

Article

Initial Coefficients Upper Bounds for Certain Subclasses of Bi-Prestarlike Functions

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- ‡ Dedicated to the memory of Professor Walter K. Hayman (1926–2000).

Abstract: In this article, we introduce and study the behavior of the modules of the first two coefficients for the classes $\mathcal{N}_{\Sigma}(\gamma, \lambda, \delta, \mu; \alpha)$ and $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ of normalized holomorphic and bi-univalent functions that are connected with the prestarlike functions. We determine the upper bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for the functions of each of these families, and we also point out some special cases and consequences of our main results. The study of these classes is closely connected with those of Ruscheweyh who in 1977 introduced the classes of prestarlike functions of order μ using a convolution operator and the proofs of our results are based on the well-known Carathéodory’s inequality for the functions with real positive part in the open unit disk. Our results generalize a few of the earlier ones obtained by Li and Wang, Murugusundaramoorthy et al., Brannan and Taha, and could be useful for those that work with the geometric function theory of one-variable functions.

Keywords: holomorphic functions; univalent functions; bi-univalent functions; convolution (Hadamard) product; prestarlike functions; coefficient estimates; Taylor–Maclaurin coefficients

MSC: 30C45; 30C50



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1. Introduction

We denote by $\mathcal{A}$ the family of functions which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and with the following normalized form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}. \quad (1)$$

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ of the functions that are univalent in $\mathbb{U}$. From the *Koebe one-quarter theorem* [1], all the functions $f \in \mathcal{S}$ have an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4} \right).$$

In addition, for every function $f \in \mathcal{S}$, there exists an inverse function $f^{-1} : f(\mathbb{U}) \rightarrow \mathbb{U}$ analytic in the domain $f(\mathbb{U})$, but it is not sure that $f(\mathbb{U}) \subseteq \mathbb{U}$. Therefore, if we denote by g the analytic continuation of f^{-1} to the unit disk $\mathbb{U}$, assuming that it exists, then

$$g(w) := f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots, w \in \mathbb{U}. \quad (2)$$

A function $f \in \mathcal{A}$ is called to be *bi-univalent* in $\mathbb{U}$ if both f and $g = f^{-1}$ are univalent in $\mathbb{U}$ and Σ denotes the class of normalized bi-univalent functions in $\mathbb{U}$. For the historical account and for many relevant examples of functions belonging to the class Σ , see the pioneering work connected with this subject of Srivastava et al. [2], which has actually been of crucial importance for studies of bi-univalent functions in recent years. According to this article of Srivastava et al. [2], we would like to recall here some examples of functions belonging to the class Σ , such as

$$\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{2} \log\left(\frac{1+z}{1-z}\right).$$

Thus, the class Σ is not empty, while the Koebe function does not belongs to Σ .

In a large number of papers which appeared after the work of Srivastava et al. [2], the authors defined and studied the different families of the bi-univalent function class Σ (as can be seen, for example, in [3–22]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor–Maclaurin expansion (1) were obtained in many of these recent papers. The problem of finding the upper bounds for the general coefficient of the power series expansion coefficients

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}, \mathbb{N} := \{1, 2, 3, \dots\})$$

for functions $f \in \Sigma$ is still not completely solved for many subclasses of the bi-univalent function class Σ (as can be seen, for example, in [11,14,15]).

For two analytic functions in $\mathbb{U}$, namely $F(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ and $G(z) = \sum_{k=0}^{\infty} \beta_k z^k$, $*$ usually denotes the *convolution (or Hadamard) product* of these functions by

$$(F * G)(z) := \sum_{k=0}^{\infty} \alpha_k \beta_k z^k, \quad z \in \mathbb{U}.$$

In [23], Ruscheweyh defined and investigated the family of *prestarlike functions of order μ* , that are the functions f with the property that $f * I_\mu$ is a starlike function of order μ in $\mathbb{U}$, where

$$I_\mu(z) := \frac{z}{(1-z)^{2(1-\mu)}}, \quad z \in \mathbb{U} \quad (0 \leq \mu < 1).$$

Remark that the function I_μ could be written in the form

$$I_\mu(z) = z + \sum_{k=2}^{\infty} \varphi_k(\mu) z^k, \quad z \in \mathbb{U},$$

where

$$\varphi_k(\mu) = \frac{\prod_{j=2}^k (j - 2\mu)}{(k-1)!}, \quad k \geq 2.$$

In addition, we note that φ_k is a decreasing function and satisfies the limit property

$$\lim_{k \rightarrow \infty} \varphi_k(\mu) = \begin{cases} \infty, & \text{if } \mu < \frac{1}{2}, \\ 1, & \text{if } \mu = \frac{1}{2}, \\ 0, & \text{if } \mu > \frac{1}{2}. \end{cases}$$

Next, we recall the following lemma that will be used as a main tool in the proofs of our two main results.

Lemma 1 ([1,24]). (Carathéodory's inequality) If $h \in \mathcal{P}$, then

$$|c_k| \leq 2 \quad (k \in \mathbb{N}),$$

where $\mathcal{P}$ is the class of all functions h analytic in $\mathbb{U}$, for which

$$\operatorname{Re} h(z) > 0, \quad z \in \mathbb{U},$$

with

$$h(z) = 1 + c_1 z + c_2 z^2 + \dots, \quad z \in \mathbb{U}.$$

2. Initial Coefficient Estimates for the Bi-Univalent Function Subclass $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$

First, we will first define the new subclass $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$ of the bi-univalent function as follows:

Definition 1. A function $f \in \Sigma$ of the form (1) belongs to the bi-univalent function class $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$ if it satisfies the conditions

$$\left| \arg \left(\left(\frac{z(f * I_\mu)'(z)}{(f * I_\mu)(z)} \right)^\gamma \left[(1 - \delta) \frac{z(f * I_\mu)'(z)}{(f * I_\mu)(z)} + \delta \left(1 + \frac{z(f * I_\mu)''(z)}{(f * I_\mu)'(z)} \right) \right]^\lambda \right) \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{U}, \quad (3)$$

and

$$\left| \arg \left(\left(\frac{w(g * I_\mu)'(w)}{(g * I_\mu)(w)} \right)^\gamma \left[(1 - \delta) \frac{w(g * I_\mu)'(w)}{(g * I_\mu)(w)} + \delta \left(1 + \frac{w(g * I_\mu)''(w)}{(g * I_\mu)'(w)} \right) \right]^\lambda \right) \right| < \frac{\alpha\pi}{2}, \quad w \in \mathbb{U}, \quad (4)$$

where

$$0 < \alpha \leq 1, \quad 0 \leq \gamma \leq 1, \quad 0 \leq \lambda \leq 1, \quad 0 \leq \delta \leq 1,$$

and $g = f^{-1}$ is given by (2).

Remark 1. The subclass $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$ generalizes some well-known families considered in earlier studies and which will be recalled below:

- (i) For $\gamma = 0$, $\lambda = 1$ and $\mu = \frac{1}{2}$, the class $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$ reduces to the class $M_\Sigma(\alpha, \delta)$, which was investigated by Li and Wang [25], that is

$$M_\Sigma(\alpha, \delta) := \left\{ f \in \Sigma : \left| \arg \left[(1 - \delta) \frac{zf'(z)}{f(z)} + \delta \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{U}, \text{ and } \left| \arg \left[(1 - \delta) \frac{wg'(w)}{g(w)} + \delta \left(1 + \frac{wg''(w)}{g'(w)} \right) \right] \right| < \frac{\alpha\pi}{2}, \quad w \in \mathbb{U} \right\},$$

where $g = f^{-1}$ is defined like in (2).

- (ii) For $\gamma = 1$, $\lambda = 0$ and $\mu = \frac{1}{2}$, the class $\mathcal{N}_{\Sigma}(\gamma, \lambda, \delta, \mu; \alpha)$ reduces to the class $S_{\Sigma^*}(\alpha)$ that was defined and studied by Brannan and Taha [26] by

$$S_{\Sigma^*}(\alpha, \delta) := \left\{ f \in \Sigma : \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2}, z \in \mathbb{U}, \text{ and } \left| \arg \frac{wg'(w)}{g(w)} \right| < \frac{\alpha\pi}{2}, w \in \mathbb{U} \right\},$$

where $g = f^{-1}$ is defined as in (2).

Remark 2. We would like to emphasize that, for appropriate parameter choices, the classes $\mathcal{N}_{\Sigma}(\gamma, \lambda, \delta, \mu; \alpha)$ are not empty. Thus, if we consider $\mu = \frac{1}{2}$, then $\operatorname{mathrm{I}}_{\frac{1}{2}}(z) = \frac{z}{1-z} = \sum_{k=1}^{\infty} z^k$, and letting $f_*(z) = \frac{z}{1-z}$, it is easy to check that $f_* \in \mathcal{S}$, and moreover, $f_* \in \Sigma$ with $g(w) = f_*^{-1}(w) = \frac{w}{1+w}$. A simple computation shows that the conditions (3) and (4) become

$$\left| \arg \left(\left(\frac{1}{1-z} \right)^{\gamma} \left(\frac{1+\delta z}{1-z} \right)^{\lambda} \right) \right| < \frac{\alpha\pi}{2}, \quad (5)$$

and

$$\left| \arg \left(\left(\frac{1}{1+w} \right)^{\gamma} \left(\frac{1-\delta w}{1+w} \right)^{\lambda} \right) \right| < \frac{\alpha\pi}{2}, \quad (6)$$

respectively. For the particular case $\gamma = \frac{1}{3}$, $\delta = \frac{1}{2}$ and $\lambda = \frac{1}{5}$, using the 2D plot of the MAPLE™ computer software, we obtain the image of the open unit disk $\mathbb{U}$ by the function

$$\Phi(z) := \left(\frac{1}{1-z} \right)^{\gamma} \left(\frac{1+\delta z}{1-z} \right)^{\lambda}$$

which is the same with those by

$$\Psi(w) := \left(\frac{1}{1+w} \right)^{\gamma} \left(\frac{1-\delta w}{1+w} \right)^{\lambda}$$

and it is shown in Figure 1:

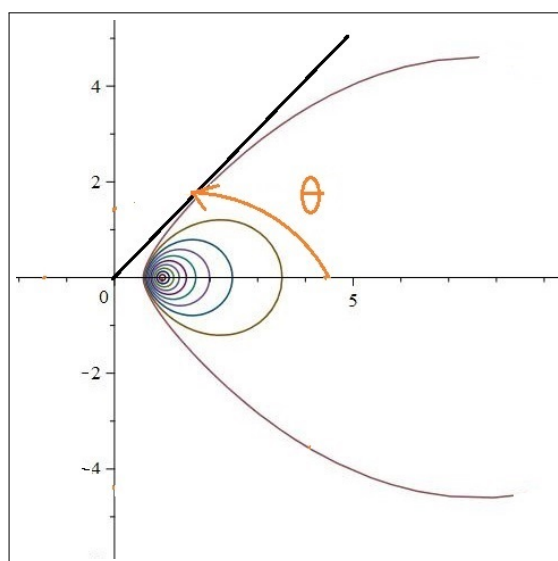


Figure 1. The image of $\Phi(\mathbb{U}) = \Psi(\mathbb{U})$.

Since $\overline{\Phi(z)} = \Phi(\bar{z})$ and similarly for Ψ , it follows that the domains $\Phi(\mathbb{U}) = \Psi(\mathbb{U})$ are symmetric with respect to the real axis. Therefore, if θ is the positive argument of the tangent starting from

the origin to the boundary of the domain $\Phi(\mathbb{U})$ that is $\Phi(\partial\mathbb{U})$, for $\alpha \geq \tan \theta$ (see also Figure 1), we obtain that the inequalities (5) and (6) are satisfied, and hence $f_*(z) = \frac{z}{1-z} \in \mathcal{N}_\Sigma\left(\frac{1}{3}, \frac{1}{5}, \frac{1}{2}, \frac{1}{2}; \alpha\right)$.

Concluding, for appropriate choices of the parameters γ , λ , δ , μ and α , the subclasses $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$ are not empty.

Our first main result is presented in the below theorem where we found upper bounds for the first two coefficients $|a_2|$ and $|a_3|$ of the power series expansion of the functions belonging to these classes.

Theorem 1. Let the function $f \in \mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$, with $0 < \alpha \leq 1$, $0 \leq \gamma \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq \delta \leq 1$ be given by (1). Then,

$$|a_2| \leq \frac{\alpha}{\sqrt{|\alpha(1-\mu)\Phi(\mu, \gamma, \lambda, \delta) + (1-\alpha)(1-\mu)^2(\gamma + \lambda(\delta + 1))^2|}}$$

and

$$|a_3| \leq \frac{\alpha^2}{8(1-\mu)^2(\gamma + \lambda(\delta + 1))^2} + \frac{\alpha}{(1-\mu)(3-2\mu)(\gamma + \lambda(2\delta + 1))},$$

where

$$\Phi(\mu, \gamma, \lambda, \delta) = (1-\mu)[\gamma(\gamma-1) + \lambda(\delta+1)(2\gamma + (\lambda-1)(\delta+1)) - 2(\gamma + \lambda(3\delta+1))] + (3-2\mu)(\gamma + \lambda(2\delta+1)). \quad (7)$$

Proof. According to the conditions (3) and (4), we have

$$\left(\frac{z(f * I_\mu)'(z)}{(f * I_\mu)(z)}\right)^\gamma \left[(1-\delta)\frac{z(f * I_\mu)'(z)}{(f * I_\mu)(z)} + \delta\left(1 + \frac{z(f * I_\mu)''(z)}{(f * I_\mu)'(z)}\right)\right]^\lambda = [p(z)]^\alpha, \quad (8)$$

and

$$\left(\frac{w(g * I_\mu)'(w)}{(g * I_\mu)(w)}\right)^\gamma \left[(1-\delta)\frac{w(g * I_\mu)'(w)}{(g * I_\mu)(w)} + \delta\left(1 + \frac{w(g * I_\mu)''(w)}{(g * I_\mu)'(w)}\right)\right]^\lambda = [q(w)]^\alpha, \quad (9)$$

where $g = f^{-1}$, with the functions $p, q \in \mathcal{P}$ having the power series representations

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots, \quad z \in \mathbb{U}, \quad (10)$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \dots, \quad w \in \mathbb{U}. \quad (11)$$

Equating the corresponding coefficients of (8) and (9), we obtain that

$$2(1-\mu)(\gamma + \lambda(\delta + 1))a_2 = \alpha p_1, \quad (12)$$

$$\begin{aligned} &2(1-\mu)(3-2\mu)(\gamma + \lambda(2\delta + 1))a_3 \\ &+ 2(1-\mu)^2[\gamma(\gamma-1) + \lambda(\delta+1)(2\gamma + (\lambda-1)(\delta+1)) - 2(\gamma + \lambda(3\delta+1))]a_2^2 \\ &= \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \end{aligned} \quad (13)$$

$$-2(1-\mu)(\gamma + \lambda(\delta + 1))a_2 = \alpha q_1, \quad (14)$$

and

$$\begin{aligned} & 2(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))(2a_2^2-a_3) \\ & +2(1-\mu)^2\left[\gamma(\gamma-1)+\lambda(\delta+1)(2\gamma+(\lambda-1)(\delta+1))-2(\gamma+\lambda(3\delta+1))\right]a_2^2 \\ & = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \end{aligned} \quad (15)$$

Using (12) and (14), it follows that

$$p_1 = -q_1, \quad (16)$$

and

$$8(1-\mu)^2(\gamma+\lambda(\delta+1))^2 a_2^2 = \alpha^2(p_1^2 + q_1^2), \quad (17)$$

and if we add (13) to (15), we obtain

$$4(1-\mu)\Phi(\mu, \gamma, \lambda, \delta)a_2^2 = \alpha(p_2 + q_2) + \frac{\alpha(\alpha-1)}{2} (p_1^2 + q_1^2), \quad (18)$$

where $\Phi(\mu, \gamma, \lambda, \delta)$ is given by (7).

Substituting the value of $p_1^2 + q_1^2$ from (17) into the right-hand side of (18), a simple computation leads to

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{4\alpha(1-\mu)\Phi(\mu, \gamma, \lambda, \delta) + 4(1-\alpha)(1-\mu)^2(\gamma+\lambda(\delta+1))^2}. \quad (19)$$

Taking the modules of both sides of (19) and using the Lemma 1 for the coefficients p_2 and q_2 , we obtain

$$|a_2| \leq \frac{\alpha}{\sqrt{|\alpha(1-\mu)\Phi(\mu, \gamma, \lambda, \delta) + (1-\alpha)(1-\mu)^2(\gamma+\lambda(\delta+1))^2|}}.$$

In order to determine the upper bound of $|a_3|$, subtracting (15) from (13), we have

$$4(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))(a_3 - a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2} (p_1^2 - q_1^2). \quad (20)$$

Substituting the value of a_2^2 from (17) into (20) and using (16), we obtain

$$a_3 = \frac{\alpha^2(p_1^2 + q_1^2)}{8(1-\mu)^2(\gamma+\lambda(\delta+1))^2} + \frac{\alpha(p_2 - q_2)}{4(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))}. \quad (21)$$

Taking the modules for both sides of (21) and once again using Lemma 1 for the coefficients p_1 , p_2 , q_1 and q_2 , it follows that

$$|a_3| \leq \frac{\alpha^2}{8(1-\mu)^2(\gamma+\lambda(\delta+1))^2} + \frac{\alpha}{(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))},$$

and the proof of our theorem is complete. $\square$

Remark 3. Note that Theorem 1 generalizes some earlier results obtained by different authors:

- (i) If, in this theorem, we choose $\gamma = 0$, $\lambda = 1$, and $\mu = \frac{1}{2}$, then we have the following result of Li and Wang ([25] Theorem 2.2):

Let f be given by (1) in the class $M_{\Sigma}(\alpha, \delta) := \mathcal{N}_{\Sigma}\left(0, 1, \delta, \frac{1}{2}; \alpha\right)$, $0 \leq \alpha < 1$, $\delta \geq 0$. Then,

$$|a_2| \leq \frac{2\alpha}{(1+\delta)(\alpha+1+\delta-\alpha\delta)}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1+\delta)^2} + \frac{\alpha}{1+2\delta}.$$

- (ii) For the special case $\gamma = 1$, $\lambda = 0$ and $\mu = \frac{1}{2}$, we obtain the result of Murugusundaramoorthy et al. ([27] Corollary 6), that is:

Let f be given by (1) be in the class $\mathcal{S}_{\Sigma}^{\alpha} := \mathcal{N}_{\Sigma}(1, 0, \delta, \frac{1}{2}; \alpha)$, $0 < \alpha \leq 1$. Then,

$$|a_2| \leq \frac{2\alpha}{\alpha+1}, \quad |a_3| \leq 4\alpha^2 + \alpha.$$

3. Initial Coefficient Estimates for the Bi-Univalent Function Subclass $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$

In the next main result of the paper, we also found the upper bounds of the two initial coefficients of the power series. Thus, we define the subclass $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ of the class of bi-univalent functions.

Definition 2. A function $f \in \Sigma$ of the form (1) is called to be in subclass $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ of the class of bi-univalent functions if it satisfies the conditions

$$\operatorname{Re} \left\{ \left(\frac{z(f * I_{\mu})'(z)}{(f * I_{\mu})(z)} \right)^{\gamma} \left[(1-\delta) \frac{z(f * I_{\mu})'(z)}{(f * I_{\mu})(z)} + \delta \left(1 + \frac{z(f * I_{\mu})''(z)}{(f * I_{\mu})'(z)} \right) \right]^{\lambda} \right\} > \beta, z \in \mathbb{U}, \quad (22)$$

and

$$\operatorname{Re} \left\{ \left(\frac{w(g * I_{\mu})'(w)}{(g * I_{\mu})(w)} \right)^{\gamma} \left[(1-\delta) \frac{w(g * I_{\mu})'(w)}{(g * I_{\mu})(w)} + \delta \left(1 + \frac{w(g * I_{\mu})''(w)}{(g * I_{\mu})'(w)} \right) \right]^{\lambda} \right\} > \beta, w \in \mathbb{U}, \quad (23)$$

where

$$0 \leq \beta < 1, \quad 0 \leq \gamma \leq 1, \quad 0 \leq \lambda \leq 1, \quad 0 \leq \delta \leq 1,$$

and $g = f^{-1}$ is given by (2).

Remark 4. The subclass $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ is a generalization of some well-known classes investigated previously, which we recall below:

1. For $\gamma = 0$, $\lambda = 1$ and $\mu = \frac{1}{2}$, the class $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ is reduced to the subclass $B_{\Sigma}(\beta, \delta)$ introduced by Li and Wang [25], as follows

$$B_{\Sigma}(\beta, \delta) := \left\{ f \in \Sigma : \operatorname{Re} \left[(1-\delta) \frac{zf'(z)}{f(z)} + \delta \left(1 + \frac{zf''(z)}{f'(z)} \right) \right] > \beta, z \in \mathbb{U}, \text{ and } \operatorname{Re} \left[(1-\delta) \frac{wg'(w)}{g(w)} + \delta \left(1 + \frac{wg''(w)}{g'(w)} \right) \right] > \beta, w \in \mathbb{U} \right\},$$

where $g = f^{-1}$ is defined as in (2).

2. For $\gamma = 1$, $\lambda = 0$, and $\mu = \frac{1}{2}$, the class $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ is reduced to the subclass $S_{\Sigma^*}(\beta)$ that was already investigated by Brannan and Taha [26], and was defined by

$$S_{\Sigma^*}(\beta) := \left\{ f \in \Sigma : \operatorname{Re} \frac{zf'(z)}{f(z)} > \beta, z \in \mathbb{U}, \text{ and } \operatorname{Re} \frac{wg'(w)}{g(w)} > \beta, w \in \mathbb{U} \right\},$$

where $g = f^{-1}$ is defined as in (2).

Remark 5. Considering the same values of the parameters $\gamma, \lambda, \delta, \mu$, and α as in the Remark 2, for the function $f_*(z) = \frac{z}{1-z}$, we obtain that the inequalities (22) and (23) become

$$\operatorname{Re} \Phi(z) > \beta, \quad z \in \mathbb{U}, \quad \operatorname{Re} \Psi(w) > \beta, \quad w \in \mathbb{U},$$

respectively. As can be seen in Figure 1, there exists a positive value of $\beta < 1$ such that the above two inequalities hold, hence $f_*(z) = \frac{z}{1-z} \in \mathcal{N}_{\Sigma^*}(\frac{1}{3}, \frac{1}{5}, \frac{1}{2}, \frac{1}{2}; \beta)$

Consequently, for appropriate choices of the parameters $\gamma, \lambda, \delta, \mu$, and β , the subclasses $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ are not empty.

Our second main result presented in the next theorem gives upper bounds for the two initial coefficients of the functions belonging to the class $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$.

Theorem 2. If the function $f \in \mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$, with $0 \leq \beta < 1, 0 \leq \gamma \leq 1, 0 \leq \lambda \leq 1, 0 \leq \delta \leq 1$, is given by (1), then

$$|a_2| \leq 2 \sqrt{\frac{1-\beta}{(1-\mu)(\gamma+2)(\gamma+1)+2\mu\lambda(2\lambda-1)}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[(1-\mu)(\gamma+1)+\mu(2\lambda-1)]^2} + \frac{2(1-\beta)}{(1-\mu)(\gamma+2)+\mu(3\lambda-1)}.$$

Proof. From the relations (22) and (23), it follows that the functions $p, q \in \mathcal{P}$ exist such that

$$\left(\frac{z(f * I_\mu)'(z)}{(f * I_\mu)(z)} \right)^\gamma \left[(1-\delta) \frac{z(f * I_\mu)'(z)}{(f * I_\mu)(z)} + \delta \left(1 + \frac{z(f * I_\mu)''(z)}{(f * I_\mu)'(z)} \right) \right]^\lambda = \beta + (1-\beta)p(z), \quad (24)$$

and

$$\left(\frac{w(g * I_\mu)'(w)}{(g * I_\mu)(w)} \right)^\gamma \left[(1-\delta) \frac{w(g * I_\mu)'(w)}{(g * I_\mu)(w)} + \delta \left(1 + \frac{w(g * I_\mu)''(w)}{(g * I_\mu)'(w)} \right) \right]^\lambda = \beta + (1-\beta)q(w), \quad (25)$$

where $g = f^{-1}$, and the functions $p, q \in \mathcal{P}$ have the series expansions given by (10) and (11), respectively. Equating the corresponding coefficients of (24) and (25), we deduce

$$2(1-\mu)(\gamma+\lambda(\delta+1))a_2 = (1-\beta)p_1, \quad (26)$$

$$\begin{aligned} & 2(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))a_3 \\ & + 2(1-\mu)^2 \left[\gamma(\gamma-1) + \lambda(\delta+1)(2\gamma+(\lambda-1)(\delta+1)) - 2(\gamma+\lambda(3\delta+1)) \right] a_2^2 \\ & = (1-\beta)p_2, \end{aligned} \quad (27)$$

$$-2(1-\mu)(\gamma+\lambda(\delta+1))a_2 = (1-\beta)q_1, \quad (28)$$

and

$$\begin{aligned} & 2(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))(2a_2^2 - a_3) \\ & + 2(1-\mu)^2 \left[\gamma(\gamma-1) + \lambda(\delta+1)(2\gamma+(\lambda-1)(\delta+1)) - 2(\gamma+\lambda(3\delta+1)) \right] a_2^2 \\ & = (1-\beta)q_2. \end{aligned} \quad (29)$$

From (26) and (28), we find that

$$p_1 = -q_1, \quad (30)$$

and

$$8(1-\mu)^2(\gamma+\lambda(\delta+1))^2a_2^2=(1-\beta)^2(p_1^2+q_1^2). \quad (31)$$

By adding (27) and (29), we obtain

$$4(1-\mu)\Phi(\mu,\gamma,\lambda,\delta)a_2^2=(1-\beta)(p_2+q_2), \quad (32)$$

where $\Phi(\mu,\gamma,\lambda,\delta)$ is given by (7). Consequently, we have

$$a_2^2=\frac{(1-\beta)(p_2+q_2)}{4(1-\mu)\Phi(\mu,\gamma,\lambda,\delta)}.$$

Applying the Lemma 1 for the coefficients p_2 and q_2 , it follows that

$$|a_2|\leq\sqrt{\frac{1-\beta}{(1-\mu)\Phi(\mu,\gamma,\lambda,\delta)}}.$$

To obtain the upper bound of $|a_3|$, by subtracting (29) from (27), we obtain

$$4(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))(a_3-a_2^2)=(1-\beta)(p_2-q_2)$$

or equivalently,

$$a_3=a_2^2+\frac{(1-\beta)(p_2-q_2)}{4(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))}. \quad (33)$$

Substituting the value of a_2^2 from (31) into (33), it follows that

$$a_3=\frac{(1-\beta)^2(p_1^2+q_1^2)}{8(1-\mu)^2(\gamma+\lambda(\delta+1))^2}+\frac{(1-\beta)(p_2-q_2)}{4(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))}.$$

Finally, applying once again the Lemma 1 for the coefficients p_1 , p_2 , q_1 , and q_2 , we obtain

$$|a_3|\leq\frac{(1-\beta)^2}{(1-\mu)^2(\gamma+\lambda(\delta+1))^2}+\frac{(1-\beta)}{(1-\mu)(3-2\mu)(\gamma+\lambda(2\delta+1))}.$$

Thus, we completed the proof of Theorem 2. $\square$

Remark 6. Theorem 2 also generalizes some previous results as follows:

- (i) If we choose, in this theorem, that $\gamma=0$, $\lambda=1$, and $\mu=\frac{1}{2}$, then we obtain the result of Li and Wang ([25] Theorem 3.2) as follows:

Let f be given by (1) in the class $B_{\Sigma^*}(\beta,\delta):=\mathcal{N}_{\Sigma^*}(0,1,\delta,\frac{1}{2};\beta)$, $0\leq\beta<1$, $0\leq\delta\leq1$. Then

$$|a_2|\leq\sqrt{\frac{2(1-\beta)}{1+\delta}}.$$

- (ii) For $\gamma=1$, $\lambda=0$ and $\mu=\frac{1}{2}$, we obtain the next result of Murugusundaramoorthy et al. ([27] Corollary 7):

Let f be given by (1) in the class $\mathcal{S}_{\Sigma}(\beta):=\mathcal{N}_{\Sigma^*}(1,0,\delta,\frac{1}{2};\beta)$, $0\leq\beta<1$, $0\leq\delta\leq1$. Then

$$|a_2|\leq\sqrt{2-2\beta}, \quad |a_3|\leq4(1-\beta)^2+(1-\beta).$$

4. Conclusions

In this article, we defined two new subclasses of bi-univalent functions, that are $\mathcal{N}_{\Sigma}(\gamma,\lambda,\delta,\mu;\alpha)$ and $\mathcal{N}_{\Sigma^*}(\gamma,\lambda,\delta,\mu;\beta)$, with the aid of the arguments and real parts' upper bounds, respectively. In these definitions, we used the convolution product with

the function I_μ first defined in [23]. For some particular cases of parameters, the classes $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$ generalize those introduced by Li and Wang [25] and Brannan and Taha [26], while $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$ extends the classes $B_\Sigma(\beta, \delta)$ of Li and Wang [25], and $S_{\Sigma^*}(\beta)$ is defined as studied by Brannan and Taha [26].

The two main results give upper bounds for the first two coefficients of the power series for the functions that belong to these families. Our main results extend those of Li and Wang ([25] Theorem 2.2), Li and Wang ([25] Theorem 3.2), Murugusundaramoorthy et al. ([27] Corollary 6) and Murugusundaramoorthy et al. ([27] Corollary 7).

We would like to mention that neither of the main theorems give the best (i.e., the lowest) upper bounds for $|a_2|$ and $|a_3|$ for the functions that belong to the subclasses $\mathcal{N}_\Sigma(\gamma, \lambda, \delta, \mu; \alpha)$ and $\mathcal{N}_{\Sigma^*}(\gamma, \lambda, \delta, \mu; \beta)$. To find the best (that is the lowest, or so-called the sharp) upper bounds of these coefficients remains an interesting open question, and could motivate researchers to find other methods for this type of study.

Moreover, another open question is to find upper bounds for the general coefficients $|a_n|$, $n \geq 4$ for the functions of these new classes. Our attempts for the coefficient $|a_4|$ fail because of the very complicated expression of this coefficient, but still remains a challenging problem; maybe another approach could give a satisfactory result in this sense.

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