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Equivalent Base Expansions in the Space of Cliffordian Functions

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Abstract: Intensive research efforts have been dedicated to the extension and development of essential aspects that resulted in the theory of one complex variable for higher-dimensional spaces. Clifford analysis was created several decades ago to provide an elegant and powerful generalization of complex analyses. In this paper, first, we derive a new base of special monogenic polynomials (SMPs) in Fréchet–Cliffordian modules, named the equivalent base, and examine its convergence properties for several cases according to certain conditions applied to related constituent bases. Subsequently, we characterize its effectiveness in various convergence regions, such as closed balls, open balls, at the origin, and for all entire special monogenic functions (SMFs). Moreover, the upper and lower bounds of the order of the equivalent base are determined and proved to be attainable. This work improves and generalizes several existing results in the complex and Clifford context involving the convergence properties of the product and similar bases.

Keywords: Clifford analysis; special monogenic polynomials; Fréchet modules; bases of polynomials; growth of bases; effectiveness

MSC: 30G35; 30D15; 41A10

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1. Introduction

The development of the theory of bases in Clifford analysis has indicated its growing relevance in various mathematics and mathematical physics fields. The concept of basic sets (bases) in one complex variable was initially discovered by Whittaker [1,2], and the effectiveness terminology was proposed. In this context, a significant contribution was made by Cannon [3,4], who proved the necessary and sufficient conditions for a base to possess a finite radius of regularity and to generate entire functions. In [5], Boas introduced several effectiveness criteria for entire functions.

Despite the fact that our current study has a theoretical framework, the theory of basic sets finds its utility in applications and, in particular, to solve differential equations for real-life phenomena, as indicated in [6–8]. Several approaches have been pursued in generalizing the theory of classical complex functions. Among these generalizations are the theory of several complex variables and the matrix approach [9–11]. The crucial development of the hypercomplex theory derived from higher-dimensional analysis involving Clifford algebra is called Clifford analysis. In the last decades, Clifford analysis has proved to have substantial influence as an elegant and powerful extension of the theory of holomorphic functions in one complex variable to the Euclidean space of more than two dimensions. The theory of monogenic functions created a solution for a Dirac equation or s generalized Cauchy–Riemann system, both of which are related to Riesz systems [12]. In a complex setting, holomorphic functions can be described by their differentiability or series expansion for approximations. Accordingly, exploring such representations of monogenic functions in higher-dimensional space is critical. Abul-Ez and Constales [13] initiated the study of extending Whittaker's base of polynomials in complex analysis into the context of Clifford analysis as a base of SMPs, which is a Hamel basis of linear space for all SMPs with Clifford-valued coefficients. In [13,14], the authors proved that the basic set is effective in the convergence domain when an SMF, f, can be represented in terms of a set of SMPs with some conditions. The characterization of the effectiveness property (Clifford–Cannon theorem) was determined for closed balls [13]. Locally representing a monogenic function in terms of a base of monogenic polynomials is a subject of great interest. Accordingly, the problem of replacing such a base without changing the radius of convergence restricts the class of monogenic functions to the so-called SMFs. Although straightforward generalizations may seem possible, the proof of the Cannon theorem regarding effectiveness (see [13]) in an n-dimensional domain is quite complicated. Abul-Ez and Constales [13,14] narrowed the study of the representation of monogenic functions to axially symmetric domains, which they called axially (special) monogenic functions.

A rich treatment of polynomial bases combining the functional and Clifford analyses was proposed [15], where a criterion of a general type for the effectiveness of bases in Fréchet modules was constructed in various regions. Accordingly, these authors of [15] studied effectiveness in open and closed balls and offered a remarkable method of application of approximation theory to expand some Clifford-valued functions in terms of an infinite series of Cannon sets of SMPs. A new extension of the well-known Ruscheweyh derivative operator was introduced in [16], where the representation of certain special monogenic functions in different regions of convergence was investigated in Fréchet modules. The previously mentioned treatment generalizes the results in the complex and Clifford settings given in [10,13,17]. In [18], the authors established an expansion of a particular monogenic function in terms of generalized monogenic Bessel polynomials (GMBPs). Additionally, they proved that the GMBPs are solutions of second-order homogeneous differential equations.

As is the case in complex analyses, it is of great importance to examine when the product of special monogenic polynomials is effective in the theory of bases in Clifford analysis. It is not very surprising that the product of two effective bases does not maintain effectiveness, as shown in [19], where the authors studied the effectiveness of the product of simple bases. Recently, in [20], a generalization of the product base for functions with bounded radii of convergence was investigated. The inverse of an effective base does not need to be effective [21]. Consequently, it is interesting to derive a new base of SMPs from given bases and examine how the convergence properties (region of effectiveness) of the derived base and the original bases are related. In alignment with this approach, researchers have studied the effectiveness of various constructed bases of SMPs, such as the inverse base [21], Hadamard product base [22], Bernoulli and Euler bases [23], general Bessel base [18], and Chebyshev base [24]. Numerous results concerning the polynomial bases in one complex variable were generalized to the Clifford context [16,25]. The notion of the mode of increase of special monogenic functions was initially introduced in [13]. In [26], the authors determined the order and type of the coefficients in the Taylor expansion of entire axially monogenic functions. Related contributions to the investigation of the order of bases can be found in [27,28].

Motivated by the previous discussion, this paper defines a new base of polynomials: the equivalent base in the Clifford setting in the sense of Fréchet modules. After constructing this base in terms of three constituents (the factors), we characterize the convergence properties of the equivalent base in closed balls, open balls, at the origin, and for all entire SMFs by considering specific types of constituent bases, such as simple monic bases, simple bases, and nonsimple bases, with some restrictions on the coefficients. Furthermore, knowing the orders of the constituent bases, the upper and lower bounds of the equivalent base are assessed, and two examples demonstrating the attainability of these bounds are provided. We establish the T_ρ property of the equivalent base of SMPs.

The structure of the paper is organized as follows. Section 2 provides the essential definitions and results on Clifford algebra and SMPs in Fréchet modules. The concept of equivalent bases is defined and constructed in Section 3. Section 4 details the effectiveness properties of the equivalent base. We study the effectiveness when the constituent bases are simple monic bases, simple bases with normalizing conditions, nonsimple bases with restrictions on the degree of the bases, or algebraic bases. The upper and lower bounds of the order of the equivalent base are determined and proved attainable in Section 5. Section 6 deals with the T_ρ property of the equivalent base of SMPs in open balls. We conclude the paper by summarizing the results and suggesting open problems for further study.

2. Preliminaries

This section collects several notations and results for Clifford analyses and functional analyses, which are essential throughout the paper. More details can be found in [13,15,29] and the references therein.

The real Clifford algebra $\mathcal{A}_m$ is a real algebra of dimension 2^m, which is freely generated by the orthogonal basis $(e_0, e_1, \ldots, e_m)$ in $\mathbb{R}^{m+1}$ according to the non-commutativity property $e_ie_j + e_je_i = -2\delta_{ij}$, where $e_0 = 1$ for $1 \leq i \neq j \leq m$ (for details on the main concepts of $\mathcal{A}_m$, see [30]). The space $\mathbb{R}^{m+1}1$ is embedded in $\mathcal{A}_m$. Let $x \in \mathcal{A}_m$; then, Rex refers to the real part of x, which represents the e_0 component of x and Im $x := x - (\text{Re}\ x)e_0$. The conjugate of x is $\bar{x}$, where $\bar{e}_0 = e_0$ and $\bar{e}_i = -e_i$ for $1 \leq i \leq m$. The relationship $\overline{xy} = \bar{y}\bar{x}$ holds for all $x, y \in \mathcal{A}_m$. Note that $\mathcal{A}_m$ is equipped with the Euclidean norm $\mid x \mid^2 := \text{Re}\ (x\bar{x})$. As $\mathcal{A}_m$ is isomorphic to $\mathbb{R}^{2^m}$, we have, for any $a, b \in \mathcal{A}_m$, $\mid ab \mid \leq 2^{\frac{m}{2}} \mid a \mid\mid b \mid$ and $|ab| = |a||b|$ if $a\bar{a} \in \mathbb{R}$ or $b\bar{b} \in \mathbb{R}$, where $a = \sum_{A \subseteq M} a_A e_A$ and $M = \{1, 2, \ldots, m\}$.

An $\mathcal{A}_m$-valued function f is called left (resp. right)-monogenic in an open set $\Omega \subset \mathbb{R}^{m+1}$ if it satisfies $Df = 0$ (resp. $fD = 0$) in Ω where

$$D = \sum_{i=0}^{m} e_i \frac{\partial}{\partial x_i}$$

is the generalized Cauchy–Riemann operator. Furthermore, a polynomial $P(x)$ is specially monogenic if and only if $DP(x) = 0$ (so $P(x)$ is monogenic) and there exists $a_{i,j} \in \mathcal{A}_m$, for which

$$P(x) = \sum_{i,j}^{finite} \overline{x}^i x^j a_{i,j}.$$

Definition 1. *Suppose that Ω is a connected open subset of $\mathbb{R}^{m+1}$ containing 0 and f is monogenic in Ω. Then, f is called special monogenic in Ω if and only if its Taylor series near zero (which exists) has the form $f(x) = \sum_{n=0}^{\infty} P_n(x)a_n$ for certain SMPs, specifically $P_n(x)$ and $a_n \in \mathcal{A}_m$.*

The space of all SMPs denoted by $\mathcal{A}_m[x]$ is the right $\mathcal{A}_m$ module defined by

$$\mathcal{A}_m[x] = \text{span}_{\mathcal{A}_m}\{\mathcal{P}_n(x) : n \in \mathbb{N}\},$$

where $\mathcal{P}_n(x)$ was defined by Abul-Ez and Constales [13] in the form

$$\mathcal{P}_n(x) = \frac{n!}{(m)_n} \sum_{r+s=n} \frac{(\frac{m-1}{2})_r(\frac{m+1}{2})_s}{r!s!} \overline{x}^r x^s, \tag{1}$$

where for $b \in \mathbb{R}$, $(b)_l = b(b+1)\ldots(b+l-1)$ is the Pochhamer symbol. Observe that $\mathbb{R}^{m+1}$ is identified with a subset of $\mathcal{A}_m$.

Let $P_n(x)$ be a homogeneous SMP of degree n in x and $P_n(x) = \mathcal{P}_n(x)\ \alpha$, where $\alpha \in \mathcal{A}_m$ is a Clifford constant (see [13]). Consequently, we obtain

$$\|\mathcal{P}_n\|_R = \sup_{\overline{B}(R)} |\mathcal{P}_n(x)| = R^n.$$

Now, we state the definition of a Fréchet module (F-module) as follows.

Definition 2. *An F-module E over $\mathcal{A}_m$ satisfies the following properties:*

(i) *E is a Hausdorff space,*
(ii) *E is a topology induced by a countable set of a proper system of semi-norms $\mathfrak{P} = \{\|.\|_k\}_{k\geq 0}$ such that $k < l \Rightarrow \|g\|_k \leq \|g\|_l$; $(g \in E)$. This implies that $V \subset E$ is open if and only if for all $g \in V$, there exists $\epsilon > 0,\ N \geq 0$ such that $\{f \in E : \|g - f\|_k) \leq \epsilon\} \subset V,\ \forall k \leq N$.*
(iii) *E is complete with respect to a countable set of a proper system of semi-norms.*

Definition 3. *A sequence $\{g_n\}$ in an F-module E converges to f in E if and only if*

$$\lim_{n\to\infty} \|g_n - f\|_k = 0$$

for all $\|.\|_k \in \mathfrak{P}$.

Remark 1. *In the following Table 1, each indicated space represents an F-module depending on the countable set of a proper system of associated semi-norms.*

Table 1. F-modules examples.

Notation	Space	The Associated Semi-Norms
$H_{[B(R)]}$	The space of SMFs in the open ball $B(R)$	$\|g\|_r = \sup_{\overline{B}(r)} \|g(x)\|, x \in \mathbb{R}^{m+1}, \forall r < R,\ g \in H_{[B(R)]}$
$H_{[\overline{B}(R)]}$	The space of SMFs in the closed ball $\overline{B}(R)$	$\|g\|_R = \sup_{\overline{B}(R)} \|g(x)\|,\ x \in \mathbb{R}^{m+1} \forall g \in H_{[\overline{B}(R)]}$
$H_{[B_+(R)]}$	The space of SMFs in $B_+(R)$, where $B_+(R)$ is any open ball enclosing the closed ball $\overline{B}(R)$	$\|g\|_r = \sup_{\overline{B}(r)} \|g(x)\|,\ x \in \mathbb{R}^{m+1} \forall R < r,\ g \in H_{[B_+(R)]}$
$H_{[\infty]}$	The space of entire SMFs on the whole of $\mathbb{R}^{m+1}$	$\|g\|_n = \sup_{\overline{B}(n)} \|g(x)\|,\ x \in \mathbb{R}^{m+1}, n < \infty\ \forall g \in H_{[\infty]}$
$H_{[0^+]}$	The space of SMFs at the origin	$\|g\|_\epsilon = \sup_{\overline{B}(\epsilon)} \|g(x)\|,\ x \in \mathbb{R}^{m+1},\ \epsilon > 0\ \forall g \in H_{[0^+]}$

Definition 4. *A sequence $\{P_n(x)\}$ of an F-module E is said to form a base if $\mathcal{P}_n(x)$ admits a right $\mathcal{A}_m$-unique representation of the form*

$$\mathcal{P}_n(x) = \sum_{k=0}^{\infty} P_k(x)\ \tilde{P}_{n,k},\ \ \tilde{P}_{n,k} \in \mathcal{A}_m. \tag{2}$$

The Clifford matrix $\tilde{P} = (\tilde{P}_{n,k})$ is the operator's matrix of the base $\{P_n(x)\}$. The base $\{P_n(x)\}$ can be written as follows:

$$P_n(x) = \sum_{k=0}^{\infty} \mathcal{P}_n(x)\ P_{n,k},\ \ P_{n,k} \in \mathcal{A}_m. \tag{3}$$

The Clifford matrix $P = (P_{n,k})$ is called the coefficient matrix of the base $\{P_n(x)\}$. According to [13], the set $\{P_n(x)\}$ will be a base if and only if

$$P\tilde{P} = \tilde{P}P = I, \tag{4}$$

where I denotes the unit matrix.

Let $g(x) = \sum_{n=0}^{\infty} \mathcal{P}_n(x)\ a_n(g)$ be any SMF of an F-module E. Substituting for $\mathcal{P}_n(x)$ from (2), we obtain the basic series

$$g(x) \sim \sum_{n=0}^{\infty} P_n(x)\ \Pi_n(g), \tag{5}$$

where

$$\Pi_n(g) = \sum_{k=0}^{\infty} \tilde{P}_{k,n}\, a_k(g)\,. \tag{6}$$

Remark 2. *Representation (5) is the most important series in Clifford analysis because, as we shall see, their sums are SMFs, and every SMF can be represented by basic series. Basic series generalize Taylor series, where $P_n(x)$ in (5) can be Legendre, Laguerre, Chebychev, Hermite, Bessel, Bernoulli, or Euler polynomials [18,23].*

Definition 5. *A base $\{P_n(x)\}$ is effective for an F-module E if the basic series (5) converges normally to every element $g(x) \in E$.*

Applying Definition 5, we can take the F-module E to be the space $H_{[\overline{B}(R)]}$. Thus, the base $\{P_n(x)\}$ will be effective for $H_{[\overline{B}(R)]}$ if the basic series converges normally to every SMF $g(x) \in H_{[\overline{B}(R)]}$ that is specially monogenic in $\overline{B}(R)$. A similar inclusion criteria can be applied for the spaces $H_{[B(R)]}$ and $H_{[B_+(R)]}$. When R tends to infinity in $H_{[B(R)]}$, the definition of effectiveness yields effectiveness for $H_{[\infty]}$, which means that the basic series converges normally to every complete SMF $g(x) \in H_{[\infty]}$ on the whole space $\mathbb{R}^{m+1}$. Moreover, when R tends to zero in $H_{[B_+(R)]}$, the definition of effectiveness yields effectiveness for $H_{[0^+]}$, which means that the basic series converges normally to every SMF $g(x) \in H_{[0^+]}$ that is specially monogenic there.

Results concerning the study of the effectiveness properties of bases in the F-modules E were presented in [15]. We can write

$$\|P_n\|_R = \sup_{\overline{B}(R)} |P_n(x)|, \tag{7}$$

$$\omega_n(R) = \sum_k \|P_k\, \tilde{P}_{n,k}\|_R, \tag{8}$$

where

$$\|P_k\, \tilde{P}_{n,k}\|_R = \sup_{\overline{B}(R)} |P_k(x)\, \tilde{P}_{n,k}|.$$

Then, the convergence properties of a base are totally determined by the value of

$$\lambda(R) = \limsup_{n\to\infty} \{\omega_n(R)\}^{\frac{1}{n}}, \tag{9}$$

where $\omega_n(R)$ is the Cannon sum and $\lambda(R)$ is the Cannon function.

Theorem 1. *A necessary and sufficient condition for a base $\{P_n(x)\}$ to be*

1. *Effective for $H_{[\overline{B}(R)]}$ is that $\lambda(R) = R$;*
2. *Effective for $H_{[B(R)]}$ is that $\lambda(r) < R \ \forall\, r < R$;*
3. *Effective for $H_{[B_+(R)]}$ is that $\lambda(R^+) = R$;*
4. *Effective for $H_{[\infty]}$ is that $\lambda(R) < \infty\, \forall\, R < \infty$;*
5. *Effective for $H_{[0^+]}$ is that $\lambda(0^+) = 0$.*

The Cauchy inequality for the base in (3) is defined as [15]

$$|P_{n,k}| \le \frac{\|P_n\|_R}{R^k}. \tag{10}$$

Definition 6. *When $\{P_n(x)\}$ is a base of polynomials, then Representation (2) is finite. If the number of non-zero terms $N(n)$ in (2) is such that*

$$\limsup_{n\to\infty}\{N(n)\}^{\frac{1}{n}} = 1, \tag{11}$$

the base $\{P_n(x)\}$ is called a Cannon base of polynomials. Moreover, when $\limsup_{n\to\infty}\{N(n)\}^{\frac{1}{n}} = a > 1$, then the base $\{P_n(x)\}$ is said to be a general base.

Definition 7. *A base $\{P_n(x)\}$ of polynomials is called a simple base if the polynomial $P_n(x)$ is of degree n. A simple base is called a simple monic base if $P_{n,n} = 1 \;\forall\; n \in \mathbb{N}$.*

Definition 8. *The order of a base $\{P_n(x\}$ in a Clifford setting was defined in [13,14] by*

$$\rho = \lim_{R\to\infty}\limsup_{n\to\infty}\frac{\log\ \omega_n(R)}{n\ \log\ n}. \tag{12}$$

Determining the order of a base allows us to realize that if the base $\{P_n(x)\}$ has a finite order, ω, then it represents every complete SMF of an order less than $\frac{1}{\omega}$ in any finite ball.

3. Equivalent Bases of SMPs

Employing the definition of the product base of polynomials in the context of the Clifford analysis introduced in [19], the equivalent base of SMPs can be defined as follows.

Definition 9. *Let $\{P_n^{(\ell)}(x)\}$ and $\ell = 1,2,3$ be three bases of polynomials, where x is a Clifford variable. Define*

$$\{E_n(x)\} = \{\tilde{P}_n^{(3)}(x)\}\{P_n^{(2)}(x)\}\{P_n^{(1)}(x)\} \tag{13}$$

where $\{\tilde{P}_n^{(3)}(x)\}$ is the inverse base of $\{P_n^{(3)}(x)\}$. The base $\{E_n(x)\}$ is called the equivalent base to the base $\{P_n^{(2)}(x)\}$.

Let $P^{(\ell)}$, $\ell = 1,2,3$, and E be the matrices of coefficients of the bases $\{P_n^{(\ell)}(x)\}$, $\ell = 1,2,3$, and $\{E_n(x)\}$, respectively. Then, (13) leads to

$$E_n(x) = \sum_k \mathcal{P}_k(x)E_{n,k}, \tag{14}$$

where

$$E_{n,k} = \sum_{i,j}\tilde{P}_{i,k}^{(3)}P_{j,i}^{(2)}P_{n,j}^{(1)},$$

where $\tilde{P}^{(3)}$ is the inverse matrix of the matrix $P^{(3)}$.

Remark 3. *Note that if $\{E_n(x)\}$ is the equivalent base of $\{P_n^{(2)}(x)\}$, then the base*

$$\{P_n^{(2)}(x)\} = \{P_n^{(3)}(x)\}\{E_n(x)\}\{\tilde{P}_n^{(1)}(x)\} \tag{15}$$

is the equivalent base of $\{E_n(x)\}$.

According to (13), we can write

$$E = \tilde{P}^{(3)}P^{(2)}P^{(1)}.$$

Suppose $\tilde{E}$ is a matrix given by $\tilde{P}^{(1)}\tilde{P}^{(2)}P^{(3)}$. It can be easily observed that

$$E\tilde{E} = \tilde{P}^{(3)}P^{(2)}P^{(1)}\cdot\tilde{P}^{(1)}\tilde{P}^{(2)}P^{(3)} = I,$$

and

$$\tilde{E}E = \tilde{P}^{(1)}\tilde{P}^{(2)}P^{(3)}P^{(2)}P^{(1)} = I,$$

where I is the unit matrix. Thus, the matrix $\tilde{E}$ is a unique inverse of E. This implies that the set $\{E_n(x)\}$ is indeed a base.

4. Effectiveness of the Equivalent Base

4.1. Effectiveness with Simple Monic Constituents

We begin by considering the three bases $\{P_n^{\ell}(x)\}$, where $\ell = 1, 2, 3$, as simple monic bases to attain the following result.

Theorem 2. *Let $\{P_n^{(\ell)}(x)\}$ where $\ell = 1, 2, 3$ be three simple monic bases of polynomials, and suppose that the bases $\{P_n^{(1)}(x)\}$ and $\{P_n^{(3)}(x)\}$ are effective for $H_{[\bar{B}(r)]}$. Then, the equivalent base $\{E_n(x)\}$ is effective for $H_{[\bar{B}(r)]}$ if and only if $\{P_n^{(2)}(x)\}$ is effective in the same space.*

Proof. Suppose that the three bases $\{P_n^{\ell}(x)\}$, where $\ell = 1, 2, 3$, are effective for $H_{[\bar{B}(r)]}$. Owing to [19,21], it follows directly that the base $\{E_n(x)\}$ is effective for $H_{[\bar{B}(r)]}$.

Conversely, suppose that the bases $\{P_n^{(1)}(x)\}$, $\{P_n^{(3)}(x)\}$, and $\{E_n(x)\}$ are effective for $H_{[\bar{B}(r)]}$. Using Equation (15), as we mentioned previously, we deduce that the base $\{P_n^{(2)}(x)\}$ is effective for $H_{[\bar{B}(r)]}$. $\square$

Effectiveness with Boas Conditions

In the following, we consider the case for which each base of the constituent bases $\{P_n^{(\ell)}(x)\}$, where $\ell = 1, 2, 3$, of the equivalent base has the Boas conditions [31] in the form

$$|P_{n,k}^{(\ell)}| \leq M_\ell a_\ell^{n-k},\ 0 \leq k \leq n-1, \tag{16}$$

where a_ℓ and M_ℓ are any finite positive numbers.

Theorem 3. *Suppose that $\{P_n^{(\ell)}(x)\}$, where $\ell = 1, 2, 3$, are three simple monic bases of SMPs and satisfy the Boas conditions (16). Then, the equivalent base $\{E_n(x)\}$ is effective for $H_{[\bar{B}(r)]}$ for $r \geq \max\{a_\ell(1 + M_\ell)$, where $\ell = 1, 2, 3\}$.*

Proof. Using the product $P^{(\ell)}\tilde{P}^{(\ell)} = I$, where $P^{(\ell)}$ denotes the matrix of coefficients of the base $\{P^{\ell}(x)\}$, $\tilde{P}^{(\ell)}$ is its inverse, and I is the unit matrix, it follows that

$$\sum_j \tilde{P}_{j,k}^{(\ell)} P_{n,j}^{(\ell)} = \delta_k^n. \tag{17}$$

Since each of $\{P_n^{(\ell)}(x)\}$, where $\ell = 1, 2, 3$, is simple, then the relationship (17) can be written in the form

$$\tilde{P}_{n,n+k}^{(\ell)} P_{n+k,n+k}^{(\ell)} = -\sum_{j=0}^{k-1} \tilde{P}_{n,n+j}^{(\ell)} P_{n+j,n+k}^{(\ell)}. \tag{18}$$

Owing to (16) and (18), we obtain

$$|\tilde{P}_{n,k}^{(\ell)}| \leq [a_\ell(1 + M_\ell)]^{n-k},\ 0 \leq k \leq n-1. \tag{19}$$

Using (14), (16) and (19), we have

$$\begin{aligned}
\|E_n\|_r &\leq 2^{\frac{m}{2}} \sum_k \|\mathcal{P}_k\|_r \, |E_{n,k}| \\
&\leq 2^{\frac{3m}{2}} \sum_k r^k \sum_{i,j} |\tilde{P}^{(3)}_{i,k}||P^{(2)}_{j,i}||P^{(1)}_{n,j}| \\
&\leq 2^{\frac{3m}{2}} M_1 M_2 r^n \sum_k \left[\frac{a_3(1+M_3)}{r}\right]^{i-k} \sum_i \left(\frac{a_2}{r}\right)^{j-i} \sum_j \left(\frac{a_1}{r}\right)^{n-j} \\
&\leq 2^{\frac{3m}{2}} M_1 M_2 r^n (n+1)^3 \qquad (20)
\end{aligned}$$

for $r \geq \max\{a_1, a_2, a_3(1+M_3)\}$.

Employing the relationships (16), (19), and (20) in the Cannon sum of $\{E_n(x)\}$ leads to

$$\begin{aligned}
\Omega_n(r) &\leq 2^{\frac{m}{2}} \sum_k \|E_k\|_r \, |\tilde{E}_{n,k}| \\
&\leq 2^{\frac{3m}{2}} \sum_k \|E_k\|_r \sum_{i,j} |\tilde{P}^{(1)}_{i,k}||\tilde{P}^{(2)}_{j,i}||P^{(3)}_{n,j}| \\
&\leq 2^{\frac{3m}{2}} M_3 r^n \sum_k \left[\frac{a_1(1+M_1)}{r}\right]^{i-k} \sum_i \left[\frac{a_2(1+M_2)}{r}\right]^{j-i} \sum_j \left[\frac{a_3}{r}\right]^{n-j} \\
&\leq 2^{\frac{3m}{2}} M_3 (n+1)^3 r^n
\end{aligned}$$

for $r \geq \max\{a_1(1+M_1), a_2(1+M_2), a_3\}$.

Therefore, the Cannon function of the equivalent base $\{E_n(x)\}$ gives

$$\lambda_E(r) = \limsup_{n\to\infty} \{\Omega_n(r)\}^{\frac{1}{n}} \leq r, \qquad (21)$$

for $r \geq \max\{a_\ell(1+M_\ell),\ \ell = 1,2,3\}$. According to [15,16], the equivalent base is effective for $H_{[\bar{B}(r)]}$, as desired. □

4.2. Effectiveness of Simple Bases with Normalizing Conditions

In this subsection, we study the convergence properties of the equivalent base whose constituent bases $\{P^\ell_n(x)\}$, where $\ell = 1,2,3$, are simple bases for which the diagonal coefficients satisfy Halim's condition [25]

$$\lim_{n\to\infty} |P^{(\ell)}_{n,n}|^{\frac{1}{n}} = 1.$$

For the sake of shortening notations, we write

$$\|P^{(\ell)}_n\|_R = \sup_{\bar{B}(R)} |P^{(\ell)}_n(x)|.$$

We will use K to denote a constant that needs not be the same as it is used.

Theorem 4. *Suppose that the simple bases $\{P^\ell_n(x)\}$, where $\ell = 1,2,3$, are effective for $H_{[\bar{B}(r)]}$ and satisfy the two conditions*

(i) $\quad P^{(\ell)}_{n,n}\bar{P}^{(\ell)}_{n,n} \in \mathbb{R}$;

(ii) $\quad \lim_{n\to\infty} |P^{(\ell)}_{n,n}|^{\frac{1}{n}} = 1.$

Then, the equivalent base $\{E_n(x)\}$ is effective for $H_{[\bar{B}(r)]}$.

Proof. Since the three bases $\{P_n^\ell(x)\}$, where $\ell = 1,2,3$, satisfy the condition $\lim_{n\to\infty} |P_{n,n}^{(\ell)}|^{\frac{1}{n}} = 1$, it follows that for all $n \in \mathbb{N}$, the following relationship holds:

$$K(1-\epsilon)^n < |P_{n,n}^{(\ell)}| < K(1+\epsilon)^n. \tag{22}$$

Moreover,

$$\frac{1}{K(1+\epsilon)^n} < |\tilde{P}_{n,n}^{(\ell)}| < \frac{1}{K(1-\epsilon)^n}, \tag{23}$$

where $P_{n,n}^{(\ell)} \tilde{P}_{n,n}^{(\ell)} = 1$.

Since $\{P_n^\ell(x)\}$ are simple bases and effective for $H_{[\bar{B}(r)]}$, then they are effective for $H_{[\bar{B}(R)]}$ for all $R \geq r$ (see [25]), which implies that

$$\lambda^{(\ell)}(R) = R, \quad \forall R \geq r. \tag{24}$$

Hence, for an increasing sequence $r_{j+1} > r_j > r$, $j = 1,2,\ldots 7$, it follows that

$$\omega_n^{(\ell)}(r_j) < k\, r_{j+1}^n \;\; \forall n \geq 0. \tag{25}$$

Since

$$P_k^{(\ell)}(x) = \sum_j \mathcal{P}_j(x) P_{k,j}^{(\ell)}, \tag{26}$$

it follows that

$$P_k^{(\ell)}(x) \cdot \tilde{P}_{n,k}^{(\ell)} = \sum_j \mathcal{P}_j(x) P_{k,j}^{(\ell)} \cdot \tilde{P}_{n,k}^{(\ell)}. \tag{27}$$

Thus, by applying Cauchy's inequality as stated in (10), we obtain

$$|P_{k,j}^{(\ell)} \tilde{P}_{n,k}^{(\ell)}| \leq \frac{\|P_k^{(\ell)} \tilde{P}_{n,k}^{(\ell)}\|_r}{r^j}. \tag{28}$$

We set $k = n$ in (28) to obtain

$$|P_{n,j}^{(\ell)} \tilde{P}_{n,n}^{(\ell)}| \leq \frac{\|P_n^{(\ell)} \tilde{P}_{n,n}^{(\ell)}\|_R}{R^j}. \tag{29}$$

Then, in view of (22) and the condition (i), we have

$$|P_{n,k}^{(\ell)}| \leq \frac{\omega_n^{(\ell)}(r_j)}{r_j^k} |P_{n,n}^{(\ell)}| < \frac{K[(1+\epsilon)r_{j+1}]^n}{r_j^k}. \tag{30}$$

Putting $j = k$ in (28) implies that

$$|P_{k,k}^{(\ell)} \tilde{P}_{n,k}^{(\ell)}| \leq \frac{\|P_k^{(\ell)} \tilde{P}_{n,k}^{(\ell)}\|_r}{r^k}. \tag{31}$$

Thus, using (23) and the condition in (i) again, we can write

$$|\tilde{P}_{n,k}^{(\ell)}| \leq \frac{\omega_n^{(\ell)}(r_j)}{r_j^k} |\tilde{P}_{n,n}^{(\ell)}| < \frac{r_{j+1}^k}{K[(1-\epsilon)r_{j+1}]^n}. \tag{32}$$

Now, relying on the relationships (14), (30) and (32), one can obtain

$$
\begin{aligned}
\|E_n\|_{r_1} &\leq 2^{\frac{m}{2}} \sum_k \|\mathcal{P}_k\|_{r_1} |E_{n,k}| \\
&\leq 2^{\frac{3m}{2}} \sum_k r_1^k \sum_{i,j} |\tilde{P}^{(3)}_{i,k}||P^{(2)}_{j,i}||P^{(1)}_{n,j}| \\
&< K 2^{\frac{3m}{2}} \sum_k r_1^k \sum_{i,j} \frac{r_2^i}{[(1-\epsilon)r_1]^k} \frac{[(1+\epsilon)r_3]^j}{r_2^i} \frac{[(1+\epsilon)r_4]^n}{r_3^j} \\
&< K 2^{\frac{3m}{2}} \frac{(1+\epsilon)^{2n}}{(1-\epsilon)^n} (n+1)^3 r_4^n.
\end{aligned} \tag{33}
$$

Using the relationships (30), (32) and (33), the Cannon sum $\Omega_n(r_1)$ of the equivalent base satisfies

$$
\begin{aligned}
\Omega_n(r_1) &\leq 2^{\frac{m}{2}} \sum_k \|E_k\|_{r_1} |\tilde{E}_{n,k}| \\
&\leq 2^{\frac{3m}{2}} \sum_k \|E_k\|_{r_1} \sum_{i,j} |\tilde{P}^{(1)}_{i,k}||\tilde{P}^{(2)}_{j,i}||P^{(3)}_{n,j}| \\
&\leq K 2^{3m} \sum_k \frac{(1+\epsilon)^{2k}}{(1-\epsilon)^k} (k+1)^3 r_4^k \sum_{i,j} \frac{r_5^j}{[(1-\epsilon)r_4]^k} \frac{r_6^j}{[(1-\epsilon)r_5]^k} \frac{[(1+\epsilon)r_7]^n}{r_6^j} \\
&< K 2^{3m} (n+1)^6 \left(\frac{1+\epsilon}{1-\epsilon}\right)^{3n} r_7^n.
\end{aligned}
$$

Therefore, the Cannon function of the equivalent base $\{E_n(x)\}$ is

$$\lambda_E(r_1) = \limsup_{n\to\infty} \{\Omega_n(r_1)\}^{\frac{1}{n}} \leq r_7.$$

Since r_7 can be chosen arbitrarily close to r, it follows that $\lambda_E(r) \leq r$; however, it is proved in [15,16] that $\lambda_E(r) \geq r$. This implies that $\lambda_E(r) = r$, which means that the equivalent base $\{E_n(x)\}$ is effective for $H_{[\bar{B}(r)]}$. $\square$

Next, we consider non-simple bases for which there are some restrictions on the degree of the bases. Let $d_n^{(\ell)}$ and $\tilde{d}_n^{(\ell)}$, where $\ell = 1,2,3$, denote the degrees of the bases $\{P_n^{(\ell)}(x)\}$ and $\{\tilde{P}_n^{(\ell)}(x)\}$, respectively, and satisfy the following conditions:

$$d_n^{(\ell)} = O(n) \quad \text{and} \quad \tilde{d}_n^{(\ell)} = O(n);\ \ell = 1,2,3. \tag{34}$$

Thus, there exist positive numbers α_ℓ and β_ℓ such that

$$d_n^{(\ell)} \leq \alpha_\ell n \quad \text{and} \quad \tilde{d}_n^{(\ell)} \leq \beta_\ell n. \tag{35}$$

Furthermore, suppose the bases $\{P_n^{(\ell)}(x)\}$ satisfy the following equality, which is recognized as Newns' condition [32]:

$$\mu_\ell(r) = \tilde{\mu}_\ell(r) = r, \tag{36}$$

where

$$
\begin{aligned}
\mu_\ell(r) &= \limsup_{n\to\infty} \{\|P_n^{(\ell)}\|_r\}^{\frac{1}{n}}, \\
\tilde{\mu}_\ell(r) &= \limsup_{n\to\infty} \{\|\tilde{P}_n^{(\ell)}\|_r\}^{\frac{1}{n}}.
\end{aligned}
$$

Obeying these conditions, we can state and prove the following result.

Theorem 5. *If each of the bases* $\{P_n^{(\ell)}(x)\}$ *satisfy Equations* (35) *and* (36)*, then the equivalent base* $\{E_n(x)\}$ *is effective for* $H_{[\bar{B}(r)]}$.

Proof. According to Equation (36), it follows that

$$\|P_n^{(\ell)}\|_r \le \|P_n^{(\ell)}\|_{r_i} < Kr_{i+1}^n \tag{37}$$

and

$$\|\tilde{P}_n^{(\ell)}\|_r \le \|\tilde{P}_n^{(\ell)}\|_{r_i} < Kr_{i+1}^n. \tag{38}$$

Owing to Cauchy's inequality and the relationship (35), we obtain

$$\begin{aligned}
\|P_n^{(3)}\|_{r_{12}} &= \sup_{\bar{B}(r_{12})} |P_n^{(3)}(x)| \\
&\le 2^{\frac{m}{2}} \sum_{i=0}^{d_n^{(3)}} \|\mathcal{P}_i\|_{r_{12}} |P_{n,i}^{(3)}| \\
&\le 2^{\frac{m}{2}} \|P_n^{(3)}\|_r \sum_{i=0}^{d_n^{(3)}} \left(\frac{r_{12}}{r}\right)^i \\
&\le 2^{\frac{m}{2}} \|P_n^{(3)}\|_r \left(\frac{r_{12}}{r}\right)^{\alpha_3 n} (1+\alpha_3 n).
\end{aligned} \tag{39}$$

Moreover, using the relationships (37) and (38) and applying Cauchy's inequality imply that

$$\begin{aligned}
\|E_n\|_r &\le 2^m \sum_{i,j} \|\tilde{P}_i^{(3)}\|_r |P_{j,i}^{(2)}||P_{n,j}^{(1)}| \\
&< K2^m \sum_{i,j} r_3^i \frac{\|P_j^{(2)}\|_{r_4}}{r_4^i} \frac{\|P_n^{(1)}\|_{r_6}}{r_6^j} \\
&< K2^m r_7^n \sum_{i,j} \left(\frac{r_3}{r_4}\right)^i \left(\frac{r_5}{r_6}\right)^j \\
&< K2^m r_7^n (1+\alpha_1\alpha_2 n).
\end{aligned} \tag{40}$$

Substituting (37)–(40) in the Cannon sum and using Cauchy's inequality, we obtain

$$\begin{aligned}
\Omega_n(r) &\le 2^{\frac{m}{2}} \sum_k \|E_k\|_r |\tilde{E}_{n,k}| \\
&\le 2^m \sum_k r_7^k (1+\alpha_1\alpha_2 k) \sum_{i,j} |\tilde{P}_{i,k}^{(1)}||\tilde{P}_{j,i}^{(2)}||P_{n,j}^{(3)}| \\
&\le 2^m \sum_k r_7^k (1+\alpha_1\alpha_2 k) \sum_{i,j} \frac{\|\tilde{P}_i^{(1)}\|_{r_8}}{r_8^k} \frac{\|\tilde{P}_j^{(2)}\|_{r_{10}}}{r_{10}^i} \frac{\|\tilde{P}_n^{(3)}\|_{r_{12}}}{r_{12}^j} \\
&< 2^m \|P_n^{(3)}\|_{r_{12}} \sum_{i,j,k} \left(\frac{r_9}{r_{10}}\right)^i \left(\frac{r_{11}}{r_{12}}\right)^j \left(\frac{r_7}{r_8}\right)^k (1+\alpha_1\alpha_2 k) \\
&< 2^{\frac{3m}{2}} \|P_n^{(3)}\|_r \left(\frac{r_{12}}{r}\right)^{\alpha_3 n} (1+\alpha_3 n)(1+\alpha_1\alpha_2\alpha_3\beta_1\beta_2 n).
\end{aligned}$$

Taking the n-th root and making n tend to infinity, the Cannon function of the equivalent base $\{E_n(x)\}$ satisfies that

$$\lambda_E(r) \le \mu_3(r)(\frac{r_{12}}{r})^{\alpha_3}.$$

Since r_{12} can be arbitrarily chosen near to r (36), we conclude that $\lambda_E(r) \le r$, but $\lambda_E(r) \ge r$; then, by applying Theorem 1, we obtain that $\lambda_E(r) = r$, which means that $\{E_n(x)\}$ is indeed effective for $H_{[\bar{B}(r)]}$. $\square$

4.3. Effectiveness with Algebraic Property

In the following case, the bases $\{P_n^{(\ell)}(x)\}$ are considered to be algebraic, satisfying the conditions [22]

$$\mu_\ell(r^+) \le r, \quad \ell = 1,2,3 \tag{41}$$

where

$$\mu_\ell(r^+) = \limsup_{n\to\infty}\{\|P_n^{(\ell)}\|_{r^+}\}.$$

For this consideration, we first provide the following result.

Lemma 1. *Let $\{P_n^{(\ell)}(x)\}$, where $\ell = 1,2,3$, be three algebraic bases of polynomials satisfying Equation (41). Then, the equivalent base $\{E_n(x)\}$ satisfies the condition*

$$\mu(r^+) \le r. \tag{42}$$

Proof. Since each of the three bases $\{P_n^{(\ell)}(x)\}$ is algebraic according to [22], the matrices of coefficients $P^{(\ell)}$ and their powers $(P^{(\ell)})^{(t)}$, where $t = 1,2,\ldots,N < \infty$, satisfy the following relationship:

$$\tilde{P}_{n,k}^{(\ell)} = \sum_{t=0}^{N} \gamma_t (P_{n,k}^{(\ell)})^{(t)} \tag{43}$$

where γ_t are constants.

Using Equation (41) and Theorem 1 in [22], we obtain

$$\|P_n^{(\ell)}\|_{r_i} < K r_{i+1}^n \tag{44}$$

and

$$\|\left(P_n^{(\ell)}\right)^{(t)}\|_{r_i} < K r_{i+1}^n. \tag{45}$$

By inserting (45) in (43) and then applying Cauchy's inequality, it follows that

$$|\tilde{P}_{n,k}^{(\ell)}| < K\gamma(N+1)\frac{r_{i+1}^n}{r_i^k}, \; n \ne k \tag{46}$$

where

$$\gamma = \sup_{0\le i\le t}|\gamma_i|.$$

From (44)–(46), and by using Cauchy's inequality, we obtain

$$\begin{aligned}
\|E_n\|_{r_1} &\le 2^m \sum_k r_1^k \sum_{i,j} |\tilde{P}_{i,k}^{(3)}||P_{ji}^{(2)}||P_{n,j}^{(1)}| \\
&< K2^m\gamma(N+1)\|P_n^{(1)}\|_{r_6} \sum_k r_1^k \sum_{i,j} \frac{r_3^i}{r_2^k} \frac{\|P_j^{(2)}\|_{r_4}}{r_4^i r_6^j} \\
&= K2^m\gamma(N+1)\|P_n^{(1)}\|_{r_6} \sum_{i,j,k} \left(\frac{r_3}{r_4}\right)^i \left(\frac{r_5}{r_6}\right)^j \left(\frac{r_1}{r_2}\right)^k \\
&< K2^m\gamma(N+1)\, r_7^n.
\end{aligned}$$

We can take the upper limit as $n \to \infty$ and make $r_7 \to r^+$ imply that $\mu(r^+) \leq r$, which means that the equivalent base $\{E_n(x)\}$ satisfies Equation (42) whenever the three constituent bases are algebraic. Therefore, the lemma is established. $\square$

The effectiveness of the equivalent bases of polynomials for $H_{[B_+(r)]}$ holds without any restrictions on the constituent bases to be effective in the same space as indicated in the following result.

Theorem 6. *If the three algebraic bases $\{P_n^{(\ell)}(x)\}$ satisfy the normalizing condition (42), then the equivalent base $\{E_n(x)\}$ is effective for $H_{[B_+(r)]}$.*

Proof. Suppose that the three bases $\{P_n^{(\ell)}(x)\}$ for $\ell = 1,2,3$ are algebraic and satisfy (42). Substituting from (44) and (46) in the Cannon sum of the equivalent base, it follows that

$$\begin{aligned}
\Omega_n(r_1) &\leq 2^{\frac{3m}{2}} \sum_k \|E_k\|_{r_1} \sum_{i,j} |\tilde{P}_{i,k}^{(1)}||\tilde{P}_{j,i}^{(2)}||P_{n,j}^{(1)}| \\
&< K 2^{\frac{3m}{2}} \gamma^2 (N+1)^2 r_{13}^n \sum_{i,j,k} \left(\frac{r_9}{r_{10}}\right)^i \left(\frac{r_{11}}{r_{12}}\right)^j \left(\frac{r_7}{r_8}\right)^k \\
&< K 2^{\frac{3m}{2}} \gamma^2 (N+1)^2 r_{13}^n.
\end{aligned}$$

from which we can deduce as before that $\lambda_E(r_1) \leq r_{13}$. By taking $r_{13} \to r^+$, we obtain $\lambda_E(r^+) \leq r$, but $\lambda_E(r^+) \geq r$. Therefore, $\lambda_E(r^+) = r$, which implies that the equivalent base $\{E_n(x)\}$ is effective for $H_{[B_+(r)]}$. $\square$

Now, letting $r \to 0$ in Theorem 6, Equation (42) will be replaced by the equation

$$\mu_\ell(0^+) = 0, \ \ell = 1,2,3. \tag{47}$$

Thus, the following result follows.

Corollary 1. *Let $\{P_n^{(\ell)}(x)\}$ where $\ell = 1,2,3$ be three algebraic bases satisfying Equation (47). Then, the equivalent base $\{E_n(x)\}$ is effective for $H_{[0^+]}$.*

Concerning the effectiveness of the equivalent base for $H_{[B(R)]}$, let $\{P_n^{(\ell)}(x)\}$ be bases of polynomials that satisfy the conditions

$$\begin{aligned}
\mu_\ell(r) &< R, \quad \forall r < R, \\
\tilde{\mu}_\ell(r) &< R, \quad \forall r < R.
\end{aligned} \tag{48}$$

We can similarly proceed as in the proof of Theorem 6 to conclude the following.

Theorem 7. *Let $\{P_n^{(\ell)}(x)\}$, where $\ell = 1,2,3$, be three bases of polynomials satisfying Equation (48). Then, the equivalent base $\{E_n(x)\}$ is effective for $H_{[B(R)]}$.*

Now, by letting $R \to \infty$ exist in Theorem 7, Equation (48) will be replaced by

$$\begin{aligned}
\mu_\ell(r) &< \infty, \\
\tilde{\mu}_\ell(r) &< \infty, \quad \forall r < \infty.
\end{aligned} \tag{49}$$

Consequently, the effectiveness of the equivalent base for the space of a complete special function, $H_{[\infty]}$, is established as follows.

Corollary 2. *Let $\{P_n^{(\ell)}(x)\}$ where $\ell = 1,2,3$ be three bases of polynomials satisfying Equation (49). Then, the equivalent base $\{E_n(x)\}$ is effective for $H_{[\infty]}$.*

5. The Order of the Equivalent Base

In this section, we determine the order ρ of the equivalent base $\{E_n(x)\}$ in relation to the orders ρ_ℓ where $\ell = 1, 2, 3$ of the constituent bases $\{P_n^{(\ell)}(x)\}$, where

$$\rho_\ell = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\log \omega_n^{(\ell)}(r)}{n \log n}. \tag{50}$$

This relationship is formulated in the following.

Theorem 8. *Let $\{P_n^{(\ell)}(x)\}$ be a simple monic base of polynomials of the receptive order ρ_ℓ, where $\ell = 1, 2, 3$. Then, the order of the equivalent base $\{E_n(x)\}$ satisfies the inequality*

$$\frac{1}{2}(\rho_2 - 2\rho_1 - 2\rho_3) \le \rho \le \rho_1 + 2\rho_2 + 2\rho_3, \tag{51}$$

and these bounds are attainable.

Proof. Since the three bases $\{P_n^{(\ell)}(x)\}$ are simple monic bases of the orders ρ_ℓ, $\ell = 1, 2, 3$, then Equation (50) yields

$$1 < r^n < \|P_n^{(\ell)}\|_r < \Omega_n^{(\ell)}(r) < K n^{\sigma_\ell n},\ n \ge 1,\ \sigma_\ell > \rho_\ell, \tag{52}$$

and

$$\|\tilde{P}_n^{(\ell)}\|_r < \Omega_n^{(\ell)}(r) < K n^{\sigma_\ell n}. \tag{53}$$

By multiplying $P_k^{(1)}(x) = \sum_j \mathcal{P}_j(x) P_{k,j}^{(1)}$ by $P_{s,k}^{(1)}$ and using Cauchy's inequality (see [13]), it follows that

$$|P_{k,j}^{(1)} \tilde{P}_{s,k}^{(1)}| \le \frac{\|P_k^{(1)}(x) \tilde{P}_{s,k}^{(1)}\|}{r^j} \le \omega_k^{(1)}(r). \tag{54}$$

Owing to Equations (40) and (52)–(54), the Cannon sum $\Omega_n(r)$ of the equivalent base satisfies

$$\begin{aligned}
\Omega_n(r) &\le 2^m \sum_k \sum_i \|\tilde{P}_i^{(3)}\|_r \sum_{j,s,t} |P_{j,i}^{(2)}||P_{k,j}^{(1)} \tilde{P}_{s,k}^{(1)}||\tilde{P}_{t,s}^{(2)}||P_{n,t}^{(3)}| \\
&\le 2^m \sum_k \sum_i \|\tilde{P}_i^{(3)}\|_r \sum_{j,s,t} \frac{\|P_j^{(2)}\|_r}{r^j} \omega_k^{(1)}(r) \frac{\|\tilde{P}_t^{(2)}\|_r}{r^s} \frac{\|P_n^{(3)}\|_r}{r^t} \\
&< K 2^m n^{\sigma_1 + 2\sigma_2 + 2\sigma_3 + 5}.
\end{aligned}$$

Since σ_ℓ can be chosen as near as possible to ρ_ℓ, where $\ell = 1, 2, 3$, an upper bound of the order ρ of the equivalent base $\{E_n(x)\}$ is given by

$$\rho = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\log \omega_n(r)}{n \log n} \le \rho_1 + 2\rho_2 + 2\rho_3. \tag{55}$$

Now, we estimate the lower bound of the order of the equivalent base. According to Theorem 3 in [21], the order $\tilde{\rho}_1$ of the inverse base $\{\tilde{P}_n^{(1)}(x)\}$ is

$$\tilde{\rho}_1 \le 2\rho_1 \tag{56}$$

Using Equations (15) and (56), it follows that

$$\begin{aligned}
\rho_2 &\le \tilde{\rho}_1 + 2\rho + 2\rho_3 \\
&\le 2\rho_1 + 2\rho + 2\rho_3.
\end{aligned}$$

Therefore,

$$\rho \geq \frac{1}{2}(\rho_2 - 2\rho_1 - 2\rho_3), \tag{57}$$

and the result is established. $\square$

The upper and lower bounds of the order of the equivalent base are attainable. This fact is illustrated by the following two examples.

Example 1. *Consider the three bases of SMPs $\{P_n^{(\ell)}(x)\}$ where $\ell = 1, 2, 3$ as follows:*

$$P_n^{(1)}(x) = \begin{cases} \mathcal{P}_n(x) + \alpha_n \mathcal{P}_{n-1}(x), & n \text{ is odd} \\ \mathcal{P}_n(x), & n \text{ is even} \end{cases}$$

$$P_n^{(2)}(x) = \begin{cases} \mathcal{P}_n(x) + \beta_n \mathcal{P}_{n-1}(x), & n \text{ is even} \\ \mathcal{P}_n(x), & n \text{ is odd} \end{cases}$$

and

$$P_n^{(3)}(x) = \begin{cases} \mathcal{P}_n(x) + \gamma_n \mathcal{P}_{n-1}(x), & n \text{ is odd} \\ \mathcal{P}_n(x), & n \text{ is even} \end{cases}$$

where $\alpha_n = n^{\alpha n}$, $\beta_n = n^{\beta n}$, and $\gamma_n = n^{\gamma n}$.

It is easy to see that $\rho_1 = \alpha$, $\rho_2 = \beta$, and $\rho_3 = \gamma$.

Now, we construct the equivalent base as follows:

$$E_n(x) = \begin{cases} \mathcal{P}_n(x) + (\gamma_n + \alpha_n)\mathcal{P}_{n-1}(x) + \alpha_n \beta_{n-1}(\mathcal{P}_{n-2}(x) + \gamma_{n-2}\mathcal{P}_{n-3}(x)), & n \text{ is odd} \\ \mathcal{P}_n(x) + \beta_n(\mathcal{P}_{n-1}(x) + \gamma_{n-1}\mathcal{P}_{n-2}(x)), & n \text{ is even.} \end{cases}$$

Hence,

$$\mathcal{P}_n(x) = \begin{cases} E_n(x) - (\alpha_n + \gamma_n)E_{n-1}(x) + \gamma_n \beta_{n-1} E_{n-2}(x) - \gamma_n \beta_{n-1} E_{n-3}(x), & n \text{ is odd} \\ E_n(x) - \beta_n E_{n-1}(x) + \beta_n \alpha_{n-1} E_{n-2}(x), & n \text{ is even} \end{cases}$$

Thus, the Cannon sum of the base $\{E_n(x)\}$ is

$$\begin{aligned} \Omega_n(r) = r^n &+ 2(\alpha_n + \gamma_n) r^{n-1} + 2((\alpha_n + \gamma_n)\beta_{n-1} + \gamma_n) r^{n-2} \\ &+ 2(\alpha_n \beta_{n-1} \gamma_{n-2} + \gamma_n \beta_{n-1} \gamma_{n-2} + \gamma_n \beta_{n-1} \alpha_{n-2} \beta_{n-1}) r^{n-3} \\ &+ 2\gamma_n \beta_{n-1} \alpha_{n-2} \beta_{n-3}(r^{n-4} + \gamma_{n-4} r^{n-5}), \ \text{ when } n \text{ is odd.} \end{aligned}$$

which means that

$$\lim_{n \to \infty} \frac{\log \Omega_{2n+1}(r)}{(2n+1)\log(2n+1)} = \alpha + 2\beta + 2\gamma.$$

Now, we observe that

$$\begin{aligned} \Omega_n(r) = r^n &+ 2\beta_n R^{n-1} + 2(\beta_n \gamma_{n-1} + \beta_n \alpha_{n-1}) r^{n-2} \\ &+ 2\beta_n \alpha_{n-1} \beta_{n-2} r^{n-3} + 2\beta_n \alpha_{n-1} \beta_{n-2} \gamma_{n-3} r^{n-4}, \ \text{ when } n \text{ is even.} \end{aligned}$$

In this case,

$$\lim_{n \to \infty} \frac{\log \Omega_{2n}(r)}{2n \log 2n} = \alpha + 2\beta + \gamma.$$

Therefore, the order ρ of the equivalent base is given by

$$\rho = \lim_{r\to\infty} \limsup_{n\to\infty} \frac{\log \Omega_n(r)}{n \log n} = \alpha + 2\beta + 2\gamma.$$

Example 2. *Let $\{P_n^{(\ell)}(x)\}$ be three simple monic bases of SMPs such that*

$$P_n^{(1)}(x) = \begin{cases} \mathcal{P}_n(x) + \alpha_n(\mathcal{P}_{n-1}(x) + \mathcal{P}_{n-2}), & n \text{ is even} \\ \mathcal{P}_n(x) + \mathcal{P}_{n-1}(x), & n \text{ is odd} \end{cases}$$

$$P_n^{(2)}(x) = \begin{cases} \mathcal{P}_n(x) + \delta_n \mathcal{P}_{n-1}(x) + (\gamma_{n-1}\delta_n - \gamma_{n-1} + \alpha_{n-1})\mathcal{P}_{n-2}(x) & \\ \quad + (\gamma_{n-1}\delta_n + \alpha_{n-1}\delta_{n-2} + \alpha_{n-1} - \gamma_{n-1})\mathcal{P}_{n-3}(x) & \\ \quad + \alpha_{n-1}\gamma_{n-3}\delta_{n-2}(\mathcal{P}_{n-4}(x) + \mathcal{P}_{n-5}(x)), & n \text{ is odd} \\ \mathcal{P}_n(x) + (\gamma_n - \alpha_n)\mathcal{P}_{n-1}(x) + (\gamma_n - \alpha_n - \alpha_n\delta_{n-1})\mathcal{P}_{n-2}(x) & \\ \quad - \alpha_n\gamma_{n-2}\delta_{n-1}(\mathcal{P}_{n-3}(x) + \mathcal{P}_{n-4}(x)), & n \text{ is even,} \end{cases}$$

and

$$P_n^{(3)}(x) = \begin{cases} \mathcal{P}_n(x) + \gamma_n(\mathcal{P}_{n-1}(x) + \mathcal{P}_{n-2}(x)), & n \text{ is even,} \\ \mathcal{P}_n(x) + \mathcal{P}_{n-1}(x), & n \text{ is odd,} \end{cases}$$

where $\alpha_n = n^{\alpha n}$, $\delta_n = n^{(\beta - 2\alpha + 2\gamma)n}$ and $\gamma_n = n^{\gamma n}$.

In this case, the equivalent base is given in the form

$$E_n(x) = \begin{cases} \mathcal{P}_n(x) + \delta_n \mathcal{P}_{n-1}(x), & n \text{ is odd,} \\ \mathcal{P}_n(x), & n \text{ is even.} \end{cases}$$

We can proceed in a similar procedure as in Example 1 to prove that the orders of the bases $\{P_n^{(1)}(x)\}$, $\{P_n^{(2)}(x)\}$, and $\{P_n^{(3)}(x)\}$ are α, β, and γ, respectively. In this case, the order of the equivalent set is $\rho = \frac{1}{2}(\beta - 2\alpha - 2\gamma)$, as required.

6. The T_ρ Property of the Equivalent Base of SMPs

In this section, we construct the T_ρ property of equivalent bases of special monogenic polynomials in the open ball $B(R)$. First, we recall the definition of the T_ρ property as given in [27], as follows.

Definition 10. *Let $0 < \rho < \infty$. Then, a base $\{P_n(x)\}$ has the T_ρ property in an open ball $B(R)$ if it represents all entire special monogenic functions of an order less than ρ in $B(R)$.*

Let

$$\omega(r) = \limsup_{n\to\infty} \frac{\log\ \omega_n(r)}{n\ \log\ n}.$$

The restriction placed on the base $\{P_n(x)\}$ of SMPs to satisfy the T_ρ property in the open ball $B(R)$ [27] is stated as follows.

Theorem 9. *Let $\{P_n(x)\}$ be a base of special monogenic polynomials and suppose that the function $f(x)$ is an entire SMF of an order less than ρ. Then, the necessary and sufficient conditions for the base $\{P_n(x)\}$ to have the property T_ρ in $B(R)$ are $\omega(r) \le \frac{1}{\rho}$ $\forall r < R$.*

In this regard, we state and prove the following result.

Theorem 10. *If the simple monic bases $\{P_n^{(\ell)}(x)\}$ have a T_{ρ_ℓ} property in $B(R)$, where $R > 0$ and $\rho \le \min\{\rho_1, \rho_2, \rho_3\}$, then the equivalent base $\{E_n(x)\}$ will have a $T_{\frac{\rho}{5}}$ property in $B(R)$.*

Proof. Since the bases $\{P_n^{(\ell)}(x)\}$ have the T_{ρ_ℓ} property where $\ell = 1, 2, 3$ in $B(R)$ for $R > 0$, then, according to Theorem 9, we have

$$\omega^{(\ell)}(r) \leq \frac{1}{\rho_\ell} \; \forall r < R, where \;\; \ell = 1, 2, 3.$$

Since $\rho \leq \min\{\rho_1, \rho_2, \rho_3\}$, then

$$\omega_n^{(\ell)}(r) < K\, n^{\frac{n}{\rho}},\; n \geq 1. \tag{58}$$

Since the bases $\{P_n^{(\ell)}(x)\}$ are simple monic, it follows that

$$1 < r^n < \|P_n^{(\ell)}\|_r < \omega_n^{(\ell)}(r) < K\, n^{\frac{n}{\rho}}, \tag{59}$$

and

$$\|\tilde{P}_n^{(\ell)}\|_r < \omega_n^{(\ell)}(r) < K\, n^{\frac{n}{\rho}}. \tag{60}$$

Using (58)–(60), and Cauchy's inequality, the Cannon sum for the equivalent base $\{E_n(x)\}$ is

$$\begin{aligned}
\Omega_n(r) &= \sum_k \|E_k \tilde{E}_{n,k}\|_r \\
&\leq 2^m \sum_k \sum_i \|\tilde{P}_i^{(3)}\|_r \sum_{j,s,t} \frac{\|P_j^{(2)}\|_r}{r^j} \omega_k^{(1)}(r) \frac{\|\tilde{P}_t^{(2)}\|_r}{r^s} \frac{\|P_n^{(3)}\|_r}{r^t} \\
&< K\, 2^m n^{\frac{5n}{\rho}+5}.
\end{aligned}$$

Taking the upper limit, we obtain the function $\Omega(r)$ for the equivalent base $\{E_n(x)\}$ in the form

$$\Omega(r) = \limsup_{n\to\infty} \frac{log\, \Omega_n(r)}{n\; log\; n} \leq \frac{1}{(\frac{\rho}{5})} \quad \forall r < R,$$

which leads to the fact that the equivalent base has the property $T_{\frac{\rho}{5}}$ in $B(R)$, as required. □

Example 3. *Consider the three simple monic bases of polynomials $\{P_n^{(\ell)}(x)\}$ where $\ell = 1, 2, 3$ as follows:*

$$P_n^{(1)}(x) = \begin{cases} \mathcal{P}_n(x) + n^{\frac{n}{2}} \mathcal{P}_{n-1}(x), & n \text{ is even} \\ \mathcal{P}_n(x), & n \text{ is odd} \end{cases}$$

$$P_n^{(2)}(x) = \begin{cases} \mathcal{P}_n(x) + n^{\frac{n}{3}} \mathcal{P}_{n-1}(x), & n \text{ is even} \\ \mathcal{P}_n(x), & n \text{ is odd} \end{cases}$$

$$P_n^{(3)}(x) = \begin{cases} \mathcal{P}_n(x) + n^{\frac{n}{4}} \mathcal{P}_{n-1}(x), & n \text{ is odd} \\ \mathcal{P}_n(x), & n \text{ is even} \end{cases}$$

It is easily seen that $\omega^{(1)}(r) = \frac{1}{2}$, $\omega^{(2)}(r) = \frac{1}{3}$, and $\omega^{(3)}(r) = \frac{1}{4}$.

Therefore, the bases $\{P_n^{(\ell)}(x)\}$ have a T_2 property, T_3 property, and T_4 property in $B(R)$ for $\ell l = 1, 2,$ and 3, respectively.

Now, we construct the equivalent base as follows:

$$E_n(x) = \begin{cases} \mathcal{P}_n(x) + (n^{\frac{n}{2}} - n^{\frac{n}{4}})\mathcal{P}_{n-1}(x) + n^{\frac{n}{2}}(n-1)^{\frac{n-1}{3}}(\mathcal{P}_{n-2}(x) - (n-2)^{\frac{n-2}{4}}\mathcal{P}_{n-3}(x)), & n \text{ is odd} \\ \mathcal{P}_n(x) + n^{\frac{n}{3}}(\mathcal{P}_{n-1}(x) - (n-1)^{\frac{n-1}{4}}\mathcal{P}_{n-2}(x)), & n \text{ is even.} \end{cases}$$

Hence,

$$\Omega(r) = \limsup_{n\to\infty} \frac{\log \Omega_n(r)}{n \log n} \leq \frac{1}{(\frac{2}{5})},$$

i.e., the equivalent base has a $T_{\frac{2}{5}}$ property in $B(R)$.

7. Conclusions and Future Work

This paper employs the definition of the product base of SMPs to construct a new base called the equivalent base in Fréchet modules in the Clifford setting. The convergence properties of the derived base were treated for different classes of bases. Within this study, we indicate which type of restrictions we should consider on the coefficients to justify the effectiveness properties of the equivalent base in various regions of convergence, such as open balls, closed balls, at the origin, and for all entire SMFs. Furthermore, given the orders of the constituent bases, we determined the lower and upper bounds of the order of the equivalent base. Moreover, the T_ρ property of the equivalent base is determined in the case of simple monic bases, which are promising for characterizing this property for more general bases.

Looking back to our constructed base,

$$\{E_n(x)\} = \{\tilde{P}_n^{(3)}(x)\}\{P_n^{(2)}(x)\}\{P_n^{(1)}(x)\}$$

and by taking $\{P_n^{(3)}(x)\} = \{P_n^{(1)}(x)\}$, a similar base $\{S_n(x)\}$ can be considered a special case of the equivalent base $\{E_n(x)\}$, reflecting that the results in the current study generalize the corresponding results in [33].

This study encourages the provision of answers to other open problems regarding the representations of entire functions in several complex variables. We believe that the results in this study are likely to hold in the setting of several complex matrices in different convergence regions, such as hyperspherical, polycylindrical, and hyperelliptical regions.

Recently, the authors of [18] proved that the Bessel special monogenic polynomials are effective for the space $H_{[\bar{B}(r)]}$, and the authors of [24] proved that the Chebychey polynomials is effective for the space $H_{[\bar{B}(1)]}$. The Bernoulli special monogenic polynomials are proved to have an order of 1 and a type $\frac{1}{2\pi}$, while the Euler special monogenic polynomials have an order of 1 and a type $\frac{1}{\pi}$ (see [23]). Demonstrating how the convergence properties involve the effectiveness, order, and type of the different constructed bases we have mentioned above, as well as the corresponding aspects of the original bases and, in particular, the well-known special polynomial bases, is one of the most challenging subjects to explore. The proposed methodological weakness is that the work lacks practical application. However, in upcoming research, it will be interesting to study concrete applications of mathematical physics problems, such as Legendre polynomials and their relation to solutions of the Dirac equation and its other formulation as the spinor functions, as well as in curved space–time, which has many applications in quantum mechanics.

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