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Applications of Shell-like Curves Connected with Fibonacci Numbers

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Abstract: We introduce a new subclass $\mathbf{J}\Sigma^{\eta,\delta,\mu}(\widetilde{p})$ of bi-univalent functions, defined by shell-like curves connected with Fibonacci numbers. Our main results in this paper include estimates of the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclass, as well as solutions to Fekete–Szegö functional problems. We also show novel outcomes resulting from the specialization of the parameters used in our main results.

Keywords: Fekete–Szegö problem; bi-univalent functions; Fibonacci numbers; analytic functions; shell-like curve

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1. Introduction

The Fibonacci numbers form a sequence of integers, which are defined recursively as follows: $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. In other words, each term is the sum of the two preceding terms [1]. The first few terms of the sequence are: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$ The Fibonacci numbers have many interesting properties and applications. For example, they appear in the growth patterns of various biological structures, such as branching in trees and the arrangement of leaves on a stem. They also appear in various mathematical and scientific contexts, such as number theory, geometry, and physics. The Fibonacci sequence can be generated in many ways, including using matrix multiplication, generating functions, and various combinatorial methods. The sequence has many interesting properties and relationships with other mathematical objects, such as the golden ratio, Lucas numbers, and Pell numbers.

Shell-like curves are a family of mathematical curves that are inspired by the shapes of shells found in nature. These curves are characterized by their smooth, spiraling shapes, which can be generated using simple mathematical equations.

One well-known example of a shell-like curve is the logarithmic spiral, which is a type of spiral that increases or decreases in size as it rotates around a central point. The logarithmic spiral is often used to model various natural phenomena, such as the spiral growth patterns of shells, horns, and even galaxies.

Another type of shell-like curve is the Archimedean spiral, which is a spiral that is created by a point moving away from a fixed point with a constant speed along a line that rotates with constant angular velocity. This type of spiral is often used in engineering and design, as it allows for the creation of smooth, continuous curves with a consistent spacing between them.

Shell-like curves can also be generated using more complex mathematical equations, such as the Fibonacci spiral, which is a spiral that is created by connecting points along a

Fibonacci sequence of rectangles. This type of spiral is often used in art and design, as it allows for the creation of visually striking patterns and shapes.

Overall, shell-like curves are a fascinating area of mathematical study with numerous applications in science, engineering, art, and design.

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \qquad (z \in \mathbb{U}). \tag{1}$$

which are analytic in the disk $\mathbb{U} = \{z : |z| < 1\}$ and gratify the normalization condition $f'(0) - 1 = 0 = f(0)$. Moreover, we use $\mathcal{S}$ to denote the subclass of $\mathcal{A}$ comprising functions of Equation (1), which are also univalent in $\mathbb{U}$.

Geometric function theory can benefit greatly from the powerful tools that the differential subordination of analytical functions provides. Miller and Mocanu [2] introduced the first differential subordination problem; additionally, see [3]. The majority of the developments in the field were compiled in Miller and Mocanu's book [4], along with the publication dates.

Additionally, every function $f \in \mathcal{A}$ that is univalent in $\mathbb{U}$ will be represented as $\mathcal{S}$. Therefore, each $f \in \mathcal{S}$ possesses an inverse denoted by f^{-1}, which is defined as follows:

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is in the class Σ of all bi-univalent functions in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. For some fascinating facts of functions in the class Σ, see ([5–9]).

Lewin [10] is one of the first researchers who studied Bi-Univalent Functions, and recently, the researcher Illafe et al. [11] explored a new topic on Bi-Univalent Functions. In our paper, we studied and worked on the subordinate relationship between Fibonacci numbers and Shell-Like Curves and found $|a_2|$, $|a_3|$ and Fekete–Szegö functional problems. In reference [11], the authors established a subordinate relationship between Gegenbauer polynomials and used a series based on the zero-truncated Poisson distribution. They also found $|a_2|$, $|a_3|$ and Fekete–Szegö functional problems. My study, on the other hand, is considered an advancement and generalization of the work of many scientists, achieving notable results in this field.

Let $\boldsymbol{B}$ denote the class of functions of the form:

$$b(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{3}$$

which are analytic with $\mathrm{Re}\{b(z)\} > 0$. In this context, the function $b(z)$ is referred to as Caratheodory functions [12]. $b \in \boldsymbol{B}$ iff $b(z) = (1 + w(z))/(1 - w(z))$, provided that a Schwarz function w exists. Recently, Sokół [13] and Dziok et al. [14] studied the classes $\mathcal{SL}(\widetilde{b})$ of shell-like functions, which are characterized by $\frac{zf'(z)}{f(z)} \prec \widetilde{b}(z)$ and $\mathcal{KSL}(\widetilde{b})$ of convex shell-like functions characterized by $1 + \frac{z^2 f''(z)}{f'(z)} \prec \widetilde{b}(z)$, where $\widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$, $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$, see [15,16]. It is worth mentioning that the function $\widetilde{b}$ is not univalent in $\mathbb{U}$ but is univalent in the disc $|z| < \frac{3-\sqrt{5}}{2} \approx 0.38$. As equation $\tau^2 = 1 + \tau$ is satisfied by τ, it is possible to express higher powers τ^n as linear combinations of lower powers, ultimately leading to a linear combination of τ and 1. Using the recurrence relationships derived from this decomposition, one can obtain the Fibonacci numbers σ_n.

$$\tau^n = \sigma_n \tau + \sigma_{n-1}.$$

Raina and Sokół [16] proved that

$$\widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} = 1 + \sum_{n=2}^{\infty}(\sigma_{n-1}+\sigma_{n+1})\tau^n z^n,$$

where

$$\sigma_n = \frac{(1-\tau)^n - (\tau)^n}{\sqrt{5}}, \quad \tau = \frac{1-\sqrt{5}}{2}, \quad n = 1, 2, \cdots.$$

This demonstrates that:

$$\sigma_0 = 0, \quad \sigma_1 = 1, \quad \sigma_{n+2} = \sigma_n + \sigma_{n+1}, \quad n = 0, 1, 2, \cdots.$$

Hence,

$$\begin{aligned}\widetilde{b}(z) &= 1 + \sum_{n=1}^{\infty} \widetilde{b}_n z^n \\ &= 1 + (\sigma_0+\sigma_2)\tau z + (\sigma_1+\sigma_3)\tau^2 z^2 + \sum_{n=3}^{\infty}(\sigma_{n-3}+\sigma_{n-2}+\sigma_{n-1}+\sigma_n)\tau^n z^n \\ &= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots.\end{aligned} \tag{4}$$

We note that the $\widetilde{b} \in \boldsymbol{B}(\boldsymbol{\alpha})$ with $\alpha = \frac{\sqrt{5}}{10} \approx 0.2236$; see [16].

Fibonacci numbers have been shown to be related to several polynomial families, such as Pell-Lucas polynomials, Gegenbauer polynomials, Chebyshev polynomials, Horadam polynomials, Fermat–Lucas polynomials, and their generalizations, which have applications in various fields, including physics, architecture, combinatorics, number theory, statistics, and engineering. Detailed information on these polynomials can be found in [17–19]. Moreover, the Fekete–Szegö functional, which is widely used for bi-univalent functions based on k-Fibonacci numbers, is discussed in ([20–23]).

A current research trend involves studying functions in the class Σ based on various polynomials such as Fibonacci, Pell–Lucas, Gegenbauer, Chebyshev, Horadam, and Fermat–Lucas polynomials, which have applications in fields such as physics, architecture, combinatorics, number theory, statistics, and engineering. Recent papers, such as ([24–37]), have focused on estimating the first two coefficient bounds and the functional of Fekete–Szegö for specific subfamilies of Σ.

2. Definitions

In this section, we introduce some new subclasses of bi-univalent functions to which the Fibonacci numbers are subordinate.

Definition 1. *A function* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbf{J}_{\Sigma}^{\kappa,\daleth,\gamma}(\widetilde{b})$, *where* $\daleth, \kappa \geq 1$, $\gamma \in \mathbb{C}$ *and* $Re(\gamma) \geq 0$, *if it satisfies:*

$$(1-\kappa)f'(z) + \kappa\big(f'(z)\big)^{\daleth}\left(\frac{f(z)}{z}\right)^{\gamma-1} \prec \widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \qquad (z \in \mathbb{U}) \tag{5}$$

and for g given by Equation (2),

$$(1-\kappa)g'(w) + \kappa\big(g'(w)\big)^{\daleth}\left(\frac{g(w)}{w}\right)^{\gamma-1} \prec \widetilde{b}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \qquad (w \in \mathbb{U}), \tag{6}$$

where $\tau = \frac{1-\sqrt{5}}{2} \approx -0.618$.

Specializing the values of $\daleth, \kappa$ and γ *where* $\tau = \frac{1-\sqrt{5}}{2}$, the class $\mathbf{J}_{\Sigma}^{\kappa,\daleth,\gamma}(\widetilde{b})$ reduces to various new subclasses. For example:

(i) For $\kappa = 1$, we can obtain the class $\mathbf{J}_{\Sigma}^{1,\daleth,\gamma}(\widetilde{b}) = \mathbf{J}_{\Sigma}^{\daleth,\gamma}(\widetilde{b})$. A function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbf{J}_{\Sigma}^{\daleth,\gamma}(\widetilde{b})$, where $\daleth \geq 1$, $\gamma \in \mathbb{C}$ and $\mathrm{Re}\{\gamma\} \geq 0$, if satisfying:

$$(f'(z))^{\daleth}\left(\frac{f(z)}{z}\right)^{\gamma-1} \prec \widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \qquad (z \in \mathbb{U}) \tag{7}$$

and for g given by Equation (2)

$$(g'(w))^{\daleth}\left(\frac{g(w)}{w}\right)^{\gamma-1} \prec \widetilde{b}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \qquad (w \in \mathbb{U}). \tag{8}$$

(ii) For $\kappa = \daleth = 1$; we get the class $\mathbf{J}_{\Sigma}^{1,1,\gamma}(\widetilde{b}) = \mathbf{J}_{\Sigma}^{\gamma}(\widetilde{b})$. A function $f(z)$ given by Equation (1) is said to be in the class $\mathbf{J}_{\Sigma}^{\gamma}(\widetilde{b})$, where $\gamma \in \mathbb{C}$ and $\mathrm{Re}\{\gamma\} \geq 0$ if satisfying:

$$f'(z)\left(\frac{f(z)}{z}\right)^{\gamma-1} \prec \widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \qquad (z \in \mathbb{U}) \tag{9}$$

and for g given by Equation (2)

$$g'(w)\left(\frac{g(w)}{w}\right)^{\gamma-1} \prec \widetilde{b}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \qquad (w \in \mathbb{U}). \tag{10}$$

(iii) For $\kappa = \daleth = \gamma = 1$; we get the class $\mathbf{J}_{\Sigma}^{1,1,1}(\widetilde{b}) = \mathbf{J}_{\Sigma}(\widetilde{b})$. A function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbf{J}_{\Sigma}(\widetilde{b})$, if the satisfied:

$$f'(z) \prec \widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \qquad (z \in \mathbb{U}) \tag{11}$$

and for g given by Equation (2)

$$g'(w) \prec \widetilde{b}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \qquad (w \in \mathbb{U}). \tag{12}$$

(iv) For $\kappa = 1$ and $\daleth = 1$, we get the class $\mathbf{J}_{\Sigma}^{1,1,\gamma}(\widetilde{b}) = \mathbf{J}_{\Sigma}^{1,\gamma}(\widetilde{b})$. A function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbf{J}_{\Sigma}^{1,\gamma}(\widetilde{b})$, where $\gamma \in \mathbb{C}$ and $\mathrm{Re}\{\gamma\} \geq 0$, if it satisfies:

$$(f'(z))\left(\frac{f(z)}{z}\right)^{\gamma-1} \prec \widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \qquad (z \in \mathbb{U}) \tag{13}$$

and for g given by (2)

$$(g'(w))\left(\frac{g(w)}{w}\right)^{\gamma-1} \prec \widetilde{b}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \qquad (w \in \mathbb{U}). \tag{14}$$

(v) For $\kappa = 1$ and $\gamma = 1$, we get the class $\mathbf{J}_{\Sigma}^{1,\daleth,1}(\widetilde{b}) = \mathbf{J}_{\Sigma}^{\daleth,1}(\widetilde{b})$. A function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbf{J}_{\Sigma}^{\daleth,\gamma}(\widetilde{b})$, where $\daleth \geq 1$, and $\mathrm{Re}\{\gamma\} \geq 0$, if satisfied:

$$(f'(z))^{\daleth} \prec \widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \qquad (z \in \mathbb{U}) \tag{15}$$

and for g given by (2)

$$(g'(w))^{\daleth} \prec \widetilde{b}(w) = \frac{1+\tau^2 w^2}{1-\tau w-\tau^2 w^2} \qquad (w \in \mathbb{U}). \tag{16}$$

(vi) For $\gamma = 1$, we get the class $\mathbf{J}_{\Sigma}^{\kappa,\daleth}(\widetilde{b})$. A function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbf{J}_{\Sigma}^{\daleth}(\widetilde{b})$, where $\daleth \geq 1$, if it satisfies:

$$(1-\kappa) f'(z) + \kappa\big(f'(z)\big)^{\daleth} \prec \widetilde{b}(z) = \frac{1+\tau^2 z^2}{1-\tau z - \tau^2 z^2} \qquad (z \in \mathbb{U}) \tag{17}$$

and for g given by (2),

$$(1-\kappa) g'(w) + \kappa\big(g'(w)\big)^{\daleth} \prec \widetilde{b}(w) = \frac{1+\tau^2 w^2}{1-\tau w - \tau^2 w^2} \qquad (w \in \mathbb{U}), \tag{18}$$

We need the following lemma to prove our results.

Lemma 1 ([38]). *If $b \in \boldsymbol{B}$, then $|b_i| \leq 2$ for each i, where $b(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$.*

3. Coefficient Bounds and Fekete–Szegö Inequality

In the following theorem, we discuss the coefficient estimate and Fekete–Szegö inequality for functions in the class $\mathbf{J}_{\Sigma}^{\kappa,\daleth,\gamma}(\widetilde{b})$.

Theorem 1. *Let $f(z)$ given by Equation (1) be in the function class $\mathbf{J}_{\Sigma}^{\kappa,\daleth,\gamma}(\widetilde{b})$, where $\daleth, \kappa \geq 1$, $\gamma \in \mathbb{C}$ and $Re\{\gamma\} \geq 0$. Then,*

$$|a_2| \leq \sqrt{\frac{2}{|\chi|}}|\tau|, \quad |a_3| \leq \frac{|\tau|[\chi + 2\tau(\kappa(3\daleth + \gamma - 4) + 3)]}{(\kappa(3\daleth + \gamma - 4) + 3)\chi} \tag{19}$$

and for $\varepsilon \in \mathbb{R}$

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{\kappa(3\daleth+\gamma-4)+3}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{4(\kappa(3\daleth+\gamma-4)+3)} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{4(\kappa(3\daleth+\gamma-4)+3)} \end{cases}, \tag{20}$$

where

$$\chi = \kappa\tau[(\gamma+2)(\gamma-3) + 4\daleth(\gamma-1) + 2\daleth(2\daleth+1)] + 6\tau + 2(1-3\tau)(\kappa(2\daleth+\gamma-3)+2)^2$$

and $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{2\chi}$.

Proof. Since $f \in \mathbf{J}_{\Sigma}^{\kappa,\daleth,\gamma}(\widetilde{b})$, from Equations (17) and (18), we have:

$$(1-\kappa) f'(z) + \kappa\big(f'(z)\big)^{\daleth}\left(\frac{f(z)}{z}\right)^{\gamma-1} = \widetilde{b}(s(z)) \tag{21}$$

and for $f^{-1} = g$,

$$(1-\kappa) g'(w) + \kappa\big(g'(w)\big)^{\daleth}\left(\frac{g(w)}{w}\right)^{\gamma-1} = \widetilde{b}(r(w)). \tag{22}$$

Using the fact that $b \prec \widetilde{b}$ and $b(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots$. Then, there exists a function $s \in \mathcal{A}$ such that $|s(z)| < 1$ in $\mathbb{U}$ and $b(z) = \widetilde{b}(s(z))$. Therefore, define the function

$$t(z) = \frac{1+s(z)}{1-s(z)} = 1 + s_1 z + s_2 z^2 + \cdots.$$

Note that $t(z) \in \boldsymbol{B}$. It follows that

$$s(z) = \frac{t(z)-1}{t(z)+1} = \frac{s_1}{2}z + \left(s_2 - \frac{s_1^2}{2}\right)\frac{z^2}{2} + \left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right)\frac{z^3}{2} + \cdots$$

and

$$\begin{aligned}
\widetilde{b}(s(z)) &= 1 + \widetilde{b}_1\left(\frac{s_1}{2}z + \left(s_2 - \frac{s_1^2}{2}\right)\frac{z^2}{2} + \left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right)\frac{z^3}{2} + \cdots\right) \\
&+ \widetilde{b}_2\left(\frac{s_1}{2}z + \left(s_2 - \frac{s_1^2}{2}\right)\frac{z^2}{2} + \left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right)\frac{z^3}{2} + \cdots\right)^2 \\
&+ \widetilde{b}_3\left(\frac{s_1}{2}z + \left(s_2 - \frac{s_1^2}{2}\right)\frac{z^2}{2} + \left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right)\frac{z^3}{2} + \cdots\right)^3 + \cdots \qquad (23) \\
&= 1 + \frac{\widetilde{b}_1 s_1}{2}z + \left(\frac{1}{2}\left(s_2 - \frac{s_1^2}{2}\right)\widetilde{b}_1 + \frac{s_1^2}{4}\widetilde{b}_2\right)z^2 \\
&+ \left(\frac{1}{2}\left(s_3 - s_1 s_2 + \frac{s_1^3}{4}\right)\widetilde{b}_1 + \frac{s_1}{2}\left(s_2 - \frac{s_1^2}{2}\right)\widetilde{b}_2 + \frac{s_1^3}{8}\widetilde{b}_3\right)z^3 + \cdots
\end{aligned}$$

Similarly, function $r \in \mathcal{A}$ exists, such that $|r(w)| < 1$ in $\mathbb{U}$ and $b(w) = \widetilde{b}(r(w))$. Therefore, the function

$$d(w) = \frac{1 + r(w)}{1 - r(w)} = 1 + r_1 w + r_2 w^2 + \cdots .$$

is in the class $\boldsymbol{B}$. It follows that

$$r(w) = \frac{d(w)-1}{d(w)+1} = \frac{r_1}{2}w + \left(r_2 - \frac{r_1^2}{2}\right)\frac{w^2}{2} + \left(r_3 - r_1 r_2 + \frac{r_1^3}{4}\right)\frac{w^3}{2} + \cdots$$

and

$$\begin{aligned}
\widetilde{b}(r(w)) &= 1 + \widetilde{b}_1\left(\frac{r_1}{2}w + \left(r_2 - \frac{r_1^2}{2}\right)\frac{w^2}{2} + \left(r_3 - r_1 r_2 + \frac{r_1^3}{4}\right)\frac{w^3}{2} + \cdots\right) \\
&+ \widetilde{b}_2\left(\frac{r_1}{2}w + \left(r_2 - \frac{r_1^2}{2}\right)\frac{w^2}{2} + \left(r_3 - r_1 r_2 + \frac{r_1^3}{4}\right)\frac{w^3}{2} + \cdots\right)^2 \\
&+ \widetilde{b}_3\left(\frac{r_1}{2}w + \left(r_2 - \frac{r_1^2}{2}\right)\frac{w^2}{2} + \left(r_3 - r_1 r_2 + \frac{r_1^3}{4}\right)\frac{w^3}{2} + \cdots\right)^3 + \cdots \qquad (24) \\
&= 1 + \frac{\widetilde{b}_1 r_1}{2}w + \left(\frac{1}{2}\left(r_2 - \frac{r_1^2}{2}\right)\widetilde{b}_1 + \frac{r_1^2}{4}\widetilde{b}_2\right)w^2 \\
&+ \left(\frac{1}{2}\left(r_3 - r_1 r_2 + \frac{r_1^3}{4}\right)\widetilde{b}_1 + \frac{r_1}{2}\left(r_2 - \frac{r_1^2}{2}\right)\widetilde{b}_2 + \frac{r_1^3}{8}\widetilde{b}_3\right)w^3 + \cdots .
\end{aligned}$$

By virtue of Equations (21)–(24), we have

$$(\kappa(2\beth + \gamma - 3) + 2)a_2 = \frac{s_1\tau}{2}, \qquad (25)$$

$$\begin{aligned}
&\left[\kappa\left(\frac{(\gamma-1)(\gamma-2)}{2} + 2\beth(\gamma-1) + 2\beth(\beth-1)\right)\right]a_2^2 + [\kappa(3\beth + \gamma - 4) + 3]a_3 \\
&= \frac{1}{2}\left(s_2 - \frac{s_1^2}{2}\right)\tau + \frac{3s_1^2}{4}\tau^2,
\end{aligned} \qquad (26)$$

$$-(\kappa(2\gimel+\gamma-3)+2)a_2=\frac{r_1\tau}{2} \tag{27}$$

and

$$\begin{aligned}&\left[\kappa\left(\frac{(\gamma-2)(\gamma+3)}{2}+2\gimel(\gamma-1)+2\gimel(\gimel+2)-4\right)+6\right]a_2^2-[\kappa(3\gimel+\gamma-4)+3]a_3\\&=\frac{1}{2}\left(r_2-\frac{r_1^2}{2}\right)\tau+\frac{3r_1^2}{4}\tau^2.\end{aligned} \tag{28}$$

From Equations (25) and (27), we get

$$s_1=-r_1 \tag{29}$$

and

$$2(\kappa(2\gimel+\gamma-3)+2)^2a_2^2=\frac{(s_1^2+r_1^2)\tau^2}{4}. \tag{30}$$

Adding Equations (26) to (28), we have

$$\begin{aligned}&[\kappa((\gamma+2)(\gamma-3)+4\gimel(\gamma-1)+2\gimel(2\gimel+1))+6]a_2^2\\&=\frac{1}{2}(s_2+r_2)\tau-\frac{1}{4}(s_1^2+r_1^2)\tau+\frac{3}{4}(s_1^2+r_1^2)\tau^2.\end{aligned} \tag{31}$$

By substituting Equations (30) in (31), we can obtain

$$a_2^2=\frac{(s_2+r_2)\tau^2}{2(\kappa\tau[(\gamma+2)(\gamma-3)+4\gimel(\gamma-1)+2\gimel(2\gimel+1)]+6\tau+2(1-3\tau)(\kappa(2\gimel+\gamma-3)+2)^2)} \tag{32}$$

Now, using Lemma 1, we have

$$|a_2|\leq\frac{\sqrt{2}|\tau|}{\sqrt{|\kappa\tau[(\gamma+2)(\gamma-3)+4\gimel(\gamma-1)+2\gimel(2\gimel+1)]+6\tau+2(1-3\tau)(\kappa(2\gimel+\gamma-3)+2)^2|}}. \tag{33}$$

Next, by subtracting Equations (28) from (26), we get

$$a_3=\frac{(s_2-r_2)\tau}{4(\kappa(3\gimel+\gamma-4)+3)}+a_2^2. \tag{34}$$

Hence, by Lemma 1, we get

$$|a_3|\leq\frac{(|s_2|+|r_2|)|\tau|}{4(\kappa(3\gimel+\gamma-4)+3)}+|a_2|^2\leq\frac{|\tau|}{\kappa(3\gimel+\gamma-4)+3}+|a_2|^2.$$

Using Equation (33), we obtain

$$|a_3|\leq\frac{|\tau|[\chi+2\tau(\kappa(3\gimel+\gamma-4)+3)]}{(\kappa(3\gimel+\gamma-4)+3)\chi},$$

where

$$\chi=\kappa\tau[(\gamma+2)(\gamma-3)+4\gimel(\gamma-1)+2\gimel(2\gimel+1)]+6\tau+2(1-3\tau)(\kappa(2\gimel+\gamma-3)+2)^2.$$

From Equation (34), we have

$$a_3-\varepsilon a_2^2=\frac{(s_2-r_2)\tau}{4(\kappa(3\gimel+\gamma-4)+3)}+(1-\varepsilon)a_2^2. \tag{35}$$

By substituting Equations (32) in (35), we have

$$
\begin{aligned}
a_3 - \varepsilon a_2^2 &= \frac{(s_2 + r_2)\tau}{4(\kappa(3\gimel + \gamma - 4) + 3)} + \frac{(1-\varepsilon)(s_2 - r_2)\tau^2}{2\chi} \\
&= \left(j(\varepsilon) + \frac{\tau}{4(\kappa(3\gimel + \gamma - 4) + 3)}\right)s_2 + \left(j(\varepsilon) - \frac{\tau}{4(\kappa(3\gimel + \gamma - 4) + 3)}\right)r_2,
\end{aligned}
\tag{36}
$$

where $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{2\chi}$.

Taking the modulus of Equation (36), we can obtain

$$
\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{\kappa(3\gimel+\gamma-4)+3}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{4(\kappa(3\gimel+\gamma-4)+3)} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{4(\kappa(3\gimel+\gamma-4)+3)}. \end{cases}
$$

□

4. Corollaries and Consequences

In this section, we provide special cases for the subclasses of $\mathbf{J}_{\Sigma}^{\kappa,\gimel,\gamma}(\widetilde{b})$.

Corollary 1. *Let* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *be in the function class* $\mathbf{J}_{\Sigma}^{\gimel,\gamma}(\widetilde{b})$, *where* $\gimel \geq 1$, $\gamma \in \mathbb{C}$ *and* $Re\{\gamma\} \geq 0$. *Then*

$$
|a_2| \leq \sqrt{\frac{2}{|\chi_1|}}|\tau|, \quad |a_3| \leq \frac{|\tau|(\chi_1 + 2\tau((3\gimel + \gamma - 4) + 3))}{((3\gimel + \gamma - 4) + 3)\chi_1},
$$

and for $\varepsilon \in \mathbb{R}$

$$
\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{(3\gimel+\gamma-4)+3}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{4((3\gimel+\gamma-4)+3)} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{4((3\gimel+\gamma-4)+3)} \end{cases},
$$

where

$$
\chi_1 = \tau[(\gamma+2)(\gamma-3) + 4\gimel(\gamma-1) + 2\gimel(2\gimel+1) + 6] + 2(1-3\tau)((2\gimel+\gamma-3)+2)^2,
$$

and $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{2\chi_1}$.

Corollary 2. *Let* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *be in the function class* $\mathbf{J}_{\Sigma}^{\gamma}(\widetilde{b})$, *where* $\gamma \in \mathbb{C}$ *and* $Re\{\gamma\} \geq 0$. *Then*

$$
|a_2| \leq \sqrt{\frac{2}{|\chi_2|}}|\tau|, \quad |a_3| \leq \frac{|\tau|(\chi_2 + 2\tau(\gamma+2))}{(\gamma+2)\chi_2},
$$

and for $\varepsilon \in \mathbb{R}$

$$
\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{\gamma+2}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{4(\gamma+2)} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{4(\gamma+2)} \end{cases},
$$

where

$$
\chi_2 = 2(\gamma+1)^2 - (\gamma+1)(5\gamma+4)\tau,
$$

and $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{2\chi_2}$.

Corollary 3. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in the function class $\mathbf{J}_{\Sigma}(\widetilde{b})$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{4-9\tau}}, \quad |a_3| \leq \frac{2(2-3\tau)|\tau|}{3(4-9\tau)},$$

and for $\varepsilon \in \mathbb{R}$

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{3}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{12} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{12} \end{cases},$$

where $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{4(4-9\tau)}$.

Corollary 4. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in the function class $\mathbf{J}_{\Sigma}^{1,\gamma}(\widetilde{b})$, where $\gamma \in \mathbb{C}$ and $Re\{\gamma\} \geq 0$. Then,*

$$|a_2| \leq \sqrt{\frac{2}{|\chi_1|}}|\tau|, \quad |a_3| \leq \frac{|\tau|(\chi_1 + 2\tau(\gamma+2))}{(\gamma+2)\chi_1},$$

and for $\varepsilon \in \mathbb{R}$

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{(\gamma-1)+3}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{4(\gamma+2)} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{4(\gamma+2)} \end{cases},$$

where

$$\chi_1 = \tau[(\gamma+2)(\gamma-3) + 4(\gamma-1) + 12] + 2(1-3\tau)(\gamma+1)^2,$$

and $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{2\chi_1}$.

Corollary 5. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in the function class $\mathbf{J}_{\Sigma}^{\beth,1}(\widetilde{b})$, where $\beth \geq 1$. Then,*

$$|a_2| \leq \sqrt{\frac{2}{|\chi_1|}}|\tau|, \quad |a_3| \leq \frac{|\tau|(\chi_1 + 2\tau((3\beth-3)+3))}{((3\beth-3)+3)\chi_1},$$

and for $\varepsilon \in \mathbb{R}$

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{(3\beth-3)+3}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{4((3\beth-3)+3)} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{4((3\beth-3)+3)} \end{cases},$$

where

$$\chi_1 = \tau[2\beth(2\beth+1)] + 2(1-3\tau)((2\beth-2)+2)^2,$$

and $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{2\chi_1}$.

Corollary 6. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be in the function class $\mathbf{J}_{\Sigma}^{\kappa,\beth}(\widetilde{b})$, where $\beth \geq 1$ and $\kappa \geq 1$. Then,*

$$|a_2| \leq \sqrt{\frac{2}{|\chi|}}|\tau|, \quad |a_3| \leq \frac{|\tau|[\chi + 6\tau\kappa\beth]}{3\kappa\beth\chi} \tag{37}$$

and for $\varepsilon \in \mathbb{R}$

$$\left|a_3 - \varepsilon a_2^2\right| \leq \begin{cases} \frac{|\tau|}{3\kappa\beth}; & 0 \leq |j(\varepsilon)| \leq \frac{|\tau|}{12\kappa\beth} \\ 4|j(\varepsilon)|; & |j(\varepsilon)| \geq \frac{|\tau|}{12\kappa\beth} \end{cases}, \tag{38}$$

where

$$\chi = \kappa\tau[-6 + 2\beth(2\beth+1)] + 6\tau + 4(1-3\tau)(\kappa\beth)^2$$

and $j(\varepsilon) = \frac{(1-\varepsilon)\tau^2}{2\chi}$.

5. Conclusions

In our research, we introduced a new subclass of normalized analytic functions and bi-univalent functions connected to the standardized shell-like curves connected with Fibonacci numbers, namely $\mathbf{J}\Sigma^{\eta,\delta,\mu}(\widetilde{p})$. We established estimates for the Taylor coefficients $|a_2|$ and $|a_3|$ and obtained solutions for Fekete–Szegö functional problems for functions that belong to this class. This study could potentially serve as a stimulus for further research on the creation of new classes of analytic and bi-univalent functions related to the Fibonacci numbers.

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