Self-Improving Properties of Continuous and Discrete Muckenhoupt Weights: A Unified Approach

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Abstract: In this paper, we develop a new technique on a time scale $\mathbb{T}$ to prove that the self-improving properties of the Muckenhoupt weights hold. The results contain the properties of the weights when $\mathbb{T} = \mathbb{R}$ and when $\mathbb{T} = \mathbb{N}$, and also can be extended to cover different spaces such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = q^{\mathbb{N}}$, etc. The results will be proved by employing some new refinements of Hardy's type dynamic inequalities with negative powers proven and designed for this purpose. The results give the exact value of the limit exponent as well as the new constants of the new classes.

Keywords: dynamic Hardy's type inequality; Muckenhoupt weights; self-improving properties; time scales

MSC: 26D07; 42B25; 42C10

1. Introduction

A weight u is a non-negative locally integrable function defined on a bounded interval $\hat{J}_0 \subset \mathbb{R}_+ = [0, \infty)$. We consider subintervals $\hat{J}$ of $\hat{J}_0$ of the form $[0, t]$, for $0 < t < \infty$ and denote by $|\hat{J}|$ the Lebesgue measure of $\hat{J}$. A weight u which satisfies

$$\frac{1}{|\hat{J}|} \int_{\hat{J}} u(t)dt \leq \mathcal{C} \operatorname*{ess\,inf}_{t \in \hat{J}} u(t), \text{ for all } t \in \hat{J}, \tag{1}$$

is called an $A^1(\mathcal{C})-$ Muckenhoupt weight, where $\mathcal{C} > 1$. In [1], the author proved that if u is a monotonic weight that satisfies the condition (1), then there exists $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$ such that

$$\frac{1}{|\hat{J}|} \int_{\hat{J}} u^p(t)dt \leq \frac{\mathcal{C}}{\mathcal{C} - p(\mathcal{C}-1)} \left(\frac{1}{|\hat{J}|} \int_{\hat{J}} u(t)dt \right)^p, \tag{2}$$

which is the reverse of Hölder's inequality. In [2], the authors improved the Muckenhoupt inequality (2) by establishing the best constant for any weight u, which is not necessarily monotonic. Their proof was obtained by using the rearrangement u^* of the function u over the interval $\hat{J}_0$. In particular, they proved that if u satisfies (1) with $\mathcal{C} > 1$, then

$$\frac{1}{|\hat{J}|} \int_{\hat{J}} u^p(t)dt \leq \frac{\mathcal{C}^{1-p}}{\mathcal{C} - p(\mathcal{C}-1)} \left(\frac{1}{|\hat{J}|} \int_{\hat{J}} u(t)dt \right)^p, \tag{3}$$

for $p < \mathcal{C}/(\mathcal{C}-1)$. A non-negative measurable weight u is called an $\mathcal{A}^p(\mathcal{C})-$Muckenhoupt weight for $p > 1$, if there exists a constant $\mathcal{C} > 1$, such that the inequality

$$\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}u(t)dt\right)\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}u^{-\frac{1}{p-1}}(t)dt\right)^{p-1} \leq \mathcal{C}, \tag{4}$$

holds for every subinterval $\hat{J} \subset \hat{J}_0$. The smallest constant $\mathcal{C}$ satisfying (1) or (4) is called the $\mathcal{A}^p-$norm of the weight u and is denoted by $[\mathcal{A}^p(u)]$. For a given fixed constant, $\mathcal{C} > 1$ if the weight $u \in \mathcal{A}^p$ then $[\mathcal{A}^p(u)] \leq \mathcal{C}$. In 1972, Muckenhoupt [1] introduced the full characterizations of $\mathcal{A}^p-$weights in connection with the boundedness of the Hardy and Littlewood maximal operator in the space $L_u^p(\mathbb{R}_+)$. In [3], the authors proved that if $u \in L^p(\mathbb{R}_+)$ and satisfies (4), then

$$\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}u(t)dt\right)\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}u^{-\frac{1}{q-1}}(t)dt\right)^{q-1} \leq \mathcal{C}_1, \tag{5}$$

for all $q < p$, where the constant $\mathcal{C}_1 = \mathcal{C}_1(q,\mathcal{C})$. In other words, Muckenhoupt's result for the *self-improving* property states that $u \in \mathcal{A}^p(\mathcal{C}) \Rightarrow \exists \epsilon > 0$ such that $u \in \mathcal{A}^{p-\epsilon}(\mathcal{C}_1)$, and then

$$\mathcal{A}^p(\mathcal{C}) \subset \mathcal{A}^{p-\epsilon}(\mathcal{C}_1). \tag{6}$$

The properties of Muckenhoupt class have been deeply investigated, especially in one dimension, and the following aspects have been considered extensively:

(h_1). *Finding the exact value of the limit exponent q for which the self-improving property holds;*

(h_2). *Finding the best constants C_1 for which the improved $\mathcal{A}^q-$ condition is satisfied.*

Some great work in the problems of finding the exact bounds of exponents for embedding (6) was achieved in many papers, see for example, [1,2,4–11]. Since it is impossible to give an exhaustive account of the results related to the problems under consideration, we shall dwell only on some of them, concerned with sharp results for a self-improving property given by Korenovskii [12]. In particular, Korenovskii found the sharp lower bound of the exponent (*self-improving property*), for which (6) holds and proved that the optimal integrability exponent q is the positive root of the equation

$$\frac{1}{x}\left(\frac{p-1}{p-x}\right)^{p-1} = \mathcal{C}, \tag{7}$$

and also found the explicit value of the constant of the new class. One of the most significant characteristics of the A^p Muckenhoupt weights is the extrapolation theorem that was announced, and a detailed proof was given, by Rubio de Francia in [13,14]. Many results related to this topic have been studied by several authors (see [15–22]).

Over the past few years, the interest in the area of discrete harmonic analysis has been renovated and it became an active field of research. This renovated interest began with an observation of M. Riesz in their work on the Hilbert transform in 1928, who proved that the Hilbert discrete operator

$$Hf(n) := \sum_{m \in \mathbb{Z}_+} \frac{f(n-m)}{m},$$

is bounded in ℓ^p-spaces if the operator

$$Hf(x) := p.v. \int_{\mathbb{R}}\left(\frac{f(x-t)}{t}\right)dt,$$

is bounded in L^p-spaces. In 1952, Alberto Calderón and Antoni Zygmund [23] extended the results to a more general class of singular integral operators with kernels. It is worth mentioning that the progress in the last years regarding discrete analogues of operators in harmonic analysis is related with Calderón–Zygmund analogues, discrete maximal

operators and related problems with number theory, translation invariant fractional integral operators, translation invariant singular Radon transforms, quasi-translation invariant operators, spherical averages and discrete rough maximal functions; we suggest the reader to consider the paper [24] and the references cited therein.

As performed by Hughes (see [25] and the references therein), the discrete operators are nicely connected to critical problems in number theory. For example, Waring's problem, which questions whether each natural number k is associated with a positive integer s satisfying that every natural number is the sum of at most s natural numbers raised to the power k. This problem has been extended to find the the operator $G(k)$, which is defined to be the smallest positive integer s so that every sufficiently large integer (i.e., every integer greater than some constant) can be illustrated as a sum of no more than s positive integers to the power of k. Throughout the paper, we assume that $1 < p < \infty$ and assume that the discrete weights are positive sequences defined on $J \subset \mathbb{Z}_+ = \{1, 2, 3, \dots\}$, where J is of the form $\{1, 2, \dots, N\}$. The notion X^d denotes the set of all nonincreasing and non-negative sequences of X. The discrete weight v is said to be in the discrete Muckenhoupt $\mathcal{A}_p$ class for $p > 1$, if there exists a constant $A > 1$ satisfying the inequality

$$\left(\frac{1}{n} \sum_{k=1}^{n} v(k) \right) \left(\frac{1}{n} \sum_{k=1}^{n} v^{\frac{-1}{p-1}}(k) \right)^{p-1} \leq A, \text{ for all } n \in J. \tag{8}$$

The discrete v is said to be in the discrete Ariňo and Muckenhoupt $\mathcal{B}_p$ class for $p > 0$, if there exists a constant $A > 1$ such that the inequality

$$\sum_{k=n}^{\infty} \frac{v(k)}{k^p} \leq \frac{A}{n^p} \sum_{k=1}^{n} v(k), \text{ for all } n \in J. \tag{9}$$

The necessary and sufficient conditions for the boundedness of a series of discrete classical operators (Hardy–Littlewood maximal operator, Hardy's operator) in the weighted spaces $\ell^p(v)$ are the $\mathcal{A}_p$−Muckenhoupt condition, $\mathcal{B}_p$−condition on the weight v. In [26], the authors proved that the discrete Hardy–Littlewood maximal operator $\mathcal{M} : \ell^p(v)^d \to \ell^p(v)$, which is defined by

$$\mathcal{M}f(n) = \sup_{n \in J} \frac{1}{n} \sum_{k=1}^{n} f(k), \text{ for all } n \in J,$$

is bounded on $\ell^p(v)^d$ for $p > 1$ if and only if $v \in \mathcal{A}_p$. In [27], Heing and Kufner proved that the Hardy operator $\mathcal{H} : \ell^p(v)^d \to \ell^p(v)$, which is defined by

$$\mathcal{H}f(n) = \frac{1}{n} \sum_{k=1}^{n} f(k), \text{ for all } n \in J,$$

is bounded in $\ell^p(v)^d$ for $1 < p < \infty$ if and only if $v \in \mathcal{B}_p$ and $\lim_{n \to \infty}(v(n+1)/v(n)) = c$ for some constant $c > 0$ and $\sum_{n=1}^{\infty} v(n) = \infty$. In [28], Bennett and Gross-Erdmann improved the result of Heing and Kufner by excluding the conditions on v. In [29], the authors proved that the discrete Hardy operator is bounded in $\ell^p(v)^d$ for $p > 1$ if and only if $v \in \mathcal{A}_p$. The discrete weight v is said to be belong to the discrete Muckenhoupt $\mathcal{A}_1$−class if there exists a constant $A > 0$ such that the inequality $\mathcal{H}u(n) \leq A \inf_{n \in J} u(n)$, or equivalently $\mathcal{M}u(n) \leq Au(n)$, holds for all $n \in J$. In [29], the authors proved the self-improving property of the weighted discrete Muckenhoupt classes. They established also the exact values of the limit exponents as well as new constants of the new classes. These values correspond to the sharp values of the continuous case that has been obtained by Nikolidakis (see [7,8]). For more details of discrete results, we refer the reader to the papers [30–34].

In [28], the authors marked that the study of discrete inequalities is not a simple mission, and it is in fact more complicated to analyze than its integral counterparts. They

discovered that the conditions do not coincide, in any natural way, with those that are obtained by discretization of the results of functions but the reverse is true. In other words, the results satisfied for sums holds, with the obvious modifications, for integrals which in fact proved the first part of basic principle of Hardy, Littlewood and Polya [35]. Obviously the proofs in the discrete form are transferred instantly and much more simpler, when applied to integrals.

The natural questions which arise now are as follows:

(Q_1). *Is it possible to find a new approach to unify the proofs of the self-improving properties of continuous and discrete Muckenhoupt weights*?

(Q_2). *Is it possible to prove the self-improving properties of Ariňo and Muckenhoupt $\mathcal{B}_p$ weights*?

Our aim in this paper is to give an answer to the first question on time scales, which has received much attention and become a major field in pure and applied mathematics today. The second question will be considered later.

The general idea on time scales is to prove a result for dynamic inequality or dynamic equation, where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. This idea has been created by Hilger [36] to unify the study of the continuous and the discrete results. He started the study of dynamic equations on time scales. The three most popular examples of calculus on time scales are differential calculus, difference calculus and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, $\mathbb{T} = h\mathbb{N}$, for $h > 0$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities. In more precise terms, we can say that the study of dynamic inequalities or dynamic equations on time scales helps avoid proving results twice—once for differential inequality and once again for difference inequality. For more details we refer to the books [37,38] and the references they have cited. Very recently, the authors in [39–43] proved the time scale versions of the Muckenhoupt and Gehring inequalities and used them to prove some higher integrability results on time scales. This also motivated us to develop a new technique on time scales to prove some new results of inequalities with weights and use the new inequalities to formulate some conditions for the boundedness of the Hardy operator with negative powers on time scales and show the applications of the obtained results.

The paper is organized as follows: In Section 2, we prove some Hardy's type inequalities and new refinements of these inequalities with negative powers. In Section 3, we will employ some of these inequalities to prove the self-improving properties of the Muckenhoupt class on a time scale $\mathbb{T}$ for non-negative and nondecreasing weights. The main results give a solution on time scales of the problem of finding the exact value of the limit exponent $q < p$, for which the self-improving property holds and also for the problem of finding the best constants $\mathcal{C}_1$ for which the improved A_q-condition satisfies (h_1) and (h_2) above.

2. Hardy's Type Inequalities with Negative Powers

In this section, we prove some Hardy's type inequalities and the new refinements of these inequalities with negative powers. First, we recall the following concepts related to the notions of time scales and for more details, we refer to the two books [44,45] which summarize and organize much of the time scale calculus. A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous (rd-continuous) if f is continuous at left-dense points and right dense-points in $\mathbb{T}$, and left-side limits exist and are finite. The set $C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$ denotes the set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$. The derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable functions f and g are given by

$$(fg)^\Delta = fg^\Delta + f^\Delta g^\sigma = f^\Delta g + f^\sigma g^\Delta, \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{g\, g^\sigma},$$

where $\sigma = \sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ is the forward jump operator on a time scale. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose that $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the two chain rules that we will use in this paper are given in the next two formulas.

$$f^\Delta(g(t)) = f'(g(\xi))g^\Delta(t), \quad for \;\; \xi \in [t, \sigma(t)], \tag{10}$$

and

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'\Big(g(t) + h\mu(t)g^\Delta(t)\Big) dh \right\} g^\Delta(t). \tag{11}$$

A special case of (11) is

$$\left[u^\lambda(t) \right]^\Delta = \lambda \int_0^1 [hu^\sigma + (1-h)u]^{\lambda-1} u^\Delta(t) dh. \tag{12}$$

In this paper, we will refer to the (delta) integral which, we can define as follows: If $G^\Delta(t) = g(t)$, then the Cauchy (delta) integral of g is defined by $\int_a^t g(x)\Delta x := G(t) - G(a)$. If $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(x)\Delta x$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^\Delta(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as $\int_a^\infty f(x)\Delta x := \lim_{b\to\infty} \int_a^b f(x)\Delta x$. The integration on discrete time scales is defined by

$$\int_a^b g(t)\Delta t = \sum_{t \in [a,b)} \mu(t)g(t).$$

The integration by parts formula on time scale is given by

$$\int_a^\infty u^\Delta(t)v^\sigma(t)\,\Delta t = u(t)v(t)\big|_a^\infty - \int_a^\infty u(t)v^\Delta(t)\Delta t. \tag{13}$$

The Hölder inequality on the time scale is given by

$$\int_a^\infty f(t)g(t)\Delta t \le \left(\int_a^\infty f^\gamma(t)\Delta t \right)^{\frac{1}{\gamma}} \left(\int_a^\infty g^\nu(t)\Delta t \right)^{\frac{1}{\nu}}, \tag{14}$$

where $\gamma > 1$, $1/\gamma + 1/\nu = 1$ and $f, g \in C_{rd}([a,\infty)_\mathbb{T}, \mathbb{R}^+)$. The inequality (14) is reversed for $0 < \gamma < 1$. In the following, we will assume that $0 \in \mathbb{T}$ and $\mathbb{I} = [0, \infty)_\mathbb{T}$. Throughout this paper, we will assume that the functions in the statements of the theorems are rd-continuous functions and the integrals considered are assumed to exist and be finite. In addition, in our proofs, we will use the convention $0.\infty = 0$ and $0/0 = 0$. Throughout the paper, we assume that $1 < p < \infty$ and $\mathbb{I}$ is a fixed finite interval from $[0, \infty)_\mathbb{T}$. We define the time scale interval $[a, b]_\mathbb{T}$ by $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$. A weight ω defined on $\mathbb{T}$ is a Δ−integrable function of non-negative real numbers. We consider the norm on $L^p(\mathbb{T})$ of the form

$$\|\omega\|_{L^p(\mathbb{T})} := \left(\int_0^\infty |\omega(s)|^p \Delta s \right)^{1/p} < \infty.$$

A non-negative Δ−integrable function ω belongs to the Muckenhoupt class $\mathbb{A}^1(\mathcal{C})$ on the fixed interval $\mathbb{I} = [0, \infty)_\mathbb{T}$ if there exists a constant $\mathcal{C} > 1$ such that the inequality

$$\frac{1}{|\hat{J}|} \int_{\hat{J}} \omega(x)\Delta x \le \mathcal{A} \operatorname*{ess\,inf}_{x \in \hat{J}} \omega(x), \text{ for all } x \in \hat{J}, \tag{15}$$

holds for every subinterval $\hat{J} \subset \mathbb{I}$. A non-negative Δ−integrable function ω belongs to the Muckenhoupt class $\mathbb{A}^p(\mathcal{C})$ for $p > 1$ if there exists a constant $\mathcal{C} > 1$ such that the inequality

$$\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}\omega(x)\Delta x\right)\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}\omega^{\frac{-1}{p-1}}(x)\Delta x\right)^{p-1}\leq C, \tag{16}$$

holds for every subinterval $\hat{J}\subset\mathbb{I}$. For a given exponent $p>1$, we define the $\mathbb{A}^p$-norm of A non-negative $\Delta-$integrable weight ω by the following quantity:

$$[\mathbb{A}^p(\omega)]:=\sup_{\hat{J}\subset\hat{\mathbb{J}}}\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}\omega(x)\Delta x\right)\left(\frac{1}{|\hat{J}|}\int_{\hat{J}}\omega^{\frac{-1}{p-1}}(x)\Delta x\right)^{p-1},$$

where the supremum is taken over all intervals $\hat{J}\subset\mathbb{I}$. Note that by Hölder's inequality $[\mathbb{A}^p(\omega)]\geq 1$ for all $1<p<\infty$, and the following inclusion is true:

$$\text{if } 1<p\leq q<\infty, \text{ then } \mathbb{A}^1\subset\mathbb{A}^p\subset\mathbb{A}^q \text{ and } [\mathbb{A}^q(\omega)]\leq[\mathbb{A}^p(\omega)].$$

For any function $f:\mathbb{I}\to\mathbb{R}^+$ which is non-negative, we define the operator $\mathcal{H}f:[0,\infty)_\mathbb{T}\to\mathbb{R}^+$ by

$$\mathcal{H}(t)=\frac{1}{t}\int_0^t f(s)\Delta s, \text{ for all } t\in\mathbb{I}. \tag{17}$$

From the definition of $\mathcal{H}$, we see that if f is nondecreasing, then

$$\mathcal{H}f(t)=\frac{1}{t}\int_0^t f(s)\Delta s\leq\frac{1}{t}\int_0^t f(t)\Delta s=f(t).$$

Additionally, we have determined by using the above inequality that

$$(\mathcal{H}f(t))^\Delta=\frac{1}{\sigma(t)}[f(t)-\mathcal{H}f(t)]\geq 0, \text{ for } t\in\mathbb{I}.$$

Furthermore, if f is nonincreasing, we have that

$$\mathcal{H}f(t)=\frac{1}{t}\int_0^t f(s)\Delta s\geq\frac{1}{t}\int_0^t f(t)\Delta s=f(t),$$

and

$$(\mathcal{H}f(t))^\Delta=\frac{1}{\sigma(t)}[f(t)-\mathcal{H}f(t)]\leq 0, \text{ for } t\in\mathbb{I}.$$

From these facts, we have the following properties of $\mathcal{H}f$.

Lemma 1.

(i).　If f is nondecreasing, then $\mathcal{H}f(t)\leq f(t)$.
(ii).　If f is nondecreasing, then so is $\mathcal{H}f$.

Lemma 2.

(i).　If f is nonincreasing, then $\mathcal{H}f(t)\geq f(t)$.
(ii).　If f is nonincreasing, then so is $\mathcal{H}f$.

Remark 1. *As a consequence of Lemma 1, we notice that if f is non-negative, and nondecreasing, then $\mathcal{H}f^q\leq f^q$. We also notice from Lemma 1 that if f is non-negative, and nondecreasing, then $\mathcal{H}f^q$ is also non-negative and nondecreasing for $q>1$.*

Remark 2. *As a consequence of Lemma 2, we notice that if f is non-negative, and nonincreasing, then $\mathcal{H}f^q\geq f^q$. We also notice from Lemma 2 that if f is non-negative, and nonincreasing, then $\mathcal{H}f^q$ is also non-negative and nonincreasing for $q>1$.*

In what follows, we will define f^σ, $\mathcal{H}^\sigma f$ and $\mathcal{H}[\mathcal{H}^\sigma f]^p$ where σ is the forward jump operator, by $f^\sigma(t) = (f \circ \sigma)(t)$,

$$\mathcal{H}^\sigma f(t) = \frac{1}{\sigma(t)} \int_0^{\sigma(t)} f(x)\Delta x, \ \text{ for } t \in \mathbb{I},$$

and

$$\mathcal{H}[\mathcal{H}^\sigma f]^p(t) = \frac{1}{t} \int_0^t \left(\frac{1}{\sigma(s)} \int_0^{\sigma(s)} f(x)\Delta x \right)^p \Delta s, \ \text{ for } t \in \mathbb{I}.$$

Theorem 1. *Assume that f is non-negative and nondecreasing on $\mathbb{I}$. If $s \geq r > 0$, then*

$$\int_0^{\sigma(t)} [f(x)]^{r/s} [\mathcal{H}^\sigma f(x)]^{-s - \frac{r}{s}} \Delta x \leq \left(\frac{s+1}{s} \right)^{r/s} \int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s} \Delta x, \tag{18}$$

for any $t \in \mathbb{I}$.

Proof. First, we consider the case when $s = r$ and prove that

$$\int_0^{\sigma(t)} f(x) [\mathcal{H}^\sigma f(x)]^{-s-1} \Delta x \leq \left(\frac{s+1}{s} \right) \int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s} \Delta x.$$

For brevity, we write $F = \mathcal{H}f$. By employing the integration by parts (13), with $u(t) = \sigma(t)$ and $v(t) = F^{-s}(t)$, we obtain

$$
\begin{aligned}
\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x &= u(x)F^{-s}(x)\big|_0^{\sigma(t)} - \int_0^{\sigma(t)} \sigma(x)(F^{-s}(x))^\Delta \Delta x \\
&= u^\sigma(t)(F^\sigma(t))^{-s} - \int_0^{\sigma(t)} \sigma(x)(F^{-s}(x))^\Delta \Delta x \qquad (19) \\
&\geq - \int_0^{\sigma(t)} \sigma(x)(F^{-s}(x))^\Delta \Delta x. \qquad (20)
\end{aligned}
$$

By the chain rule (12), we see that

$$
\begin{aligned}
(F^{-s})^\Delta &= -sF^\Delta \int_0^1 \frac{dh}{(hF^\sigma + (1-h)F)^{s+1}} \\
&\leq -sF^\Delta \int_0^1 (hF^\sigma + (1-h)F^\sigma)^{-s-1} dh = -sF^\Delta(F^\sigma)^{-s-1}.
\end{aligned}
$$

Substituting the last inequality into (20), we obtain

$$
\begin{aligned}
\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x &\geq s \int_0^{\sigma(t)} \sigma(x)F^\Delta(x)(F^\sigma(x))^{-s-1} \Delta x \\
&\geq s \int_0^{\sigma(t)} xF^\Delta(x)(F^\sigma(x))^{-s-1} \Delta x. \qquad (21)
\end{aligned}
$$

Moreover, since

$$tF(t) = \int_0^t f(x)\Delta x,$$

the product rule gives

$$tF^\Delta(t) + F^\sigma(t) = f(t). \tag{22}$$

Substituting (22) into (21), we obtain

$$\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x \geq s \int_0^{\sigma(t)} f(x)(F^\sigma(x))^{-s-1} \Delta x - s \int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x.$$

By combining like terms, we obtain

$$\int_0^{\sigma(t)} f(x)(F^{\sigma}(x))^{-s-1}\Delta x \leq \left(\frac{s+1}{s}\right) \int_0^{\sigma(t)} (F^{\sigma}(x))^{-s}\Delta x,$$

which proves the inequality (18) when $s = r$. Now, consider the case when $s \neq r$ and fix $r \in (0, s)$. Then by applying Hölder's inequality (14) with s/r and $s/(s-r)$, we obtain

$$\int_0^{\sigma(t)} [f(x)]^{r/s} (F^{\sigma}(x))^{-r-\frac{r}{s}} (F^{\sigma}(x))^{-s+r}\Delta x$$

$$\leq \left[\int_0^{\sigma(t)} f(x)(F^{\sigma}(x))^{-s-1}\Delta x\right]^{r/s} \left[\int_0^{\sigma(t)} (F^{\sigma}(x))^{-s}\Delta x\right]^{1-\frac{r}{s}}$$

$$\leq \left(\frac{s+1}{s}\right)^{r/s} \left[\int_0^{\sigma(t)} (F^{\sigma}(x))^{-s}\Delta x\right]^{r/s} \left[\int_0^{\sigma(t)} (F^{\sigma}(x))^{-s}\Delta x\right]^{1-\frac{r}{s}}$$

$$= \left(\frac{s+1}{s}\right)^{r/s} \int_0^{\sigma(t)} (F^{\sigma}(x))^{-s}\Delta x,$$

which is the desired inequality (18). The proof is complete. $\square$

Theorem 2. *Assume that f is non-negative and nondecreasing on $\mathbb{I}$. If $s \geq r > 0$, then*

$$\int_0^{\sigma(t)} [\mathcal{H}^{\sigma} f(x)]^{-s}\Delta x \leq \left(\frac{s+1}{s}\right)^r \int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^{\sigma} f(x)]^{-s+r}\Delta x. \tag{23}$$

Proof. From the elementary inequality (see Elliott [46]),

$$sy^{s+1} - (s+1)y^s \geq -1, \tag{24}$$

for every $y \geq 0$ and $s > 0$, we deduce by using $y = y_1/y_2$, where $y_1, y_2 > 0$, that

$$y_1^{-s} + sy_1 y_2^{-s-1} - (s+1)y_2^{-s} \geq 0. \tag{25}$$

Now, by defining

$$y_1 := \left(\frac{s}{s+1}\right)^{1+\frac{r}{s}} f^{r/s}(t)[\mathcal{H}^{\sigma} f(t)]^{1-\frac{r}{s}}, \text{ and } y_2 := \left(\frac{s}{s+1}\right)\mathcal{H}^{\sigma} f(t),$$

we obtain

$$y_1^{-s} := \left(\frac{s}{s+1}\right)^{-s-r} f^{-r}(t)[\mathcal{H}^{\sigma} f(t)]^{-s+r}, \text{ and } y_2^{-s} := \left(\frac{s}{s+1}\right)^{-s} [\mathcal{H}^{\sigma} f(t)]^{-s},$$

and then

$$y_1 y_2^{-s-1} := \left(\frac{s}{s+1}\right)^{-s+r/s} f^{r/s}(t)[\mathcal{H}^{\sigma} f(t)]^{-s-\frac{r}{s}}.$$

By using these values in (25), we have

$$\left(\frac{s}{s+1}\right)^{-s-r} f^{-r}(t)[\mathcal{H}^{\sigma} f(t)]^{-s+r} + s\left(\frac{s}{s+1}\right)^{-s+\frac{r}{s}} f^{r/s}(t)[\mathcal{H}^{\sigma} f(t)]^{-s-\frac{r}{s}}$$

$$\geq (s+1)\left(\frac{s}{s+1}\right)^{-s} [\mathcal{H}^{\sigma} f(t)]^{-s}. \tag{26}$$

By integrating (26) from 0 to $\sigma(t)$, we obtain

$$
\left(\frac{s+1}{s}\right)^r \int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^\sigma f(x)]^{-s+r}\Delta x
$$

$$
+s\left(\frac{s}{s+1}\right)^{\frac{r}{s}} \int_0^{\sigma(t)} f^{r/s}(x)[\mathcal{H}^\sigma f(x)]^{-s-\frac{r}{s}}\Delta x
$$

$$
\geq \quad (s+1)\int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s}\Delta x. \tag{27}
$$

Now, by applying Theorem 1 on the term

$$
\int_0^{\sigma(t)} f^{r/s}(x)[\mathcal{H}^\sigma f(x)]^{-s-\frac{r}{s}}\Delta x,
$$

we obtain

$$
\int_0^{\sigma(t)} f^{r/s}(x)[\mathcal{H}^\sigma f(x)]^{-s-\frac{r}{s}}\Delta x \leq \left(\frac{s+1}{s}\right)^{r/s} \int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s}\Delta x. \tag{28}
$$

Comparing (27) and (28), we have

$$
\left(\frac{s+1}{s}\right)^r \int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^\sigma f(x)]^{-s+r}\Delta x
$$

$$
+s\left(\frac{s}{s+1}\right)^{r/s}\left(\frac{s+1}{s}\right)^{r/s} \int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s}\Delta x
$$

$$
\geq \quad (s+1)\int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s}\Delta x. \tag{29}
$$

By combining like terms in the last inequality, we conclude that

$$
\int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s}\Delta x \leq \left(\frac{s+1}{s}\right)^r \int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^\sigma f(x)]^{-s+r}\Delta x, \tag{30}
$$

which is the desired inequality (18). The proof is complete. □

Theorem 3. *Assume that f is non-negative and nondecreasing on $\mathbb{I}$. If $0 < r_1 < r_2 < s$, then*

$$
\int_0^{\sigma(t)} f^{-r_1}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_1}\Delta x \leq \left(\frac{s+1}{s}\right)^{r_2-r_1} \int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_2}\Delta x. \tag{31}
$$

Proof. By applying Hölder's inequality (14) with r_2/r_1 and $r_2/(r_2 - r_1)$ on the left-hand side of (31), we obtain

$$
\int_0^{\sigma(t)} f^{-r_1}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_1}\Delta x \quad \leq \quad \left(\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_2}\Delta x\right)^{\frac{r_1}{r_2}}
$$

$$
\times\left(\int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s}\Delta x\right)^{1-\frac{r_1}{r_2}}. \tag{32}
$$

Now, by replacing r with r_2 in (30), we obtain

$$
\int_0^{\sigma(t)} [\mathcal{H}^\sigma f(t)]^{-s}\Delta x \leq \left(\frac{s+1}{s}\right)^{r_2} \int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_2}\Delta x. \tag{33}
$$

By combining (32) and (33), we see that

$$\int_0^{\sigma(t)} f^{-r_1}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_1}\Delta x$$

$$\leq \left(\frac{s+1}{s}\right)^{r_2-r_1} \int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_2}\Delta x,$$

which is the desired inequality (31). The proof is complete. $\square$

Theorem 4. *Assume that f is non-negative and nondecreasing on $\mathbb{I}$. If $s \geq r > 0$, then*

$$\frac{1}{\sigma(t)}\int_0^{\sigma(t)} f^{r/s}(x)[\mathcal{H}^\sigma f(x)]^{-s-\frac{r}{s}}\Delta x \leq \left(\frac{s+1}{s}\right)^{r/s}\frac{1}{\sigma(t)}\int_0^{\sigma(t)}[\mathcal{H}^\sigma f(x)]^{-s}\Delta x$$

$$-\frac{r}{s^2}\left(\frac{s+1}{s}\right)^{r/s-1}[\mathcal{H}^\sigma f(t)]^{-s}. \tag{34}$$

Proof. We proceed as in the proof of Theorem 1 (without removing the term $\sigma^\sigma(t)(F^\sigma(t))^{-p}$) to obtain

$$\int_0^{\sigma(t)}(F^\sigma(x))^{-s}\Delta x = \sigma^\sigma(t)(F^\sigma(t))^{-s} - \int_0^{\sigma(t)}\sigma(x)(F^{-s}(x))^\Delta\Delta x$$

$$\geq \sigma(t)(F^\sigma(t))^{-s} - \int_0^{\sigma(t)}\sigma(x)(F^{-s}(x))^\Delta\Delta x$$

$$\geq \sigma(t)(F^\sigma(t))^{-s} + s\int_0^{\sigma(t)}\sigma(x)F^\Delta(x)(F^\sigma(x))^{-s-1}\Delta x$$

$$\geq \sigma(t)(F^\sigma(t))^{-s} + s\int_0^{\sigma(t)}xF^\Delta(x)(F^\sigma(x))^{-s-1}\Delta x$$

$$\geq \sigma(t)(F^\sigma(t))^{-s} + s\int_0^{\sigma(t)}f(x)(F^\sigma(x))^{-s-1}\Delta x - s\int_0^{\sigma(t)}(F^\sigma(x))^{-s}\Delta x.$$

By combining like terms, we obtain

$$\int_0^{\sigma(t)}f(x)(F^\sigma(x))^{-s-1}\Delta x \leq \left(\frac{s+1}{s}\right)\int_0^{\sigma(t)}(F^\sigma(x))^{-s}\Delta x - \frac{1}{s}\sigma(t)(F^\sigma(t))^{-s}. \tag{35}$$

If we fix $r \in (0,s)$ then by applying Hölder's inequality with s/r and $s/(s-r)$, we obtain

$$\int_0^{\sigma(t)}f^{r/s}(x)(F^\sigma(x))^{-r-r/s}(F^\sigma(x))^{-s+r}\Delta x$$

$$\leq \left[\int_0^{\sigma(t)}f(x)(F^\sigma(x))^{-s-1}\Delta x\right]^{r/s}\left[\int_0^{\sigma(t)}(F^\sigma(x))^{-s}\Delta x\right]^{1-\frac{r}{s}}$$

$$\leq \left[\left(\frac{s+1}{s}\right)\int_0^{\sigma(t)}(F^\sigma(x))^{-s}\Delta x - \frac{1}{s}\sigma(t)(F^\sigma(t))^{-s}\right]^{r/s}$$

$$\times \left[\int_0^{\sigma(t)}(F^\sigma(x))^{-s}\Delta x\right]^{1-\frac{r}{s}}. \tag{36}$$

Now, in order to complete the proof, we shall utilize the inequality

$$(u+v)^\gamma \leq u^\gamma + pu^{\gamma-1}v, \quad \text{where } 0 < \gamma < 1. \tag{37}$$

which is a variant of the well-known Bernoulli inequality. This inequality is valid for all $u \geq 0$ and $u+v \geq 0$ or $u > 0$ and $u+v > 0$ and equality holds if only if $v = 0$. Now, by employing (37) with $\gamma = r/s < 1$,

$$u := \left(\frac{s+1}{s}\right) \int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x, \text{ and } v := -\frac{1}{s}\sigma(t)(F^\sigma(t))^{-s},$$

and noting that

$$\left(\frac{s+1}{s}\right) \int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x - \frac{1}{s}\sigma(t)(F^\sigma(t))^{-s} > 0,$$

we obtain

$$\left[\left(\frac{s+1}{s}\right) \int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x - \frac{1}{s}\sigma(t)(F^\sigma(t))^{-s}\right]^{r/s}$$

$$\leq \left(\frac{s+1}{s}\right)^{r/s} \left[\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x\right]^{r/s}$$

$$-\frac{r}{s}\left(\frac{s+1}{s}\right)^{r/s-1} \times \frac{1}{s}\sigma(t)(F^\sigma(t))^{-s} \left[\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x\right]^{r/s-1}$$

$$= \left(\frac{s+1}{s}\right)^{r/s} \left[\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x\right]^{r/s}$$

$$-\frac{r}{s^2}\left(\frac{s+1}{s}\right)^{r/s-1} \left[\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x\right]^{r/s-1} \sigma(t)(F^\sigma(t))^{-s}.$$

Substituting the last inequality into (36), we obtain

$$\int_0^{\sigma(t)} f^{r/s}(x)(F^\sigma(x))^{-s-r/s} \Delta x$$

$$\leq \left(\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x\right)^{1-\frac{r}{s}} \left[\left(\frac{s+1}{s}\right)^{r/s} \left(\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x\right)^{r/s}\right.$$

$$\left.-\frac{r}{s^2}\left(\frac{s+1}{s}\right)^{r/s-1} \left[\int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x\right]^{r/s-1} \sigma(t)(F^\sigma(t))^{-s}\right]$$

$$= \left(\frac{s+1}{s}\right)^{r/s} \int_0^{\sigma(t)} (F^\sigma(x))^{-s} \Delta x - \frac{r}{s^2}\left(\frac{s+1}{s}\right)^{r/s-1} \sigma(t)(F^\sigma(t))^{-s},$$

which is the desired inequality (34). The proof is complete. □

Theorem 5. *Assume that f is non-negative and nondecreasing on $\mathbb{I}$. If $s \geq r > 0$, then*

$$\int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s} \Delta x$$

$$\leq \left(\frac{s+1}{s}\right)^r \int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^\sigma f(x)]^{-s+r} \Delta x - \frac{r}{s+1}\sigma(t)[\mathcal{H}^\sigma f(t)]^{-s}. \quad (38)$$

Proof. We proceed as in the proof of Theorem 2, so we have from (27) that

$$\left(\frac{s+1}{s}\right)^r \int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^\sigma f(x)]^{-s+r} \Delta x$$

$$+s\left(\frac{s}{s+1}\right)^{\frac{r}{s}} \int_0^{\sigma(t)} f^{r/s}(x)[\mathcal{H}^\sigma f(x)]^{-s-\frac{r}{s}} \Delta x$$

$$\geq (s+1) \int_0^{\sigma(t)} [\mathcal{H}^\sigma f(x)]^{-s} \Delta x.$$

By applying Theorem 4, we obtain

$$\int_0^{\sigma(t)} f^{r/s}(x)[\mathcal{H}^\sigma f(x)]^{-s-\frac{r}{s}}\Delta x \;\leq\; \left(\frac{s+1}{s}\right)^{r/s}\int_0^{\sigma(t)}[\mathcal{H}^\sigma f(x)]^{-s}\Delta x$$

$$-\frac{r}{s^2}\left(\frac{s+1}{s}\right)^{r/s-1}\sigma(t)[\mathcal{H}^\sigma f(t)]^{-s},$$

and then

$$\left(\frac{s+1}{s}\right)^r\int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^\sigma f(x)]^{-s+r}\Delta x$$

$$+s\left(\frac{s}{s+1}\right)^{\frac{r}{s}}\left(\frac{s+1}{s}\right)^{r/s}\int_0^{\sigma(t)}[\mathcal{H}^\sigma f(x)]^{-s}\Delta x$$

$$-s\left(\frac{s}{s+1}\right)^{\frac{r}{s}}\frac{r}{s^2}\left(\frac{s+1}{s}\right)^{r/s-1}\sigma(t)[\mathcal{H}^\sigma f(t)]^{-s}$$

$$\geq\; (s+1)\int_0^{\sigma(t)}[\mathcal{H}^\sigma f(x)]^{-s}\Delta x.$$

By combining like terms, we obtain

$$\left(\frac{s+1}{s}\right)^r\int_0^{\sigma(t)} f^{-r}(x)[\mathcal{H}^\sigma f(x)]^{-s+r}\Delta x - \frac{r}{(s+1)}\sigma(t)[\mathcal{H}^\sigma f(t)]^{-s}$$

$$\geq\; \int_0^{\sigma(t)}[\mathcal{H}^\sigma f(x)]^{-s}\Delta x, \tag{39}$$

which is the desired inequality (38). The proof is complete. $\quad\square$

Theorem 6. *Assume that f is non-negative and nondecreasing on $\mathbb{I}$. If $0 < r_1 < r_2 < s$, then*

$$\int_0^{\sigma(t)} f^{-r_1}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_1}\Delta x + \frac{(r_2-r_1)s^{r_1}}{(s+1)^{1+r_1}}\sigma(t)[\mathcal{H}^\sigma f(t)]^{-s}$$

$$\leq\; \left(\frac{s+1}{s}\right)^{r_2-r_1}\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_2}\Delta x. \tag{40}$$

Proof. By applying Hölder's inequality with r_2/r_1 and $r_2/(r_2-r_1)$ on the left hand side of (40), we obtain

$$\int_0^{\sigma(t)} f^{-r_1}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_1}\Delta x \;\leq\; \left(\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_2}\Delta x\right)^{\frac{r_1}{r_2}}$$

$$\times\left(\int_0^{\sigma(t)}[\mathcal{H}^\sigma f(x)]^{-s}\Delta x\right)^{1-\frac{r_1}{r_2}}. \tag{41}$$

Now, by replacing r with r_2 in (39), we obtain

$$\int_0^{\sigma(t)}[\mathcal{H}^\sigma f(x)]^{-s}\Delta x \;\leq\; \left(\frac{s+1}{s}\right)^{r_2}\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^\sigma f(x)]^{-s+r_2}\Delta x$$

$$-\frac{r_2}{(s+1)}\sigma(t)[\mathcal{H}^\sigma f(t)]^{-s}. \tag{42}$$

By combining (41) and (42), we obtain

$$\int_0^{\sigma(t)} f^{-r_1}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_1}\Delta x \;\leq\; \left(\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right)^{\frac{r_1}{r_2}} \times$$

$$\left[\left(\frac{s+1}{s}\right)^{r_2}\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right.$$

$$\left.-\frac{r_2}{(s+1)}\sigma(t)[\mathcal{H}^{\sigma}f(t)]^{-s}\right]^{1-\frac{r_1}{r_2}}. \tag{43}$$

Now, by employing (37), with $\gamma = 1 - (r_1/r_2) < 1$,

$$u = \left(\frac{s+1}{s}\right)^{r_2}\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x, \text{ and } v = -\frac{r_2}{(s+1)}\sigma(t)[\mathcal{H}^{\sigma}f(t)]^{-s},$$

we obtain

$$\left[\left(\frac{s+1}{s}\right)^{r_2}\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x - \frac{r_2}{(s+1)}\sigma(t)[\mathcal{H}^{\sigma}f(t)]^{-s}\right]^{1-\frac{r_1}{r_2}}$$

$$\leq \quad \left(\frac{s+1}{s}\right)^{r_2-r_1}\left[\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right]^{1-\frac{r_1}{r_2}} - \frac{r_2-r_1}{r_2}$$

$$\times \left(\frac{s+1}{s}\right)^{-r_1}\left[\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right]^{-\frac{r_1}{r_2}}\frac{r_2}{(s+1)}\sigma(t)[\mathcal{H}^{\sigma}f(t)]^{-s}$$

$$= \quad \left(\frac{s+1}{s}\right)^{r_2-r_1}\left[\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right]^{1-\frac{r_1}{r_2}}$$

$$-\frac{(r_2-r_1)s^{r_1}}{(s+1)^{1+r_1}}\left[\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right]^{-\frac{r_1}{r_2}}\sigma(t)[\mathcal{H}^{\sigma}f(t)]^{-s}.$$

Substituting the last inequality into (43), we obtain

$$\int_0^{\sigma(t)} f^{-r_1}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_1}\Delta x$$

$$\leq \quad \left(\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right)^{\frac{r_1}{r_2}}$$

$$\times \left[\left(\frac{s+1}{s}\right)^{r_2-r_1}\left[\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right]^{1-\frac{r_1}{r_2}}\right.$$

$$\left.-\frac{(r_2-r_1)s^{r_1}}{(s+1)^{1+r_1}}\left[\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right]^{-\frac{r_1}{r_2}}\sigma(t)[\mathcal{H}^{\sigma}f(t)]^{-s}\right]$$

$$= \quad \left(\frac{s+1}{s}\right)^{r_2-r_1}\left(\int_0^{\sigma(t)} f^{-r_2}(x)[\mathcal{H}^{\sigma}f(x)]^{-s+r_2}\Delta x\right)$$

$$-\frac{(r_2-r_1)s^{r_1}}{(s+1)^{1+r_1}}\sigma(t)[\mathcal{H}^{\sigma}f(t)]^{-s},$$

which is the desired inequality (40). The proof is complete. □

Theorem 7. *Assume that ω is non-negative and nondecreasing and $q > 1$. Then we have for every $t \in \mathbb{I}$ that*

$$\frac{1}{\sigma(t)} \int_0^{\sigma(t)} \left[(\omega^\sigma(x))^{\frac{-1}{q-1}} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\gamma-1} - \frac{(\gamma-1)}{\gamma} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^\gamma \right] \Delta x$$

$$\leq \quad \frac{1}{\gamma} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(t) \right]^\gamma \tag{44}$$

for any $\gamma \geq 1$.

Proof. Let $x \in \mathbb{I}$. Since $\omega^{\frac{-1}{q-1}}(x) = \left[x \mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right]^\Delta$, it follows that

$$(\omega^\sigma(x))^{\frac{-1}{q-1}} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\gamma-1} - \frac{(\gamma-1)}{\gamma} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^\gamma$$

$$\leq \quad \omega^{\frac{-1}{q-1}}(x) \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\gamma-1} - \frac{(\gamma-1)}{\gamma} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^\gamma$$

$$= \quad \left[x \mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right]^\Delta \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\gamma-1} - \frac{(\gamma-1)}{\gamma} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^\gamma. \tag{45}$$

Moreover, utilizing the well-known product rule

$$(fg)^\Delta = f g^\Delta + f^\Delta g^\sigma,$$

for $f = x \mathcal{H} \omega^{\frac{-1}{q-1}}$ and $g^\sigma = \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}} \right]^{\gamma-1}$, we have that

$$\left[x \mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right]^\Delta \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\gamma-1}$$

$$= \quad \left[x \left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta - \left[x \mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right] \left[\left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^{\gamma-1} \right]^\Delta, \tag{46}$$

and for $f = x$ and $g^\sigma = \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}} \right]^\gamma$, we have that

$$\left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^\gamma = \left[x \left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta - x \left[\left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta. \tag{47}$$

By comparing (46) and (47) with (45), we obtain

$$(\omega^\sigma(x))^{\frac{-1}{q-1}} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\gamma-1} - \frac{(\gamma-1)}{\gamma} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^\gamma$$

$$\leq \quad \left[x \left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta - \left[x \mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right] \left[\left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^{\gamma-1} \right]^\Delta$$

$$\quad - \frac{(\gamma-1)}{\gamma} \left[x \left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta + \frac{(\gamma-1)}{\gamma} x \left[\left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta$$

$$= \quad \frac{1}{\gamma} \left[x \left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta \tag{48}$$

$$\quad - x \mathcal{H} \omega^{\frac{-1}{q-1}}(x) \left[\left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^{\gamma-1} \right]^\Delta + \frac{(\gamma-1)}{\gamma} x \left[\left(\mathcal{H} \omega^{\frac{-1}{q-1}}(x) \right)^\gamma \right]^\Delta.$$

On the other hand, since $\omega^{\frac{-1}{q-1}}$ is nonincreasing, then so is $\mathcal{H}\omega^{\frac{-1}{q-1}}(x)$, or equivalently, $\left[\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right]^{\Delta} < 0$, then we have

$$
\begin{aligned}
&-x\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma-1}\right]^{\Delta} + \frac{(\gamma-1)}{\gamma}x\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma}\right]^{\Delta} \\
&\leq -x\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma-1}\right]^{\Delta} + x\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma}\right]^{\Delta}.
\end{aligned}
\tag{49}
$$

Consequently, yet another application of the product rule, with $f = \mathcal{H}\omega^{\frac{-1}{q-1}}(x)$ and $g = \left[\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right]^{\gamma-1}$, yields that

$$
\begin{aligned}
&\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma}\right]^{\Delta} - \mathcal{H}\omega^{\frac{-1}{q-1}}(x)\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma-1}\right]^{\Delta} \\
&= \left(\mathcal{H}^{\sigma}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma-1}\left[\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right]^{\Delta},
\end{aligned}
$$

by substituting the last equation in (49), we have

$$
\begin{aligned}
&-x\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma-1}\right]^{\Delta} + \frac{(\gamma-1)}{\gamma}x\left[\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma}\right]^{\Delta} \\
&\leq x\left(\mathcal{H}^{\sigma}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma-1}\left[\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right]^{\Delta} \leq 0.
\end{aligned}
\tag{50}
$$

Now, taking into account relations (48) and (50), we have that

$$
\begin{aligned}
&(\omega^{\sigma}(x))^{\frac{-1}{q-1}}\left[\mathcal{H}^{\sigma}\omega^{\frac{-1}{q-1}}(x)\right]^{\gamma-1} - \frac{(\gamma-1)}{\gamma}\left[\mathcal{H}^{\sigma}\omega^{\frac{-1}{q-1}}(x)\right]^{\gamma} \\
&\leq \frac{1}{\gamma}\left[x\left(\mathcal{H}\omega^{\frac{-1}{q-1}}(x)\right)^{\gamma}\right]^{\Delta}.
\end{aligned}
$$

Finally, integrating the last inequality from 0 to $\sigma(t)$ and dividing by $\sigma(t)$, we obtain

$$
\begin{aligned}
&\frac{1}{\sigma(t)}\int_0^{\sigma(t)}\left[(\omega^{\sigma}(x))^{\frac{-1}{q-1}}\left[\mathcal{H}^{\sigma}\omega^{\frac{-1}{q-1}}(x)\right]^{\gamma-1} - \frac{(\gamma-1)}{\gamma}\left[\mathcal{H}^{\sigma}\omega^{\frac{-1}{q-1}}(x)\right]^{\gamma}\right]\Delta x \\
&\leq \frac{1}{\gamma}\left[\mathcal{H}^{\sigma}\omega^{\frac{-1}{q-1}}(t)\right]^{\gamma}.
\end{aligned}
$$

The proof is complete □

3. Self-Improving Properties of Muckenhoupt's Weights

In this section, we will prove the self-improving properties of the Muckenhoupt class on a time scale $\mathbb{T}$ for non-negative and nondecreasing weights.

Theorem 8. *Assume that ω is non-negative and nondecreasing on $\mathbb{I}$ and $q > 1$ such that $\omega \in \mathbb{A}^q(\mathcal{C})$. Then for any $\eta \geq 1$ satisfying that $\omega^{\sigma}(t) \leq \eta\omega(t)$, we have that $\omega \in \mathbb{A}^p(\mathcal{C}_1)$ for any $p \in (p_0, q]$ where p_0 is the unique root of the equation*

$$
\frac{q-p_0}{q-1}(\mathcal{C}\eta p_0)^{\frac{1}{q-1}} = 1.
\tag{51}
$$

Furthermore, the constant C_1 is given by

$$C_1 := \left(\frac{p-1}{q-1} \frac{C^{\frac{1}{q-1}}}{\Psi^{q,p}(C)} \right)^{p-1}, \text{ where } \Psi^{q,p}(C) := \left(1 - \frac{q-p}{q-1}(C\eta p)^{\frac{1}{q-1}} \right) \eta^{\frac{-1}{p-1}} > 0.$$

Proof. By Lemma 7 with $\gamma = (q-1)/(p-1) > 1$ for $q > p > 1$, we obtain

$$\frac{q-1}{p-1} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\frac{q-p}{p-1}} \Delta x - \frac{q-p}{p-1} \int_0^{\sigma(t)} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\frac{q-1}{p-1}} \Delta x \tag{52}$$

$$\leq \sigma(t) \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(t) \right]^{\frac{q-1}{p-1}}.$$

Since $\omega \in \mathbb{A}^q(C)$, we see that

$$\mathcal{H}^\sigma \omega(t) \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(t) \right]^{q-1} \leq C, \text{ for } C > 1. \tag{53}$$

Substituting the last inequality into (52), we obtain

$$\frac{q-1}{q-p} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\frac{q-p}{p-1}} \Delta x - \int_0^{\sigma(t)} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\frac{q-1}{p-1}} \Delta x \tag{54}$$

$$\leq \frac{p-1}{q-p} C^{\frac{1}{p-1}} \sigma(t) [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}}.$$

Define

$$g_\xi(\rho) = \frac{q-1}{q-p} \xi \rho^{\frac{q-p}{p-1}} - \rho^{\frac{q-1}{p-1}},$$

with

$$\rho = \mathcal{H}^\sigma \omega^{\frac{-1}{q-1}} \text{ and } \xi = (\omega^\sigma)^{\frac{-1}{q-1}}.$$

Since ω^σ is nondecreasing, then we have $(\omega^\sigma)^{\frac{-1}{q-1}}$ is nonincreasing, then by Lemma 2, we have $(\omega^\sigma)^{\frac{-1}{q-1}} \leq \mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}$, that is $\xi < \rho$. From the definition of $g_\xi(\rho)$, we see that

$$\frac{d}{d\rho} g_\xi(\rho) = \frac{q-1}{p-1} \xi \rho^{\frac{q-p}{p-1}-1} - \frac{q-1}{p-1} \rho^{\frac{q-p}{p-1}} = \frac{q-1}{p-1} \rho^{\frac{q-p}{p-1}-1} [\xi - \rho] < 0,$$

and so we can recognize that $g_\xi(\rho)$ is nonincreasing. By defining

$$\zeta = C^{\frac{1}{q-1}} [\mathcal{H}^\sigma \omega]^{\frac{-1}{q-1}},$$

and using $\rho \leq \zeta$, we have that

$$g_\xi(\rho) \geq g_\xi(\zeta),$$

and then we obtain

$$\frac{q-1}{q-p} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\frac{q-p}{p-1}} \Delta x - \int_0^{\sigma(t)} \left[\mathcal{H}^\sigma \omega^{\frac{-1}{q-1}}(x) \right]^{\frac{q-1}{p-1}} \Delta x$$

$$\geq \frac{q-1}{q-p} C^{\frac{q-p}{(p-1)(q-1)}} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} [\mathcal{H}^\sigma \omega(x)]^{\frac{1}{q-1}-\frac{1}{p-1}} \Delta x$$

$$-C^{\frac{1}{p-1}} \int_0^{\sigma(t)} [\mathcal{H}^\sigma \omega]^{\frac{-1}{p-1}} \Delta x.$$

Compare last inequality and (54) we obtain

$$\frac{q-1}{q-p}\mathcal{C}^{\frac{q-p}{(p-1)(q-1)}}\int_0^{\sigma(t)}(\omega^\sigma(x))^{\frac{-1}{q-1}}[\mathcal{H}^\sigma\omega(x)]^{\frac{1}{q-1}-\frac{1}{p-1}}\Delta x$$

$$\leq \frac{p-1}{q-p}\mathcal{C}^{\frac{1}{p-1}}\sigma(t)[\mathcal{H}^\sigma\omega(t)]^{\frac{-1}{p-1}}+\mathcal{C}^{\frac{1}{p-1}}\int_0^{\sigma(t)}[\mathcal{H}^\sigma\omega]^{\frac{-1}{p-1}}\Delta x.$$

Cancel a suitable power of $\mathcal{C}$ to obtain

$$\frac{q-1}{q-p}\int_0^{\sigma(t)}(\omega^\sigma(x))^{\frac{-1}{q-1}}[\mathcal{H}^\sigma\omega(x)]^{\frac{1}{q-1}-\frac{1}{p-1}}\Delta x$$

$$\leq \frac{p-1}{q-p}\mathcal{C}^{\frac{1}{q-1}}\sigma(t)[\mathcal{H}^\sigma\omega(t)]^{\frac{-1}{p-1}}+\mathcal{C}^{\frac{1}{q-1}}\int_0^{\sigma(t)}[\mathcal{H}^\sigma\omega]^{\frac{-1}{p-1}}\Delta x. \tag{55}$$

Replace s and r with $\frac{1}{p-1}$ and $\frac{1}{q-1}$ in the inequality (23), respectively, we obtain

$$\int_0^{\sigma(t)}[\mathcal{H}^\sigma\omega(x)]^{-\frac{1}{p-1}}\Delta x \leq p^{\frac{1}{q-1}}\int_0^{\sigma(t)}\omega^{\frac{-1}{q-1}}(x)[\mathcal{H}^\sigma\omega(x)]^{-\frac{1}{p-1}+\frac{1}{q-1}}\Delta x. \tag{56}$$

By combining (55) and (56), we see immediately that

$$\frac{q-1}{q-p}\int_0^{\sigma(t)}(\omega^\sigma(x))^{\frac{-1}{q-1}}[\mathcal{H}^\sigma\omega(x)]^{\frac{1}{q-1}-\frac{1}{p-1}}\Delta x$$

$$-(\mathcal{C}p)^{\frac{1}{q-1}}\int_0^{\sigma(t)}\omega^{\frac{-1}{q-1}}(x)[\mathcal{H}^\sigma\omega(x)]^{-\frac{1}{p-1}+\frac{1}{q-1}}\Delta x$$

$$\leq \frac{p-1}{q-p}\mathcal{C}^{\frac{1}{q-1}}\sigma(t)[\mathcal{H}^\sigma\omega(t)]^{\frac{-1}{p-1}}. \tag{57}$$

Since $\omega^\sigma(t)\leq\eta\omega(t)$, so we can see that

$$\omega^{\frac{-1}{q-1}}(t)\leq\eta^{\frac{1}{q-1}}(\omega^\sigma(t))^{\frac{-1}{q-1}}.$$

Substituting the last inequality into (57) we see that

$$\frac{q-1}{q-p}\int_0^{\sigma(t)}(\omega^\sigma(x))^{\frac{-1}{q-1}}[\mathcal{H}^\sigma\omega(x)]^{\frac{1}{q-1}-\frac{1}{p-1}}\Delta x$$

$$-(\mathcal{C}\eta p)^{\frac{1}{q-1}}\int_0^{\sigma(t)}(\omega^\sigma(x))^{\frac{-1}{q-1}}[\mathcal{H}^\sigma\omega(x)]^{-\frac{1}{p-1}+\frac{1}{q-1}}\Delta x$$

$$\leq \frac{p-1}{q-p}\mathcal{C}^{\frac{1}{q-1}}\sigma(t)[\mathcal{H}^\sigma\omega(t)]^{\frac{-1}{p-1}},$$

which gives us that

$$\left[1-\frac{q-p}{q-1}(\mathcal{C}\eta p)^{\frac{1}{q-1}}\right]\left(\frac{1}{\sigma(t)}\int_0^{\sigma(t)}(\omega^\sigma(x))^{\frac{-1}{q-1}}[\mathcal{H}^\sigma\omega(x)]^{\frac{1}{q-1}-\frac{1}{p-1}}\Delta x\right)$$

$$\leq \frac{p-1}{q-1}\mathcal{C}^{\frac{1}{q-1}}[\mathcal{H}^\sigma\omega(t)]^{\frac{-1}{p-1}}. \tag{58}$$

The constant

$$K:=1-\frac{q-p}{q-1}(\mathcal{C}\eta p)^{\frac{1}{q-1}},$$

is positive for every $p\in(p_0,q]$, where p_0 is the unique positive root of the equation

$$\frac{q-p_0}{q-1}(\mathcal{C}\eta p_0)^{\frac{1}{q-1}}=1.$$

Since ω is nondecreasing then we obtain (from Lemma 1) that

$$\mathcal{H}^\sigma \omega(x) \leq \omega^\sigma(x).$$

This implies, since $p - 1 < q - 1$, that

$$[\mathcal{H}^\sigma \omega(x)]^{\frac{1}{q-1} - \frac{1}{p-1}} \geq (\omega^\sigma)^{\frac{1}{q-1} - \frac{1}{p-1}}(x). \tag{59}$$

which gives us

$$\left[1 - \frac{q-p}{q-1}(\mathcal{C}\eta p)^{\frac{1}{q-1}} \right] \left(\frac{1}{\sigma(t)} \int_0^{\sigma(t)} (\omega^\sigma)^{\frac{-1}{p-1}}(x) \Delta x \right)$$
$$\leq \quad \frac{p-1}{q-1} \mathcal{C}^{\frac{1}{q-1}} [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}}. \tag{60}$$

Since $\omega^\sigma(t) \leq \eta \omega(t)$, so we can see that

$$(\omega^\sigma(t))^{\frac{-1}{p-1}} \geq (\eta \omega(t))^{\frac{-1}{p-1}}.$$

Substituting the last inequality into (60) we obtain

$$\left[1 - \frac{q-p}{q-1}(\mathcal{C}\eta p)^{\frac{1}{q-1}} \right] \eta^{\frac{-1}{p-1}} \left(\frac{1}{\sigma(t)} \int_0^{\sigma(t)} \omega^{\frac{-1}{p-1}}(x) \Delta x \right)$$
$$\leq \quad \frac{p-1}{q-1} \mathcal{C}^{\frac{1}{q-1}} [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}}.$$

which implies that

$$(\mathcal{H}^\sigma \omega(t)) \left(\mathcal{H}^\sigma \omega^{\frac{-1}{p-1}}(t) \right)^{p-1} \leq \mathcal{C}_1,$$

where $\mathcal{C}_1 = \mathcal{C}_1(p, q, \mathcal{C}, \eta)$ is positive constant. The proof is complete. $\square$

Now, we will refine the result above by improving the constant that appears as following.

Theorem 9. *Assume that ω is non-negative and nondecreasing on $\mathbb{I}$ and $q > 1$ such that $\omega \in \mathbb{A}^q(\mathcal{C})$. Then $\omega \in \mathbb{A}^p(\bar{\mathcal{C}}_1)$ for any $p \in (p_0, q]$ where p_0 is the unique root of the equation*

$$\frac{q - p_0}{q - 1}(\mathcal{C}\eta p_0)^{\frac{1}{q-1}} = 1. \tag{61}$$

Furthermore the constant $\bar{\mathcal{C}}_1$ is given by

$$\bar{\mathcal{C}}_1 \quad : \quad = \left(\frac{q}{p} \left(\frac{p-1}{q-1} \right)^2 \frac{\mathcal{C}^{\frac{1}{q-1}}}{\Psi^{q,p}(\mathcal{C})} \right)^{p-1},$$

$$\Psi^{q,p}(\mathcal{C}) \quad : \quad = \left(1 - \frac{q-p}{q-1}(\mathcal{C}\eta p)^{\frac{1}{q-1}} \right) \eta^{\frac{-1}{p-1}} > 0.$$

Proof. We will apply the same technique we use in Theorem 8 but we will replace s and r with $1/(p-1)$ and $1/(q-1)$ in (39), respectively to obtain

$$\int_0^{\sigma(t)} [\mathcal{H}^\sigma \omega(x)]^{-\frac{1}{p-1}} \Delta x \tag{62}$$
$$\leq \quad p^{\frac{1}{q-1}} \int_0^{\sigma(t)} \omega^{\frac{-1}{q-1}}(x) [\mathcal{H}^\sigma \omega(x)]^{-\frac{1}{p-1} + \frac{1}{q-1}} \Delta x - \frac{p-1}{p(q-1)} \sigma(t) [\mathcal{H}^\sigma \omega(t)]^{-\frac{1}{p-1}}.$$

Now, combine (55) and (62), we see immediately that

$$\frac{q-1}{q-p} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} [\mathcal{H}^\sigma \omega(x)]^{\frac{1}{q-1} - \frac{1}{p-1}} \Delta x$$

$$-(\mathcal{C}p)^{\frac{1}{q-1}} \int_0^{\sigma(t)} \omega^{\frac{-1}{q-1}}(x) [\mathcal{H}^\sigma \omega(x)]^{-\frac{1}{p-1} + \frac{1}{q-1}} \Delta x$$

$$\leq \frac{p-1}{q-p} \mathcal{C}^{\frac{1}{q-1}} \sigma(t) [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}} - \mathcal{C}^{\frac{1}{q-1}} \frac{p-1}{p(q-1)} \sigma(t) [\mathcal{H}^\sigma \omega(t)]^{-\frac{1}{p-1}}, \qquad (63)$$

Since $\omega^\sigma(t) \leq \eta \omega(t)$, so we can see that

$$\omega^{\frac{-1}{q-1}}(t) \leq \eta^{\frac{1}{q-1}} (\omega^\sigma(t))^{\frac{-1}{q-1}}.$$

Substituting the last inequality into (63) we see that

$$\frac{q-1}{q-p} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} [\mathcal{H}^\sigma \omega(x)]^{\frac{1}{q-1} - \frac{1}{p-1}} \Delta x$$

$$-(\mathcal{C}\eta p)^{\frac{1}{q-1}} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} [\mathcal{H}^\sigma \omega(x)]^{-\frac{1}{p-1} + \frac{1}{q-1}} \Delta x$$

$$\leq \frac{p-1}{q-p} \mathcal{C}^{\frac{1}{q-1}} \sigma(t) [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}} - \mathcal{C}^{\frac{1}{q-1}} \frac{p-1}{p(q-1)} \sigma(t) [\mathcal{H}^\sigma \omega(t)]^{-\frac{1}{p-1}},$$

which gives us

$$\left[1 - \frac{q-p}{q-1} (\mathcal{C}\eta p)^{\frac{1}{q-1}}\right] \left(\frac{1}{\sigma(t)} \int_0^{\sigma(t)} (\omega^\sigma(x))^{\frac{-1}{q-1}} [\mathcal{H}^\sigma \omega(x)]^{\frac{-1}{p-1} + \frac{1}{q-1}} \Delta x\right)$$

$$\leq \left[\frac{p-1}{q-1} - \frac{(q-p)(p-1)}{p(q-1)^2}\right] \mathcal{C}^{\frac{1}{q-1}} [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}}$$

$$= \frac{q}{p} \left(\frac{p-1}{q-1}\right)^2 \mathcal{C}^{\frac{1}{q-1}} [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}}. \qquad (64)$$

Since ω is nondecreasing then we obtain (from Lemma 1) that

$$\mathcal{H}^\sigma \omega(x) \leq \omega^\sigma(x).$$

This implies, since $p - 1 < q - 1$, that

$$[\mathcal{H}^\sigma \omega(x)]^{\frac{1}{q-1} - \frac{1}{p-1}} \geq (\omega^\sigma)^{\frac{1}{q-1} - \frac{1}{p-1}}(x),$$

then, we obtain

$$\left[1 - \frac{q-p}{q-1} (\mathcal{C}\eta p)^{\frac{1}{q-1}}\right] \frac{1}{\sigma(t)} \int_0^{\sigma(t)} (\omega^\sigma(x))^{-\frac{1}{p-1}} \Delta x \leq \frac{q}{p} \left(\frac{p-1}{q-1}\right)^2 \mathcal{C}^{\frac{1}{q-1}} [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}}. \qquad (65)$$

Since $\omega^\sigma(x) \leq \eta \omega(x)$ so we can see that

$$[\omega^\sigma(x)]^{\frac{-1}{p-1}} \geq [\eta \omega(x)]^{\frac{-1}{p-1}},$$

Substituting the last inequality into (65) we obtain

$$\left[1 - \frac{q-p}{q-1} (\mathcal{C}\eta p)^{\frac{1}{q-1}}\right] \eta^{\frac{-1}{p-1}} \left(\frac{1}{\sigma(t)} \int_0^{\sigma(t)} \omega^{-\frac{1}{p-1}}(x) \Delta x\right) \leq \frac{q}{p} \left(\frac{p-1}{q-1}\right)^2 \mathcal{C}^{\frac{1}{q-1}} [\mathcal{H}^\sigma \omega(t)]^{\frac{-1}{p-1}}.$$

which implies that

$$(\mathcal{H}^\sigma \omega(t)) \left(\mathcal{H}^\sigma \omega^{\frac{-1}{p-1}}(t)\right)^{p-1} \leq \bar{\mathcal{C}}_1,$$

where $\bar{\mathcal{C}}_1 = \bar{\mathcal{C}}_1(q, p, \mathcal{C}, \eta)$ is positive constant,which proves that $\omega \in \mathbb{A}^p(\bar{\mathcal{C}}_1)$. The proof is complete. $\square$

Remark 3. *We note that Equation (61) can be written as*

$$\frac{1}{p_0}\left(\frac{q-1}{q-p_0}\right)^{q-1} = \mathcal{C}\eta. \tag{66}$$

When $\mathbb{T} = \mathbb{R}$*, we see that* $\eta = 1$ *and then (66) becomes the Equation (7) which is given by*

$$\frac{1}{p_0}\left(\frac{q-1}{q-p_0}\right)^{q-1} = \mathcal{C}. \tag{67}$$

When $\mathbb{T} = \mathbb{N}$*, we can choose* $\eta = 2$ *and then (66) becomes*

$$\frac{1}{p_0}\left(\frac{q-1}{q-p_0}\right)^{q-1} = 2\mathcal{C}, \tag{68}$$

for the discrete weights.

4. Conclusions

In this paper, we proved some Hardy's type inequalities on time scales and the new refinements of these inequalities with negative powers that are needed to prove the main results. Next, we used these inequalities to design and prove some new additional inequalities by using the Bernoulli inequality that will be also needed in the proof of the main results. These results are the self-improving results for the Muckenhoupt weights on time scales. The self-improving properties used in harmonic analysis to prove one of the important theorems, which is the extrapolation theorem. We also expect that the new theory on time scales will also play the same act in proving extrapolation theory on time scales via the $\mathbb{A}^q(\mathcal{C})-$Muckenhoupt weights. The results as special cases contain the results for the classical results obtained for integrals and the discrete results obtained for the discrete weights. The technique that we have applied in this paper give a unified approach in proving a general results and avoiding the proof of integrals and again for sums. The results in the discrete case that we have derived contain an additional constant which is different from the case in the integral forms, see (67) and (68). We have checked the results with some values and concluded that these equations has unique positive roots.

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