

Article

A Mathematical Model for an Inventory Management and Order Quantity Allocation Problem with Nonlinear Quantity Discounts and Nonlinear Price-Dependent Demand

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Abstract: This article focuses on solving the order quantity allocation problem for retailers. It considers factors such as quality constraints, nonlinear quantity discounts, and price-dependent demand. By formulating it as a nonlinear maximization problem, the article aims to find the best combination of suppliers and order quantity out of infinite solutions to maximize the retailer's profit. The main contribution of this research is a new mathematical model that can solve the problem of quality constraint and demand in a single step. This problem is complex due to the number of equations, their nonlinear nature, and the various trade-offs given by the market. Additionally, this research considers demand as output and includes price-dependent demand, which is more realistic for retailers. The proposed model was tested using an example from the recent literature and showed better results than the previously published best solution regarding profit maximization.

Keywords: perfect rate; inventory management; supply chain; order quantity



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1. Introduction

Efficient inventory management is crucial for large companies and retailers as it directly affects profitability [1,2]. A retailer's profits rely on factors such as the cost of purchasing items, the selling price, and the number of items sold in a given period. Therefore, to maximize profits, retailers can increase the number of items sold, the difference between the purchasing and selling costs, or both.

This study focuses on the Economic Order Quantity (EOQ) problem, which involves choosing the appropriate supplier, selecting the order size or percentage of total demand, and determining the order cycle. The EOQ problem can optimize two problems: minimizing costs for manufacturing companies with predetermined production quantities or maximizing profits for final retailers with unpredictable demand. This article belongs to the second group, where the selling price is determined in a single step during problem-solving. This approach is ideal for elastic markets where demand is price-sensitive.

These types of problems can be complex due to various factors, including the number of equations, the nonlinear nature of the problem, and market trade-offs. For example, a single supplier may suffice if supplier capacity exceeds demand, resulting in a relatively simple solution. However, multiple suppliers may be required if demand exceeds capacity, leading to a more complex optimization model. Additionally, discounts offered by suppliers for purchasing larger quantities can add complexity to the problem.

When making decisions, quality is a crucial factor to consider. A study by Mendoza and Ventura [3] defined quality as the “perfect rate”, which is the percentage of items from a supplier that meet all quality standards. For example, a supplier with a perfect rate of 0.98 means that two out of every one hundred items supplied are non-conforming. This

affects the total cost, as high-quality suppliers may have higher prices. However, sometimes suppliers with lower perfect rates may be preferred if the cost per item offsets the losses. It is important to note that how defective items are handled may differ from company to company, and this article assumes that the retailer must absorb the cost of defective items without any refunds.

Purchasing is a vital but complicated part of the supply chain. It involves selecting suppliers, determining the number of items to order from each supplier, and deciding how long the order cycle should be. A new mathematical model was introduced in [4] that addresses these challenges recently. This model, referred to as the previous model, considers the demand as output and considers the selling price.

Numerous investigations have been conducted to enhance inventory management. Some of these studies factor in demand as a variable that relies on other parameters or variables. For instance, stock-dependent demand [5,6] assumes that many customers are enticed by the availability of large stock quantities with a wide range of diversity. This can be applied to supermarkets or super-shops. On the other hand, alternative studies postulate that customers are attracted by lower selling prices, which applies to retailers specializing in particular products. Our work belongs to this second category, where demand is price-dependent.

The relationship between demand and selling price can be explained as a trade-off. Retailers have some freedom to choose the selling price. While selling items at a high price may seem logical to make a significant profit, the retailers may not be able to sell the products if the price is too high. There has been research on the mathematical relationship between demand and selling price in the literature [7–10].

This paper assumes that the retailer buys and sells the same number of items each month, which is crucial for planning and avoiding losses. If the retailer purchases more items than they sell, they can compensate by buying fewer items the following month, but this incurs a stock cost and does not change the average monthly demand. On the other hand, if the retailer fails to meet demand due to purchasing fewer items from the supplier(s), it causes losses and is unacceptable in the decision-making process.

This study examines the impact of a quality constraint, nonlinear quantity discounts on purchasing cost, and a nonlinear price-dependent demand on a retailer's profit. The demand equation used is the same as in a previous study for easy comparison. The goal is to propose a new mathematical model that improves the retailer's profit compared to the previously published solution. The problem is formulated as a nonlinear maximization problem that aims to maximize a retailer's profit by choosing a combination of suppliers and ordering quantities. The article's main contribution is a new mathematical model that solves the quality constraint and demand in a single step, which is a significant advancement. The problem is complex due to several factors, including nonlinear equations, market trade-offs, and price-dependent demand. Furthermore, unlike previous research, demand is considered an output rather than a given parameter. The proposed model was tested against a recent example from the literature and resulted in a better profit-maximizing solution than the previously published best solution.

The proposed model is solved in MATLAB with the particle swarm optimization (PSO) algorithm because commercial software such as Lingo may not be efficient for such complex problems. To simplify the comparison of results, the numerical example was taken from the state-of-the-art literature. The results showed that the proposed model satisfactorily maximized the benefits while meeting all constraints.

This article is organized as follows: Section 2 provides a literature review on how other works have addressed demand and its relationship to the selling price. Section 3 introduces the problem and its feasibility conditions. Section 4 presents the proposed model and the metaheuristic used to solve the problem. Section 5 analyzes the results, and Section 6 presents the conclusions.

2. Literature Review

This topic has been studied extensively by various authors. Whitin [11] first connected inventory theory and economic price theory by examining how demand is affected by product price. He found that suppliers can offer price discounts to incentivize larger orders. References [12,13] explored how suppliers can offer optimal quantities with quantity discounts. Later, models were developed to determine lot size and optimal price for purchasing a product with all-unit quantity discounts [12,13]. Other researchers have extended the relationship between demand and selling price to include time in their analysis [14,15]. Additionally, supplier selection has been studied with uncertain demand, quantity discounts, and fixed costs [16].

Pricing policies and optimal replenishment schedules have also been explored [17]. Maiti and Giri [18] looked at this topic in a two-echelon supply chain, while others have examined it as a means of coordinating supplier selection and order quantity allocation [19–21].

Finally, some researchers have analyzed lot-sizing decisions with price-sensitive demand, including exploring all-quantity discounts to generate an optimal lot-sizing decision [22–25]. Recent studies have also looked at product demand sensitivity variations using discounts and price patterns [26–30].

3. Description of the Problem and Its Analysis

This section will discuss the problem under study and provide some parameters for a numerical example. Our goal is to help retailers maximize their profits by focusing on a single item that can be purchased from various suppliers. It is worth noting that the optimization algorithm can be run multiple times, depending on the number of items being considered.

Table 1 shows the notation that will be used to describe the problem.

Table 1. Notation of variables and parameters of the problem.

Parameters	
c_i	Maximum monthly capacity of supplier i (given in units per month)
k_i	Ordering or setup cost of supplier i (given in dollars per order), $\forall i = 1, \dots, n$.
r	Storage or holding cost rate (given in dollars per item per month)
q_i	Perfect rate of supplier i , $\forall i = 1, \dots, n$.
q_a	Minimum required perfect rate
v_i	Per unit cost (depends on the order quantity assigned Q_i)
M	The maximum number of orders assigned per cycle
Decision Variables	
j_i	The number of orders assigned to supplier i per order cycle ($\sum_j j_{ij}$), $\forall i = 1, \dots, n$.
Q_i	Ordered quantity assigned to supplier i (in units), $\forall i = 1, \dots, n$.
P	Selling price
Additional Variables	
T_c	Order cycle period (in months)
R_i	The total number of items ordered to supplier i during the order cycle period ($R_i = j_i Q_i$), $\forall i = 1, \dots, n$.
d	Demand per time unit

As a retailer, it is important to maintain inventory levels without any shortages. Therefore, the decision maker must purchase a specific type of item and can choose to buy it from one or more of the n suppliers available. Each supplier has its own setup cost (k_i), perfect rate (q_i), and capacity (c_i), $\forall i = 1, \dots, n$. It is also essential to establish a

minimum perfect rate (q_a). Please refer to Table 1 for a summary of all the parameters and decision variables.

The cost per unit of the item from different suppliers is based on a quantity discount structure. Three suppliers are considered in this example, and their costs are presented in Tables 2–4, respectively. The discount structure for each supplier is shown in their respective tables.

Table 2. The per-unit cost of items—Supplier 1.

Supplier 1		
Interval	Bound (Units)	Unit Cost (USD)
1	0–50	$v_1 = 9$
2	50–100	$v_1 = 8.9$
3	100–150	$v_1 = 8.8$
4	150–200	$v_1 = 8.7$
5	200– ∞	$v_1 = 8.6$

Table 3. The per-unit cost of items—Supplier 2.

Supplier 2		
Interval	Bound (Units)	Unit Price (USD)
1	0–75	$v_2 = 9.8$
2	75–150	$v_2 = 9.6$
3	150–225	$v_2 = 9.4$
4	225– ∞	$v_2 = 9.2$

Table 4. The per-unit cost of items—Supplier 3.

Supplier 3		
Interval	Bound (Units)	Unit Price (USD)
1	0–100	$v_3 = 10.5$
2	100–200	$v_3 = 10.4$
3	200– ∞	$v_3 = 10.3$

Please refer to Table 2 for the pricing information. If the retailer purchases less than 50 items from Supplier 1, the cost is 9 per item. The setup costs for the three suppliers are as follows: Supplier 1 is \$500, Supplier 2 is \$250, and Supplier 3 is \$450. The perfect rates for each supplier are $q_1 = 0.92$, $q_2 = 0.95$, and $q_3 = 0.98$. Suppose the setup or order cost of \$500 is dominant for purchases of a few items. In that case, the average cost can be reduced for purchases of larger quantities due to the volume discount offered by the supplier.

After analyzing the suppliers, we found that supplier 1 has the lowest per-unit costs, but their setup cost is the highest. This is because they prefer to sell in larger quantities. On the other hand, Supplier 3 has the most expensive items, and their setup cost is not low either, but they have the highest perfect rate.

Moving on, we express the relationship between selling price and demand through the function in Equation (1):

$$D = \alpha P^{-e}, \quad (1)$$

where α and e are the scaling factor and price elasticity index, respectively. For this article’s numerical example, α is equal to 3,375,000 and e is equal to 3.

3.1. The Previous Model

The total average (monthly) profit is formulated using a mixed-integer nonlinear model and depends upon the order quantity assigned to the selected suppliers. This section briefly explains the model introduced by [4], called hereafter the reference model. The solution to the problem consists of finding the number of orders assigned to each supplier j_i during the order cycle and the order size for each supplier Q_i .

First, the order cycle period is defined as follows:

$$T_c = \frac{\sum_{i=1}^r R_i}{d} = \frac{\sum_{i=1}^r R_i}{\alpha P^{-e}} = \frac{R}{\alpha P^{-e}}. \quad (2)$$

Therefore, the complete mixed-integer nonlinear programming model used as our reference model is as follows:

(Reference Model)

$$\text{Max } Z = \alpha P^{1-e} - \frac{1}{R} \left[\alpha P^{-e} \sum_{i=1}^n j_i k_i + \frac{1}{2} r \sum_{i=1}^n R_i^2 j_i v_i + \alpha P^{-e} \sum_{i=1}^n R_i v_i \right]. \quad (3)$$

Subject to:

$$R \leq \sum_{i=1}^n Q_i j_i, \quad (4)$$

$$\frac{\alpha P^{-e} \sum_{i=1}^n Q_i j_i}{R} \leq c_i, \quad \forall i = 1 \dots n \quad (5)$$

$$\sum_{i=1}^n Q_i (q_i - q_a) j_i \geq 0, \quad (6)$$

$$Q_i \leq \sum_{j=1}^{a_i} u_{ij} Y_{ij}, \quad \forall i = 1 \dots n \text{ and } \forall j = 1 \dots a_i \quad (7)$$

$$Q_i \geq \sum_{j=1}^{a_i} u_{i,j-1} Y_{ij}, \quad \forall i = 1 \dots n \text{ and } \forall j = 1 \dots a_i \quad (8)$$

$$\sum_{j=1}^{a_i} Y_{ij} \leq 1, \quad \forall i = 1 \dots n \quad (9)$$

$$\sum_{i=1}^n j_i = M, \quad (10)$$

$$j_i \leq M, \quad \forall i = 1 \dots n \quad (11)$$

$$j_i \geq 0, \text{ integer}, \quad \forall i = 1 \dots n \quad (12)$$

$$P \geq 0, \quad (13)$$

$$Q_i \geq 0, \quad \forall i = 1 \dots n \quad (14)$$

$$Y_{ij} \in (0, 1), \quad \forall i = 1 \dots n \text{ and } \forall j = 1 \dots a_i. \quad (15)$$

Constraint (4) represents the sum of the total ordered quantities considering all selected suppliers. Constraint (5) confirms that the order quantity per supplier does not exceed the

supplier's capacity. Constraint (6) ensures that the supplier's average quality level is greater than the minimum acceptable quality of the retailer. Constraints (7) and (8) guarantee that the order quantity per selected supplier is within the correct quantity discount interval. Constraint (9) ensures that an interval per selected supplier is selected. Constraint (10) establishes the maximum number of orders per cycle that can be assigned. Constraint (11) ensures that each selected supplier has a total assigned order less than or equal to the total orders per cycle. Constraint (12) ensures that the total number of orders per supplier is an integer. Constraints (13) and (14) help to control non-negative conditions, and constraint (15) establishes the requirements for the binary variable.

The objective of the Reference Model is to maximize the profit function per time unit. The total profit equals the total sales per time unit, $\alpha P^{(1-e)}$, minus the total cost per time unit. The profit function considers the ordering cost per time unit, holding cost per time unit, and purchasing cost per time unit.

The total ordering cost per time unit is calculated as follows:

$$\text{Ordering cost} = \frac{1}{T_c} \sum_{i=1}^n j_i k_i. \quad (16)$$

The holding cost per time unit from supplier i is calculated by multiplying the holding cost r by the average inventory level, expressed as

$$\text{Average inventory level} = \frac{1}{T_c} \left(\frac{Q_i j_i}{2} \right). \quad (17)$$

To incorporate it into the objective function, the average inventory cost is calculated as follows:

$$\text{Inventory cost} = \frac{1}{R} \left(\frac{1}{2} r \sum_{i=1}^n R_i^2 j_i v_i \right). \quad (18)$$

Finally, the purchasing cost per time unit is calculated as follows:

$$\text{Purchasing cost} = \frac{1}{T_c} \left(\sum_{i=1}^n Q_i j_i v_i \right). \quad (19)$$

3.2. Reference Model's Analysis: The Feasibility of the Solution Considering a New Focus on the Quality Parameters

To analyze the reference model, we need to measure the demand based on the number of pieces without defects. We can define the demand for perfect items, the number of perfect items required considering the demand, and it can be expressed as the demand multiplied by the minimum perfect rate of the retailer (see Equation (20)):

$$D_a = \alpha P^{-e} q_a. \quad (20)$$

The solution will provide the following:

- (i) The number of orders and the order quantity assigned to each supplier. The number of orders to supplier i is called j_i ; it is assumed that all orders assigned to a specific supplier are of the same size, and the size of the order assigned to supplier i is referred to as Q_i .
- (ii) The order cycle period is in months. This, along with the number of orders and order quantity, determines how many items are purchased each month or each order cycle (the order cycle is expressed in months, it does not need to be an integer). In other words, this can be used to calculate the demand. Then, it is not necessary to provide the demand explicitly. Notice that the percentage of the demand covered by each supplier is not explicitly provided but can also be calculated.

- (iii) The selling price. This is also implicit since the demand can be calculated, and the selling price can be determined from Equation (1). Still, it is an important variable. In this work, we will consider it to be part of the solution to the problem.

There are countless solutions available, especially if there are no limits to the order cycle or the number of orders. However, some solutions may lead to lower profits or even losses if the selling price is too low. Before we delve into those details, we must differentiate between feasible and infeasible solutions. An infeasible solution is one where the perfect demand is not met or there is a planned shortage. To avoid this, we must set the selling price based on the demand, ensuring that it matches the established demand and minimum perfect rate. If these values do not match up, it will reduce expected profits, which is essentially the same as a shortage.

To meet the demand, you must purchase a quantity of items that is at least equal to the monthly demand multiplied by the order cycle in months:

$$\sum_{i=1}^r j_i Q_i \geq d T_C. \quad (21)$$

To meet the retailer's minimum perfect rate, we need a second equation. In other words, the demand can be covered by low-quality suppliers, but we must not only ensure that all the items are in demand but also consider that some items will be defective. We can compensate for this by purchasing more items or shortening the order cycle. Equation (22) tells us no planned shortage will be caused by the quality of items:

$$\sum_{i=1}^r R_i q_i \geq d q_a T_C. \quad (22)$$

It is important to ensure that the solutions do not exceed the suppliers' monthly capacity:

$$\frac{R_i}{T_C} \leq c_i. \quad (23)$$

A solution is feasible if it meets the requirements of Equations (21)–(23), regardless of the method used to arrive at it or the mathematical model. The monthly profits are another parameter that needs to be assessed.

4. The Proposed Model

4.1. Model Analysis

We have presented a revised version of the reference model in Section 3.2. This update aims to expand the range of solutions that can be examined using the reference model, potentially leading to improved outcomes, as evidenced in the results section. It is important to note that the original reference model only accounted for the traditional order cycle period T_c :

$$T_c = \sum_{i=1}^n \frac{Q_i j_i}{\alpha P^{-e}}. \quad (24)$$

Additionally, the reference model is subject to several constraints. Recall the quality constraint is as follows:

$$\sum_{i=1}^n Q_i (q_i - q_a) j_i \geq 0. \quad (25)$$

This formulation ensures that the quality of selected suppliers is greater than or equal to the minimum acceptable quality of the retailer q_a .

The perfect rate from each supplier (Equation (25)) presents a lack of possibility of finding a solution in scenarios when the perfect rates of suppliers will be less than the minimum acceptable quality. For instance, let us consider the suppliers in the numerical example with perfect rates of 0.92, 0.95, and 0.98, respectively. However, let us consider

that the retailer establishes 0.99 (instead of 0.95) as a minimum acceptable quality and the order quantity per supplier is 1000 units within one order. A feasible solution cannot be found:

$$Q_1 j_1 (0.92 - 0.99) + Q_2 j_2 (0.95 - 0.99) + Q_3 j_3 (0.98 - 0.99) \geq 0. \quad (26)$$

Since Q_i and j_i are positive numbers, no combination can satisfy Equation (26). Since this situation (low perfect rate situation) might be possible in real life, it is highly desirable that models can deal with it. An intuitive solution would be to shorten the order cycle to have more items to compensate for the number of imperfect items and satisfy the demand. Then, a new equation to calculate the order cycle may be required.

If the model allows purchasing more (or the same quantity in a shorter time), the lack of perfect items can be compensated. For example, if a retailer requires 1000 pieces a month and establishes a 0.99 minimum acceptable quality, the company needs 990 perfect pieces each month. Even the lower perfect rate supplier can provide them if 1077 pieces are ordered with a perfect rate of 0.92, resulting in 990.84 perfect parts.

One of the main differences in this article is the reformulation of the order cycle, considering the perfect rate. Let us consider the order quantity per supplier with the required quality as $Q_{ij_i} q_i$, and the total ordered quantity from supplier i as follows:

$$R = \sum_{i=1}^n Q_{ij_i} q_i. \quad (27)$$

Demand is calculated as shown in Equation (28), where the minimum acceptable quality of the retailer is considered:

$$Dq_a = \alpha P^{-e} q_a. \quad (28)$$

Then, the order cycle can be formulated as follows:

$$T_c = \frac{\sum_{i=1}^n Q_{ij_i} q_i}{\alpha P^{-e} q_a}. \quad (29)$$

One of the key components of this work is the reformulation of the order cycle process. Quality parameters are now included in the calculation of the order cycle, rather than being treated solely as constraints. Our proposed model has been shown to yield higher profits than the reference model. Table 5 provides a comparison of various works on the EOQ/EPQ problem using traditional order cycle calculations.

Table 5. Comparison with other works in the literature.

Reference	EPQ/EOQ	Discounts	Demand Depends on	Methodology	Considering Quality
Moon I. et al. [31]	EPQ	No	Production cost	Geometric programming	No
Kugele ASH. et al. [32]	Smart	No	N.A.	Geometric programming	No
Pando et al. [33]	EOQ	No	stock	Analytic	No
Pando et al. [34]	EOQ	No	stock	Analytic	No
Cárdenas-Barrón et al. [35]	EOQ	No	stock	Analytic	No
Ventura et al. [9]	EOQ	No	price	Analytic	No
Venegas et al. [2]	EOQ	All units	price	Game theoretic model	No
Adeinat and Ventura [4]	EOQ	All units	price	Analytic	Yes
This study	EOQ	All units	price	Metaheuristic	Yes

In Table 6, a comparison is made between works using traditional order cycle calculation and quality parameters. All the works listed in Tables 5 and 6 use nonlinear models, with most of them considering demand as a dependent function based on production cost, stock, or price. Specific works consider all quantity discounts in purchasing costs. Analytic methods and geometric programming are used to find solutions in many works, but this paper explores the use of metaheuristic methods as a fast and efficient alternative. The novelty of this work lies in considering quality parameters as part of the order cycle and demand as a function of price, as shown in Tables 5 and 6.

Table 6. Comparison with works that used quality parameters.

Reference	Quality Parameters Are Considered as	Demand Depends on	Parameters of the Order Cycle
Mendoza A. et al. [3]	Quality constraint	Constant	Demand, order quantity, number of orders
Subramanian P. et al. [36]	Quality constraint	Constant	Demand, order quantity, number of orders
Mendoza A. et al. [37]	Quality constraint	Constant	Demand, order quantity, number of orders
Adeinat and Ventura [4]	Quality constraint	Price	Demand, order quantity, number of orders
This study	Part of the order cycle	Price	Demand, order quantity, number of orders, quality parameters

4.2. Reformulated Model

After incorporating the redefinition of the order cycle into the reference model, the reformulated model is as follows:

$$\text{Max } Z = \alpha P^{1-e} - \frac{1}{\sum_{i=1}^n Q_i j_i q_i} \left[\alpha P^{-e} q_a \sum_{i=1}^n j_i k_i + \frac{1}{2} r \sum_{i=1}^n R_i^2 j_i v_i + \alpha P^{-e} q_a \sum_{i=1}^n R_i v_i \right]. \quad (30)$$

Subject to:

$$\frac{\alpha P^{-e} q_a \sum_{i=1}^n Q_i j_i}{R} \leq c_i, \quad \forall i = 1 \dots n \quad (31)$$

$$Q_i \leq \sum_{j=1}^{a_i} u_{ij} Y_{ij}, \quad \forall i = 1 \dots n \text{ and } \forall j = 1 \dots a_i \quad (32)$$

$$Q_i \geq \sum_{j=1}^{a_i} u_{i,j-1} Y_{ij}, \quad \forall i = 1 \dots n \text{ and } \forall j = 1 \dots a_i \quad (33)$$

$$\sum_{j=1}^{a_i} Y_{ij} \leq 1, \quad \forall i = 1 \dots n \quad (34)$$

$$\sum_{i=1}^n j_i = M, \quad (35)$$

$$j_i \leq M, \quad \forall i = 1 \dots n \quad (36)$$

$$j_i \geq 0, \text{ integer}, \quad \forall i = 1 \dots n \quad (37)$$

$$P \geq 0, \quad (38)$$

$$Q_i \geq 0, \quad \forall i = 1 \dots n \quad (39)$$

$$Y_{ij} \in (0, 1), \quad \forall i = 1 \dots n \text{ and } \forall j = 1 \dots a_i. \quad (40)$$

The profit function considers the ordering, holding, and purchasing costs per time unit. The order cycle is calculated using Equation (29). The total ordering cost is expressed as

$$\text{Total ordering cost} = \frac{1}{T_c} \sum_{i=1}^n j_i k_i. \quad (41)$$

The holding cost per time unit from supplier i is calculated by multiplying the holding cost r by the average inventory level:

$$\text{Average inventory level} = \frac{1}{T_c} \left(\frac{Q_i j_i}{2} \right). \quad (42)$$

To calculate the objective function, this is incorporated as

$$\text{Inventory cost} = \frac{1}{R} \left(\frac{1}{2} r \sum_{i=1}^n R_i^2 j_i v_i \right). \quad (43)$$

Finally, the purchasing cost per cycle is

$$\text{Average purchasing cost} = \frac{1}{T_c} \left(\sum_{i=1}^n Q_i j_i v_i \right). \quad (44)$$

Constraint (31) ensures that the order quantity per supplier does not exceed the supplier's capacity. Constraints (4) and (6) are unnecessary in this formulation. Equations (32) to (40) are the same as in the reference model.

4.3. Particle Swarm Optimization (PSO)

The PSO algorithm is used to improve possible solutions by optimizing a function iteratively. This is achieved by adjusting the position and velocity parameters of particles within a search space. The algorithm simulates the behavior of birds and begins by initializing a population of random particles, each representing a possible solution. At each iteration, the particles evolve, and their velocity is updated. The PSO process is further explained using a presentation of the pseudocode.

Pseudocode. Particle Swarm Optimization

```

P = P0; /*Generating the initial population*/
t = 0;
Repeat
  Update (pik and gk);
  Update velocity (vik+1);
  Update position (xik+1);
  t = t + 1
Until /*Stopping criteria*/
Output /*Best solution so far*/

```

4.3.1. Initialization

The algorithm starts by initializing the population. The population represents all the solutions as particles. Decision variables establish particles and are randomly initialized and distributed over the search space of the objective function, following uniform probability distribution. When particles (solutions) are initialized, this decision vector is evaluated in the objective function, and the algorithm calculates the local and global particles. The velocity and position are updated, and particles are modified over iterations.

4.3.2. Velocity of Particles

In order to obtain a new solution vector, the velocity of particles must be updated. This process is based on the local and global influence of the best particles and the last velocity value. The new velocity can be adjusted using the cognitive (c_1) and social (c_2) factors. The velocity \mathbf{v}_i^k of each particle or decision vector \mathbf{x}_i^k is updated by employing Equation (45):

$$v_i^{k+1} = v_i^k + c_1(r_1^k(p_i^k - x_i^k)) + c_2(r_2^k(g^k - x_i^k)), \quad (45)$$

where the velocity vector of the particle \mathbf{x}_i^k for the next iteration $k + 1$ is denoted by \mathbf{v}_i^{k+1} . \mathbf{v}_i^k is the current velocity, \mathbf{g}^k is the global best current particle so far, and \mathbf{p}_i^k is the best local so far (up to the current iteration). Additionally, \mathbf{r}_1^k and \mathbf{r}_2^k are random uniform values usually in the interval $[0, 1]$.

4.3.3. Movement of Particles

The particles are changed to their new position. This action allows the exploration of new candidate solutions in the search space. The new position is calculated as shown in Equation (46):

$$x_i^{k+1} = x_i^k + v_i^{k+1}, \quad (46)$$

where \mathbf{x}_i^{k+1} is the updated position of the particle \mathbf{x}_i^k with its new velocity \mathbf{v}_i^{k+1} .

After the particles are updated, they are evaluated in the objective function. The local and global solutions are updated. The entire process is repeated until the maximum number of iterations is reached.

5. Solution of the Numerical Example

Table 7 introduces the parameters for an illustrative numerical example. Tables 2–4 show the item's per-unit cost from each supplier, respectively.

Table 7. Parameters of the instance.

Variable	Numerical Example Data
r	3 suppliers
r	0.3 per month
k_i	$k_1 = 500, k_2 = 250, k_3 = 450$, dollars per order
q_i	$q_1 = 0.92, q_2 = 0.95, q_3 = 0.98$
q_a	0.95
c_i	$c_1 = 300, c_2 = 350, c_3 = 250$, units a month
α	3,375,000
e	3

The experiments are implemented using MATLAB R2017a in a computer with a processor Intel(R) core(TM)i5-4200ucpu@1.60GHz2.30GHz.

The reference solution is the best solution presented in [4], in which several solutions have been presented, but the best of them is $j_1 = 4, j_2 = 9, j_3 = 5, Q_1 = 608.68, Q_2 = 378.74, Q_3 = 486.95$, and $P = 15.84$. This solution led to a monthly profit of \$4179.91.

To solve the proposed model, we utilized the Particle Swarm Optimization (PSO) algorithm. Our population consisted of 200 individuals ($m = 200$), and we limited the number of iterations to 300 ($k_{max} = 300$). By balancing the number of individuals and iterations, we were able to conduct a thorough search for solutions within a reasonable timeframe. The problem involved nine decision variables, which means that it had nine dimensions ($n = 9$). Our cognitive and social factors were set at $c_1 = 2$ and $c_2 = 2$, respectively. To ensure accuracy and consistency, we repeated the optimization process 30 times for

each metaheuristic algorithm. PSO yielded a total of 30 results, and we have provided a summary of the top 10 in Table 8.

Table 8. Results using Particle Swarm Optimization in the proposed model.

j_1	j_2	j_3	Q_1	Q_2	Q_3	P	Profit	T_c
5	10	0	625.2024	357.5136	0	17.48964	\$4195.2628	10.4658
5	10	0	460.4654	263.152	0	17.48778	\$4195.5702	7.7031
10	10	5	447.4173	507.6502	399.8	16.40596	\$4199.647	15.0089
4	9	0	618.4156	317.4855	0	17.47399	\$4200.1496	8.3043
9	10	0	395.3861	415.1201	0	17.419	\$4200.6401	11.8975
7	10	0	481.8841	397.9198	0	17.46638	\$4211.5892	11.3869
6	10	0	517.472	372.3754	0	17.48854	\$4212.0876	10.6669
7	10	0	438.2362	361.4035	0	17.46361	\$4213.3668	10.3913
5	10	2	668.2426	387.1903	409.3942	16.84653	\$4219.4276	11.2654
6	10	0	491.0763	346.7467	0	17.43933	\$4223.2641	9.9333
3	5	1	566.374	394.1614	352.0102	16.89048	\$4236.1505	5.6816

Table 9 shows three solutions to the problem, two of them obtained using the proposed model and the third is the solution presented along with the reference model.

Table 9. Best results using both the previous and the proposed model.

Method	j_1	j_2	j_3	Q_1	Q_2	Q_3	P	Profit	T_c
Proposed	6	10	0	491.0763	346.7467	0	17.43933	\$4223.26	9.93
Proposed	3	5	1	566.374	394.1614	352.0102	16.89048	\$4236.1505	5.68
Reference	4	9	5	608.6800	378.7400	486.9500	15.8400	\$4179.91	9.74

The solutions for the reference model obtained through the PSO strategy to solve the problem are listed in Table 10. It is important to mention that these results are not as good as the ones obtained by the proposed model.

Table 10. Results using the PSO algorithm in the previous model.

j_1	j_2	j_3	Q_1	Q_2	Q_3	P	Profit	T_c
3	10	2	611.43482	302.2221	1000	16.422054	\$3991.35	8.9973
2	4	2	603.3485	549.5823	631.8927	17.07176	\$3998.97	6.8828
2	5	2	447.7809	252.1053	547.7207	16.58173	\$4016.19	4.3924
3	10	5	1000	397.1731	603.42911	16.259653	\$4045.13	12.7226
4	10	10	1000	598.94822	401.12309	16.079424	\$4046.64	17.246
4	10	10	781.919	500.231	357.0862	16.27514	\$4066.50	14.9458
4	9	5	703.4059	377.2035	570.511	16.5024	\$4070.85	12.0655
6	9	7	446.8211	507.0697	464.7826	16.34089	\$4085.31	13.5725
6	10	10	599.3858	521.8324	370.7003	15.91724	\$4133.20	14.9621
6	10	8	462.4226	416.3043	368.7471	15.98633	\$4134.55	11.9691

Figure 1 shows that the reference model yields results ranging from \$4000 to around \$4130, while the new reformulation (see Figure 2) generates results ranging from \$4197 to

approximately \$4235. It is worth noting that all solutions obtained through the proposed model are superior to those obtained through the reference model.

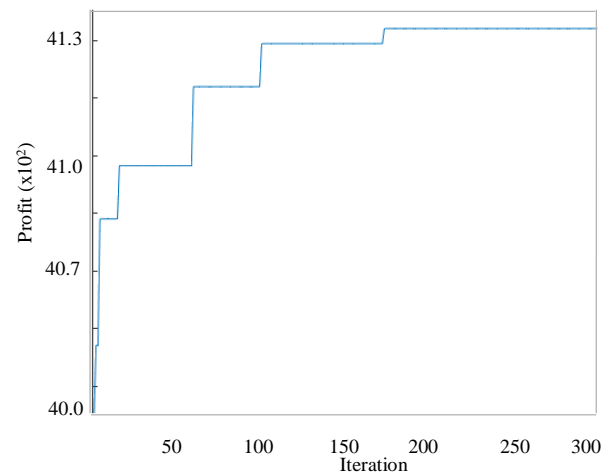


Figure 1. Convergence graph using PSO to solve the reference model.

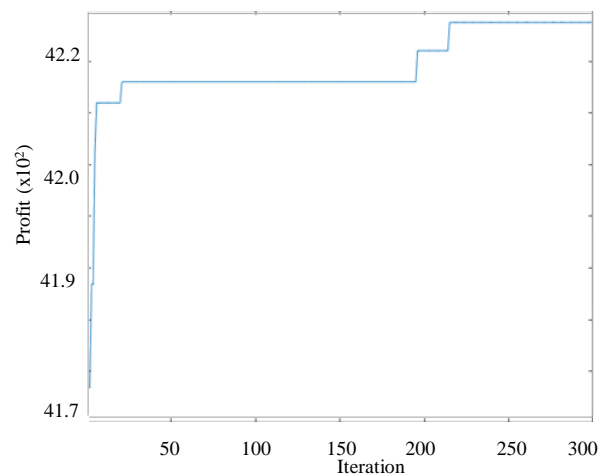


Figure 2. Convergence graph using PSO to solve the proposed model.

6. Conclusions

This article explores the problem of selecting suppliers and allocating order quantities to maximize profit while considering the effect of the selling price on demand. Previous studies have focused on minimizing inventory costs. Still, this article presents a new mathematical model that includes quality constraints, nonlinear quantity discounts, price-dependent demand, and supplier capacity constraints. The model differs from previous ones in that it considers the perfect rate of suppliers in the order cycle calculation equation and solves the quality constraint in a single step. The Particle Swarm Optimization (PSO) algorithm was used on a numerical example from the literature to test the model. The proposed model outperformed the previous model, leading to a higher profit.

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