf{m}_{i,k}^{(v)} \Big|_{v=0} = \mathbf{m}_{i,k}^{(v)} \Big|_{v=10} = 0, \quad \mathbf{p}_{i,k}^{(v)} \Big|_{v=0} = \mathbf{p}_{i,k}^{(v)} \Big|_{v=10} = 0, \quad \forall k \in \mathbb{Z}_0, i = 1, 2.$$

Taking $h = 0.1$ and $\hbar = 0.5$. Corresponding to GRNs Equation (1),

$$\begin{aligned} a_{1,k} &= 9, \quad a_{2,k} = 10, \quad c_{1,k} = 12, \quad c_{2,k} = 15, \quad \Theta_{11} = \Theta_{22} = 0.2, \quad \Pi_{11} = \Pi_{22} = 0.1, \\ \Theta_{12} &= \Theta_{21} = \Pi_{12} = \Pi_{21} = 0, \quad b_{11,k} = 1.2 \cos(k\pi + \frac{\pi}{4}) + e^{-|k|}, \quad b_{12,k} = 0.5, \quad b_{21,k} = 0.3, \\ b_{22,k} &= 1.8 \sin(k\pi + \frac{\pi}{4}) + e^{-|k|}, \quad \sigma_{1,k} = 2, \quad \sigma_{2,k} = 1, \quad \gamma_{11,k} = 0.2, \quad \gamma_{12,k} = 0.1, \quad \gamma_{21,k} = 0, \\ \gamma_{22,k} &= 0.15, \quad I_{1,k} = 1 + 0.2 \cos(k\pi + \frac{\pi}{4}), \quad I_{2,k} = 0.5 + 0.5 \sin(k\pi + \frac{\pi}{4}), \quad d_{1,k} = d_{2,k} = 0.1, \\ \omega_{11,k} &= \cos(k\pi + \frac{\pi}{3}) + e^{-|k|}, \quad \omega_{12,k} = 0.1, \quad \omega_{21,k} = 0, \quad \omega_{22,k} = \sin(k\pi + \frac{\pi}{5}) + e^{-|k|}, \\ f_1(\mathbf{p}_{1,k-2}^{(v)}) &= \frac{\left(\frac{\mathbf{p}_{1,k-2}^{(v)}}{15}\right)^2}{1 + \left(\frac{\mathbf{p}_{1,k-2}^{(v)}}{15}\right)^2}, \quad f_2(\mathbf{p}_{2,k-1}^{(v)}) = \frac{\left(\frac{\mathbf{p}_{2,k-1}^{(v)}}{15}\right)^2}{1 + \left(\frac{\mathbf{p}_{2,k-1}^{(v)}}{15}\right)^2}, \quad g_i(\mathbf{p}_{i,k-1}^{(v)}) = \frac{\left(\frac{\mathbf{p}_{i,k-1}^{(v)}}{20}\right)^2}{1 + \left(\frac{\mathbf{p}_{i,k-1}^{(v)}}{20}\right)^2}, \\ \eta_1(\mathbf{m}_{1,k-1}^{(v)}) &= \frac{\left(\frac{\mathbf{m}_{1,k-1}^{(v)}}{10}\right)^2}{1 + \left(\frac{\mathbf{m}_{1,k-1}^{(v)}}{10}\right)^2}, \quad \eta_2(\mathbf{m}_{2,k-2}^{(v)}) = \frac{\left(\frac{\mathbf{m}_{2,k-2}^{(v)}}{10}\right)^2}{1 + \left(\frac{\mathbf{m}_{2,k-2}^{(v)}}{10}\right)^2}, \quad i = 1, 2, \forall (v, k) \in (0, 10) \times \mathbb{Z}_0. \end{aligned}$$

Obviously, $L_1^f = L_2^f = \frac{1}{15}$, $L_1^g = L_2^g = 0.05$, $L_1^\eta = L_2^\eta = 0.1$. It follows from the direct calculation that $\max\{\zeta_1, \zeta_2\} < 1$. Therefore, assumptions $(\mathbf{g}_1)$ – $(\mathbf{g}_4)$ in Theorem 2 are valid, i.e., GRNs Equation (22) possesses a unique weighted pseudo- or α -pseudo-almost periodic sequence solution, see Figures 1 and 2. Let $\hat{a}_1 = 1.25$, $\hat{a}_2 = 12$, $\hat{c}_1 = 14$, and $\hat{c}_2 = 7$. Then, the state feedback controllers corresponding to Equation (14) are listed as follows:

$$\rho_{1,k}^{(\cdot)} = 0.4983\mathbf{e}_{1,k}^{(\cdot)}, \quad \rho_{2,k}^{(\cdot)} = -0.0667\mathbf{e}_{2,k}^{(\cdot)}, \quad \varrho_{1,k}^{(\cdot)} = -0.0546\mathbf{w}_{1,k}^{(\cdot)}, \quad \varrho_{2,k}^{(\cdot)} = 0.2735\mathbf{w}_{2,k}^{(\cdot)}, \quad (23)$$

where $k \in \mathbb{Z}_0$. Moreover, assumptions $(\mathbf{g}_2)$ and $(\mathbf{g}_5)$ – $(\mathbf{g}_6)$ in Theorem 3 hold. Then, the state feedback controller Equation (23) is a finite-time guaranteed cost controller for GRNs Equation (22) with the settling time K satisfying $K < 4.0294$, see Figures 3–6. Finally, the trajectories of the finite-time exponential stability of GRNs Equation (22) in three-dimensional and two-dimensional spaces are shown in Figures 7–10.

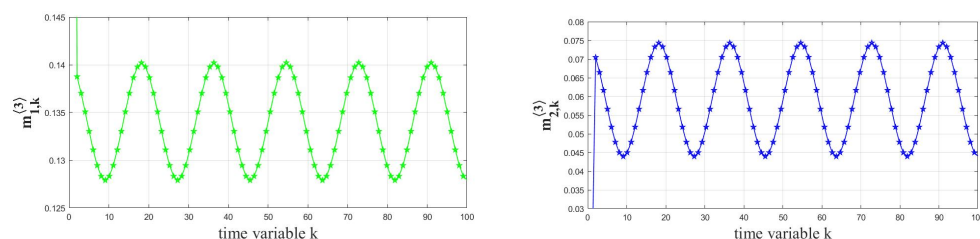


Figure 1. Weighted pseudo-almost periodic sequence solution of $\mathbf{m}_{1,k}^{(3)}$ and $\mathbf{m}_{2,k}^{(3)}$.

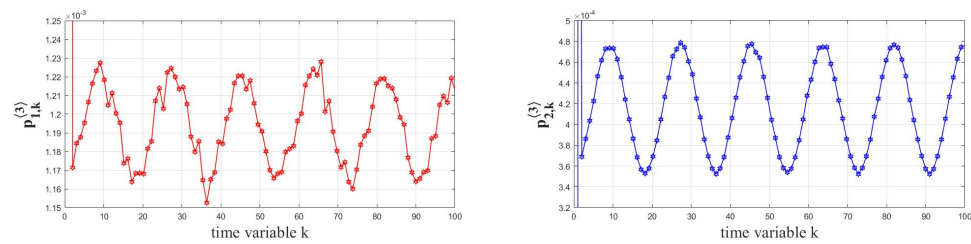


Figure 2. Weighted pseudo-almost periodic sequence solution of $p_{1,k}^{(3)}$ and $p_{2,k}^{(3)}$.

In Figures 1 and 2, the pictures show the weighted pseudo-almost periodicity of $\mathbf{m}$ and $\mathbf{p}$ in GRNs Equation (22). From these pictures, we can observe that the solution of GRNs Equation (22) is not weighted pseudo-almost periodic at the beginning of the time, but it becomes almost periodic as the time increases.

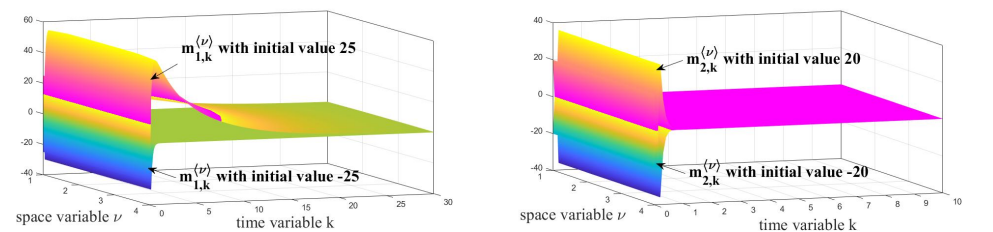


Figure 3. Finite-time guaranteed cost controller for $m_{1,k}^{(v)}$ and $m_{2,k}^{(v)}$ with the settling time K satisfying $K < 4.0294$.

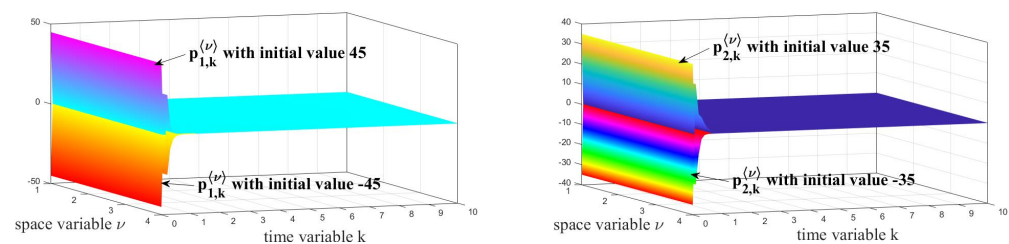


Figure 4. Finite-time guaranteed cost controller for $p_{1,k}^{(v)}$ and $p_{2,k}^{(v)}$ with the settling time K satisfying $K < 4.0294$.

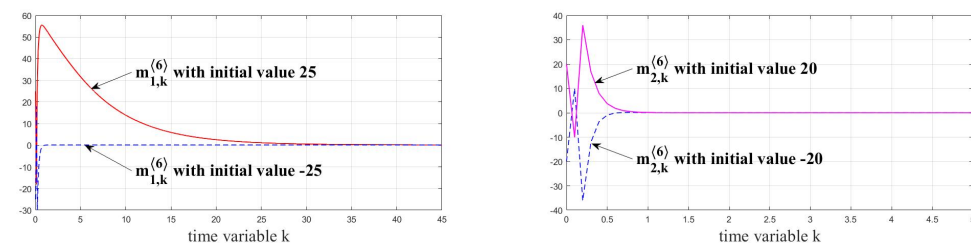


Figure 5. Finite-time guaranteed cost controller for $m_{1,k}^{(6)}$ and $m_{2,k}^{(6)}$ with the settling time K satisfying $K < 4.0294$.

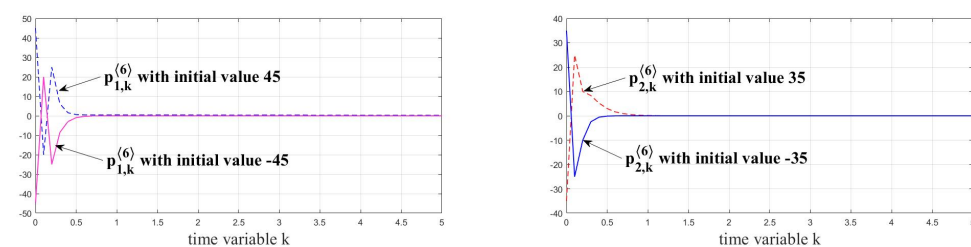


Figure 6. Finite-time guaranteed cost controller for $p_{1,k}^{(6)}$ and $p_{2,k}^{(6)}$ with the settling time K satisfying $K < 4.0294$.

In Figures 3 and 4, the pictures show the trajectories of $\mathbf{m}$ and $\mathbf{p}$ in GRNs Equation (22) with feedback controls in the closed loop. By observing these pictures, we can observe that the solutions of GRNs Equation (22) with different initial values realise finite-time exponential stability in three-dimensional space. Figures 5 and 6 give the trajectories of $\mathbf{m}$ and $\mathbf{p}$ of GRNs Equation (22) with feedback controls in the closed loop when $\nu = 6$.

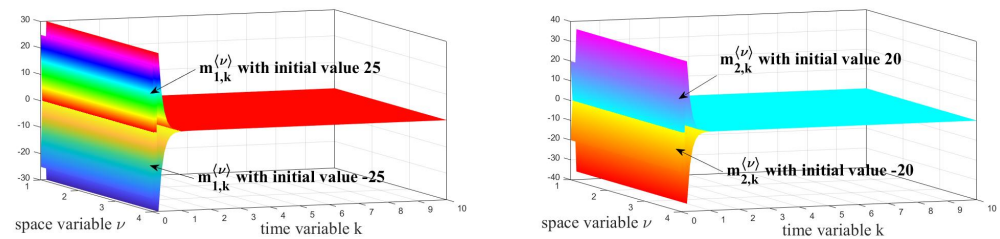


Figure 7. Finite-time exponential stability of $\mathbf{m}_{1,k}^{(\nu)}$ and $\mathbf{m}_{2,k}^{(\nu)}$.

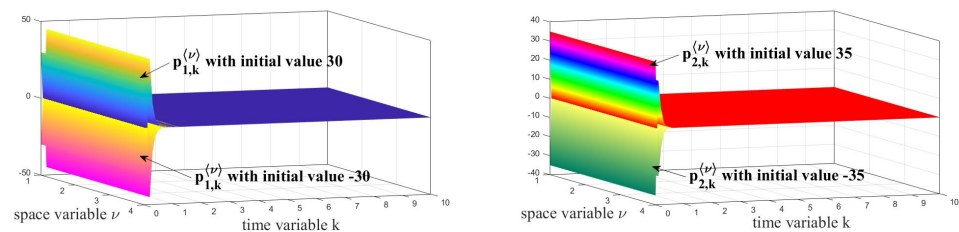


Figure 8. Finite-time exponential stability of $\mathbf{p}_{1,k}^{(\nu)}$ and $\mathbf{p}_{2,k}^{(\nu)}$.

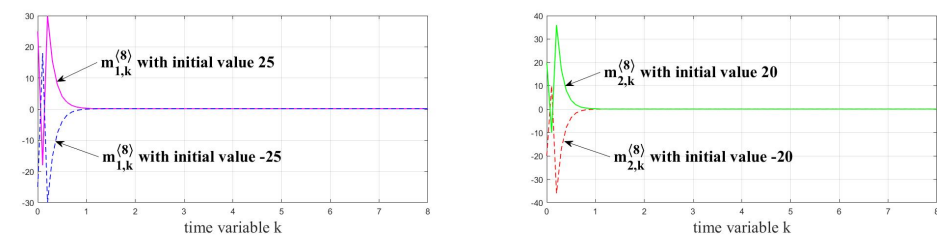


Figure 9. Finite-time exponential stability of $\mathbf{m}_{1,k}^{(8)}$ and $\mathbf{m}_{2,k}^{(8)}$.

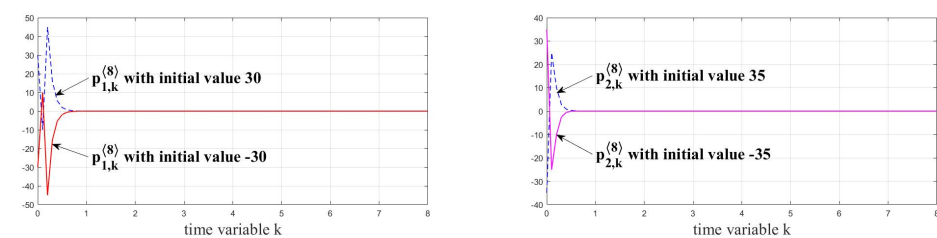


Figure 10. Finite-time exponential stability of $\mathbf{p}_{1,k}^{(8)}$ and $\mathbf{p}_{2,k}^{(8)}$.

Figures 7 and 8 show the solutions of GRNs Equation (22) without feedback control, realising finite-time exponential stability in three-dimensional space. Figures 9 and 10 draw the solutions of GRNs Equation (22) without feedback control, realising finite-time exponential stability when $\nu = 8$.

6. Conclusions and Perspectives

Utilizing EED and CFT techniques, discrete stochastic genetic regulatory networks with discrete spatial diffusion are presented, which can be considered as fully discrete configurations of stochastic genetic regulatory networks with reaction diffusion. Based on the constant variable formulation in discrete form, the existence uniqueness, the finite-time guaranteed cost control, and the exponential stability of the weighted pseudo- θ -almost

periodic sequence of such discrete stochastic genetic regulatory networks in the mean-square sense are discussed. In addition, Lemmas 2 and 3, among others, have been crucial to the discussion in this paper over the course of the study. Notably, the work in this paper will initiate the development of qualitative problems in discrete-time and discrete-space models, laying the theoretical and practical foundations for future work in this area.

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