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Modeling Long Memory and Regime Switching with an MRS-FIEGARCH Model: A Simulation Study

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Abstract: Recent research suggests that long memory can be caused by regime switching and is easily confused with it. However, if the causes of confusion were properly controlled, they could be distinguished. Motivated by this idea, our study aims to distinguish between the long memory and regime switching of financial volatility. We firstly modeled the long memory and regime switching of volatility using the Fractionally Integrated Exponential GARCH (FIEGARCH) and Markov Regime-Switching EGARCH (MRS-EGARCH) frameworks, respectively, and performed a simulation study on their finite-sample properties when innovations followed a non-normal distribution. Subsequently, we demonstrated the confusion between the FIEGARCH and MRS-EGARCH processes using simulations. A recent study theoretically proved that the time-varying smoothing probability series can induce the presence of significant long memory in the regime-switching process. To control for its effect, the two-stage two-state FIEGARCH and MRS-FIEGARCH frameworks are proposed. The Monte Carlo studies showed that both frameworks can effectively distinguish between the pure FIEGARCH and pure MRS-EGARCH processes. When the MRS-FIEGARCH model was further employed to fit series generated with the MRS-FIEGARCH process, it outperformed the ordinary FIEGARCH model. Finally, an empirical study of NASDAQ index return was conducted to demonstrate that our MRS-FIEGARCH model can provide potentially more reliable long-memory estimates, identify the volatility states and outperform both the FIEGARCH and MRS-EGARCH models.

Keywords: long memory; regime switching; FIEGARCH; MRS-FIEGARCH

MSC: 37M10; 91G15

Citation: Zhang, C.; Shi, Y. Modeling Long Memory and Regime Switching with an MRS-FIEGARCH Model: A Simulation Study. *Axioms* **2023**, *12*, 446. https://doi.org/10.3390/axioms12050446

Academic Editors: Conghua Wen, Yi Hong, Xianming Sun, Fei Ma and Maria Letizia Guerra

Received: 22 February 2023
Revised: 24 April 2023
Accepted: 27 April 2023
Published: 30 April 2023

1. Introduction

Long-memory persistence describes the property of financial series whose sample autocorrelations are significantly different from zero, even for large lags [1–4]. In many recent studies, long-memory persistence is extensively observed, especially in financial return series [5–8]. However, apart from the actual everlasting effects of autocorrelations, it is well known that regime switching can also cause long memory [3,9–13]. Diebold and Inoue [3] give a theoretical explanation of this phenomenon, and their simulation study further demonstrates that when structural breaks or stochastic regime switching exist, they are related to long memory and are easily confused with it (a regime-switching process, such as Markov switching, can be identified as non-stationary [14]; therefore, it is expected that it could be confused with long memory from a technical point of view). Further, they argue that long memory and regime switching are interchangeable concepts and should not be studied separately.

However, in an influential study, Perron and Qu [15] propose a test to effectively distinguish the long- and short-memory processes with mean shifts in the first moment of financial return series. Motivated by their work, it is expected that if the effects of regime switching can be appropriately controlled, the pure long-memory process can be distinguished from the pure regime-switching process.

This paper aims to distinguish between long memory and regime switching based on the second moment of the financial return series (it is widely recognized that the definitions of long memory and regime switching may be referred to much broader non-linear concepts; as explained in the paper, we focus on fractional integration (long memory) and Markov switching-type non-linearity (regime switching) of financial data). Among the existing models of financial volatility, the GARCH family models [16] have enjoyed great popularity because of their ability to capture the properties of financial volatility, such as time-varying heteroskedasticity and volatility clustering [8,17–23]. In particular, to incorporate long-memory persistence in the GARCH framework, the Fractionally Integrated GARCH (FIGARCH) model has been proposed [1]. This model is based on the application of the fractional differencing operator to the autoregressive structure of conditional variance by assuming that it follows hyperbolic rather than exponential decay [1]. Despite the model's popularity, Davidson [24] argues that FIGARCH cannot measure the real long memory based on the second moment and proposes to use the Fractionally Integrated Exponential GARCH (FIEGARCH) model developed by Bollerslev and Mikkelsen [2] as an alternative. In contrast, the Markov Regime-Switching GARCH (MRS-GARCH) model is developed in the seminar work by Hamilton [25] by including regime-switching parameters into the GARCH framework to make jumps between state spaces possible. In particular, the MRS-GARCH model discussed in Haas et al. [26] outperforms other specifications by appropriately modeling the path dependency of conditional volatility [8]. To consider modeling the real long memory and compare its counterpart with the regime-switching framework, we employ FIEGARCH and MRS-EGARCH, respectively, in this paper.

Regarding the distribution of innovations, it is originally assumed to be Gaussian, and the Quasi-Maximum Likelihood Estimation (QMLE) of GARCH family models is developed based on the Gaussian distribution. However, significant evidence suggests that the financial return series is rarely Gaussian but typically leptokurtic and exhibits heavy-tail behavior [27–30]. In terms of the FIEGARCH model, it is argued that even if the true innovation does not follow a normal distribution, QMLE based on the normal distribution is still asymptotically consistent [2]. Nevertheless, MLE based on the true distribution is expected to be more efficient than its QMLE counterpart. With respect to the MRS-GARCH model, however, Klaassen [31], Ardia [32] and Haas [33] notice that if regimes (states) are not normal but leptokurtic, the use of within-regime normality can seriously affect the identification of the regime process. The details can be found in Haas and Paolella [34], who further argue that QMLE based on normal components does not provide a consistent estimator of the MRS-GARCH model if the true distribution of innovations is not normal. The same argument may also apply to the MRS-EGARCH extension. We, therefore, performed a Monte Carlo study on the QMLE property of the FIEGARCH and MRS-EGARCH models, where simulated data actually followed Student's t-distribution. Our results suggest that the QMLE of FIEGARCH is consistent but not efficient and that the QMLE of MRS-EGARCH is neither consistent nor efficient. Those results are consistent with the existing literature. As a result, this indicates that a non-Gaussian distribution of innovations should always be used to analyze the long memory and regime switching of the volatility of financial return series with GARCH-type models.

To demonstrate the confusion between FIEGARCH and MRS-EGARCH, we performed a simulation study and conducted the usual residual diagnostics. Similarly to Diebold and Inoue [3], it was found that the MRS-EGARCH Data Generation Process (DGP) can lead to significant estimates of long memory. In addition, all of the usual residual diagnostics, including the Brock–Dechert–Scheinkman (BDS) test, the Ljung–Box test, the ARCH Lagrange multiplier (LM) test and the square root of the mean square error tended to favor the FIEGARCH model, even if the true DGP was MRS-EGARCH. Li and Mak [35] proposed portmanteau statistics for the time series goodness-of-fit test to examine whether the ARCH specification of financial volatility is adequate. Based on their work, Fisher and Gallagher [36] developed a more powerful weighted version of this test. However, the results of both tests are similar to those of the usual residual diagnostics, which still prefer the FIEGARCH model to the MRS-EGARCH model, even if the true DGP is MRS-EGARCH. Thus, residual diagnostics cannot distinguish between long memory and regime switching.

Diebold and Inoue [3] theoretically analyze the causality of long memory in the regime-switching DGP based on the assumption that transition probability is time-varying. However, in the standard MRS model, it is actually constant over time [25]. Thus, a recent study by Shi [37] gives an improved theoretical proof where transition probability is a non-time-varying constant and suggests that the existing long memory in the MRS process is caused by smoothing probability, which indicates the specific state that the financial series lies in over time. Based on this proof, Shi [37] further shows that if the effects of smoothing probability can be appropriately controlled, the long memory of the MRS process should disappear. Although the above results were developed based on the first moment, it is expected that they also hold for the MRS-EGARCH DGP.

We firstly verified this argument based on the second moment by proposing a two-stage FIEGARCH framework. This two-stage framework is motivated by a similar model fitting of the first moment of financial series studied by Shi [37]. The intercept in the FIEGARCH framework is allowed to switch between states, which are identified using the smoothing probability extracted from the MRS-EGARCH model. Similar to Shi [37], a Monte Carlo study demonstrated that with the pure MRS-EGARCH DGP, the estimates of long memory with the two-stage FIEGARCH framework were mostly insignificant and close to 0, while those with the pure FIEGARCH DGP were mostly significant and not far from the true values.

Based on this, to further incorporate the effects of smoothing probability into the FIEGARCH model, we propose an MRS-FIEGARCH framework. Another Monte Carlo study was performed, where three DGPs were considered: pure FIEGARCH, pure MRS-EGARCH and MRS-FIEGARCH DGP. We observed the following facts according to the results: when the true DGP was pure MRS-EGARCH, the estimates of long memory in the MRS-FIEGARCH model were mostly insignificant and close to 0; when the true DGP was pure FIEGARCH, the estimates were mostly significant and close to the true values; when the true DGP was MRS-FIEGARCH, the estimates were smaller than those of FIGARCH and were close to the true values. Therefore, the MRS-FIEGARCH framework can be used to distinguish between long memory and regime switching, as well as to model the data at the same time.

To empirically compare the model evaluations, we fitted the FIEGARCH, MRS-EGARCH and MRS-FIEGARCH frameworks to the daily NASDAQ Composite Index ranging from 1 January 2001 to 31 December 2022. According to the estimates, it was demonstrated that the MRS-FIEGARCH framework could estimate the true transition probabilities and identify the volatility states. Compared with the FIEGARCH model, it could generate smaller estimates of long-memory persistence. In terms of model evaluations, the MRS-FIEGARCH framework outperformed both the FIEGARCH and MRS-EGARCH models.

The contributions of this paper to the existing literature are fourfold: First, we adopted a recently developed fast Fourier transformation algorithm [38] to calculate the fractionally integrated component of the (MRS-)FIEGARCH model. Compared with the widely used truncation strategy [1], it is more accurate and is still computationally efficient. Hence, our simulation and empirical results are more reliable, especially for data of large size. Second, using a comprehensive simulation study, we demonstrated that the QMLE of MRS-EGARCH is not consistent for fat-tailed data. To the best of our knowledge, although this problem is mentioned in various research studies, no systematic study has been performed to support the theoretical argument. Therefore, our simulation results provide further evidence for relevant literature. Third, our proposed MRS-FIEGARCH model can effectively distinguish between the long memory and regime switching of financial series based on the second moment. Although related studies, such as Beine et al. [39], Lux and Morales-Arias [40] and Raggi and Bordignon [41], consider the incorporation of regime switching into the long-memory model, they mostly focus on estimating the long-memory parameter and forecasting the volatility of data with long memory. Additionally, despite those studies mentioning that the estimated long-memory parameter from this incorporation approach is potentially more reliable, no evidence or reason is provided. Following the cause of the confusion between long memory and regime switching found by Shi [37], our study additionally sheds light on the capability of the MRS-FIEGARCH model to distinguish between them based on the second moment of financial series. Hence, this can further explain the reliability of estimates obtained with the MRS-FIEGARCH framework, which significantly complements the works by Beine et al. [39], Lux and Morales-Arias [40] and Raggi and Bordignon [41]. Finally, this paper also supplements existing studies on long memory and regime switching based on the second moment. Compared with Shi and Yang [42], and Ho and Shi [43], FIEGARCH may be able to capture the real long memory, rather than the hyperbolic memory, based the second moment. With respect to Gao et al. [44] and Shi [45], the FIEGARCH model can provide an asymmetric measure of shocks to the conditional volatility.

Additionally, our results can provide useful implications for financial studies where long memory is the main concern. In real financial data, due to reasons such as structural breaks in the real economy, regime switching is very likely to exist [46–48]. Thus, the long memory observed in such data is 'spurious', as it could be caused by regime switching [11,49,50]. Therefore, the MRS-FIEGARCH framework, which can control for the effects of regime switching, is a competitive option to estimate the 'true' (not caused by regime switching) long memory. For instance, it can be used to enhance the accuracy of dynamic hedging strategies and derivative pricing models, as the long-memory persistence of asset volatility is a key input in these strategies and models [51–53].

The remainder of this paper proceeds as follows: Section 2 describes the FIEGARCH model and studies its QMLE property using simulations. Section 3 explains MRS-EGARCH with a simulation study on its QMLE property. In Section 4, the results from the simulations demonstrate the confusion between long memory and regime switching. In Section 5, we propose the MRS-FIEGARCH model with a Monte Carlo study. Section 6 presents an empirical study using the daily NASDAQ Composite Index. We conclude the paper in Section 7.

2. Long Memory and the FIEGARCH Model

As described in Diebold and Inoue [3], long memory is defined using the rate of growth variances of partial sums as $var(S_T) = O(T^{2d+1})$, where $S_T = \sum_{t=1}^{T} y_t$, $\{y_t\}$ is a sequence of interested financial series, T is the number of observations and S_t is the summation of y_t. Then, d is the long-memory parameter.

For the study of time-varying financial volatility, the FIGARCH model is widely employed to estimate the long-memory characteristic. The FIGARCH model was proposed by Baillie et al. [1], and it was extended from GARCH family models. As concluded by Marcucci [46], GARCH family models have enjoyed popularity among academics because of their ability to capture some of the typical stylized facts of financial return series, such as volatility clustering. French et al. [17], and Franses and van Dijk [18] show that GARCH family models take into account the feature of time-varying volatility over a long period and provide good in-sample estimates. Despite the effectiveness of the model, Davidson [24] points out that the FIGARCH model cannot measure the real long memory. Thus, an alternative for this aim is the FIEGARCH model proposed by Bollerslev and Mikkelsen [2]. The original FIEGARCH(1,d,1) model, where 1, d and 1 correspond to the AR, long memory and MA factors in the conditional variance equation, respectively, is described as

$$
\begin{aligned}
r_t &= \mu + \varepsilon_t \\
\varepsilon_t &= \eta_t \sqrt{h_t} \text{ where } \eta_t \overset{iid}{\sim} N(0,1) \\
\log h_t &= \omega + \frac{1-\phi L}{1-\beta L}(1-L)^{-d}(\eta_{t-1} + \gamma|\eta_{t-1}|)
\end{aligned} \tag{1}
$$

where ε_t is the error at time t, h_t is the conditional variance of ε_t at time t, η_t is an identical and independent sequence following a Gaussian distribution, L is the lag operator, γ measures the asymmetric effect of shocks to conditional volatility and $(1-L)^d$ is the fractional differencing operator as defined by Hosking [54], i.e.,

$$
(1-L)^d = \sum_{k=0}^{\infty} \delta_k(d) L^k,\ \delta_k(d) = \frac{k-1-d}{k}\delta_{k-1}(d) \text{ and } \delta_0(d) = 1 \tag{2}
$$

where d is the long-memory parameter. We have a stationary long-memory process when $0 < d < 1/2$. If $d = 0$, the process reduces to an ordinary EGARCH process without the long-memory property [55].

The infinity process in Equation (2) is replaced by $(1-L)^d = \sum_{k=0}^{t-1} \delta_k(d) L^k$ in practice. However, this recursive summation process can be very slow for large sample sizes T. Hence, a widely employed strategy is to truncate it at 1000 lags (replace $t-1$ with 1000) as performed by Baillie et al. [1]. Despite its computational efficiency, the accuracy of this approach is questionable, especially for large-sized data. In order to enhance the accuracy and keep the computation efficient, we employed the fast Fourier transformation algorithm developed in a recent study by [38]. Compared with results obtained with the truncation strategy, our results are more reliable, as they cover the entire dataset without truncation, and are still computationally efficient.

We notice that in Equation (1), the distribution of innovations is assumed to be Gaussian. However, significant evidence suggests that the financial return series is rarely Gaussian but typically leptokurtic and exhibits heavy-tail behavior [27–30]. In addition, Student's t-distribution is a widely used alternative, which can accommodate the excess kurtosis of innovations [27].

To estimate the parameters of the FIEGARCH models, Bollerslev and Mikkelsen [2] suggested a QMLE method based on the Gaussian distribution. QMLE estimators are argued to be asymptotically consistent, even if the true distribution is not Gaussian. However, Student's t-distribution is generally expected to result in more efficient estimations than the Gaussian distribution in GARCH family models [28].

We performed a Monte Carlo study to investigate this argument. Altogether, nine sets of simulations with different d, ϕ and β were generated, where $\mu = 0$, $\omega = 0.1$, $\gamma = 0$ and $T = 5000$ (we set T to 5000, because in Diebold and Inoue [3], the confusion between long memory and regime switching was the most significant when T = 5000). Moreover, the simulated data followed Student's t-distribution with 3 degrees of freedom. To avoid the starting bias, 10,000 simulated points were generated for each simulation; then, the first 5000 points were truncated. Finally, we produced 500 replicates for each set of parameters, where the first 200 replicates were discarded to avoid simulation bias.

The simulated data were fitted into FIEGARCH models with a Gaussian distribution (FIEGARCH-N) and Student's t (FIEGARCH-t)-distribution, respectively. In Table 1, the bias, root mean square error (RMSE) and standard error (SE) of d, ϕ and β are reported. Bias is the mean difference between the true parameter and its estimate; RMSE is the square root of the mean of the squared difference between the true parameter and its estimate; and SE is the standard error of the estimates. It was shown that the biases of all parameters of FIEGARCH-N were quite small and similar to those of FIEGARCH-t. In contrast, the SE, which is the indicator of estimation efficiency, suggests that the FIEGARCH-t model outperformed the FIEGARCH-N model. Thus, we demonstrated that the QMLE estimators of the FIEGARCH model are consistent but less efficient than the estimators of the FIEGARCH-t model. As a result, the FIEGARCH-t model was employed to fit the dataset in the rest of this study.

Table 1. Report of simulation results of FIEGARCH(1,d,1) models.

d	ϕ	β	$Bias_d$	$RMSE_d$	SE_d	$Bias_\phi$	$RMSE_\phi$	SE_ϕ	$Bias_\beta$	$RMSE_\beta$	SE_β
Panel A: FIEGARCH(1,d,1) Model with Normal Distribution											
0.25	0.20	0.30	−0.0181	0.0808	0.0792	−0.0065	0.1949	0.1957	0.0192	0.2339	0.2343
	0.25	0.25	−0.0251	0.0623	0.0573	0.1104	0.2951	0.2751	0.1263	0.3209	0.2965
	0.30	0.20	−0.0152	0.0509	0.0488	0.0313	0.2390	0.2382	0.0558	0.2737	0.2693
0.30	0.20	0.30	−0.0318	0.0527	0.0422	0.0326	0.2438	0.2429	0.0657	0.2515	0.2440
	0.25	0.25	−0.0171	0.0551	0.0526	0.0548	0.2520	0.2472	0.0331	0.2507	0.2498
	0.30	0.20	−0.0140	0.0677	0.0666	0.0618	0.2448	0.2381	0.0815	0.2598	0.2479
0.40	0.20	0.30	−0.0320	0.0748	0.0679	0.0970	0.2741	0.2577	0.1301	0.2925	0.2633
	0.25	0.25	−0.0348	0.0498	0.0359	0.0805	0.2546	0.2428	0.1079	0.2807	0.2604
	0.30	0.20	−0.0078	0.1226	0.1229	0.0555	0.2063	0.1997	0.0628	0.2308	0.2232
Panel B: FIEGARCH(1,d,1) Model with Student's t-Distribution											
0.25	0.20	0.30	−0.0324	0.0387	0.0213	0.0243	0.1814	0.1807	0.0419	0.1906	0.1869
	0.25	0.25	−0.0144	0.0298	0.0262	−0.0588	0.2073	0.1998	−0.0547	0.2268	0.2212
	0.30	0.20	−0.0179	0.0266	0.0197	−0.0027	0.1810	0.1819	−0.0061	0.1658	0.1665
0.30	0.20	0.30	−0.0263	0.0361	0.0250	−0.0250	0.1566	0.1554	−0.0006	0.1890	0.1899
	0.25	0.25	−0.0220	0.0355	0.0281	−0.0070	0.2465	0.2477	−0.0133	0.2580	0.2590
	0.30	0.20	−0.0218	0.0311	0.0223	0.0034	0.2030	0.2040	0.0158	0.2031	0.2035
0.40	0.20	0.30	−0.0297	0.0366	0.0215	−0.0223	0.1710	0.1704	−0.0034	0.1956	0.1965
	0.25	0.25	−0.0221	0.0344	0.0265	−0.0423	0.2270	0.2241	−0.0391	0.2361	0.2340
	0.30	0.20	−0.0198	0.0278	0.0196	0.0288	0.1786	0.1771	0.0132	0.1595	0.1598

This table presents the report of the simulation results of FIEGARCH(1,d,1) models with normal and Student's t-distributions. d, ϕ and β are true values of parameters. $Bias$ is the Monte Carlo bias. $RMSE$ is the square root of the mean square error. SE is the standard error. The Monte Carlo study was based on 300 replications, and the sample size was 5000.

3. MRS-EGARCH Model

The main weakness of the GARCH family model is that it assumes that the conditional volatility has only one regime over the entire period. Unfortunately, this is not always true. Marcucci [46] argues that due to reasons such as structural breaks in the real economy and changes in operators' expectations about the future, financial returns may exhibit sudden jumps and do not stay in the same regime over a long period.

Hamilton [25] proposes the inclusion of regime-switching parameters to make jumps between state spaces possible. To retain the advantages of GARCH models and make structural breaks possible at the same time, Cai [56], and Hamilton and Susmel [57] apply this seminal idea to the ARCH specification. However, the extension to the GARCH specification is difficult because of the path dependency of the conditional volatility term, and researchers such as Gray [58], Dueker [59], Lin [60] and Klaassen [31] have generate various algorithms to overcome this issue. The details can be found in Marcucci [46].

In this paper, to be comparable with the FIEGARCH model, we employed a two-state MRS-EGARCH(1,1) model with Student's t-innovations (MRS-EGARCH-t), which is based on the MRS-EGARCH model investigated in Haas et al. [26], as follows:

$$\begin{aligned} r_t &= \mu_{s_t} + \varepsilon_{s_t,t} \\ \varepsilon_{s_t,t} &= \eta_t \sqrt{h_{s_t,t}} \text{ where } \eta_{s_t,t} \overset{iid}{\sim} t(0,1,v) \\ \log h_{s_t,t} &= \begin{cases} \omega_1 + \alpha_1(\eta_{1,t-1} + \gamma_1|\eta_{1,t-1}|) + \beta_1 \log h_{1,t-1} \text{ when } s_t = 1 \\ \omega_2 + \alpha_2(\eta_{2,t-1} + \gamma_2|\eta_{2,t-1}|) + \beta_2 \log h_{2,t-1} \text{ when } s_t = 2 \end{cases} \end{aligned} \tag{3}$$

where $\varepsilon_{s_t,t}$ is the residual at time t in state s_t; s_t is the state that the stock lies in at time t; $\eta_{s_t,t}$ is an identical and independent sequence following Student's t-distribution, with mean of 0 and standard deviation of 1; v is the degree of freedom of Student's t-distribution; and $h_{s_t,t}$ is the conditional variance in state s_t at time t. Further, the sequence $\{s_t\}$ is assumed to be a stationary, irreducible Markov process with discrete state space $\{1,2\}$ and transition matrix $P = [p_{jk}]$, where $p_{jk} = P(s_{t+1} = k|s_t = j)$ is the transition probability of moving from state j to state k ($j,k \in \{1,2\}$).

As argued by Mullen et al. [61], Equation (3) can also capture the volatility clustering as in the GARCH model, as well as making structural breaks in unconditional variance possible. In the jth regime, the unconditional logged variance is

$$\overline{\log \sigma_j^2} = \frac{\omega_j}{1-\beta_j} \tag{4}$$

as long as $|\beta_j| < 1$, that is, the process is covariance stationary [16,26]. In this paper, we indicate state 1 as the calm (low-volatility) state and state 2 as the turbulent (high-volatility) state, so that $\overline{\log \sigma_1^2} < \overline{\log \sigma_2^2}$. In addition, we use the notations $P_1 = \beta_1$ and $P_2 = \beta_2$ to measure persistence (of logged volatility) in the calm and turbulent states, respectively.

We estimated the parameters of the MRS-EGARCH model using MLE. The conditional density of ε_t is given by Mullen et al. [61] as

$$\begin{aligned} \Omega_{t-1} &= \{\varepsilon_{s_t,t-1}, \varepsilon_{s_t,t-2}, \ldots, \varepsilon_{s_t,1},\} \\ \theta &= (\mu_1, \mu_2, \omega_1, \omega_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, p_{11}, p_{22}, v)' \\ f(\varepsilon_{s_t,t}|s_t &= j, \theta, \Omega_{t-1}) = \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)\sqrt{\pi(v-2)h_{j,t}}}\left[1 + \frac{\varepsilon_{j,t}^2}{(v_j-2)h_{j,t}}\right]^{\frac{v+1}{2}} \end{aligned} \tag{5}$$

By inserting the filtered probability in state j at time $t-1$, $\rho_{j,t-1} = P(s_{t-1} = j|\theta, \Omega_{t-1})$, into Equation (5) and integrating out the state variable s_{t-1}, the density function in Equation (5) becomes

$$f(\varepsilon_{s_t,t}|\theta, \Omega_{t-1}) = \sum_{j=1}^{2}\sum_{k=1}^{2} p_{jk}\rho_{j,t-1} f(\varepsilon_{s_t,t}|s_t = j, \theta, \Omega_{t-1}). \tag{6}$$

$\rho_{j,t-1}$ can be obtained using an integrative algorithm given in Hamilton [25]. The log-likelihood function corresponds to Equation (3) as

$$L(\theta|\varepsilon) = \sum_{t=2}^{T} \ln f(\varepsilon_{s_t,t}|\theta, \Omega_{t-1}), \text{ where } \varepsilon = (\varepsilon_{s_t,1}, \varepsilon_{s_t,2}, \ldots, \varepsilon_{s_t,T})', \tag{7}$$

and the MLE estimator $\hat{\theta}$ is obtained by maximizing Equation (7).

To identify which state the financial return series lies in at time t, the smoothing probability of the calm state is constructed as follows [25]:

$$P(s_t = 1|\theta, \Omega_T) = \rho_{1,t}\left[\frac{p_{11}P(s_{t+1} = 1|\theta, \Omega_T)}{P(s_{t+1} = 1|\theta, \Omega_t)} + \frac{p_{12}P(s_{t+1} = 2|\theta, \Omega_T)}{P(s_{t+1} = 2|\theta, \Omega_t)}\right] \tag{8}$$

Using the fact that $P(s_T = 1|\theta, \Omega_T) = \rho_{1,T}$, the smoothing probability series $P(s_t = 1|\theta, \Omega_T)$ can be generated by iterating Equation (8) backwards from T to 1. As suggested by Hamilton [25], a widely recognized rule to identify the state of r_t is that if $P(s_t = 1|\theta, \Omega_T)$ is less than 0.5, r_t is assumed to lie in the turbulent state at time t, and otherwise, in the calm state.

We did not use QMLE and MRS-EGARCH with the Gaussian distribution (MRS-EGARCH-N) for the following reasons: As noted by Klaassen [31], Ardia [32] and Haas [33], if regimes are not Gaussian but leptokurtic, the use of within-regime normality can seriously affect the identification of the regime process. The details can be found in Haas and Paolella [34], who further argue that the QMLE based on Gaussian components does not provide a consistent estimator of the MRS-GARCH model if the true distribution of innovations is not Gaussian. This argument may also apply to the MRS-EGARCH extension.

A Monte Carlo study was performed to verify this argument. We constructed simulations for 12 sets of different p_{11}, p_{22} (in Diebold and Inoue [3], the confusion between long memory and regime switching is much more significant when transition probabilities are greater than or equal to 0.99; thus, we only selected the transition probabilities studied in their work that are greater than or equal to 0.99; we also investigated the case where transition probabilities are smaller (e.g., 0.90 and 0.95), and the results were generally consistent), P_1 and P_2, where $\mu_1 = \mu_2 = 0$, $\omega_1 = 0.1$, $\omega_2 = 0.5$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\gamma_1 = \gamma_2 = 0$ and $T = 5000$. Note that although the original MRS-EGARCH model makes regime switching in the mean possible, it is not our focus in this study. Thus, we set both μ_1 and μ_2 to 0, which is the widely observed mean for financial return. Moreover, the simulated data followed Student's t-distribution with 3 degrees of freedom. The replicates and each simulation were further truncated as described in Section 2 to avoid simulation bias. All simulated data were fitted to the MRS-EGARCH-N and MRS-EGARCH-t models, and the results are presented in Table 2.

The bias in Table 2 suggests that the MRS-EGARCH-N model is not consistent in either transition probabilities or volatility persistence. The SE further indicates that the MRS-EGARCH-N model is much less efficient than the MRS-EGARCH-t model. Thus, we demonstrated that the QMLE estimator of the MRS-EGARCH model is neither consistent nor efficient when the true distribution of innovations is not Gaussian. As a result, the MRS-EGARCH-t model was employed in the rest of this study.

Table 2. Report of simulation results of MRS-EGARCH(1,1) models.

p_{11}	p_{22}	P_1	P_2	$Bias_{p_{11}}$	$RMSE_{p_{11}}$	$SE_{p_{11}}$	$Bias_{p_{22}}$	$RMSE_{p_{22}}$	$SE_{p_{22}}$	$Bias_{P_1}$	$RMSE_{P_1}$	SE_{P_1}	$Bias_{P_2}$	$RMSE_{P_2}$	SE_{P_2}
Panel A: MRS-EGARCH(1,1) Model with Normal Distribution															
0.99	0.999	0.70	0.90	−0.1018	0.1085	0.0379	−0.7786	0.7847	0.0982	0.1056	0.1188	0.0548	0.0938	0.0951	0.0158
		0.80	0.80	−0.1070	0.1136	0.0383	−0.7865	0.7963	0.1251	−0.0615	0.1118	0.0938	0.1818	0.1866	0.0420
		0.90	0.70	−0.0982	0.1037	0.0333	−0.8242	0.8311	0.1072	−0.2640	0.2748	0.0768	0.2455	0.2809	0.1372
0.999	0.99	0.70	0.90	−0.1045	0.1085	0.0290	−0.7037	0.7112	0.1038	0.0484	0.1006	0.0886	0.0920	0.0928	0.0116
		0.80	0.80	−0.0979	0.1032	0.0328	−0.7592	0.7697	0.1274	−0.0686	0.0827	0.0464	0.1835	0.1965	0.0706
		0.90	0.70	−0.0960	0.1005	0.0299	−0.7538	0.7622	0.1139	−0.1510	0.1588	0.0494	0.2891	0.2903	0.0272
0.99	0.99	0.70	0.90	−0.0934	0.0970	0.0260	−0.7198	0.7337	0.1426	0.1364	0.1606	0.0852	0.0903	0.0944	0.0279
		0.80	0.80	−0.1034	0.1068	0.0272	−0.7490	0.7557	0.1009	0.0121	0.0370	0.0352	0.1929	0.1931	0.0080
		0.90	0.70	−0.1098	0.1136	0.0292	−0.7757	0.7798	0.0805	−0.1355	0.1407	0.0378	0.2654	0.2932	0.1252
0.999	0.999	0.70	0.90	−0.0870	0.0901	0.0235	−0.6683	0.6912	0.1771	0.1759	0.1904	0.0734	0.0896	0.0964	0.0358
		0.80	0.80	−0.0971	0.1010	0.0282	−0.7000	0.7218	0.1772	0.0583	0.0910	0.0702	0.1912	0.1918	0.0151
		0.90	0.70	−0.1152	0.1186	0.0284	−0.7696	0.7766	0.1040	−0.1273	0.1417	0.0627	0.2854	0.2867	0.0274
Panel B: MRS-EGARCH(1,1) Model with Student's t-Distribution															
0.99	0.999	0.70	0.90	−0.0065	0.0202	0.0192	−0.0002	0.0008	0.0008	−0.0567	0.2206	0.2143	−0.0008	0.0337	0.0339
		0.80	0.80	−0.0032	0.0148	0.0145	−0.0003	0.0009	0.0009	−0.0661	0.1747	0.1625	−0.0098	0.0475	0.0467
		0.90	0.70	−0.0049	0.0201	0.0196	−0.0003	0.0010	0.0010	−0.0490	0.1261	0.1168	−0.0071	0.0550	0.0548
0.999	0.99	0.70	0.90	−0.0003	0.0009	0.0009	−0.0034	0.0118	0.0114	−0.0207	0.0606	0.0572	−0.0445	0.1606	0.1551
		0.80	0.80	−0.0005	0.0015	0.0014	−0.0046	0.0162	0.0156	−0.0046	0.0524	0.0524	−0.0594	0.2087	0.2010
		0.90	0.70	−0.0010	0.0027	0.0025	−0.0070	0.0204	0.0193	−0.0092	0.0388	0.0379	−0.0278	0.1886	0.1875
0.99	0.99	0.70	0.90	-0.0009	0.0030	0.0028	−0.0008	0.0033	0.0032	−0.0953	0.1620	0.1317	−0.0058	0.0491	0.0490
		0.80	0.80	−0.0021	0.0049	0.0044	−0.0004	0.0034	0.0034	−0.0682	0.1772	0.1644	−0.0007	0.0533	0.0536
		0.90	0.70	−0.0028	0.0077	0.0073	−0.0017	0.0068	0.0066	−0.0392	0.0875	0.0786	-0.0359	0.1188	0.1138
0.999	0.999	0.70	0.90	−0.0004	0.0014	0.0014	−0.0007	0.0021	0.0020	−0.0205	0.0898	0.0879	−0.0015	0.0472	0.0474
		0.80	0.80	−0.0003	0.0015	0.0014	−0.0004	0.0024	0.0024	−0.0018	0.0529	0.0531	−0.0002	0.0506	0.0508
		0.90	0.70	−0.0011	0.0078	0.0078	−0.0016	0.0100	0.0100	−0.0161	0.0499	0.0474	−0.0107	0.0975	0.0974

This table presents the report of the simulation results of MRS-EGARCH(1,1) models with normal and Student's t-distributions. p_{11} and p_{22} are true values of the transition probabilities in calm and turbulent states, respectively. P_1 is the persistence of the calm state ($P_1 = \alpha_1 + \beta_1$). P_2 is the persistence of the turbulent state ($P_2 = \alpha_2 + \beta_2$). For explanations of other variables, please see Table 1.

4. Confusion between Long Memory and Regime Switching

Diebold and Inoue [3] argue that under certain conditions, even if the true DGP is MRS, it can cause long memory and is easily confused with it. We demonstrate this in Tables 3 and 4, where the previously generated FIEGARCH and MRS-EGARCH simulations are fitted into MRS-EGARCH-t and MRS-FIEGARCH-t models, respectively.

In Table 3, it can be seen that although the true DGP was MRS-EGARCH, the mean estimate of d obtained with the FIEGARCH model is considerably large in all cases. Additionally, the fraction of rejection with $H_0 : d = 0$ at 5% is large and close to 1 in all cases, which significantly suggests that the MRS-EGARCH DGP can be confused with the FIEGARCH DGP. In Table 4, the mean estimates of p_{11} and p_{22} are all greater than 0.9. Thus, the true FIEGARCH DGP can also lead to confused MRS-EGARCH estimates (we also extended our study to the FIEGARCH DGP where d is greater than 0.5; the estimations led to the same conclusion and are available upon request). Nevertheless, it is interesting to note that the mean estimates of the degree of freedom with Student's t-distribution are very close to 3. This suggests that even if the DGP is fitted into the 'wrong' model, the distribution of innovations can be properly fitted and identified.

To distinguish these two DGPs, we performed the regular residual diagnostics for (squared) standardized residuals (in the MRS-EGARCH model, estimated volatility is the weighted average of estimated volatility in the calm and turbulent states; the weights are set to the corresponding update probability $P(s_t|\theta, \Omega_{t-1})$; the details can be found in Hamilton [25], Hamilton [62] and Haas et al. [26]) for both the FIEGARCH and MRS-EGARCH models, including the BDS test, the ARCH LM test, the Ljung–Box test and RMSE (where r_t^2 is assumed to be the true volatility at time t). In Table 5, it can be seen that both the FIEGARCH and MRS-EGARCH models generated similar residual diagnostics when the true DGP was FIEGARCH. More specifically, it appears that residual diagnostics from the FIEGARCH model were slightly better. In relation to the MRS-EGARCH DGP, in Table 6, it can be observed that the same conclusions still hold. In summary, the regular residual diagnostics cannot distinguish between the FIEGARCH and MRS-EGARCH DGPs. In addition, the FIEGARCH model tends to generate comparatively better residual diagnostics, even if the true DGP is MRS-EGARCH.

In addition to the traditional residual diagnostics, Li and Mak [35] proposed portmanteau statistics for the time series goodness-of-fit test to detect whether the fitted ARCH process is adequate. Based on their work, Fisher and Gallagher [36] proposed a weighted version of this test and argued that their test is more powerful. We here generated the original portmanteau statistics and their weighted version for the FIEGARCH and MRS-EGARCH DGPs. However, in Table 7, no matter which DGP was true, the FIEGARCH and MRS-EGARCH models led to similar results. In addition, FIEGARCH still tended to generate slightly better portmanteau statistics. Thus, we demonstrated that the (weighted) portmanteau statistics cannot distinguish between the FIEGARCH and MRS-EGARCH DGPs either.

The above conclusions are consistent with those by Diebold and Inoue [3], who argue that long memory and regime switching are interchangeable concepts and should not be distinguished. In addition, they further argue that the long memory in regime switching is caused by the time-varying transition probabilities p_{11} and p_{22}. However, in the standard MRS(-EGARCH) framework, transition probabilities are not time dependent. Thus, based on proposition three in Diebold and Inoue [3], Shi [37] gives a refined proof, arguing that the significant long memory of the MRS DGP is caused by the smoothing probability series $P(s_t = 1|\Omega_T)$. Further, the author demonstrates that if the effects of $P(s_t = 1|\Omega_T)$ can be appropriately controlled for, the long memory of MRS DGP will disappear. Although the results obtained by Shi [37] were derived for the first moment, the same idea could be straightforwardly extended to the case of the MRS-EGARCH DGP. In the next section, we will test this argument by proposing different approaches to control for the effect of $P(s_t = 1|\Omega_T)$.

Table 3. Report of FIEGARCH(1,d,1) models fitted for simulated MRS-EGARCH(1,1) data.

p_{11}	p_{22}	P_1	P_2	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_\phi$	SE_ϕ	$Mean_\beta$	SE_β	$Mean_v$	SE_v
0.99	0.999	0.7	0.9	0.8395	0.1457	1.0000	0.0361	0.0726	0.5350	0.1051	2.9584	0.1834
		0.8	0.8	0.6930	0.2382	1.0000	0.0436	0.2125	0.3854	0.2298	2.7952	0.2064
		0.9	0.7	0.6593	0.2932	0.9889	−0.0125	0.3756	0.2498	0.3971	2.5941	0.2275
0.999	0.99	0.7	0.9	0.6956	0.2140	1.0000	0.0783	0.2281	0.4051	0.2292	2.6630	0.1511
		0.8	0.8	0.7811	0.2045	1.0000	0.0242	0.1928	0.4253	0.2232	2.6982	0.1029
		0.9	0.7	0.9008	0.1190	1.0000	0.0198	0.0730	0.5369	0.1113	2.7589	0.0849
0.99	0.99	0.7	0.9	0.8827	0.1098	1.0000	0.0619	0.0672	0.6158	0.1081	3.0658	0.1139
		0.8	0.8	0.8017	0.1376	1.0000	0.0688	0.0922	0.5226	0.1350	2.9318	0.1054
		0.9	0.7	0.7733	0.1729	1.0000	0.0747	0.1450	0.4837	0.1776	2.7564	0.1083
0.999	0.999	0.7	0.9	0.5659	0.1620	1.0000	0.1725	0.2833	0.4438	0.2305	3.0655	0.2113
		0.8	0.8	0.5164	0.1882	1.0000	0.0404	0.1580	0.2703	0.2199	3.0241	0.1915
		0.9	0.7	0.5998	0.2889	1.0000	0.1027	0.3688	0.3820	0.3579	2.8984	0.1972

This table presents the report of FIEGARCH(1,d,1) models fitted to simulated MRS-EGARCH(1,1) data. v is the degree of freedom of Student's t-distribution. *Mean* is the mean of simulated data. *Frac. Rej.* is the Z-test fraction of rejection with the hypotheses $H_0 : d = 0$ against $H_1 : d \neq 0$ at the 5% level. For explanations of other variables, please see Tables 1 and 2.

Table 4. Report of MRS-EGARCH(1,1) models fitted for simulated FIEGARCH(1,d,1) data.

d	ϕ	β	$Mean_{p_{11}}$	$SE_{p_{11}}$	$Mean_{p_{22}}$	$SE_{p_{22}}$	$Mean_{P_1}$	SE_{P_1}	$Mean_{P_2}$	SE_{P_2}	$Mean_v$	SE_v
0.25	0.20	0.30	0.9590	0.1433	0.9437	0.1510	0.9100	0.0664	0.8214	0.2269	3.0826	0.2381
	0.25	0.25	0.9424	0.1456	0.9377	0.1224	0.7925	0.1957	0.7951	0.2062	3.0923	0.2318
	0.30	0.20	0.9554	0.1269	0.9502	0.1067	0.7894	0.1474	0.7784	0.1800	3.1204	0.1874
0.30	0.20	0.30	0.9135	0.1795	0.9077	0.1558	0.9259	0.0756	0.8002	0.2018	3.1227	0.2959
	0.25	0.25	0.9616	0.1135	0.9511	0.1140	0.8247	0.1351	0.8239	0.1633	3.0969	0.1802
	0.30	0.20	0.9703	0.0898	0.9686	0.0769	0.7042	0.1857	0.8540	0.1597	3.0916	0.1970
0.40	0.20	0.30	0.9578	0.1338	0.9258	0.1102	0.8909	0.1278	0.8092	0.2501	3.1506	0.2758
	0.25	0.25	0.9619	0.1304	0.9567	0.1220	0.8023	0.1580	0.8988	0.1096	3.1118	0.1826
	0.30	0.20	0.9610	0.1125	0.9578	0.0914	0.7706	0.1413	0.8600	0.1500	3.1242	0.2096

This table presents the report of MRS-EGARCH(1,1) models fitted to simulated FIEGARCH(1,d,1) data. For explanations of other variables, please see Tables 1–3.

Table 5. Residual diagnostics of simulated FIEGARCH(1,d,1) data.

d	ϕ	β	$T.S.$	$Mean_F$	SE_F	$Mean_M$	SE_M
0.25	0.20	0.30	BDS_{10}	0.0133	0.7662	0.1410	0.8988
			Q^2_{10}	5.2108	11.2434	5.5899	12.1978
			$ARCH_{10}$	5.2340	11.5583	5.6299	12.6512
			$RMSE$	4.2066	6.8830	4.1902	6.8768
	0.25	0.25	BDS_{10}	−0.0752	0.7469	0.0000	1.4380
			Q^2_{10}	4.9148	8.6911	8.5754	15.1184
			$ARCH_{10}$	4.9179	8.7818	8.4879	15.0239
			$RMSE$	13.7959	55.1468	13.6060	54.7151
	0.30	0.20	BDS_{10}	−0.0734	0.7924	−0.1212	0.9680
			Q^2_{10}	6.3342	15.5035	8.6708	21.9205
			$ARCH_{10}$	6.3570	15.7050	8.7277	22.4546
			$RMSE$	13.4668	73.0038	13.1959	70.4409
0.30	0.20	0.30	BDS_{10}	0.0233	0.7576	0.0524	1.0280
			Q^2_{10}	5.6352	6.6167	6.0118	6.6788
			$ARCH_{10}$	5.6528	6.6078	6.0537	6.7141
			$RMSE$	44.8645	293.6618	44.1183	289.2863
	0.25	0.25	BDS_{10}	−0.1220	0.8157	−0.1836	0.9124
			Q^2_{10}	4.6718	6.5992	5.8339	8.8904
			$ARCH_{10}$	4.6670	6.6617	5.8326	9.0970
			$RMSE$	5.5702	9.7724	5.5939	9.8677
	0.30	0.20	BDS_{10}	−0.1757	0.8408	−0.1794	0.9776
			Q^2_{10}	11.2824	34.2011	11.6563	31.1345
			$ARCH_{10}$	11.2057	34.2707	11.3387	31.0032
			$RMSE$	9.6294	25.0198	9.5893	24.7331
0.40	0.20	0.30	BDS_{10}	0.0491	0.7625	−0.1545	1.0405
			Q^2_{10}	5.3454	9.1005	7.2278	12.2329
			$ARCH_{10}$	5.4062	9.2654	7.2497	12.4573
			$RMSE$	11.7022	29.0003	11.7131	29.2845
	0.25	0.25	BDS_{10}	−0.0368	0.6686	−0.0700	0.9103
			Q^2_{10}	9.5683	40.3191	10.8221	35.5858
			$ARCH_{10}$	9.9554	44.0926	11.0835	38.2291
			$RMSE$	20.6759	118.8271	20.6013	117.8379
	0.30	0.20	BDS_{10}	−0.0318	0.6903	0.1064	0.7356
			Q^2_{10}	5.1676	13.8613	7.4817	14.8697
			$ARCH_{10}$	5.2513	14.1162	7.5215	14.8597
			$RMSE$	16.0734	53.5696	16.2058	53.3645

This table presents the residual diagnostics of simulated FIEGARCH(1,d,1) data fitted by the FIEGARCH(1,d,1) and MRS-EGARCH(1,1) models. $T.S.$ is the test statistics. BDS_{10} is the Brock–Dechert–Scheinkman test statistics of 2 times the standard error of the standardized residuals at embedding dimension 10. Q^2_{10} is the Ljung–Box test statistics of the squared standardized residuals at lag 10. $ARCH_10$ is the ARCH LM test statistics at lag 10. Subscripts F and M indicate that the data were fitted by the FIEGARCH and MRS-EGARCH models, respectively. For explanations of other variables, please see Tables 1 and 3.

Table 6. Residual diagnostics of simulated MRS-EGARCH(1,1) data.

p_{11}	p_{22}	P_1	P_2	$T.S.$	$Mean_M$	SE_M	$Mean_F$	SE_F
0.99	0.999	0.7	0.9	BDS_{10}	0.3789	0.7057	−0.7888	0.7851
				Q^2_{10}	4.9945	8.7739	4.0924	6.3254
				$ARCH_{10}$	4.9792	8.7360	4.0842	6.2429
				$RMSE$	75.8941	336.8425	77.5118	342.6671
		0.8	0.8	BDS_{10}	0.3576	0.8572	−1.0068	0.8686
				Q^2_{10}	8.5721	25.0754	6.6365	21.2849
				$ARCH_{10}$	8.5611	24.9695	6.6462	21.1483
				$RMSE$	17.5032	17.8837	18.3780	19.0081
		0.9	0.7	BDS_{10}	0.2964	0.8398	−1.0093	0.9724
				Q^2_{10}	14.9499	75.2543	11.7877	61.6853
				$ARCH_{10}$	15.0995	75.5511	11.8934	61.8627
				$RMSE$	16.5833	39.8978	17.8228	43.1337
0.999	0.99	0.7	0.9	BDS_{10}	−0.5348	0.7244	−0.5901	0.8186
				Q^2_{10}	5.6235	8.1374	3.8480	5.8484
				$ARCH_{10}$	5.6175	8.1308	3.9184	5.8912
				$RMSE$	6.6577	8.7166	6.8684	8.8018
		0.8	0.8	BDS_{10}	0.0647	0.8045	−0.6146	0.9179
				Q^2_{10}	5.4688	10.0268	3.8438	6.9547
				$ARCH_{10}$	5.4655	10.0327	3.8459	6.9297
				$RMSE$	10.3642	26.7462	10.8935	28.2904
		0.9	0.7	BDS_{10}	0.4511	0.7632	−0.5370	0.7882
				Q^2_{10}	9.2003	21.0549	7.0127	15.6293
				$ARCH_{10}$	8.8097	20.7949	6.8447	15.5942
				$RMSE$	18.0351	74.4141	18.6012	76.1240
0.99	0.99	0.7	0.9	BDS_{10}	1.3985	0.9943	−0.6543	0.8123
				Q^2_{10}	7.4908	10.7930	3.2657	4.7852
				$ARCH_{10}$	7.5250	10.7864	3.2891	4.7723
				$RMSE$	29.3711	87.7145	30.3360	92.6413
		0.8	0.8	BDS_{10}	1.6589	0.9200	−0.7349	0.7260
				Q^2_{10}	8.8385	27.2079	3.3795	5.1603
				$ARCH_{10}$	8.8320	27.3229	3.3994	5.1391
				$RMSE$	15.0225	30.7302	15.6088	31.9617
		0.9	0.7	BDS_{10}	1.4306	0.8139	−0.9010	0.6738
				Q^2_{10}	7.0922	14.3596	3.7008	4.0653
				$ARCH_{10}$	7.1238	14.3857	3.7407	4.0581
				$RMSE$	8.4510	7.5836	8.8776	7.8413
0.999	0.999	0.7	0.9	BDS_{10}	−0.2073	0.9299	−0.2667	0.8951
				Q^2_{10}	7.3378	21.9069	3.4260	3.7045
				$ARCH_{10}$	7.3352	21.8138	3.4409	3.6902
				$RMSE$	23.8958	37.4255	23.8092	37.5175
		0.8	0.8	BDS_{10}	0.2222	0.7140	−0.2229	0.8439
				Q^2_{10}	9.2715	37.2369	5.3637	17.0294
				$ARCH_{10}$	9.6403	40.5428	5.4710	17.8718
				$RMSE$	19.3006	35.2455	19.7992	36.5110
		0.9	0.7	BDS_{10}	0.4342	0.7573	−0.2745	0.9140
				Q^2_{10}	8.2538	20.8893	6.7830	27.8055
				$ARCH_{10}$	8.3362	21.6387	6.9858	29.8967
				$RMSE$	12.7156	31.9479	13.0580	32.3474

This table presents the residual diagnostics of simulated MRS-EGARCH(1,1) data fitted by the MRS-EGARCH(1,1) and FIEGARCH(1,d,1) models. For explanations of other variables, please see Tables 2, 3 and 5.

Table 7. Portmanteau statistics of simulated data.

Simulated FIEGARCH(1,*d*,1) Data										
d		ϕ	β	$Mean_F^W$	SE_F^W	$Mean_M^W$	SE_M^W	$Mean_F$	SE_F	$Mean_M$	SE_M
0.25		0.20	0.30	3.0678	8.3217	3.4293	9.5178	5.0218	11.2059	5.3218	12.1739
		0.25	0.25	2.9581	7.4701	4.8434	10.2001	4.5253	8.5798	8.0232	14.7383
		0.30	0.20	3.1981	10.4140	4.3097	14.7415	5.7398	15.4295	7.7122	21.2627
0.30		0.20	0.30	3.0578	4.3022	3.1437	4.3853	4.8892	6.0705	5.1166	6.2266
		0.25	0.25	2.8924	5.3889	3.2516	7.0844	4.3713	6.5793	5.1175	8.5981
		0.30	0.20	5.9332	20.4833	5.4874	17.4369	10.1563	33.6301	9.9749	29.4303
0.40		0.20	0.30	2.8901	5.8113	3.5237	6.1205	4.6193	8.2164	5.5376	8.4298
		0.25	0.25	6.0485	32.0432	6.3216	27.9729	8.1199	39.9561	8.7933	35.0051
		0.30	0.20	3.2342	12.2935	3.5299	6.5648	4.9023	13.7975	6.9829	14.7758
Simulated MRS-EGARCH(1,1) Data											
p_{11}	p_{22}	P_1	P_2	$Mean_M^W$	SE_M^W	$Mean_F^W$	SE_F^W	$Mean_M$	SE_M	$Mean_F$	SE_F
0.99	0.999	0.7	0.9	2.9800	6.6697	2.3693	4.6585	4.6091	8.6743	3.7827	6.2437
		0.8	0.8	4.4845	14.8655	3.6094	12.8819	7.6088	24.4392	6.1025	21.1678
		0.9	0.7	7.2082	29.0092	5.5365	23.6750	14.6289	75.1146	11.4641	61.5562
0.999	0.99	0.7	0.9	2.7065	3.6826	1.9050	3.0922	4.5501	5.914	3.2845	5.1376
		0.8	0.8	3.0007	7.0769	2.0661	4.5270	4.8775	9.8641	3.4009	6.7012
		0.9	0.7	5.1750	12.7450	4.0420	11.3556	8.5149	21.0259	6.4298	15.4619
0.99	0.99	0.7	0.9	4.0323	5.7687	1.7833	2.6296	6.6398	9.5846	2.9948	4.6574
		0.8	0.8	4.6478	13.9927	1.7534	2.6817	8.4526	27.1193	3.1012	5.1196
		0.9	0.7	4.1918	10.3562	1.9848	2.4181	6.6259	14.3723	3.3306	3.9616
0.999	0.999	0.7	0.9	3.5769	10.8329	1.8607	2.5056	6.9607	21.8346	3.1524	3.5797
		0.8	0.8	6.2002	35.1877	3.2757	15.5554	8.2514	36.7399	4.6226	16.2494
		0.9	0.7	5.8826	18.5949	4.9015	24.9273	7.8856	20.9005	6.4630	27.7951

This table presents the summary of portmanteau statistics (from the original Li and Mak [35] test and the new weighted version extended by Fisher and Gallagher [36]) of simulated FIEGARCH(1,*d*,1) and MRS-EGARCH(1,1) data. The lag was set to 10. The number of $ARCH$ parameters was set to 1. The reported test statistics are for correlations. Superscript W indicates the new weighted portmanteau statistics. Superscript W stands for the weighted portmanteau statistics. Subscripts F and M stand for the FIEGARCH and MRS-EGARCH models, respectively. For explanations of other variables, please see Tables 1–3.

5. MRS-FIEGARCH Framework

To control for the effect of $P(s_t = 1|\Omega_T)$, as studied in Shi [37], we firstly propose a two-stage two-state FIEGARCH (2S-FIEGARCH) model.

5.1. 2S-FIEGARCH(1,d,1) Model

Adopting the idea of time-varying FIEGARCH family models [63,64], we allow the intercept of the FIEGARCH process to be time dependent with the following steps: First, a MRS-EGARCH model is fitted to the data to estimate the smoothing probability series $P(s_t = 1|\Omega_T)$. Using the criteria of Hamilton [25], when $P(s_t = 1|\Omega_T)$ is greater than 0.5, r_t is assumed to lie in the calm state, and otherwise, in the turbulent state. Second, the intercept ω in Equation (1) is set to ω_1 if r_t lies in the calm state and is set to ω_2 if r_t lies in the turbulent state. Moreover, we require $\omega_1 < \omega_2$ so that the calm state has a smaller volatility. In this way, the time-varying intercept ω should capture the variation in $P(s_t = 1|\Omega_T)$. As a result, if long memory is purely caused by regime switching, we expect the estimate of d to be close to 0 when the data are fitted into the 2S-FIEGARCH model.

To verify this, we fitted both the previously simulated FIEGARCH and MRS-FIEGARCH datasets into the 2S-FIEGARCH model with Student's t-distribution. The essential estimates are summarized in Table 8. It can be seen that when the true DGP was FIEGARCH, the mean of the estimated d was mostly significant (indicated by the fraction of rejection with $H_0 : d = 0$) and not far from the true value. Further, when the true DGP was MRS-EGARCH, the mean estimate of d was mostly insignificant and close to 0. Hence, we demonstrated that the 2S-FIEGARCH framework is effective in controlling for $P(s_t = 1|\Omega_T)$ and distinguishes between FIEGARCH and MRS-EGARCH DGPs. In addition, the mean estimates of v were close to the true value of 3 in all cases, indicating the capability of this model of identifying and estimating the underlying distribution.

Table 8. Summary of 2S-FIEGARCH(1,d,1) models.

Simulated FIEGARCH(1,d,1) Data											
d		ϕ	β	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{\omega_1}$	SE_{ω_1}	$Mean_{\omega_2}$	SE_{ω_2}	$Mean_v$	SE_v
0.25		0.20	0.30	0.1691	0.1067	0.9800	0.0591	0.0443	0.2138	0.3139	3.0271	0.1626
		0.25	0.25	0.2478	0.0871	1.0000	0.1679	0.4801	0.1057	0.1473	3.0154	0.1884
		0.30	0.20	0.2112	0.1571	0.9600	0.0745	0.0393	0.2427	0.1907	3.0198	0.1816
0.30		0.20	0.30	0.2838	0.1103	1.0000	0.2107	0.8536	0.3553	0.2817	2.9876	0.1669
		0.25	0.25	0.2891	0.1178	0.9900	0.2015	0.3423	0.4944	0.4932	2.9509	0.1936
		0.30	0.20	0.2475	0.1725	0.9900	0.0921	0.0480	0.3147	0.3123	3.0278	0.2199
0.40		0.20	0.30	0.3908	0.1262	1.0000	0.2375	1.0061	0.9295	1.5330	2.9758	0.1679
		0.25	0.25	0.4157	0.1973	0.9800	0.0922	0.0573	0.3489	0.3163	2.9486	0.2088
		0.30	0.20	0.4863	0.1931	0.9900	0.1558	0.0718	0.6042	0.4183	2.8983	0.1836
Simulated MRS-EGARCH(1,1) Data												
p_{11}	p_{22}	P_1	P_2	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{\omega_1}$	SE_{ω_1}	$Mean_{\omega_2}$	SE_{ω_2}	$Mean_v$	SE_v
0.99	0.999	0.7	0.9	0.0057	0.0163	0.0000	0.0760	0.0790	0.5680	0.0811	2.9745	0.1314
		0.8	0.8	0.0044	0.0084	0.0000	0.0964	0.0263	0.5405	0.0806	3.0309	0.1492
		0.9	0.7	0.0063	0.0109	0.0000	0.1806	0.2766	0.5069	0.0912	2.9662	0.1502
0.999	0.99	0.7	0.9	0.0028	0.0064	0.0100	0.1045	0.0168	0.9268	0.2750	2.9831	0.1377
		0.8	0.8	0.0071	0.0224	0.0208	0.1025	0.0180	0.5871	0.1739	2.9869	0.1527
		0.9	0.7	0.0167	0.0832	0.0204	0.1091	0.0200	0.4926	0.3023	2.9953	0.1557
0.99	0.99	0.7	0.9	0.0012	0.0025	0.0000	0.1066	0.0193	0.9467	0.1909	2.9775	0.1575
		0.8	0.8	0.0007	0.0020	0.0000	0.1306	0.0211	0.7257	0.1333	3.0422	0.1677
		0.9	0.7	0.0026	0.0063	0.0100	0.1396	0.0224	0.5155	0.1034	3.0571	0.1912
0.999	0.999	0.7	0.9	0.0030	0.0061	0.0103	0.0871	0.0172	0.6986	0.1464	2.9574	0.1351
		0.8	0.8	0.0043	0.0088	0.0000	0.1042	0.0168	0.5419	0.0949	3.0040	0.1326
		0.9	0.7	0.0137	0.0827	0.0208	0.1230	0.0278	0.4207	0.1163	3.0012	0.1587

This table presents the summary of 2S-FIEGARCH(1,d,1) models fitted to simulated FIEGARCH(1,d,1) and MRS-EGARCH(1,1) data. For explanations of other variables, please see Tables 1–3.

5.2. 2S-V-FIEGARCH(1,d,1) Model

To be more flexible, the ϕ, β and γ of the FIEGARCH process in Equation (1) can also be time-varying. In this paper, we only considered the case where the long-memory term is constant for a finite sample, that is, the long-memory parameter d is not allowed to change along with time. Since the long-memory parameter d is defined by $var(S_T) = O(T^{2d+1})$, allowing it to change over time for a finite period could lead to problematic interpretation. Therefore, we extended the 2S-FIEGARCH model to the two-state time-varying FIEGARCH (2S-V-FIEGARCH) model by setting ϕ, β and γ to ϕ_1, β_1 and γ_1 for the calm state and to ϕ_2, β_2 and γ_2 for the turbulent state.

We further fitted the simulated FIEGARCH and MRS-EGARCH datasets into the 2S-V-FIEGARCH model, and the results can be used to check the robustness of our previous conclusions. In Table 9, the summarized estimates suggest that the mean estimates of d were mostly significant and not far from the true value with the FIEGARCH DGP and were mostly insignificant and close to 0 with the MRS-EGARCH DGP. In addition, the mean estimates of v were still similar to the true value of 3.

As a result, we demonstrated that when ϕ and β are time-varying, the more flexible 2S-V-FIEGARCH model can also distinguish between the FIEGARCH and MRS-EGARCH DGPs.

Table 9. Summary of 2S-V-FIEGARCH(1,d,1) models.

Simulated FIEGARCH(1,d,1) Data											
d	ϕ	β	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{\omega_1}$	SE_{ω_1}	$Mean_{\omega_2}$	SE_{ω_2}	$Mean_v$	SE_v
0.25	0.20	0.30	0.1650	0.1116	0.9400	0.0500	0.0568	0.2789	0.4052	3.0271	0.1869
	0.25	0.25	0.2565	0.0924	1.0000	0.1983	0.3863	0.1348	0.1336	3.0406	0.2358
	0.30	0.20	0.1912	0.1456	0.9495	0.0722	0.0591	0.2974	0.2809	3.0527	0.2033
0.30	0.20	0.30	0.2976	0.1239	1.0000	0.1166	0.0994	0.2749	0.3647	2.9871	0.1734
	0.25	0.25	0.2743	0.1357	0.9800	0.2080	0.4098	0.3091	0.3214	2.9892	0.1904
	0.30	0.20	0.2557	0.2012	0.9697	0.1002	0.0547	0.3001	0.3196	3.0364	0.2167
0.40	0.20	0.30	0.4075	0.2041	0.9798	0.1511	0.3152	2.0066	2.0958	2.8172	0.4020
	0.25	0.25	0.4596	0.2184	0.9700	0.0920	0.0611	0.3921	0.4710	2.9473	0.2321
	0.30	0.20	0.5343	0.1824	1.0000	0.1513	0.0783	0.6857	0.4919	2.8758	0.1776
Simulated MRS-EGARCH(1,1) Data											
p_{11}	p_{22}	P_1	P_2	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{\omega_1}$	SE_{ω_1}	$Mean_{\omega_2}$	SE_{ω_2}	$Mean_v$	SE_v
0.99	0.999	0.7	0.9	0.0048	0.0151	0.0000	0.1449	0.0612	0.5083	0.0621	3.0103	0.1315
		0.8	0.8	0.0041	0.0083	0.0217	0.1352	0.1039	0.5028	0.0783	3.0253	0.1419
		0.9	0.7	0.0076	0.0147	0.0333	0.2011	0.4379	0.5209	0.1067	2.9649	0.1544
0.999	0.99	0.7	0.9	0.0026	0.0065	0.0200	0.1098	0.0195	0.8820	0.5270	2.9944	0.1472
		0.8	0.8	0.0101	0.0477	0.0313	0.1018	0.0183	0.8369	0.4900	2.9907	0.1660
		0.9	0.7	0.0063	0.0132	0.0107	0.1048	0.0208	0.9144	0.6240	2.9777	0.1625
0.99	0.99	0.7	0.9	0.0009	0.0020	0.0101	0.1490	0.0342	0.6406	0.1301	3.0514	0.1653
		0.8	0.8	0.0016	0.0088	0.0000	0.1444	0.0348	0.6167	0.1331	3.0600	0.2060
		0.9	0.7	0.0030	0.0096	0.0108	0.1370	0.0422	0.7413	0.2604	2.9356	0.2845
0.999	0.999	0.7	0.9	0.0024	0.0049	0.0000	0.1120	0.0284	0.5515	0.1231	3.0094	0.1495
		0.8	0.8	0.0042	0.0083	0.0102	0.1053	0.0262	0.5470	0.1350	3.0010	0.1394
		0.9	0.7	0.0077	0.0244	0.0220	0.1341	0.1848	0.5581	0.2044	2.9193	0.2245

This table presents the summary of 2S-V-FIEGARCH(1,d,1) models fitted to simulated FIEGARCH(1,d,1) and MRS-EGARCH(1,1) data. For explanations of other variables, please see Tables 1–3.

5.3. MRS-FIEGARCH(1,d,1) Model

Instead of controlling for the effect of smoothing probability in separate stages, we can incorporate it in the integrated one-stage framework. Based on this idea, we tried to model the regime-switching process into the FIEGARCH framework, and we propose the following MRS-FIEGARCH(1,d,1) model with Student's t-distribution:

$$\begin{aligned} r_t &= \mu_{s_t} + \varepsilon_{s_t,t} \\ \varepsilon_{s_t,t} &= \eta_t \sqrt{h_{s_t,t}} \text{ where } \eta_{s_t,t} \overset{iid}{\sim} t(0,1,v) \\ \log h_{s_t,t} &= \begin{cases} \omega_1 + \dfrac{1-\phi L}{1-\beta L}(1-L)^{-d}(\eta_{1,t-1} + \gamma|\eta_{1,t-1}|) \text{ when } s_t = 1 \\ \omega_2 + \dfrac{1-\phi L}{1-\beta L}(1-L)^{-d}(\eta_{2,t-1} + \gamma|\eta_{2,t-1}|) \text{ when } s_t = 2 \end{cases} \end{aligned} \tag{9}$$

where only μ and ω can switch between states and s_t is defined in the same way as in Equation (3). We can further allow ϕ and β to be time-varying and construct the MRS-V-FIEGARCH(1,d,1) model with Student's t-distribution. Thus, the new conditional variance equations are

$$\log h_{s_t,t} = \begin{cases} \omega_1 + \dfrac{1-\phi_1 L}{1-\beta_1 L}(1-L)^{-d}(\eta_{1,t-1} + \gamma_1|\eta_{1,t-1}|) \text{ when } s_t = 1 \\ \omega_2 + \dfrac{1-\phi_2 L}{1-\beta_2 L}(1-L)^{-d}(\eta_{2,t-1} + \gamma_2|\eta_{2,t-1}|) \text{ when } s_t = 2 \end{cases} \tag{10}$$

where ϕ, β and γ can also switch between states. In addition, the MRS-(V-)FIEGARCH model can be estimated with the MLE in the same way as the MRS-EGARCH model. Further, as the unconditional volatility of the FIGARCH model does not exist [65], which may also be applicable to FIEGARCH, we require that for finite T, the mean of conditional volatility in the calm state is smaller than that in the turbulent state.

We here argue that the MRS-FIEGARCH framework can also distinguish between the FIEGARCH and MRS-EGARCH DGPs. To verify this, previously simulated data were fitted into the MRS-FIEGARCH and MRS-V-FIEGARCH models, and the estimates are summarized in Tables 10 and 11, respectively.

The mean estimates of d in Table 10 were all significant and close to the true values with the FIEGARCH DGP and were mostly insignificant and close to 0 with the MRS-EGARCH DGP in all cases. As its robustness check, in Table 11, the estimated d with the MRS-V-FIEGARCH model led to almost the same conclusion. Turning to the simulated MRS-EGARCH data, the mean estimates of p_{11} and p_{22} in Tables 10 and 11 were close to the corresponding true values. Therefore, we further argue that MRS-FIEGARCH can consistently identify the states of the MRS-EGARCH DGP. In addition, we note that the mean estimates of v were still close to the true value of 3 in all cases.

Finally, it is interesting to see how the MRS-FIEGARCH framework performs when the true DGP contains both long memory and regime switching. The MRS-V-FIEGARCH(1,d,1) DGP (we also conducted a simulation study with the MRS-FIEGARCH DGP (only ω was allowed to change), and the results were robust and are available upon request) was employed to generate the simulation, where μ, ω_1, ω_2, ϕ_1, β_1, ϕ_2 and β_2 were set to 0, 0.1, 1, 0.2, 0.3, 0.3 and 0.2, respectively. Additionally, Student's t-distributed innovation was used, with $v = 3$. There were 12 combinations of different p_{11}, p_{22} and d, the specific values of which were consistent with those in previous simulations. We still generated 300 replicates with 5000 simulated points for each combination. To avoid starting and simulation biases, we adopted the same strategies described in Section 2. Each simulated dataset was then fitted into the original FIEGARCH and MRS-V-FIEGARCH models, and the estimates are summarized in Table 12.

Table 10. Summary of MRS-FIEGARCH(1,d,1) models.

Simulated FIEGARCH(1,d,1) Data												
d		ϕ	β	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{p_{11}}$	$SE_{p_{11}}$	$Mean_{p_{22}}$	$SE_{p_{22}}$	$Mean_v$	SE_v
0.25		0.20	0.30	0.2456	0.0664	1.0000	0.8990	0.2247	0.5304	0.3741	3.0311	0.1888
		0.25	0.25	0.2373	0.0898	1.0000	0.8972	0.2226	0.4898	0.3534	3.0699	0.2835
		0.30	0.20	0.2271	0.0791	1.0000	0.7167	0.3685	0.6113	0.3687	3.0586	0.2027
0.30		0.20	0.30	0.3029	0.0952	1.0000	0.8548	0.2718	0.5159	0.3685	3.0258	0.1946
		0.25	0.25	0.2673	0.0798	1.0000	0.9577	0.1320	0.5130	0.3502	3.0904	0.2046
		0.30	0.20	0.2627	0.0846	1.0000	0.9615	0.1192	0.5008	0.3845	3.0966	0.1812
0.40		0.20	0.30	0.4034	0.0783	1.0000	0.9525	0.1221	0.5682	0.3585	3.0174	0.1283
		0.25	0.25	0.3747	0.1097	1.0000	0.9000	0.2184	0.5841	0.3673	3.0390	0.1582
		0.30	0.20	0.3574	0.0944	1.0000	0.8728	0.2284	0.6150	0.3640	3.0234	0.1544
Simulated MRS-EGARCH(1,1) Data												
p_{11}	p_{22}	P_1	P_2	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{p_{11}}$	$SE_{p_{11}}$	$Mean_{p_{22}}$	$SE_{p_{22}}$	$Mean_v$	SE_v
0.99	0.999	0.7	0.9	0.0115	0.0274	0.0000	0.9863	0.0147	0.9987	0.0008	2.9753	0.1242
		0.8	0.8	0.0061	0.0100	0.0000	0.9884	0.0113	0.9988	0.0007	3.0337	0.1512
		0.9	0.7	0.0081	0.0129	0.0000	0.9822	0.0204	0.9969	0.0093	2.9797	0.1453
0.999	0.99	0.7	0.9	0.0041	0.0074	0.0100	0.9988	0.0008	0.9866	0.0114	2.9995	0.1346
		0.8	0.8	0.0108	0.0243	0.0306	0.9968	0.0123	0.9853	0.0160	2.9888	0.1506
		0.9	0.7	0.0129	0.0274	0.0208	0.9939	0.0152	0.9805	0.0230	3.0079	0.1521
0.99	0.99	0.7	0.9	0.0067	0.0160	0.0204	0.9907	0.0022	0.9896	0.0032	2.9054	0.1464
		0.8	0.8	0.0078	0.0141	0.0000	0.9883	0.0036	0.9894	0.0034	2.9799	0.1574
		0.9	0.7	0.0099	0.0181	0.0000	0.9819	0.0114	0.9868	0.0082	3.0409	0.1781
0.999	0.999	0.7	0.9	0.0042	0.0074	0.0102	0.9986	0.0015	0.9983	0.0018	2.9684	0.1308
		0.8	0.8	0.0072	0.0128	0.0000	0.9977	0.0100	0.9985	0.0025	2.9899	0.1389
		0.9	0.7	0.0101	0.0237	0.0208	0.9979	0.0043	0.9978	0.0048	3.0205	0.1585

This table presents the summary of MRS-FIEGARCH(1,d,1) models fitted to simulated FIEGARCH(1,d,1) and MRS-EGARCH(1,1) data. For explanations of other variables, please see Tables 1–3.

Table 11. Summary of MRS-V-FIEGARCH(1,d,1) models.

Simulated FIEGARCH(1,*d*,1) Data												
d		ϕ	β	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{p_{11}}$	$SE_{p_{11}}$	$Mean_{p_{22}}$	$SE_{p_{22}}$	$Mean_v$	SE_v
0.25		0.20	0.30	0.2538	0.0596	1.0000	0.9484	0.1397	0.6532	0.3624	3.0097	0.1434
		0.25	0.25	0.2516	0.0885	1.0000	0.9643	0.1208	0.5828	0.3218	3.0223	0.1561
		0.30	0.20	0.2369	0.0732	1.0000	0.9710	0.0843	0.5298	0.3377	3.0647	0.1831
0.30		0.20	0.30	0.3008	0.0933	1.0000	0.9542	0.1315	0.6362	0.3819	3.0632	0.2791
		0.25	0.25	0.2834	0.0969	1.0000	0.9523	0.1617	0.6577	0.3385	3.0594	0.1583
		0.30	0.20	0.2766	0.0780	1.0000	0.9470	0.1655	0.5648	0.3496	3.1050	0.1917
0.40		0.20	0.30	0.4057	0.0745	1.0000	0.9749	0.0787	0.7098	0.3113	3.0231	0.1081
		0.25	0.25	0.3854	0.0996	1.0000	0.9325	0.1699	0.5717	0.3502	3.0334	0.1402
		0.30	0.20	0.3688	0.0915	1.0000	0.9445	0.1370	0.6975	0.3326	3.0515	0.2049
Simulated MRS-EGARCH(1,1) Data												
p_{11}	p_{22}	P_1	P_2	$Mean_d$	SE_d	*Frac. Rej.*	$Mean_{p_{11}}$	$SE_{p_{11}}$	$Mean_{p_{22}}$	$SE_{p_{22}}$	$Mean_v$	SE_v
0.99	0.999	0.7	0.9	0.0087	0.0209	0.0000	0.9851	0.0165	0.9986	0.0013	2.9945	0.1308
		0.8	0.8	0.0056	0.0097	0.0000	0.9877	0.0117	0.9978	0.0090	3.0407	0.1520
		0.9	0.7	0.0094	0.0128	0.0000	0.9878	0.0106	0.9972	0.0102	2.9714	0.1474
0.999	0.99	0.7	0.9	0.0038	0.0070	0.0100	0.9987	0.0008	0.9865	0.0113	3.0108	0.1333
		0.8	0.8	0.0118	0.0283	0.0309	0.9974	0.0084	0.9856	0.0161	2.9932	0.1552
		0.9	0.7	0.0139	0.0342	0.0109	0.9976	0.0038	0.9822	0.0214	3.0166	0.1567
0.99	0.99	0.7	0.9	0.0023	0.0042	0.0000	0.9890	0.0029	0.9891	0.0032	2.9842	0.1400
		0.8	0.8	0.0090	0.0187	0.0000	0.9878	0.0045	0.9894	0.0035	2.9947	0.1671
		0.9	0.7	0.0137	0.0219	0.0220	0.9872	0.0070	0.9882	0.0073	3.0050	0.1673
0.999	0.999	0.7	0.9	0.0031	0.0061	0.0000	0.9983	0.0028	0.9982	0.0023	2.9892	0.1304
		0.8	0.8	0.0077	0.0147	0.0000	0.9987	0.0014	0.9984	0.0032	2.9921	0.1395
		0.9	0.7	0.0114	0.0309	0.0105	0.9982	0.0026	0.9970	0.0081	3.0269	0.1607

This table presents the summary of MRS-V-FIEGARCH(1,d,1) models fitted to simulated FIEGARCH(1,d,1) and MRS-EGARCH(1,1) data. For explanations of other variables, please see Tables 1–3.

The mean estimates of d with the FIEGARCH model in Table 12 were all significant, and all led to positive bias. This is consistent with our previous findings that regime switching can lead to 'spurious' long memory in the FIEGARCH model. In addition, it is interesting to note that the biases of d were generally greater in the cases where p_{11} and p_{22} were smaller. Transition probabilities measure the expected length of remaining in the interested state [8]. Hence, a smaller value of p_{jj} means that r_t tends to leave state j quicker, and the transition between states is more frequent. Our results suggest that a greater 'spurious' long-memory parameter would be generated in such case. Additionally, the biases of d were relatively larger when the true value of d was smaller.

As demonstrated above, the MRS-V-FIEGARCH model can properly control for the effect of smoothing probability and should generate more reliable estimates of the long-memory parameter. In Table 12, it can be seen that the estimates of d were all significant, with considerably smaller biases, compared with those of the FIEGARCH model. The results of $RMSE$ further confirm that the MRS-V-FIEGARCH model overall outperformed the FIEGARCH counterpart in almost all cases. In relation to the transition probabilities, the biases and SEs obtained with the MRS-V-FIEGARCH model were all quite small. Hence, this suggests that the MRS-V-FIEGARCH model can correctly identify the underlying volatility states.

In conclusion, with a Monte Carlo study, we demonstrated that the proposed MRS-FIEGARCH framework can distinguish between the pure FIEGARCH and pure MRS-EGARCH DGPs. More specifically, if the estimated d is significantly different from 0 in the FIEGARCH model and is insignificant in the MRS-FIEGARCH model, we can conclude that long memory is purely caused by regime switching. If the estimated d in the MRS-FIEGARCH model is significant, it suggests that at least the detected long memory is not 'spurious'. That is, long memory is not purely caused by regime switching. In addition, when the true DGP contains both long memory and regime switching, the

MRS-FIEGARCH framework is capable of estimating the true transition probabilities and long-memory persistence.

Table 12. Summary of simulated MRS-V-FIEGARCH(1,d,1) data.

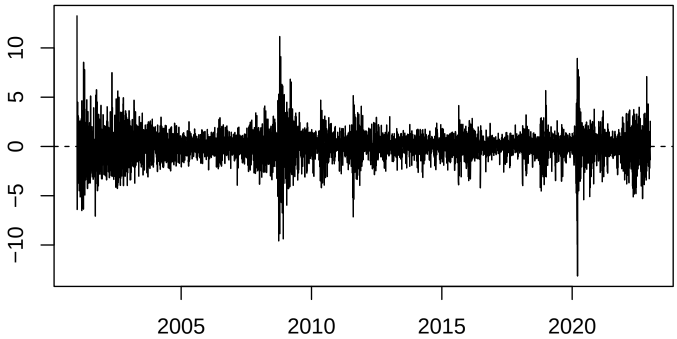
p_{11}	p_{22}	d	$Bias_d$	SE_d	$RMSE_d$	$Frac. Rej.$	$Bias_{p_{11}}$	$SE_{p_{11}}$	$RMSE_{p_{11}}$	$Bias_{p_{22}}$	$SE_{p_{22}}$	$RMSE_{p_{22}}$
Panel A: FIEGARCH model												
0.99	0.999	0.25	0.0524	0.0894	0.1036	1.0000						
		0.30	0.0595	0.0864	0.1049	1.0000						
		0.40	0.0385	0.0871	0.0952	1.0000						
0.999	0.99	0.25	0.1097	0.0635	0.1267	1.0000						
		0.30	0.0958	0.0706	0.1190	1.0000						
		0.40	0.0713	0.0668	0.0977	1.0000						
0.99	0.99	0.25	0.1231	0.0878	0.1512	1.0000						
		0.30	0.1263	0.0771	0.1480	1.0000						
		0.40	0.0924	0.0848	0.1254	1.0000						
0.999	0.999	0.25	0.1045	0.0818	0.1327	1.0000						
		0.30	0.0862	0.0720	0.1123	1.0000						
		0.40	0.0359	0.0716	0.0801	1.0000						
Panel B: MRS-V-FIEGARCH model												
0.99	0.999	0.25	0.0243	0.0891	0.0923	1.0000	−0.0173	0.0344	0.0385	−0.0055	0.0177	0.0186
		0.30	0.0213	0.0835	0.0862	1.0000	−0.0224	0.0362	0.0426	−0.0064	0.0175	0.0186
		0.40	0.0286	0.0947	0.0989	1.0000	−0.0191	0.0340	0.0390	−0.0082	0.0187	0.0204
0.999	0.99	0.25	0.0281	0.1016	0.1054	1.0000	−0.0011	0.0030	0.0033	−0.0086	0.0226	0.0242
		0.30	0.0345	0.0834	0.0902	1.0000	−0.0017	0.0048	0.0051	−0.0084	0.0254	0.0268
		0.40	0.0408	0.0840	0.0934	1.0000	−0.0038	0.0117	0.0123	−0.0127	0.0275	0.0303
0.99	0.99	0.25	0.0399	0.1035	0.1109	1.0000	−0.0053	0.0219	0.0225	−0.0054	0.0221	0.0228
		0.30	0.0432	0.0873	0.0975	1.0000	−0.0029	0.0179	0.0181	−0.0040	0.0217	0.0221
		0.40	0.0412	0.0994	0.1076	1.0000	−0.0029	0.0164	0.0167	−0.0064	0.0227	0.0236
0.999	0.999	0.25	0.0284	0.0853	0.0899	1.0000	−0.0029	0.0126	0.0129	−0.0020	0.0102	0.0104
		0.30	0.0424	0.0751	0.0863	1.0000	−0.0020	0.0061	0.0064	−0.0022	0.0096	0.0099
		0.40	0.0254	0.0827	0.0865	1.0000	−0.0026	0.0104	0.0107	−0.0012	0.0052	0.0053

This table presents the summary of simulated MRS-V-FIEGARCH(1,d,1) data fitted by FIEGARCH(1,d,1) and MRS-V-FIEGARCH(1,d,1) models. In both regimes, ϕ_1 (ϕ_2) and β_1 (β_2) were set to 0.20 (0.30) and 0.30 (0.20), respectively. ω_1 and ω_2 were set to 0.1 and 1, respectively. For explanations of other variables, please see Tables 1–3.

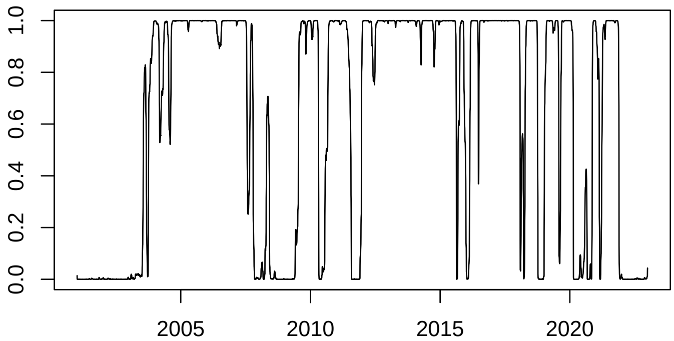
6. Empirical Results

To empirically compare our MRS-V-FIEGARCH framework with the FIEGARCH and MRS-EGARCH models, we fitted them to the daily NASDAQ Composite Index (NASDAQ). The daily closing prices of NASDAQ over the period from 1 January 2001 to 31 December 2022 were obtained from the Thomson Reuters Tick History (TRTH) database, which contains microsecond-time-stamped tick data dating back to January 1996 (the empirical dataset covers data starting from the 21st century for illustration purposes; consistent results can hold if data are sourced from 1996). The database covers 35 million over-the-counter (OTC) and exchange-traded instruments worldwide, which are provided by the Securities Industry Research Centre of Australasia (SIRCA). The corresponding return in the percentage series is defined as the logarithm of the daily closing price differences times 100, that is, $r_t = 100 \times \log(S_t/S_{t-1})$, where r_t is the return and S_t is the daily closing price.

The return series of NASDAQ is plotted in Figure 1a. It can be seen that the volatility of return was relatively greater between 2001 and 2003 and that it then decreased. From 2008 to 2010, the return became much more volatile again. After 2010, the volatility tended to be smaller, with some turbulence around the end of 2010 and the start of 2022. In addition, using the sample measures, we obtained that the mean and standard errors of the NASDAQ return were 0.0009 and 0.5309, respectively. The skewness was 0.2223, indicating that the NASDAQ return was slightly positively skewed. The kurtosis was 14.1575, suggesting a non-Gaussian distribution. Thus, we performed the Kolmogorov–Smirnov and Jarque–Bera normality tests (not presented), where the null hypotheses indicating normality were rejected in both cases. We then fitted the NASDAQ data into the FIEGARCH, MRS-EGARCH and MRS-V-FIEGARCH models, all with Student's t-distribution.



(**a**) Return of NASDAQ index



(**b**) Smoothing probability of calm state of NASDAQ index

Figure 1. Return and smoothing probability of calm state of NASDAQ index.

The estimates are presented in Table 13. The estimated d with the FIEGARCH model was 0.5374—slightly greater than 0.5—potentially suggesting an overestimated long-memory persistence. v was estimated to be significant and around 3, indicating a significant non-Gaussian distribution. Turning to the MRS-EGARCH model, the estimates of p_{11} and p_{22} were significant and greater than 0.99. This suggests a significant regime-switching process, with a small frequency to switch between states. In addition, β_1 was around 0.13, while β_2 was 0.98. This indicates that the volatility persistence in the calm state was much smaller than in the turbulent state. In addition, the estimate of v was close to 3. In terms of model performance evaluations, the logarithm of likelihood, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) all suggest that the MRS-EGARCH model outperformed the FIEGARCH model.

As demonstrated in Section 5.3, the MRS-V-FIEGARCH model can properly control for the effect of smoothing probability and should generate more reliable estimates of d. In Table 13, the estimated d with the MRS-V-FIEGARCH model was 0.3029. As it was not close to 0, the long memory is expected to exist for the NASDAQ return volatility. Further, it confirms our argument that after controlling for the regime-switching effect, the long-memory persistence should be smaller. In terms of p_{11} and p_{22}, the MRS-V-FIEGARCH model generated estimates similar to those of the MRS-EGARCH model. The estimated smoothing probability $P(s_t = 1|\Omega_T)$ with the MRS-V-FIEGARCH model is plotted in Figure 1b. The patterns of $P(s_t = 1|\Omega_T)$ are consistent with our observation mentioned above: between 2001 and 2003, the NASDAQ return lay in the turbulent state; between 2004 and 2008, it mostly lay in the calm state; from 2008 to 2010, the return series tended to switch back to the turbulent state and stayed there; after 2010, although the calm state was dominant, the turbulent state was observed in 2016 and was persistent after 2020. This is generally consistent with the real macro-economic situation: between 2001 and

2003, the macro-economy suffered the effects of Information Technology (IT) bubble cracks, and the Global Financial Crisis (GFC) affected the macro-economy from 2008 to 2010. In addition, 2016 Brexit, COVID-19 and the 2020 US presidential election contributed to the turbulence after 2010. In addition, the estimated v was similar to those of the FIEGARCH and MRS-EGARCH models and was close to 3. Finally, the logarithm of likelihood, AIC and BIC all show that the MRS-V-FIEGARCH model outperformed both the FIEGARCH and MRS-EGARCH models. Specifically, the MRS-V-FIEGARCH model had the largest likelihood and the smallest AIC and BIC across all competing models.

Table 13. Summary of daily NASDAQ index.

	FIEGARCH		**MRS-EGARCH**		**MRS-V-FIEGARCH**
ϕ	0.3487	α_1	0.0642	ϕ_1	0.5273
	(0.0339)		(0.0159)		(0.0200)
β	0.8264	β_1	0.1320	β_1	0.7792
	(0.0190)		(0.0229)		(0.0043)
		α_2	0.0179	ϕ_2	0.3851
			(0.0019)		(0.0185)
		β_2	0.9816	β_2	0.6670
			(0.0019)		(0.0052)
		p_{11}	0.9960	p_{11}	0.9981
			(0.0011)		(0.0003)
		p_{22}	0.9981	p_{22}	0.9962
			(0.0005)		(0.0004)
v	3.1496	v	3.0240	v	2.9471
	(0.0516)		(0.0519)		(0.0264)
d	0.5374			d	0.3029
	(0.0523)				(0.0119)
$log.lik$	−11456	$log.lik$	−11353	$log.lik$	−11344
AIC	22924	AIC	22728	AIC	22711
BIC	22972	BIC	22817	BIC	22808

This table presents the summary of daily NASDAQ index data fitted by the FIEGARCH(1,d,1), MRS-EGARCH(1,1) and MRS-V-FIEGARCH(1,d,1) models. The data range from 1 January 2001, to 31 December 2022. $log.lik$ is the logarithm of likelihood. AIC and BIC are Akaike Information Criterion and Bayesian Information Criterion. Values in brackets are the corresponding standard errors.

In conclusion, using the empirical results of the daily return of NASDAQ, we demonstrated that the MRS-FIEGARCH framework is capable of estimating the true transition probabilities and identifying the volatility states. Compared with the FIEGARCH model, it can generate smaller and potentially better estimates of long-memory persistence. In terms of model evaluations, the MRS-FIEGARCH framework performs better than both the FIEGARCH and MRS-EGARCH models. Thus, the MRS-FIEGARCH framework outperforms both the FIEGARCH and MRS-EGARCH models. As long memory and heteroskedasticity are important features of financial time series, the proposed MRS-FIEGARCH frameworks could be a widely useful tool for modeling volatility in other contexts.

7. Conclusions

Diebold and Inoue [3] argue that regime switching and long memory are easily confused with each other and should be studied as interchangeable concepts. However, a recent study by Shi [37] argues that if the cause of this confusion is properly controlled, they can be distinguished based on the first moment of time series. Following this idea, we propose an effective method to distinguish between long memory and regime switching based on the second moment of financial return series. The FIEGARCH and MRS-EGARCH models are used to estimate the long memory and regime switching of the conditional volatility of financial return series, respectively. Using a Monte Carlo study, we firstly demonstrated that when the true distribution of innovations is non-Gaussian, the QMLE of FIEGARCH is consistent but not efficient and the QMLE of MRS-EGARCH is neither consistent nor efficient. As financial returns are rarely Gaussian, it is suggested that a non-Gaussian distribution of innovations should always be used when a EGARCH-type model is employed.

Second, the confusion between the FIEGARCH and MRS-EGARCH DGPs was shown using simulations. The regular residual diagnostics and portmanteau statistics were produced and compared. However, none of them could distinguish between these two DGPs. Following the proof and idea provided by Shi [37], we then controlled for the effect of the smoothing probability in separate stages, which led to the 2S-FIEGARCH framework. Using a Monte Carlo study, it was demonstrated that the 2S-FIEGARCH framework can distinguish between the FIEGARCH and MRS-EGARCH DGPs.

Instead of separately controlling for the smoothing probability, we incorporated it in the FIEGARCH process by proposing the MRS-FIEGARCH framework. Another Monte Carlo study demonstrated that the MRS-FIEGARCH framework can also effectively distinguish between the pure FIEGARCH and pure MRS-EGARCH DGPs. More specifically, if the estimated d in the MRS-FIEGARCH model is significant, we can safely conclude that the long memory is not caused only by regime switching and that it does exist. In addition, when the true DGP contains both long memory and regime switching, the MRS-FIEGARCH framework is capable of identifying the volatility states and providing more reliable long-memory persistence.

An empirical study of the daily return of the NASDAQ Composite Index was conducted to compare the model performance of the FIEGARCH, MRS-EGARCH and MRS-FIEGARCH frameworks. According to the estimates, it was demonstrated that the MRS-FIEGARCH framework could identify the volatility states, the structure of which was consistent with the macro-economic situation. Compared with the FIEGARCH model, it could generate smaller and potentially more reliable estimates of long-memory persistence. In terms of model evaluations, the MRS-FIEGARCH framework outperformed both the FIEGARCH and MRS-EGARCH models. As a result, the MRS-FIEGARCH framework could be a widely useful tool for modeling the long-memory characteristic of volatility in other contexts.

For instance, our findings could be extended to enhance the accuracy of dynamic hedging strategies and derivative pricing models, as asset volatility is a key input in these strategies and models [51–53]. As noted by Hyung et al. [52], the FIEGARCH specification dominates various short- and long-memory volatility models in terms of its out-of-sample forecasting performance in forecast horizons of 10 days and beyond. The long-memory characteristic has important implications for volatility forecasting and option pricing. Option pricing in a stochastic volatility setting requires a risk premium for the unhedgeable volatility risk. Fractionally integrated series lead to volatility forecasts larger than those obtained with short-memory models, which immediately translates into higher option prices. This could be an explanation for the better pricing performance of the FIEGARCH model [8]. Stentoft [53] further notes that incorporating FIEGARCH features in option pricing models may help to explain some empirically well-documented systematic pricing errors. In addition, out-of-sample performance shows that FIEGARCH effects are important when pricing options on individual stocks and that they lead to improvements over the constant volatility model.

Apart from the proposed model, other methodologies, such as the STAR model, can also incorporate the long-memory process [66]. More recently, with the advance of machine learning methodologies, multi-layer perceptron neural networks have been developed to accommodate regime switching and long memory in volatility modeling. Examples of those models with applications in financial data can be found in Bildirici and Ersin [67], Bildirici and Ersin [68], and Bildirici and Ersin [69], among others. A systematic comparison of the proposed MRS-FIEGACH and those frameworks remains for the future.

Author Contributions: All authors contributed equally to this work. All authors wrote, reviewed and commented on the manuscript. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

GARCH	Generalized AutoRegressive Conditional Heteroskedasticity
FIEGARCH	Fractionally Integrated Exponential GARCH
MRS-EGARCH	Markov Regime-Switching EGARCH
2S-FIEGARCH	Two-state FIEGARCH with state-invariant long memory parameter
2S-V-FIEGARCH	Two-state FIEGARCH with state-variant long-memory parameter
MRS-FIEGARCH	MRS FIEGARCH with state-invariant long-memory parameter
MRS-V-FIEGARCH	MRS FIEGARCH with state-variant long-memory parameter

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