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A q-Analog of the Class of Completely Convex Functions and Lidstone Series

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Abstract: This paper introduces a q-analog of the class of completely convex functions. We prove specific properties, including that q-completely convex functions have convergent q-Lidstone series expansions. We also provide a sufficient and necessary condition for a real function to have an absolutely convergent q-Lidstone series expansion.

Keywords: quantum calculus; q-series; q-Lidstone polynomials; completely convex functions

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1. Introduction

In 1929, Lidstone [1] introduced a generalization of Taylor's theorem that approximates an entire function f in a neighborhood of two points instead of one. That is

$$f(x) = \sum_{n=0}^{\infty} \Big[f^{(2n)}(1)\Lambda_n(x) + f^{(2n)}(0)\Lambda_n(1-x) \Big], \tag{1}$$

where $\Lambda_n(x)$ is a unique polynomial of degree $2n+1$, and called a Lidstone polynomial. In [2], Whittaker proved that an entire function of an exponential type of less than π has a convergent Lidstone series expansion in any compact set of the complex plane. Buckholtz and Shaw [3] provided some conditions for (1) to hold. Other authors worked on this problem (see, e.g., [4–10]). They presented different sufficient and necessary conditions for the representation of functions by this series. We mention, in particular, the result of Widder [10]. He proved that if f is a real-valued function satisfying

$$(-1)^k f^{(2k)}(x) \geq 0 \quad (k \in \mathbb{N}_0) \tag{2}$$

in an interval of length greater than π, then it has a Lidstone series expansion (1) (such a function is known as completely convex). Furthermore, he defined the class of minimal completely convex functions, and then he proved that a real-valued function $f(x)$ could be expanded in an absolutely convergent Lidstone series if and only if it is the difference of two minimal completely convex functions.

Recently, the Lidstone expansion theorem was generalized in quantum calculus (as can be seen in [11–17]). The quantum calculus (Jackson calculus or q-calculus [18]) is an extension of the traditional calculus, and it has been used by many researchers in different branches of science and engineering (as can be seen in, e.g., [19–24]). It has a lot of applications in different mathematical areas such as orthogonal polynomials, number theory, hypergeometric functions, theory of finite differences, gamma function theory, Sobolev spaces, Bernoulli and Euler polynomials, operator theory, and quantum mechanics. For the basic definitions and notations applicable in the q-calculus, see Section 2.

In [11], Ismail and Mansour proved the following q-analog of the Lidstone expansion theorem.

Theorem 1. *Assume that the function $f(z)$ is an entire function of q^{-1}-exponential growth of order 1 and a finite type α less than ξ_1, or it is an entire function of q^{-1}-exponential growth of an order of less than 1. Then, $f(z)$ has a convergent q-Lidstone representation*

$$f(z) = \sum_{n=0}^{\infty} \Big[D_{q^{-1}}^{2n} f(1) A_n(z) - D_{q^{-1}}^{2n} f(0) B_n(z) \Big], \tag{3}$$

where $(A_n)_n$ and $(B_n)_n$ are the q-Lidstone polynomials defined, respectively, by the generating functions

$$\frac{E_q(zw) - E_q(-zw)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} A_n(z) w^{2n}, \tag{4}$$

$$\frac{E_q(zw)E_q(-w) - E_q(-zw)E_q(w)}{E_q(w) - E_q(-w)} = \sum_{n=0}^{\infty} B_n(z) \frac{w^n}{[n]_q!}. \tag{5}$$

Moreover, $A_0(z) = z$, $B_0(z) = 1 - z$, and for $n \in \mathbb{N}$, $A_n(z)$ and $B_n(z)$ satisfy the q-difference equation

$$D_{q^{-1}}^2 y_n(z) = y_{n-1}(z) \quad \textit{with} \quad y_n(0) = y_n(1) = 0. \tag{6}$$

In [16], AL-Towailb and Mansour proved that the condition

$$D_{q^{-1}}^n f(0) = o(\xi_1^n) \quad \text{as } n \to \infty \tag{7}$$

is both sufficient and necessary for expanding an entire function $f(z)$ in the q-Lidstone series

$$f(1)A_0(z) - f(0)B_0(z) + D_{q^{-1}}^2 f(1)A_1(z) - D_{q^{-1}}^2 f(0)B_1(z) + \ldots,$$

and we noted that Condition (7) is insufficient for the convergence of the following arrangement of the q-Lidstone series:

$$\sum_{n=0}^{\infty} D_{q^{-1}}^{2n} f(1) A_n(z) - \sum_{n=0}^{\infty} D_{q^{-1}}^{2n} f(0) B_n(z),$$

and not necessary for the convergence of (3). This paper aimed to obtain a sufficient and necessary condition for a real-valued function to have an absolutely convergent q-Lidstone series expansion (3). To achieve this aim, we introduced generalizations for the class of completely convex functions (2) on a closed interval of form $[0, a]$ $(a > 0)$, and the class of minimal completely convex functions on the interval $[0, 1]$. This paper is organized as follows. The following section gives the essential notions and basic definitions of q-calculus. Section 3 contains some properties and basic results on q-Lidstone polynomials, which we need in our investigation. In Section 4, we define a q-analog of the class of completely convex functions for the difference operator $D_{q^{-1}}$. Then, we study the relation of this class to a problem of the representation of functions by the q-Lidstone series. In Section 5, we provide a necessary and sufficient condition for a real function to have an absolutely convergent q-Lidstone series expansion.

2. Preliminaries

In this section, we recall some definitions, notations, and results in the q-calculus, which we need in our investigations (see [25]).

Throughout this paper, q is a positive number less than one, and we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 := \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}.$$

The sets A_q and A_q^* are defined by $A_q := \{q^n : n \in \mathbb{N}_0\}$ and $A_q^* := A_q \cup \{0\}$. For $a \in \mathbb{C}$, $n \in \mathbb{N}_0$,

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j), \quad (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty},$$

and the q-numbers $[n]_q$ and q-factorial $[n]_q!$ are defined by

$$[n]_q = \frac{1-q^n}{1-q}, \quad [n]_q! = \prod_{k=1}^{n}[k]_q.$$

Let $\mu \in \mathbb{C}$. A set $A \subset \mathbb{C}$ is called μ-geometric set if $\mu z \in A$ for any $z \in A$. If f is a function defined on a q-geometric set A, then Jackson's q-difference operator is defined by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \in A - \{0\}; \\ f'(0), & z = 0, \end{cases} \tag{8}$$

provided that f is differentiable at zero. Furthermore, Jackson [26] introduced the following q-integrals for a function f defined on a q-geometric set A:

$$\int_a^b f(t)\, d_q t := \int_0^b f(t) d_q t - \int_0^a f(t)\, d_q t \quad (a, b \in \mathbb{R}),$$

where

$$\int_0^z f(t)\, d_q t := (1-q)\sum_{n=0}^{\infty} z q^n f(zq^n),$$

provided that the series converges at $z = a$ and $z = b$.

Jackson's q-trigonometric functions $\mathrm{Sin}_q z$ and $\mathrm{Cos}_q z$ are defined by

$$\begin{aligned} \mathrm{Sin}_q z &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(2n+1)}}{(q;q)_{2n+1}}(z(1-q))^{2n+1}, \\ \mathrm{Cos}_q z &:= \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(2n-1)}}{(q;q)_{2n}}(z(1-q))^{2n}, \end{aligned} \tag{9}$$

where $E_q(\cdot)$ is one of Jackson's q-exponential function defined by

$$E_q(z) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(z(1-q))^n}{(q;q)_n} = (-z(1-q);q)_\infty \quad (z \in \mathbb{C}). \tag{10}$$

We use $\{\xi_k\}_{k\in\mathbb{N}}$ to denote the positive zeros of $\mathrm{Sin}_q z$ arranged in increasing order of magnitude. One can verify that $\mathrm{Sin}_q z$ has no zeroes on $|z| < q^{-3/2}$, i.e., the first positive zeros $\xi_1 > q^{-3/2}$.

Lemma 1. *For any $x \in [0,1]$, we have*

$$\mathrm{Sin}_q \xi_1 x \le \xi_1 x. \tag{11}$$

Proof. Let $f(x) = \xi_1 x - \mathrm{Sin}_q \xi_1 x$, $x \in [0,1]$. Then, $D_{q^{-1}} f(x) = \xi_1(1 - \mathrm{Cos}_q \xi_1 x) \ge 0$. Therefore, by using (8), we obtain

$$f(x) \le f(\frac{x}{q}) \quad (x \in [0,1]),$$

which implies $f(x) \ge \lim_{n\to\infty} f(q^n x) = 0$. Then, Inequality (11) holds. □

3. Some Results on q-Lidstone Polynomials

We start this section by recalling some properties of the q-Lidstone polynomials $A_n(x)$ and $B_n(x)$ from [14,16,17], for which we need to prove the main results.

Proposition 1 ([16]). *Let $\{\xi_k\}_{k\in\mathbb{N}}$ be the sequence of the positive zeros of $\mathrm{Sin}_q(x)$ and $m \in \mathbb{N}_0$. Then,*

$$\begin{aligned} (-1)^{n-1}A_n(x) &= \frac{2\mathrm{Sin}_q(\xi_1 x)}{\xi_1^{2n+1}\mathrm{Sin}'_q(\xi_1)} + \mathcal{O}(\xi_2^{-(2n+1)}); & (12)\\ (-1)^{n-1}B_n(x) &= \frac{\mathrm{Sin}_q(\xi_1 x)\mathrm{Cos}_q(\xi_1)}{(1-q)(\xi_1)^{2n+1}\mathrm{Sin}'_q(\xi_1)} + \mathcal{O}(\xi_1^{-2n}(2n)^{-m}), & (13)\end{aligned}$$

for a sufficiently large n.

Proposition 2 ([17]). *If $f \in C_q^{2n}([0,1])$, then*

$$f(x) = \sum_{m=0}^{n-1}\left[D_{q^{-1}}^{2m} f(1)A_m(x) - D_{q^{-1}}^{2m} f(0)B_m(x)\right] + \int_0^1 G_n(x,qt)D_{q^{-1}}^{2n} f(q^2t)\, d_qt, \quad (14)$$

where

$$G(x,t) = G_1(x,t) = \begin{cases} -qt(1-x), & 0 \le t < x \le 1; \\ -qx(1-t), & 0 \le x < t \le 1, \end{cases} \quad (15)$$

$$G_n(x,qt) = \int_0^1 G(x,qy)\, G_{n-1}(qy,qt)\, d_qy \quad (n \in \mathbb{N}). \quad (16)$$

Moreover,

$$\int_0^1 G_n(x,qt)\, d_qt = A_n(x) - B_n(x) \quad (n \in \mathbb{N}). \quad (17)$$

Remark 1 ([14]). *For $x \in [0,1]$ and $n \in \mathbb{N}_0$, we have*

$$(-1)^n A_n(x) \ge 0 \quad \text{and} \quad (-1)^{n-1}B_n(x) \ge 0. \quad (18)$$

Proposition 3. *Let ξ_1 be the smallest positive zero of $\mathrm{Sin}_q(x)$. Then, there exist some constants M_1 and M_2 and a positive integer n_0 such that the following inequalities hold*

$$0 \le (-1)^n A_n(x) \le \frac{M_1}{\xi_1^{2n}}; \quad (19)$$

$$0 \le (-1)^{n-1}B_n(x) \le \frac{M_2}{\xi_1^{2n}}, \quad (20)$$

for all $x \in [0,1]$ and $n \ge n_0$.

Proof. From (12), there is a positive real number C_1 and $n_0 \in \mathbb{N}$ such that

$$\left|(-1)^{n-1}A_n(x) - 2\frac{\mathrm{Sin}_q(\xi_1 x)}{\xi_1^{2n+1}\mathrm{Sin}'_q(\xi_1)}\right| \le \frac{C_1}{\xi_2^{2n}}, \quad (21)$$

for all $x \in [0,1]$ and $n \ge n_0$. Consequently,

$$0 \le (-1)^n A_n(x) \le \frac{C_1}{\xi_2^{2n}} - 2\frac{\mathrm{Sin}_q(\xi_1 x)}{\xi_1^{2n+1}\mathrm{Sin}'_q(\xi_1)}. \quad (22)$$

Note that $\xi_1 < \xi_2$ and $\text{Sin}_q(\xi_1 x)$ is bounded on $[0,1]$. Then, from (22), we obtain

$$\begin{aligned} 0 \leq (-1)^n A_n(x) &\leq \frac{C_1}{\xi_1^{2n}} + \frac{2}{\xi_1^{2n+1}} \left| \frac{\text{Sin}_q(\xi_1 x)}{\text{Sin}'_q(\xi_1)} \right| \\ &\leq \frac{C_1}{\xi_1^{2n}} + \frac{C_2}{\xi_1^{2n}} = \frac{M_1}{\xi_1^{2n}}. \end{aligned} \tag{23}$$

Similarly, we obtain (20) from (13). □

Proposition 4. *There exists a constant M such that*

$$0 \leq \int_0^1 (-1)^n G_n(x, qt)\, d_q t \leq \frac{M}{\xi_1^{2n}}.$$

Proof. The proof follows immediately from Equation (17) and Proposition 3. □

Proposition 5. *For any fixed point $x_0 \in (0,1)$ and sufficiently large n, there exist some constants M_1 and M_2 such that*

$$(-1)^n A_n(x_0) \geq \frac{M_1}{\xi_1^{2n}}; \tag{24}$$

$$(-1)^{n-1} B_n(x_0) \geq \frac{M_2}{\xi_1^{2n}}. \tag{25}$$

Proof. From (12), we obtain

$$(-1)^n A_n(x)\xi_1^{2n+1} = L(x) + \mathcal{O}\left(\left(\frac{\xi_1}{\xi_2}\right)^{2n+1}\right) \quad (n \to \infty),$$

where $L(x) = \frac{-2\text{Sin}_q(\xi_1 x)}{\text{Sin}'_q(\xi_1)}$. Notice, for any fixed $x_0 \in (0,1)$, $L(x_0) > 0$ and

$$\lim_{n\to\infty} (-1)^n A_n(x_0)\xi_1^{2n+1} = L(x_0).$$

This implies that the sequence $(-1)^n A_n(x_0)\xi_1^{2n+1}$ is bounded below by a positive number. I.e., (24) holds. Similarly, we obtain the Inequality (25) from (13). □

Now, using the previous results, we prove the following theorem.

Theorem 2. *If the series*

$$S = a_0 A_0(x) + b_0 B_0(x) + a_1 A_1(x) + b_1 B_1(x) + \dots \tag{26}$$

converges for a single value $x_0 \in (0,1)$, then the series $\sum_{n=0}^{\infty} (-1)^n \left[\frac{a_n + b_n}{\xi_1^{2n}}\right]$ is absolutely convergent.

Proof. Since the series (26) converges for $x_0 \in (0,1)$, we have

$$\lim_{n\to\infty} a_n A_n(x_0) = 0, \quad \lim_{n\to\infty} b_n B_n(x_0) = 0.$$

Then, from the inequalities (24) and (25), we obtain

$$a_n = \mathcal{O}(\xi_1^{2n}) \quad \text{and} \quad b_n = \mathcal{O}(\xi_1^{2n}). \tag{27}$$

From (12), (13), and (27), we conclude that the series

$$S_1 = \sum_{n=0}^{\infty}\left\{a_n\left[A_n(x_0) + \frac{2(-1)^n \mathrm{Sin}_q(\xi_1 x_0)}{\xi_1^{2n+1}\mathrm{Sin}_q'(\xi_1)}\right] + b_n\left[B_n(x_0) + \frac{(-1)^n \mathrm{Cos}_q\xi_1 \mathrm{Sin}_q(\xi_1 x_0)}{(1-q)\xi_1^{2n+1}\mathrm{Sin}_q'(\xi_1)}\right]\right\}$$

converges absolutely. This implies that $S_1 - S$ is also convergent. Notice that

$$\begin{aligned} S_1 - S &= \sum_{n=0}^{\infty}\left[\frac{2\mathrm{Sin}_q(\xi_1 x_0)}{\xi_1 \mathrm{Sin}_q'(\xi_1)}\frac{(-1)^n}{\xi_1^{2n}}a_n + \frac{\mathrm{Cos}_q\xi_1\mathrm{Sin}_q(\xi_1 x_0)}{(1-q)\xi_1\mathrm{Sin}_q'(\xi_1)}\frac{(-1)^n}{\xi_1^{2n}}b_n\right] \\ &> \frac{2\mathrm{Sin}_q(\xi_1 x_0)}{\xi_1\mathrm{Sin}_q'(\xi_1)}\sum_{n=0}^{\infty}\left[\frac{(-1)^n}{\xi_1^{2n}}a_n + \frac{(-1)^n}{\xi_1^{2n}}b_n\right]. \end{aligned}$$

Therefore, we obtain the result. □

4. A q-Analog of Completely Convex Function

In this section, by $C_q^{\infty}[0,a]$, we mean the space of all functions defined on $[0,a]$ such that $D_{q^{-1}}^n f(x)$ is defined and continuous at zero.

Definition 1. *A real-valued function f, defined on the interval $[0,a]$ $(a > 0)$, is said to be a q-completely convex function if $f \in C_q^{\infty}[0,a]$ and*

$$(-1)^n D_{q^{-1}}^{2n} f(aq^k) \geq 0 \quad (\text{for all } \{n,k\} \subset \mathbb{N}_0). \tag{28}$$

Example 1. *The functions $f(x) = Sin_q\xi_1 x$, defined in (9), are q-completely convex on the interval $[0,1]$. Indeed, one can verify that*

$$(-1)^n D_{q^{-1}}^{2n} f(x) = (-1)^n D_{q^{-1}}^{2n} Sin_q\xi_1 x = \xi_1^{2n}\, Sin_q(\xi_1\, x) > 0, \tag{29}$$

for all $x \in [0,1]$ and $n \in \mathbb{N}_0$.

In the following, we prove certain properties of q-completely convex functions.

Proposition 6. *If a function $f \in C_q^{\infty}[0,a]$ is q-completely convex, then*

$$(-1)^n D_{q^{-1}}^{2n} f(0) \geq 0 \quad (n \in \mathbb{N}_0). \tag{30}$$

Proof. The proof follows directly by taking the limit as $k \to \infty$ in (28) and using that $D_{q^{-1}}^{2n} f$ is continuous at zero for all $n \in \mathbb{N}_0$. □

Proposition 7. *Let $f \in C_q^{\infty}(0,1)$ be a q-completely convex function on $[0,1]$. Then, for a sufficiently large n, we have*

$$D_{q^{-1}}^{2n} f(0) = \mathcal{O}(\xi_1^{2n}); \tag{31}$$

$$D_{q^{-1}}^{2n} f(1) = \mathcal{O}(\xi_1^{2n}). \tag{32}$$

Proof. From Proposition 1 and Inequality (28), every term of (14) is non-negative. Therefore,

$$0 \leq A_n(x)\, D_{q^{-1}}^{2n} f(0) \leq f(x); \tag{33}$$

$$0 \leq (-B_n(x)) D_{q^{-1}}^{2n} f(1) \leq f(x) \quad (x \in [0,1];\ n \in \mathbb{N}_0). \tag{34}$$

Thus, by using (24) and (33), we obtain

$$0 \leq (-1)^n D_{q^{-1}}^{2n} f(0) \leq \frac{f(x_0)}{(-1)^n A_n(x_0)} \leq K \xi_1^{2n} \quad (n \to \infty),$$

for some constant $K > 0$ and $x_0 \in (0,1)$. Then, we have (31). Similarly, we obtain the asymptotic behavior in (32). $\square$

Proposition 8. *Let f be a q-completely convex function on $[0,1]$. Then, there exists a positive constant C such that for all $x \in A_q$*

$$0 \leq (-1)^n D_{q^{-1}}^{2n} f(x) \leq C\left(\frac{\xi_1}{x}\right)^{2n}, \tag{35}$$

where ξ_1 is the smallest positive zero of $\operatorname{Sin}_q(x)$.

Proof. If f is q-completely convex on $[0,1]$, then it is q-completely convex on $[0,x]$ for all $x \in A_q$. Consequently, the function $\widetilde{f}(t) := f(xt)$ is q-completely convex on $[0,1]$. Therefore, from Proposition (7), we have

$$0 \leq (-1)^n D_{q^{-1}}^{2n} \widetilde{f}(1) = (-1)^n x^{2n} D_{q^{-1}}^{2n} f(x) = \mathcal{O}(\xi_1^{2n}),$$

which is nothing else but (35). $\square$

Lemma 2. *Let $f(x)$ and $-D_{q^{-1}}^2 f(x)$ be non-negative on A_q^*, and continuous at 0. Assume that there exists a number $x_0 \in A_q$ such that $f(x_0) \leq \alpha$ ($\alpha \in \mathbb{R}$). Then,*

$$f(x) \leq \frac{(1+q)\alpha}{(1-q)x_0}, \quad \textit{for all} \quad x \in A_q^*.$$

Proof. First, let $x \in A_q^*$ and $x \geq x_0$. Then, by using the assumption $D_{q^{-1}}^2 f(x) \leq 0$, we have

$$\int_{x_0}^{x} D_q^2 f(\frac{t}{q^2})\, d_q t \leq 0.$$

Therefore, $D_q f(x) \leq D_q f(x_0)$, and

$$\int_{x_0}^{x} D_q f(t)\, d_q t \leq (x - x_0) D_q f(x_0) \quad (x \in A_q^*,\ x_0 \leq x). \tag{36}$$

Since $f(x) \geq 0$ on A_q^*, from (8) and Inequality (36), we obtain

$$f(x) \leq f(x_0) + \frac{(x-x_0)}{(1-q)x_0} f(x_0) = \frac{x - x_0 q}{(1-q)x_0} f(x_0) < \frac{\alpha}{(1-q)x_0}, \tag{37}$$

for all $x \in A_q^*$ and $x_0 \leq x$. Similarly, if $x \in A_q^*$ and $x < x_0$, then

$$f(x) \leq \frac{x_0 - x}{(1-q)x_0} f(qx_0) < \frac{f(qx_0)}{(1-q)x_0}. \tag{38}$$

On the other hand, since $D_{q^{-1}}^2 f(x) \leq 0$, we have

$$(1+q) f(qx) \geq q f(x) + f(q^2 x) \quad (x \in A_q^*).$$

Therefore, from the condition $f(x) \geq 0$, we obtain

$$(1+q)f(qx) \geq qf(\frac{x}{q}) + f(qx) > f(qx) \quad (x \in A_q^*). \tag{39}$$

So, from the inequalities (38) and (39), we obtain

$$f(x) < \frac{(1+q)\alpha}{(1-q)x_0} \quad (x \in A_q^*,\ x < x_0). \tag{40}$$

Hence, the relations (37) and (40) yield the required result. □

Corollary 1. *If $f \in C_q^\infty[0,1]$ is a q-completely convex function, then there exists a positive constant M such that*

$$0 \leq (-1)^n D_{q^{-1}}^{2n} f(x) \leq M\xi_1^{2n} \quad (n \in \mathbb{N}_0,\ x \in A_q^*). \tag{41}$$

Proof. The proof follows from Proposition 8 and Lemma 2 by taking $x_0 = 1$ and $M = \frac{1+q}{1-q}C$. □

Lemma 3. *If $f \in C_q^\infty[0,1]$ is a q-completely convex function on $[0,1]$, then there exists a constant $K > 0$ such that*

$$|D_{q^{-1}}^{n} f(x)| \leq K\xi_1^{n} \ (x \in A_q^*), \tag{42}$$

where ξ_1 is the smallest positive zero of $\mathrm{Sin}_q(z)$.

Proof. From Corollary 1, it suffices to prove (42) when n is an odd integer. We set $g(x) = (-1)^n D_{q^{-1}}^{2n} f(x)$. Since $f(x)$ is a q-completely convex on $0 \leq x \leq 1$, again from Corollary 1, there exists the constant $M > 0$ (independent of n) such that for all $x \in A_q^*$

$$0 \leq g(x) \leq M\xi_1^{2n}, \tag{43}$$

$$0 \leq -D_{q^{-1}}^{2} g(x) \leq M\xi_1^{2n+2}.$$

Therefore, for every $x \in A_q^* - \{1\}$, we have

$$0 \leq \int_{qx}^{q^2} -D_{q^{-1}}^{2} g(t)\, d_qt \leq Mq(q-x)\xi_1^{2n+2}.$$

So, by using the fundamental theorem of the q-calculus, we obtain

$$0 \leq (-1)^n D_{q^{-1}}^{2n+1} f(x) - (-1)^n D_{q^{-1}}^{2n+1} f(1) \leq M\xi_1^{2n+2},$$

and hence,

$$(-1)^n D_{q^{-1}}^{2n+1} f(1) \leq (-1)^n D_{q^{-1}}^{2n+1} f(x) \leq (-1)^n D_{q^{-1}}^{2n+1} f(1) + M\xi_1^{2n+2},$$

for all $x \in A_q^* - \{1\}$. Consequently,

$$|D_{q^{-1}}^{2n+1} f(x)| \leq |D_{q^{-1}}^{2n+1} f(1)| + M\xi_1^{2n+2}. \tag{44}$$

On the other hand, since $D_{q^{-1}}^{2} g(x) < 0$, one can verify that for all $x \in A_q^*$

$$(1+q)g(\frac{x}{q}) \geq g(x) + qg(\frac{x}{q^2}),$$

and then

$$qg(\frac{x}{q}) \leq (1+q)g(x) - g(qx) \quad (x \in A_q^*).$$

Thus, if $x = 1$, we obtain

$$\left|(D_{q^{-1}}^{2n} f)(\frac{1}{q})\right| = (-1)^n (D_{q^{-1}}^{2n} f)(\frac{1}{q}) \leq \frac{(1+q)}{q} \left|D_{q^{-1}}^{2n} f(1)\right|. \tag{45}$$

Hence, from (8), (43) and (45), we have

$$\left|D_{q^{-1}}^{2n+1} f(1)\right| = |D_{q^{-1}} g(1)| \leq \frac{|g(1)| + |g(1/q)|}{1/q - 1} \leq \frac{2q+1}{q} M \xi_1^{2n}. \tag{46}$$

However, $\xi_1 > q^{-3/2}$, this implies

$$\left|D_{q^{-1}}^{2n+1} f(1)\right| \leq \sqrt{q}(2q+1) M \xi_1^{2n+1}. \tag{47}$$

By substituting (47) in (44), we obtain

$$|D_{q^{-1}}^{2n+1} f(x)| \leq \sqrt{q}(2q+1) M \xi_1^{2n+1} + M \xi_1^{2n+2} \leq M_1 M \xi_1^{2n+1},$$

for all $n \in \mathbb{N}$ and $x \in A_q^*$, where $M_1 = \sqrt{q}(2q+1) + q^{-3/2}$.

Since $D_{q^{-1}}^{2n+1} f(x)$ is continuous at zero, then we obtain $D_{q^{-1}}^{2n+1} f(x) = \mathcal{O}(\xi_1^{2n+1})$ for a sufficiently large n. This completes the proof. □

Theorem 3. *Let $f \in C_q^\infty[0,1]$ be a q-completely convex on $[0,1]$. If f is analytic at zero, then the following q-Lidstone series expansion holds for all $x \in [0,1]$.*

$$f(x) = \sum_{n=0}^{\infty} \left[D_{q^{-1}}^{2n} f(1) A_n(x) - D_{q^{-1}}^{2n} f(0) B_n(x)\right]. \tag{48}$$

Moreover, $f(x)$ is the restriction of an entire function of q^{-1}-exponential growth of order 1 and a finite type less than ξ_1 and the expansion (48) holds for all x on the entire complex plane.

Proof. Since f is analytic at 0, there exists $0 < c < 1$ and the open interval $\Omega_c = (-c, c)$ such that $f(x)$ has the Maclaurin series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{D_{q^{-1}}^n f(0)}{[n]_q!} x^n \quad (x \in \Omega_c). \tag{49}$$

From Lemma 3, there exists a constant K such that

$$\left|f(x)\right| \leq \sum_{n=0}^{\infty} \left|q^{\frac{n(n-1)}{2}} \frac{D_{q^{-1}}^n f(0)}{[n]_q!} x^n\right| \leq K \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{(\xi_1 x)^n}{[n]_q!} = K\, E_q(\xi_1 x), \tag{50}$$

where $E_q(.)$ is Jackson's q-exponential function defined in (10). Notice that, by the known properties of $E_q(.)$ (see [11]), $E_q(x)$ is an entire function that has a q^{-1}-exponential growth of order 1, and it converges everywhere in the complex plane. Therefore, $f(x)$ is the restriction of an entire function of q^{-1}-exponential growth of order 1 and a finite type less than ξ_1. So, according to Theorem 1, we obtain the result. □

5. A q-Analog of Minimal Completely Convex Function

Definition 2. *A real-valued function $f \in C_q^\infty[0,1]$ is a minimal q-completely convex on $[0,1]$ if it is q-completely convex in the interval $[0,1]$, and if the function $g(x) = f(x) - \epsilon\, Sin_q \xi_1 x$ is not q-completely convex for any $\epsilon > 0$.*

For example, the function $f(x) = \mathrm{Sin}_q x$ is a minimal q-completely convex in $0 \le x \le 1$ while the function $f(x) = \mathrm{Sin}_q \xi_1 x$ is not because for any $0 < \epsilon < 1$ and $x \in (0,1)$,

$$(-1)^n D^{2n}_{q^{-1}} \left(\mathrm{Sin}_q \xi_1 x - \epsilon\, \mathrm{Sin}_q \xi_1 x\right) = (1-\epsilon)\xi_1^{2n} \mathrm{Sin}_q(\xi_1 x) > 0.$$

Theorem 4. *Let $n \in \mathbb{N}_0$, $(a_n)_n$ and $(b_n)_n$ be two sequences of non-negative integers. Assume that the series*

$$\sum_{n=0}^{\infty} \Big[(-1)^n a_n\, A_n(x) - (-1)^n b_n\, B_n(x)\Big]$$

converges to a function $f(x)$, $0 \le x \le 1$. Then, $f(x)$ is a minimal q-completely convex on the interval $[0,1]$.

Proof. From the assumption, we have

$$f(x) = \sum_{n=0}^{\infty} \Big[(-1)^n a_n\, A_n(x) - (-1)^n b_n\, B_n(x)\Big],\ 0 \le x \le 1. \tag{51}$$

Taking the q^{-1}-derivative for (51) $2k$ times and using (6), we obtain

$$\begin{aligned} (-1)^k D^{2k}_{q^{-1}} f(x) &= \sum_{n=k}^{\infty} (-1)^{n-k} a_n\, A_{n-k}(x) - (-1)^{n-k} b_n\, B_{n-k}(x) \\ &= \sum_{m=0}^{\infty} (-1)^m a_{m+k}\, A_m(x) - (-1)^m b_{m+k}\, B_m(x). \end{aligned} \tag{52}$$

From Proposition 5, since $(a_n)_n$ and $(b_n)_n$ are positive sequences, the right-hand side of Equation (52) is non-negative, and $f(x)$ is q-completely convex in $[0,1]$. On the other hand, from Proposition 3 and Equation (52), there exists a constant $M > 0$ such that

$$(-1)^k D^{2k}_{q^{-1}} f(x) \le M \sum_{m=0}^{\infty} \Big[a_{m+k} + b_{m+k}\Big]\, \xi_1^{-2m} = M\xi_1^{2k} \sum_{n=k}^{\infty} \frac{a_n + b_n}{\xi_1^{2n}}. \tag{53}$$

According to Theorem 2, the power series $T_k = \sum_{n=k}^{\infty} \frac{a_n + b_n}{\xi_1^{2n}}$ converges to zero as $k \to \infty$. Hence, for given $\epsilon > 0$ and $x_0 \in A_q$, there exists an integer $k_0 \in \mathbb{N}$ such that

$$MT_k - \epsilon\, \mathrm{Sin}_q(\xi_1 x_0) < 0\ (k \ge k_0).$$

This implies from (53) that the function

$$(-1)^k D^{2k}_{q^{-1}} \Big(f(x) - \epsilon\, \mathrm{Sin}_q(\xi_1 x)\Big) = (-1)^k D^{2k}_{q^{-1}} f(x) - \epsilon\, \xi_1^{2k}\, \mathrm{Sin}_q(\xi_1 x)$$

is negative at x_0. Therefore, the function f is a minimal q-completely convex in $[0,1]$. □

Theorem 5. *If $f(x)$ is a minimal q-completely convex function on $[0,1]$, then it can be expanded into a convergent q-Lidstone series:*

$$f(x) = f(1)A_0(x) - f(0)B_0(x) + D^2_{q^{-1}} f(1)A_1(x) - D^2_{q^{-1}} f(0)B_1(x) + \dots. \tag{54}$$

Proof. We denote by $S_n(x)$ the nth partial sum of the series (54). Then, from the hypothesis on $f(x)$ and Equation (14), we obtain

$$S_n(x) \le f(x) \quad (0 \le x \le 1,\ n \in \mathbb{N}_0).$$

Moreover, for each x, $S_n(x)$ is a non-decreasing function of n. Thus, $\lim_{n\to\infty} S_n(x)$ exists and tends towards some function. To prove the result, we prove that

$$\lim_{n\to\infty} S_n(x) = f(x) \quad (x \in [0,1]).$$

Suppose the contrary, and assume that for some $x_0 \in [0,1]$

$$f(x_0) - \lim_{n\to\infty} S_n(x_0) = \triangle > 0.$$

Then, by using Equation (14), we have

$$f(x_0) - S_{2n}(x_0) = \int_0^1 G_n(x_0, qt) D_{q^{-1}}^{2n} f(q^2 t)\, d_q t \geq \triangle \qquad (n \in \mathbb{N}). \tag{55}$$

Since $f(x)$ is a minimal q-completely convex function on $[0,1]$, then $f(x) - \epsilon\, \mathrm{Sin}_q \xi_1 x$ is not q-completely convex in $0 \leq x \leq 1$ for any $\epsilon > 0$. That is, there exists $n_0 \in \mathbb{N}$ and $t_0 \in A_q$,

$$(-1)^{n_0} D_{q^{-1}}^{2n_0} f(t_0) - \epsilon\, \xi_1^{2n_0}\, \mathrm{Sin}_q(\xi_1 t_0) < 0.$$

From Inequality (11), we have

$$(-1)^{n_0} D_{q^{-1}}^{2n_0} f(t_0) < \epsilon\, \xi_1^{2n_0+1}\, t_0.$$

By applying Lemma 2 on the function $g(x) = (-1)^{n_0} D_{q^{-1}}^{2n_0} f(x)$, we obtain

$$(-1)^{n_0} D_{q^{-1}}^{2n_0} f(t) \leq \frac{1+q}{1-q}\, \epsilon\, \xi_1^{2n_0+1} \quad (t \in A_q).$$

Therefore, by choosing $\epsilon < \frac{1-q}{(1+q)\xi_1 M} \triangle$, where M is the constant of Proposition 4, we obtain

$$0 \leq \int_0^1 G_{n_0}(x_0, qt) D_{q^{-1}}^{2n_0} f(q^2 t)\, d_q t < \triangle,$$

which contradicts Inequality (55), and then the result is proved. □

The following theorem is the main result of this section.

Theorem 6. *A real function $f(x)$ can be represented by an absolutely convergent q-Lidstone series if and only if it is the difference of two minimal q-completely convex functions on $[0,1]$.*

Proof. First, assume that $f(x) = g(x) - h(x)$, where $g(x)$ and $h(x)$ are both minimal q-completely convex functions on $[0,1]$. According to Theorem 5, we have

$$g(x) = \sum_{n=0}^{\infty} \Big[D_{q^{-1}}^{2n} g(1) A_n(x) - D_{q^{-1}}^{2n} g(0) B_n(x) \Big], \tag{56}$$

$$h(x) = \sum_{n=0}^{\infty} \Big[D_{q^{-1}}^{2n} h(1) A_n(x) - D_{q^{-1}}^{2n} h(0) B_n(x) \Big]. \tag{57}$$

Notice that each series only has positive terms. Thus, by subtracting (57) from (56), we obtain an absolutely convergent q-Lidstone series whose sum is $f(x)$.

Conversely, assume that $f(x)$ can be represented by an absolutely convergent q-Lidstone series

$$f(x) = \sum_{n=0}^{\infty} \Big[D_{q^{-1}}^{2n} f(1) A_n(x) - D_{q^{-1}}^{2n} f(0) B_n(x) \Big]. \tag{58}$$

Set $a_n = D^{2n}_{q^{-1}} f(1)$, $b_n = D^{2n}_{q^{-1}} f(0)$, and

$$g(x) = \sum_{n=0}^{\infty} \Big[(-1)^n \{|a_n| - (-1)^n a_n\} A_n(x) + (-1)^{n+1} \{|b_n| - (-1)^n b_n\} B_n(x)\Big], \tag{59}$$

$$h(x) = \sum_{n=0}^{\infty} \Big[(-1)^n |a_n| A_n(x) + (-1)^{n+1} |b_n| B_n(x)\Big]. \tag{60}$$

Since series in (58) is absolutely convergent, then the two series in (59) and (60) both converge. Furthermore, note that every term of these series is positive. Hence, by using Theorem 4, $g(x)$ and $h(x)$ are minimal q-completely convex functions on $[0,1]$. Since $f(x) = h(x) - g(x)$, the proof is complete. □

6. Conclusions

We introduced the class of q-completely convex functions in the interval $[0, a]$, with the functions satisfying the inequality

$$(-1)^n D^{2n}_{q^{-1}} f(aq^k) \geq 0 \quad (\{n, k\} \subset \mathbb{N}_0)).$$

This class of functions is a generalization of the class of completely convex functions introduced by Widder [10]. First, we presented some properties of a q-completely convex function, and then we proved that such a function could be expanded in a convergent q-Lidstone series:

$$f(x) = \sum_{n=0}^{\infty} \Big[D^{2n}_{q^{-1}} f(1) A_n(x) - D^{2n}_{q^{-1}} f(0) B_n(x)\Big].$$

Furthermore, we obtained a necessary and sufficient condition for a function $f(x)$ to have an absolutely convergent q-Lidstone series expansion by introducing the class of minimal q-completely convex functions.

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