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Article

Partial Sums of the Normalized Le Roy-Type Mittag-Leffler Function

Basem Aref Frasin [1] and Luminiţa-Ioana Cotîrlă [2,*]

[1] Department of Mathematics, Faculty of Science, Al al-Bayt University, Mafraq 25113, Jordan; bafrasin@yahoo.com
[2] Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania
* Correspondence: luminita.cotirla@math.utcluj.ro

Abstract: Recently, some researchers determined lower bounds for the normalized version of some special functions to its sequence of partial sums, e.g., Struve and Dini functions, Wright functions and Miller–Ross functions. In this paper, we determine lower bounds for the normalized Le Roy-type Mittag-Leffler function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z) = z + \sum_{n=1}^{\infty} A_n z^{n+1}$, where $A_n = \left[\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right]^{\gamma}$ and its sequence of partial sums $(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z))_m(z) = z + \sum_{n=1}^{m} A_n z^{n+1}$. Several examples of the main results are also considered.

Keywords: partial sums; analytic functions; Le Roy-type Mittag-Leffler function

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1. Introduction and Preliminaries

Geometric Function Theory is an important branch of complex analysis. It deals with the geometric properties of analytic functions. Special functions are very important in the study of geometric function theory, applied mathematics, physics, statistics and many other subjects. One of these functions is the Mittag-Leffler funciton [1], widely used in the solution of fractional-order integral equations or fractional-order differential equations.

The family of the two-parameter Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0,$$

is named after the great Swedish mathematician Gösta Magnus Mittag-Leffler (1846–1927), who defined the function in one parameter [1], given by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha, z \in \mathbb{C}, \Re(\alpha) > 0.$$

The Mittag-Leffler function and its generalizations has many applications in physics, biology, chemistry, engineering, and other applied sciences, making it better known among scientists. Very recently, the study of the Mittag–Leffler function has become an interesting topic in Geometric Function Theory. Geometric properties, including starlikeness, convexity and close-to-convexity, of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ were investigated by Bansal and Prajapat in [2] and by Srivastava and Bansal (see [3]). In fact, the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its extensions and generalizations continue to be used in many different contexts in geometric function theory (see [4–9]).

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$ and hold the normalization condition $f(0) = f'(0) - 1 = 0$. Furthermore, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$, which are univalent in $\mathfrak{U}$.

The integral transformation $\mathbb{I}[f] : \mathfrak{U} \to \mathbb{C}$ of f is defined by

$$\mathbb{I}[f] = \int_0^z \frac{f(t)}{t} dt = z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n.$$

This is called the Alexander Transformation, and it was introduced by Alexander in [10]. Alexander was the first to observe and prove that the integral transformation $\mathbb{I}$ maps the class $\mathcal{S}^*$of starlike functions onto the class $\mathcal{K}$ of convex functions in a one-to-one fashion.

Recently, Gerhold [11] and Garra and Polito [12] independently introduced the Le Roy-type Mittag-Leffler function, defined as

$$F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[\Gamma(\alpha n + \beta)]^{\gamma}}, \ \ \alpha, \beta, \gamma > 0, \ \ z \in \mathbb{C}.$$

In particular, when $\alpha = \beta = 1$, $F_{\alpha,\beta}^{(\gamma)}(z)$ leads to the following function studied by É. The Le Roy-type function [13] is defined as

$$R_{\gamma}(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^{\gamma}} z^n, \quad \gamma > 0, z \in \mathbb{C}.$$

It can be easily noted that $F_{\alpha,\beta}^{(\gamma)}(z)$ is a generalization of the familiar Mittag-Leffler function $E_{\alpha,\beta}(z)$.

It is clear that the Le Roy-type Mittag-Leffler function $F_{\alpha,\beta}^{(\gamma)}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of $F_{\alpha,\beta}^{(\gamma)}(z)$:

$$\begin{aligned}
\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z) &= z[\Gamma(\beta)]^{\gamma} F_{\alpha,\beta}^{(\gamma)}(z) \\
&= z + \sum_{n=1}^{\infty} \left[\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1) + \beta)} \right]^{\gamma} z^{n+1}, \ \ \alpha, \beta, \gamma > 0, \ \ z \in \mathbb{C}.
\end{aligned} \tag{2}$$

In this paper, we shall restrict our attention to the case of positive real valued α, β, γ and $z \in \mathfrak{U}$.

Observe that the function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}$ contains many well-known functions as its special case, for example,

$$\begin{cases}
\mathbb{F}_{0,1}^{(1)}(z) = z/(1-z), \\
\mathbb{F}_{1,1}^{(1)}(z) = ze^z, \\
\mathbb{F}_{1,2}^{(1)}(z) = e^z - 1, \\
\mathbb{F}_{1,3}^{(1)}(z) = 2(e^z - z - 1)/z, \\
\mathbb{F}_{1,4}^{(1)}(z) = \left(6(e^z - z - 1) - 3z^2\right)/z^2, \\
\mathbb{F}_{2,1}^{(1)}(z) = z\cosh(\sqrt{z}), \\
\mathbb{F}_{2,2}^{(1)}(z) = \sqrt{z}\sinh(\sqrt{z}), \\
\mathbb{F}_{2,3}^{(1)}(z) = 2[\cosh(\sqrt{z}) - 1], \\
\mathbb{F}_{2,4}^{(1)}(z) = 6[\sinh(\sqrt{z}) - \sqrt{z}]/\sqrt{z}.
\end{cases}$$

Geometric properties, including starlikeness, convexity and close-to-convexity, for the normalized geometric properties of the Le Roy-type Mittag-Leffler function $\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)$ were recently investigated by Mehrez and Das in [13].

The Taylor polynomial $f_m(z)$ of f in $\mathcal{A}$, defined by

$$f_m(z) = z + \sum_{n=2}^{m} a_n z^n, \tag{3}$$

is called the m-th section/partial sum of f.

The partial sums of analytic functions play an important role in Geometric Function Theory in finding the largest disk $\mathfrak{U}_r = \{z \in \mathbb{C} : |z| < r\}$, which makes the partial sum $f_m(z)$ univalent. In 1928, Szegö [14] proved that if $f \in \mathcal{S}$, then each partial sum $f_m(z)$ of f is univalent in the disk $\mathfrak{U}_{1/4}$. Clearly, the partial sums of $f \in \mathcal{S}$ are not necessarily univalent throughout the unit disk $\mathfrak{U}$, as the convex univalent function $f(z) = z/(1-z)$ demonstrates. Moreover, the second partial sum $f_2(z) = z + 2z^2$ of the Koebe function $k(z) = z/(1-z)^2$ is univalent in $\mathfrak{U}_{1/4}$, and the radius $1/4$ is the best possible. The radius of starlikeness of the partial sum $f_m(z)$ of $f \in \mathcal{S}^*$ was proven by Robertson [15].

Determining the exact (largest) radius of univalence r_m of $f_m(z)$ ($f \in \mathcal{S}$) remains an open problem. However, many other related problems on sections have been solved for various geometric subclasses of $\mathcal{S}$, for example, the classes $\mathcal{S}^*$, $\mathcal{K}$ and C of starlike, convex and close-to-convex functions, respectively (see [15–20] and the survey articles [21,22]).

Furthermore, several researchers determined the lower bound of the real part of the ratio of the partial sum of analytic functions to its infinite series sum. The concept of finding this lower bound was first introduced by Silvia [23]. In [24], Silverman found the partial sums of convex and starlike functions by developing more useful techniques. After that, several researchers investigated such partial sums for different subclasses of analytic functions. For more work on partial sums, the interested readers are referred to [20,25–33].

Recently, some researchers have studied on partial sums of special functions. For example, Orhan and Yagmur in [34] determined lower bounds for the normalized Struve functions to its sequence of partial sums. Some lower bounds for the quotients of normalized Dini functions and their partial sum, as well as for the quotients of the derivative of normalized Dini functions and their partial sums, were obtained by Aktaş and Orhan in [35]. Din et al. [36] found the partial sums of two kinds of normalized Wright functions and the partial sums of the Alexander transform of these normalized Wright functions. Meanwhile, Kazımoğlu in [37] studied the partial sums of the normalized Miller–Ross Function.

In this paper, we study the ratio of a function of the form (2) to its sequence of partial sums

$$(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))_m(z) = z + \sum_{n=1}^{m} \left[\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right]^{\gamma} z^{n+1}, m \in \mathbb{N}, \tag{4}$$

and for $m = 0$, we have $(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z))_0(z) = z$.

The purpose of the present paper is to determine lower bounds for

$$\Re\left\{\frac{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}{(\mathbb{F}_{\alpha,\beta}^{(\gamma)})_m(z)}\right\}, \Re\left\{\frac{(\mathbb{F}_{\alpha,\beta}^{(\gamma)})_m(z)}{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)}\right\}, \Re\left\{\frac{\left(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)\right)'}{(\mathbb{F}_{\alpha,\beta}^{(\gamma)})'_m(z)}\right\}, \Re\left\{\frac{(\mathbb{F}_{\alpha,\beta}^{(\gamma)})'_m(z)}{\left(\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)\right)'}\right\},$$

$$\Re\left\{\frac{\mathbb{I}\left[\mathbb{F}_{\alpha,\beta}^{(\gamma)}\right](z)}{(\mathbb{I}\left[\mathbb{F}_{\alpha,\beta}^{(\gamma)}\right])_m(z)}\right\}, \Re\left\{\frac{(\mathbb{I}\left[\mathbb{F}_{\alpha,\beta}^{(\gamma)}\right])_m(z)}{\mathbb{I}\left[\mathbb{F}_{\alpha,\beta}^{(\gamma)}\right](z)}\right\},$$

where $\mathbb{I}\left[\mathbb{F}_{\alpha,\beta}^{(\gamma)}\right]$ is the Alexander transform of $\mathbb{F}_{\alpha,\beta}^{(\gamma)}$.

In order to obtain our results, we need the following lemmas.

Lemma 1 ([13])**.** *Let α, β, γ be positive real numbers.*

(i) If $\alpha\gamma \geq 1$ and $\alpha^2\gamma > \beta$, then the sequence $(b_n)_{n\geq 1}$ defined by

$$b_n = \frac{\Gamma(n+1)[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha n+\beta)]^\gamma}$$

is decreasing for $n \geq 1$.
(ii) If $\alpha\gamma \geq 1$ and $2\alpha^2\gamma > \beta$, then the sequence $(c_n)_{n\geq 1}$ defined by

$$c_n = \frac{\Gamma(n+2)[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha n+\beta)]^\gamma}$$

is decreasing for $n \geq 1$.

Lemma 2. *Let α, β, γ be positive real numbers. Then, the function $\mathbb{F}^{(\gamma)}_{\alpha,\beta} : \mathfrak{U} \to \mathbb{C}$, defined by (2), satisfies the following inequalities:*
(i) If $\alpha\gamma \geq 1$ and $\alpha^2\gamma > \beta$, then

$$\left|\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right| \leq \frac{[\Gamma(\alpha+\beta)]^\gamma + [\Gamma(\beta)]^\gamma(e-1)}{[\Gamma(\alpha+\beta)]^\gamma} \quad (z \in \mathfrak{U}),$$

(ii)If $\alpha\gamma \geq 1$ and $2\alpha^2\gamma > \beta$, then

$$\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right| \leq \frac{[\Gamma(\alpha+\beta)]^\gamma + 2[\Gamma(\beta)]^\gamma(e-1)}{[\Gamma(\alpha+\beta)]^\gamma} \quad (z \in \mathfrak{U}),$$

(iii) If $\alpha\gamma \geq 1$ and $\alpha^2\gamma > \beta$, then

$$\left|\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)](z)]\right| \leq \frac{[\Gamma(\alpha+\beta)]^\gamma + [\Gamma(\beta)]^\gamma(e-2)}{[\Gamma(\alpha+\beta)]^\gamma} \quad (z \in \mathfrak{U}).$$

Proof. (i) Let $\alpha\gamma \geq 1$ and $\alpha^2\gamma > \beta$. It follows from Lemma 1 that

$$\begin{aligned}
\left|\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right| &= \left|z + \sum_{n=1}^{\infty} \frac{[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha n+\beta)]^\gamma} z^{n+1}\right| \\
&= \left|z + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha n+\beta)]^\gamma n!} z^{n+1}\right| \\
&\leq 1 + \sum_{n=1}^{\infty} \frac{b_1(\alpha,\beta,\gamma)}{n!} \\
&\leq 1 + \frac{[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma}(e-1).
\end{aligned}$$

(ii) Using Lemma 1 for $\alpha\gamma \geq 1$ and $2\alpha^2\gamma > \beta$, we have

$$\begin{aligned}
\left|\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right| &= \left|1 + \sum_{n=1}^{\infty} \frac{(n+1)[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha n+\beta)]^\gamma} z^n\right| \\
&= \left|1 + \sum_{n=1}^{\infty} \frac{\Gamma(n+2)[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha n+\beta)]^\gamma n!} z^n\right| \\
&\leq 1 + \sum_{n=1}^{\infty} \frac{c_1(\alpha,\beta,\gamma)}{n!} \\
&\leq 1 + \frac{2[\Gamma(\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma}(e-1).
\end{aligned}$$

(iii) Let $\alpha\gamma \geq 1$ and $\alpha^2\gamma > \beta$. It follows from Lemma 1 that

$$\begin{aligned}
\left|\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}](z)]\right| &= \left|z + \sum_{n=1}^{\infty} \frac{[\Gamma(\beta)]^{\gamma}}{(n+1)[\Gamma(\alpha n+\beta)]^{\gamma}} z^{n+1}\right| \\
&= \left|z + \sum_{n=1}^{\infty} \frac{\Gamma(n+1)[\Gamma(\beta)]^{\gamma}}{(n+1)[\Gamma(\alpha n+\beta)]^{\gamma} n!} z^{n+1}\right| \\
&\leq 1 + \sum_{n=1}^{\infty} \frac{b_1(\alpha,\beta,\gamma)}{(n+1)!} \\
&\leq 1 + \frac{[\Gamma(\beta)]^{\gamma}}{[\Gamma(\alpha+\beta)]^{\gamma}}(e-2).
\end{aligned}$$

□

Let $w(z)$ be an analytic function in $\mathfrak{U}$. In the sequel, we will use the following well-known result:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0,\ z \in \mathfrak{U} \text{ if and only if } |w(z)| < 1,\ z \in \mathfrak{U}.$$

Theorem 1. *If* $\alpha\gamma \geq 1, \alpha^2\gamma > \beta$ *and* $[\Gamma(\alpha+\beta)]^{\gamma} \geq [\Gamma(\beta)]^{\gamma}(e-1)$, *then*

$$\Re\left\{\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})_m(z)}\right\} \geq \frac{[\Gamma(\alpha+\beta)]^{\gamma} - [\Gamma(\beta)]^{\gamma}(e-1)}{[\Gamma(\alpha+\beta)]^{\gamma}}, \qquad z \in \mathfrak{U}, \tag{5}$$

and

$$\Re\left\{\frac{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})_m(z))}{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}\right\} \geq \frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\alpha+\beta)]^{\gamma} + [\Gamma(\beta)]^{\gamma}(e-1)}, \qquad z \in \mathfrak{U}. \tag{6}$$

Proof. From inequality (i) of Lemma 2, we obtain

$$1 + \sum_{n=1}^{\infty} |A_n| \leq \frac{[\Gamma(\alpha+\beta)]^{\gamma} + [\Gamma(\beta)]^{\gamma}(e-1)}{[\Gamma(\alpha+\beta)]^{\gamma}},$$

or equivalently

$$\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-1)}\right) \sum_{n=1}^{\infty} |A_n| \leq 1,$$

where

$$A_n = \left[\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}\right]^{\gamma}. \tag{7}$$

In order to prove the inequality (5), define the function $w(z)$ in $\mathfrak{U}$ as follows:

$$\begin{aligned}
\frac{1+w(z)}{1-w(z)} &= \left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-1)}\right)\left[\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})_m(z)} - \frac{[\Gamma(\alpha+\beta)]^{\gamma} - [\Gamma(\beta)]^{\gamma}(e-1)}{[\Gamma(\alpha+\beta)]^{\gamma}}\right] \\
&= \frac{1 + \sum_{n=1}^{m} A_n z^n + \left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=m+1}^{\infty} A_n z^n}{1 + \sum_{n=1}^{m} A_n z^n}.
\end{aligned} \tag{8}$$

Now, from (8), we can write

$$w(z) = \frac{\left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} A_n z^n}{2+2\sum_{n=1}^{m} A_n z^n + \left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} A_n z^n};$$

thus, we have

$$|w(z)| \leq \frac{\left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} |A_n|}{2-2\sum_{n=1}^{m} |A_n| - \left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} |A_n|}.$$

This implies that $|w(z)| \leq 1$ if and only if

$$2\left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} |A_n| \leq 2 - 2\sum_{n=1}^{m} |A_n|.$$

This further implies that

$$\sum_{n=1}^{m} |A_n| + \left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} |A_n| \leq 1. \tag{9}$$

It suffices to show that the left hand side of (9) is bounded above by

$$\left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=1}^{\infty} |A_n|,$$

which is equivalent to

$$\left(\frac{[\Gamma(\alpha+\beta)]^\gamma - [\Gamma(\beta)]^\gamma(e-1)}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=1}^{m} |A_n| \geq 0.$$

To prove (6), we write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} &= \left(\frac{[\Gamma(\alpha+\beta)]^\gamma + [\Gamma(\beta)]^\gamma(e-1)}{[\Gamma(\beta)]^\gamma(e-1)}\right)\left[\frac{(\mathbb{F}_{\alpha,\beta}^{(\gamma)})_m(z)}{\mathbb{F}_{\alpha,\beta}^{(\gamma)}(z)} - \frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\alpha+\beta)]^\gamma + [\Gamma(\beta)]^\gamma(e-1)}\right] \\ &= \frac{1+\sum_{n=1}^{m} A_n z^n + \left(\frac{[\Gamma(\alpha+\beta)]^\gamma + [\Gamma(\beta)]^\gamma(e-1)}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} A_n z^n}{1+\sum_{n=1}^{m} A_n z^n}.\end{aligned}$$

Therefore,

$$|w(z)| \leq \frac{\left(\frac{[\Gamma(\alpha+\beta)]^\gamma + [\Gamma(\beta)]^\gamma(e-1)}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} |A_n|}{2-2\sum_{n=1}^{m} |A_n| - \left(\frac{[\Gamma(\alpha+\beta)]^\gamma - [\Gamma(\beta)]^\gamma(e-1)}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} |A_n|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m} |A_n| + \left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=m+1}^{\infty} |A_n| \leq 1. \tag{10}$$

Since the left hand side of (10) is bounded above by $\left(\frac{[\Gamma(\alpha+\beta)]^\gamma}{[\Gamma(\beta)]^\gamma(e-1)}\right)\sum_{n=1}^{\infty} |A_n|$, this completes the proof of our theorem. □

Theorem 2. *If* $\alpha\gamma \geq 1, 2\alpha^2\gamma > \beta$ *and* $[\Gamma(\alpha+\beta)]^\gamma \geq 2[\Gamma(\beta)]^\gamma(e-1)$*, then*

$$\Re\left\{\frac{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})'_m(z)}\right\} \geq \frac{[\Gamma(\alpha+\beta)]^{\gamma} - 2[\Gamma(\beta)]^{\gamma}(e-1)}{[\Gamma(\alpha+\beta)]^{\gamma}}, \quad z \in \mathfrak{U}, \tag{11}$$

and

$$\Re\left\{\frac{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})'_m(z)}{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}\right\} \geq \frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\alpha+\beta)]^{\gamma} + 2[\Gamma(\beta)]^{\gamma}(e-1)}, \quad z \in \mathfrak{U}. \tag{12}$$

Proof. From part (ii) of Lemma 2, we observe that

$$1 + \sum_{n=1}^{\infty}(n+1)|A_n| \leq \frac{[\Gamma(\alpha+\beta)]^{\gamma} + 2[\Gamma(\beta)]^{\gamma}(e-1)}{[\Gamma(\alpha+\beta)]^{\gamma}},$$

where A_n as given in (7). This implies that

$$\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=1}^{\infty}(n+1)|A_n| \leq 1.$$

Consider

$$\begin{aligned}
&\frac{1+w(z)}{1-w(z)} \\
&= \left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\left[\frac{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})'_m(z)} - \frac{[\Gamma(\alpha+\beta)]^{\gamma} - 2[\Gamma(\beta)]^{\gamma}(e-1)}{[\Gamma(\alpha+\beta)]^{\gamma}}\right] \\
&= \frac{1 + \sum_{n=1}^{m}(n+1)A_n z^n + \left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=m+1}^{\infty}(n+1)A_n z^n}{1 + \sum_{n=1}^{m}(n+1)A_n z^n}.
\end{aligned}$$

Therefore,

$$|w(z)| \leq \frac{\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=m+1}^{\infty}(n+1)|A_n|}{2 - 2\sum_{n=1}^{m}(n+1)|A_n| - \left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=m+1}^{\infty}(n+1)|A_n|} \leq 1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}(n+1)|A_n| + \left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=m+1}^{\infty}(n+1)|A_n| \leq 1. \tag{13}$$

It suffices to show that the left hand side of (13) is bounded above by

$$\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=1}^{\infty}(n+1)|A_n|$$

which is equivalent to

$$\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma} - 2[\Gamma(\beta)]^{\gamma}(e-1)}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=1}^{m}(n+1)|A_n| \geq 0.$$

To prove the result (12), we write

$$\frac{1+w(z)}{1-w(z)}$$
$$=\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}+2[\Gamma(\beta)]^{\gamma}(e-1)}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\left[\frac{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})'_m(z)}{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}-\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\alpha+\beta)]^{\gamma}+2[\Gamma(\beta)]^{\gamma}(e-1)}\right]$$

where

$$|w(z)|\leq\frac{\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}+2[\Gamma(\beta)]^{\gamma}(e-1)}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=m+1}^{\infty}(n+1)|A_n|}{2-2\sum_{n=1}^{m}(n+1)|A_n|-\frac{[\Gamma(\alpha+\beta)]^{\gamma}-2[\Gamma(\beta)]^{\gamma}(e-1)}{2[\Gamma(\beta)]^{\gamma}(e-1)}\sum_{n=m+1}^{\infty}(n+1)|A_n|}\leq 1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}(n+1)|A_n|+\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=m+1}^{\infty}(n+1)|A_n|\leq 1. \tag{14}$$

It suffices to show that the left hand side of (14) is bounded above by

$$\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{2[\Gamma(\beta)]^{\gamma}(e-1)}\right)\sum_{n=1}^{\infty}(n+1)|A_n|.$$

The proof is complete. □

Theorem 3. *If* $\alpha\gamma\geq 1,\alpha^2\gamma>\beta$ *and* $[\Gamma(\alpha+\beta)]^{\gamma}\geq[\Gamma(\beta)]^{\gamma}(e-2)$, *then*

$$\Re\left\{\frac{\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}](z)}{(\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}])_m(z)}\right\}\geq\frac{[\Gamma(\alpha+\beta)]^{\gamma}-[\Gamma(\beta)]^{\gamma}(e-2)}{[\Gamma(\alpha+\beta)]^{\gamma}},\qquad z\in\mathfrak{U}, \tag{15}$$

and

$$\Re\left\{\frac{(\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}])_m(z)}{\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}](z)}\right\}\geq\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\alpha+\beta)]^{\gamma}+[\Gamma(\beta)]^{\gamma}(e-2)},\qquad z\in\mathfrak{U}. \tag{16}$$

where $\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}]$ *is the Alexander transform of* $\mathbb{F}^{(\gamma)}_{\alpha,\beta}$.

Proof. To prove (15), we consider from part (iii) of Lemma 2 that

$$1+\sum_{n=1}^{\infty}\frac{|A_n|}{n+1}\leq\frac{[\Gamma(\alpha+\beta)]^{\gamma}+[\Gamma(\beta)]^{\gamma}(e-2)}{[\Gamma(\alpha+\beta)]^{\gamma}},$$

or equivalently

$$\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=1}^{\infty}\frac{|A_n|}{n+1}\leq 1,$$

where A_n as given in (7). Now, we write

$$\frac{1+w(z)}{1-w(z)}=\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\left[\frac{\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}](z)}{(\mathbb{I}[\mathbb{F}^{(\gamma)}_{\alpha,\beta}])_m(z)}-\frac{[\Gamma(\alpha+\beta)]^{\gamma}-[\Gamma(\beta)]^{\gamma}(e-2)}{[\Gamma(\alpha+\beta)]^{\gamma}}\right]$$
$$=\frac{1+\sum_{n=1}^{m}\frac{|A_n|}{n+1}z^n+\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1}z^n}{1+\sum_{n=1}^{m}\frac{|A_n|}{n+1}z^n}. \tag{17}$$

Now, from (17), we can write

$$w(z) = \frac{\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}z^n}{2+2\sum_{n=1}^{m}\frac{A_n}{n+1}z^n+\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}z^n}.$$

Using the fact that $|w(z)| \leq 1$, we obtain

$$\frac{\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1}}{2-2\sum_{n=1}^{m}\frac{|A_n|}{n+1}-\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1}} \leq 1.$$

The last inequality is equivalent to

$$\sum_{n=1}^{m}\frac{|A_n|}{n+1}+\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=m+1}^{\infty}\frac{|A_n|}{n+1} \leq 1. \tag{18}$$

It suffices to show that the left hand side of (18) is bounded above by $\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}}{[\Gamma(\beta)]^{\gamma}(e-2)}\right)\sum_{n=1}^{\infty}\frac{|A_n|}{n+1}$, which is equivalent to

$$\left(\frac{[\Gamma(\alpha+\beta)]^{\gamma}-[\Gamma(\beta)]^{\gamma}(e-2)}{[\Gamma(\beta)]^{\gamma}(e-2}\right)\sum_{n=1}^{m}\frac{|A_n|}{n+1} \geq 0.$$

The proof of (16) is similar to the proof of Theorem 1 and is thus omitted. □

2. Special Cases

In this section, we obtain the following corollaries for special cases of m, α, β and γ for the functions listed in Remark 1.

Choosing $m = 0$, $\gamma = 1$, $\alpha = 2$ and $\beta = 1$ in Theorem 1, we obtain the following result.

Corollary 1. *The following inequalities hold true:*

$$\Re\{\cosh(\sqrt{z})\} \geq \frac{3-e}{2} \approx 0.140\,86, \qquad z \in \mathfrak{U},$$

and

$$\Re\left\{\frac{1}{\cosh(\sqrt{z})}\right\} \geq \frac{2}{1+e} \approx 0.537\,88, \qquad z \in \mathfrak{U}.$$

If we take $m = 0$, $\gamma = 1$, $\alpha = 2$ and $\beta = 2$ in Theorem 1 and Theorem 2, we obtain the following corollary.

Corollary 2. *The following inequalities hold true:*

$$\Re\left\{\frac{\sqrt{z}\sinh(\sqrt{z})}{z}\right\} \geq \frac{7-e}{6} \approx 0.713\,62, \qquad z \in \mathfrak{U},$$

$$\Re\left\{\frac{z}{\sqrt{z}\sinh(\sqrt{z})}\right\} \geq \frac{6}{5+e} \approx 0.777\,38, \qquad z \in \mathfrak{U},$$

$$\Re\left\{\frac{1}{2}\cosh\sqrt{z}+\frac{1}{2\sqrt{z}}\sinh\sqrt{z}\right\} \geq \frac{4-e}{3} \approx 0.427\,24, \quad z \in \mathfrak{U},$$

and

$$\Re\left\{\frac{1}{\frac{1}{2}\cosh\sqrt{z}+\frac{1}{2\sqrt{z}}\sinh\sqrt{z}}\right\} \geq \frac{3}{2+e} \approx 0.635\,82, \quad z \in \mathfrak{U}.$$

If we take $m = 0$, $\gamma = 1$, $\alpha = 2$ and $\beta = 3$ in Theorem 1 and Theorem 2, we obtain the following corollary.

Corollary 3. *The following inequalities hold true:*

$$\Re\left\{\frac{2[\cosh(\sqrt{z}) - 1]}{z}\right\} \geq \frac{13 - e}{12} \approx 0.856\,81, \qquad z \in \mathfrak{U},$$

$$\Re\left\{\frac{z}{2[\cosh(\sqrt{z}) - 1]}\right\} \geq \frac{12}{11 + e} \approx 0.874\,75, \qquad z \in \mathfrak{U},$$

$$\Re\left\{\frac{1}{\sqrt{z}}\sinh\sqrt{z}\right\} \geq \frac{7 - e}{6} \approx 0.713\,62, \quad z \in \mathfrak{U},$$

and

$$\Re\left\{\frac{\sqrt{z}}{\sinh\sqrt{z}}\right\} \geq \frac{6}{5 + e} \approx 0.777\,38, \quad z \in \mathfrak{U}.$$

Choosing $m = 0$, $\gamma = 1$, $\alpha = 2$ and $\beta = 4$ in Theorem 2, we obtain the following result.

Corollary 4. *The following inequalities hold true:*

$$\Re\left\{\frac{3\sqrt{z}\cosh\sqrt{z} - 3\sinh\sqrt{z}}{z\sqrt{z}}\right\} \geq \frac{11 - e}{10} \approx 0.828\,17, \quad z \in \mathfrak{U},$$

and

$$\Re\left\{\frac{z\sqrt{z}}{3\sqrt{z}\cosh\sqrt{z} - 3\sinh\sqrt{z}}\right\} \geq \frac{10}{9 + e} \approx 0.853\,37, \quad z \in \mathfrak{U}.$$

Remark 1. *Putting $m = 0$ in inequality (11), we obtain $\Re\left\{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'\right\} > 0$. In view of the Noshiro–Warschawski Theorem (see [17]), we have that the normalized Le Roy-type Mittag-Leffler function is univalent in $\mathfrak{U}$ for $[\Gamma(\alpha + \beta)]^{\gamma} \geq 2[\Gamma(\beta)]^{\gamma}(e - 1)$, where $\alpha\gamma \geq 1$ and $2\alpha^2\gamma > \beta$.*

3. Conclusions

In our present investigation, we have considered normalized Le Roy-type Mittag-Leffler function $\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)$ and determined lower bounds for

$$\Re\left\{\frac{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})_m(z)}\right\}, \Re\left\{\frac{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})_m(z)}{\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)}\right\}, \Re\left\{\frac{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})'_m(z)}\right\}, \Re\left\{\frac{(\mathbb{F}^{(\gamma)}_{\alpha,\beta})'_m(z)}{\left(\mathbb{F}^{(\gamma)}_{\alpha,\beta}(z)\right)'}\right\}.$$

Furthermore, we give lower bounds for $\Re\left\{\frac{\mathbb{I}\left[\mathbb{F}^{(\gamma)}_{\alpha,\beta}\right](z)}{(\mathbb{I}\left[\mathbb{F}^{(\gamma)}_{\alpha,\beta}\right])_m(z)}\right\}$ and $\Re\left\{\frac{(\mathbb{I}\left[\mathbb{F}^{(\gamma)}_{\alpha,\beta}\right])_m(z)}{\mathbb{I}\left[\mathbb{F}^{(\gamma)}_{\alpha,\beta}\right](z)}\right\}$, where $\mathbb{I}\left[\mathbb{R}_{\alpha,\beta,\gamma}\right]$ is the Alexander transform of $\mathbb{R}_{\alpha,\beta,\gamma}$. Several examples of the main results are also considered.

Making use of Le Roy-type Mittag-Leffler function $F^{(\gamma)}_{\alpha,\beta}(z)$ defined in (2) could inspire researchers to introduce and study new subclasses of analytic functions and obtain new properties of these classes, such as coefficient properties, distortion theorems, extreme points, etc. Furthermore, one can introduce new subclasses of bi-univalent functions and estimate the second and the third coefficients in the Taylor–Maclaurin expansions of functions belonging to these classes.

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