axioms

MDPI

Article

On Bochner Flat Kähler B-Manifolds [†]

Cornelia-Livia Bejan [1,‡], Galia Nakova [2,‡] and Adara M. Blaga [3,*]

1 Department of Mathematics, "Gh. Asachi" Technical University of Iaşi, B-dul Carol I, nr. 11, 700506 Iaşi, Romania; bejanliv@yahoo.com
2 Department of Algebra and Geometry, Faculty of Mathematics and Informatics, "St. Cyril and St. Methodius" University of Veliko Tarnovo, 2 Teodosii Tarnovski Str., 5003 Veliko Tarnovo, Bulgaria; gnakova@gmail.com or g.nakova@ts.uni-vt.bg
3 Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, B-dul V. Pârvan, nr. 4-6, 300223 Timişoara, Romania
* Correspondence: adarablaga@yahoo.com or adara.blaga@e-uvt.ro
† Dedicated to the memory of Prof. Dr. Krishan Lal Duggal (1929–2022).
‡ These authors contributed equally to this work.

Abstract: We obtain on a Kähler B-manifold (i.e., a Kähler manifold with a Norden metric) some corresponding results from the Kählerian and para-Kählerian context concerning the Bochner curvature. We prove that such a manifold is of constant totally real sectional curvatures if and only if it is a holomorphic Einstein, Bochner flat manifold. Moreover, we provide the necessary and sufficient conditions for a gradient Ricci soliton or a holomorphic η-Einstein Kähler manifold with a Norden metric to be Bochner flat. Finally, we show that a Kähler B-manifold is of quasi-constant totally real sectional curvatures if and only if it is a holomorphic η-Einstein, Bochner flat manifold.

Keywords: Bochner curvature tensor; gradient Ricci soliton; Kähler manifold; Norden metric; holomorphic Einstein manifold

MSC: 32Q15; 53B35

check for updates

Citation: Bejan, C.-L.; Nakova, G.; Blaga, A.M. On Bochner flat Kähler B-manifolds. *Axioms* **2023**, *12*, 336. https://doi.org/10.3390/axioms12040336

Academic Editor: Mica Stankovic

Received: 8 March 2023
Accepted: 28 March 2023
Published: 30 March 2023

1. Introduction

Any Riemannian manifold of constant sectional curvature is called a space form. As it is well known, the only complete, connected smooth space forms are the Euclidean spaces, the spheres, and the hyperbolic spaces. This classification shows that the condition of a constant sectional curvature on a Riemannian manifold is a strong one. This is why a generalization of the space forms was given for the Riemannian manifolds endowed with a non-zero vector field ξ. More precisely, Riemannian manifolds endowed with a non-zero vector field ξ of quasi-constant sectional curvature generalize the space forms because they satisfy a relaxed condition, meaning that not necessarily all planes have the same curvature, but only those which make a certain angle with ξ; see [1] and the references therein. In the same way as the space forms are characterized in terms of their Riemannian curvature $(3,1)$-tensor field R, Ganchev and Mihova characterized in [2] (see also the references therein) the Riemannian manifolds endowed with a non-zero vector field ξ, which are of quasi-constant sectional curvature in terms of R. Later on, a corresponding result was obtained in the Kählerian context independently by Bejan, Benyounes in [3] and by Ganchev and Mihova in [4]. In the para-Kähler context, a corresponding characterization result was given by Bejan and Ferrara in [5]. The present paper aims to fill a gap in the literature by studying a corresponding notion in the case of Kähler B-metric manifolds (introduced by Norden, which are also called Kähler manifolds with a Norden metric). In this context, we study the manifolds of quasi-constant totally real sectional curvatures.

The Bochner curvature tensor, introduced in 1949 by Bochner [6,7], plays a similar role in Kähler geometry to the Weyl curvature tensor on Riemannian manifolds. For instance, analogously to the fact that a Riemannian manifold is of constant sectional curvature if and only if it is Einstein with vanishing Weil curvature tensor, a Kähler manifold is of constant holomorphic sectional curvature if and only if it is Einstein and Bochner flat. The local expression of the Bochner curvature tensor, in real coordinates, was given by Tachibana in [8]. In [9], Chen and Yano characterized the Kähler manifolds with vanishing Bochner curvature tensor. Furthermore, Blair showed that a totally geodesic and totally real submanifold of a Kähler manifold of real dimension ≥ 8 with vanishing Bochner curvature tensor is conformally flat [10].

The Bochner flatness, studied extensively on Kähler manifolds, was not so much researched in the context of Kähler B-metric manifolds, which is the task of the present paper. In the context of complex geometry, we study some classes of Kähler B-metric manifolds with a special view towards the Bochner curvature. We prove that a Kähler B-metric manifold is of constant totally real sectional curvatures if and only if it is a holomorphic Einstein, Bochner flat manifold. We also find the necessary and sufficient conditions for a gradient Ricci soliton or a holomorphic η-Einstein Kähler B-metric manifold to be Bochner flat. In this context, the manifolds of quasi-constant totally real sectional curvatures are characterized.

2. B-Manifolds

This section provides some preliminaries, containing some basic notions, formulas, and notations that we use later on.

From now on, by (M^n, J), we denote an almost complex manifold of real dimension $n = 2m$.

Definition 1 ([11]). *A manifold (M^{2m}, J, g) endowed with an almost complex structure J and a semi-Riemannian metric g of neutral signature (m, m) is called a B-manifold (or a Kähler manifold with a Norden metric, or a Kähler B-metric manifold) if J is skew-compatible with respect to g (i.e., $g(JX, JY) = -g(X, Y)$ for any $X, Y \in \Gamma(TM)$), and J is parallel with respect to the Levi-Civita connection ∇ of g (i.e., $\nabla J = 0$).*

Let R be the Riemannian curvature tensor field defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \ \forall\, X, Y \in \Gamma(TM).$$

The Ricci curvature tensor field S defined by

$$S(X, Y) = \mathrm{Trace}\{Z \mapsto R(Z, X)Y\}, \ \forall\, X, Y \in \Gamma(TM),$$

yields the Ricci operator Q defined by

$$g(QX, Y) = S(X, Y), \ \forall\, X, Y \in \Gamma(TM).$$

We denote the scalar and holomorphic scalar curvatures, respectively, by

$$r = \mathrm{Trace}\, Q, \ \ r^* = \mathrm{Trace}(QJ). \tag{1}$$

Remark 1. *For any $X, Y \in \Gamma(TM)$, we have*

$$S(JX, JY) = -S(X, Y), \ \ QJ = JQ.$$

We denote by $X \wedge_A Y$ the operator acting on $\Gamma(TM)$ by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \ \forall\, X, Y, Z \in \Gamma(TM),$$

where A is a symmetric $(0, 2)$-tensor field on M.

In [12], the Bochner tensor field B on a B-manifold (M^n, g, J), $n \geq 6$, is defined by

$$B(X,Y) = R(X,Y) - V(X,Y), \ \forall\, X, Y \in \Gamma(TM),$$

where

$$V(X,Y) = \frac{1}{n-4}[X \wedge_g QY + QX \wedge_g Y - JX \wedge_g QJY - QJX \wedge_g JY] \\ - \frac{1}{(n-2)(n-4)}[r(X \wedge_g Y - JX \wedge_g JY) - r^*(JX \wedge_g Y + X \wedge_g JY)].$$

Let us recall that on a (semi-)Riemannian manifold (N^k, g), the gradient, Hessian, divergence, and Laplace operator are, respectively, given by

$$g(\nabla f, X) = df(X), \ \ \mathrm{Hess}(f) = \nabla\nabla f, \ \ \mathrm{div}(X) = \sum_{i=1}^{k} \epsilon_i g(\nabla_{e_i} X, e_i), \ \ \Delta f = \mathrm{div}(\nabla f)$$

for any smooth function f and any vector field X on N, where $\{e_i\}_{1 \leq i \leq k}$ is an orthonormal basis on N with respect to g and $\epsilon_i = \mathrm{sgn}\, g(e_i, e_i)$.

Moreover, a function f is called harmonic if $\Delta f = 0$ and a vector field X is called solenoidal if $\mathrm{div}(X) = 0$.

3. Holomorphic Einstein Condition

Corresponding to the Einstein condition in the Riemannian context and the Kähler–Einstein condition in the Kählerian case, we recall the following notion in the context of a B-manifold.

Definition 2. *A B-manifold (M, J, g) is called holomorphic [13] (respectively, almost [14,15]) Einstein if there exist two real constants (respectively, two smooth functions) α, β such that*

$$S(X,Y) = \alpha g(X,Y) + \beta g(JX,Y), \ \forall\, X, Y \in \Gamma(TM). \tag{2}$$

In particular, if α is a constant and $\beta = 0$, then M is Einstein.

Remark 2. *Obviously, any holomorphic Einstein B-manifold is almost Einstein.*

Moreover, on an n-dimensional almost Einstein B-manifold, the scalar and holomorphic scalar curvatures, given by (1), are, respectively,

$$r = n\alpha, \ \ r^* = -n\beta. \tag{3}$$

In particular, if M is an Einstein B-manifold, then the scalar and holomorphic scalar curvatures are, respectively,

$$r = n\alpha, \ \ r^* = 0.$$

For any $X, Y, Z, W \in \Gamma(TM)$, one sets

$$R(X,Y,Z,W) = g(R(X,Y)Z,W), \ \ B(X,Y,Z,W) = g(B(X,Y)Z,W).$$

Proposition 1. *Let (M^n, J, g) be a B-manifold. If M is almost Einstein, then*

$$
\begin{aligned}
B(X,Y,Z,W) = R(X,Y,Z,W) & \\
& - \frac{r}{n(n-2)}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
& - g(JX,W)g(JY,Z) + g(JX,Z)g(JY,W)] \\
& + \frac{r^*}{n(n-2)}[g(JX,W)g(Y,Z) - g(X,Z)g(JY,W) \\
& + g(X,W)g(JY,Z) - g(JX,Z)g(Y,W)], \ \forall\, X,Y,Z,W \in \Gamma(TM). \quad (4)
\end{aligned}
$$

Proof. From (2), we have

$$
\begin{aligned}
B(X,Y,Z,W) = R(X,Y,Z,W) & \\
& - \frac{2}{n(n-4)}\{r[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
& - g(JX,W)g(JY,Z) + g(JX,Z)g(JY,W)] \\
& - r^*[g(X,W)g(JY,Z) - g(X,Z)g(JY,W) \\
& + g(JX,W)g(Y,Z) - g(JX,Z)g(Y,W)]\} \\
& + \frac{1}{(n-2)(n-4)}\{r[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
& - g(JX,W)g(JY,Z) + g(JX,Z)g(JY,W)] \\
& - r^*[g(X,W)g(JY,Z) - g(X,Z)g(JY,W) \\
& + g(JX,W)g(Y,Z) - g(JX,Z)g(Y,W)]\},
\end{aligned}
$$

for any $X,Y,Z,W \in \Gamma(TM)$, which, by a straightforward computation, yields (4). □

Definition 3 ([16]). *Let (M, J, g) be a B-manifold.*

(i) A non-degenerate plane $\pi \subset T_pM$, $p \in M$, is said to be a totally real section of T_pM if $J\pi \perp \pi$.

(ii) The manifold M is said to be of constant totally real sectional curvatures a and b if, for any non-degenerate totally real section π generated by X and Y, its sectional curvatures,

$$
a = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - [g(X,Y)]^2}, \quad b = \frac{R(X,Y,Y,JX)}{g(X,X)g(Y,Y) - [g(X,Y)]^2},
$$

are two constant real numbers.

Theorem 1 ([16]). *Let (M, J, g) be a B-manifold. Then, M is of constant totally real sectional curvatures a and b if and only if*

$$
\begin{aligned}
R(X,Y,Z,W) = a[& g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
& - g(JX,W)g(JY,Z) + g(JX,Z)g(JY,W)] \\
& + b[g(X,JW)g(Y,Z) - g(X,JZ)g(Y,W) \\
& + g(X,W)g(Y,JZ) - g(X,Z)g(Y,JW)], \ \forall\, X,Y,Z,W \in \Gamma(TM). \quad (5)
\end{aligned}
$$

Remark 3 ([11]). *On a B-manifold (M^{2m}, J, g), a basis $\{e_i = k_i, e_{m+i} = Jk_i\}_{1 \le i \le m}$ is called an orthonormal J-basis with respect to g if $g(k_i, Jk_i) = 0$ and $g(k_i, k_j) = \epsilon_i \delta_{ij}$, where δ_{ij} is the Kronecker delta and $\epsilon_i \in \{\pm 1\}$. Additionally, in [11] it is proved that such a basis always exists on any local chart of a B-manifold.*

Proposition 2. *Let (M^n, J, g) be a B-manifold. If M is of constant totally real sectional curvatures, then M is holomorphic Einstein.*

Proof. If in (5) we take the trace, we obtain

$$\begin{aligned} S(X,W) &= \sum_{i=1}^{n} \epsilon_i R(X,e_i,e_i,W) \\ &= a[\sum_{i=1}^{n} \epsilon_i g(X,W)g(e_i,e_i) - \sum_{i=1}^{n} \epsilon_i g(X,e_i)g(e_i,W) \\ &- \sum_{i=1}^{n} \epsilon_i g(JX,W)g(Je_i,e_i) + \sum_{i=1}^{n} \epsilon_i g(JX,e_i)g(Je_i,W)] \\ &+ b[\sum_{i=1}^{n} \epsilon_i g(X,JW)g(e_i,e_i) - \sum_{i=1}^{n} \epsilon_i g(X,Je_i)g(e_i,W) \\ &+ \sum_{i=1}^{n} \epsilon_i g(X,W)g(e_i,Je_i) - \sum_{i=1}^{n} \epsilon_i g(X,e_i)g(e_i,JW)] \\ &= a[ng(X,W) - g(X,W) + g(JX,JW)] \\ &+ b[ng(X,JW) - g(X,JW) - g(X,JW)] \\ &= (n-2)[ag(X,W) + bg(JX,W)], \end{aligned}$$

for any $X, W \in \Gamma(TM)$, where $\{e_i\}_{1 \le i \le n}$ is an orthonormal basis on M. □

Remark 4. *If (M^n, J, g) is a B-manifold of constant totally real sectional curvatures a and b, then* (2) *is satisfied for*

$$\alpha = a(n-2); \;\; \beta = b(n-2).$$

Moreover, from (3)*, it follows that*

$$a = \frac{r}{n(n-2)}; \;\; b = -\frac{r^*}{n(n-2)}. \tag{6}$$

By direct computations, in which we use Proposition 1, Theorem 1, Proposition 2, and Remark 4, we obtain the following:

Theorem 2. *If (M, J, g) is a B-manifold, then the following statements are equivalent:*

(i) *M is of constant totally real sectional curvatures;*
(ii) *M is a holomorphic Einstein, Bochner flat manifold.*

Proposition 3. *Let (M^n, J, g) be a B-manifold. If M is Einstein, then M is Bochner flat if and only if*

$$\begin{aligned} R(X,Y)Z = \frac{r}{n(n-2)}&[g(Y,Z)X - g(X,Z)Y \\ &- g(JY,Z)JX + g(JX,Z)JY], \;\; \forall\, X,Y,Z \in \Gamma(TM). \end{aligned} \tag{7}$$

Proof. From (2), we have

$$\begin{aligned} B(X,Y,Z,W) = R&(X,Y,Z,W) \\ &- \frac{2}{n(n-4)}\{r[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ &- g(JX,W)g(JY,Z) + g(JX,Z)g(JY,W)] \\ &- r^*[g(X,W)g(JY,Z) - g(X,Z)g(JY,W) \\ &+ g(JX,W)g(Y,Z) - g(JX,Z)g(Y,W)]\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(n-2)(n-4)}\{r[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\
&-g(JX,W)g(JY,Z)+g(JX,Z)g(JY,W)]\\
&-r^*[g(X,W)g(JY,Z)-g(X,Z)g(JY,W)\\
&+g(JX,W)g(Y,Z)-g(JX,Z)g(Y,W)]\}\\
&=-\frac{2r}{n(n-4)}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\
&-g(JX,W)g(JY,Z)+g(JX,Z)g(JY,W)]\\
&+\frac{r}{(n-2)(n-4)}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\
&-g(JX,W)g(JY,Z)+g(JX,Z)g(JY,W)],
\end{aligned}
$$

for any $X, Y, Z, W \in \Gamma(TM)$, which, by a straightforward computation, yields (7). □

Definition 4 ([17]). *A (semi-)Riemannian manifold (N, g) is called a gradient Ricci soliton if there exist a smooth function f on N and a constant $\alpha \in \mathbb{R}$ such that*

$$\operatorname{Hess}(f)(X,Y)+S(X,Y)=\alpha g(X,Y),\ \ \forall\ X,Y\in\Gamma(TM). \tag{8}$$

Proposition 4. *Let (M^n, J, g) be a B-manifold. If M is a gradient Ricci soliton, then M is Bochner flat if and only if*

$$
\begin{aligned}
R(X,Y,Z,W)&=T(X,Y,Z,W)-T(X,Y,W,Z)\\
&\quad-T(JX,JY,Z,W)+T(JX,JY,W,Z),\ \ \forall\ X,Y,Z,W\in\Gamma(TM),
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
T(X,Y,Z,W)&=\frac{2\alpha(n-2)-r}{(n-2)(n-4)}g(X,W)g(Y,Z)+\frac{r^*}{(n-2)(n-4)}g(Y,Z)g(JX,W)\\
&\quad-\frac{1}{n-4}[g(X,W)\operatorname{Hess}(f)(Y,Z)+g(Y,Z)\operatorname{Hess}(f)(X,W)].
\end{aligned}
$$

Proof. From (8), we have

$$
\begin{aligned}
B(X,Y,Z,W)&=R(X,Y,Z,W)\\
&\quad-\frac{2\alpha(n-2)-r}{(n-2)(n-4)}[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\
&\quad-g(JX,W)g(JY,Z)+g(JX,Z)g(JY,W)]\\
&\quad-\frac{r^*}{(n-2)(n-4)}[g(JX,W)g(Y,Z)-g(JX,Z)g(Y,W)\\
&\quad+g(X,W)g(JY,Z)-g(X,Z)g(JY,W)]\\
&\quad+\frac{1}{n-4}[g(X,W)\operatorname{Hess}(f)(Y,Z)-g(X,Z)\operatorname{Hess}(f)(Y,W)\\
&\quad+g(Y,Z)\operatorname{Hess}(f)(X,W)-g(Y,W)\operatorname{Hess}(f)(X,Z)\\
&\quad-g(JX,W)\operatorname{Hess}(f)(JY,Z)+g(JX,Z)\operatorname{Hess}(f)(JY,W)\\
&\quad-g(JY,Z)\operatorname{Hess}(f)(JX,W)+g(JY,W)\operatorname{Hess}(f)(JX,Z)],
\end{aligned}
\tag{10}
$$

for any $X, Y, Z, W \in \Gamma(TM)$, which, by a straightforward computation, yields (9). □

Remark 5. *On an n-dimensional B-manifold which is a gradient Ricci soliton, the scalar and holomorphic scalar curvatures are, respectively, given by*

$$r = n\alpha - \Delta f, \;\; r^* = -\operatorname{div}(J(\nabla f)). \tag{11}$$

Corollary 1. *Let (M^n, J, g) be a B-manifold. If M is a gradient Ricci soliton with trace-free Bochner tensor (in particular, Bochner flat), then $r^* = 0$ and $J(\nabla f)$ is a solenoidal vector field.*

Proof. If in (10) we take the trace, we obtain

$$\begin{aligned} S(X,W) = {} & \frac{2\alpha(n-2) - r\Delta f}{n-4} g(X,W) + \frac{2r^*}{n-4} g(JX,W) \\ & - \frac{n-3}{n-4}\operatorname{Hess}(f)(X,W) - \frac{1}{n-4}\operatorname{Hess}(f)(JX,JW), \end{aligned}$$

for any $X, W \in \Gamma(TM)$, which, from (11) and (8), we deduce

$$2r^* g(JX,W) + \operatorname{Hess}(f)(X,W) + \operatorname{Hess}(f)(JX,JW) = 0. \tag{12}$$

If in (12) we replace X with JX and W with JW, we obtain

$$-2r^* g(JX,W) + \operatorname{Hess}(f)(JX,JW) + \operatorname{Hess}(f)(X,W) = 0,$$

from which we obtain the conclusion. □

4. Holomorphic η-Einstein Condition

The results obtained in the previous section are extended in this section in a new context given by the following:

Definition 5. *Let (M, J, g) be a B-manifold endowed with a non-zero 1-form η. We say that M is a holomorphic η-Einstein manifold if there exist three real constants α, β, and γ such that*

$$S(X,Y) = \alpha g(X,Y) + \beta g(JX,Y) + \gamma[\eta(X)\eta(Y) + \eta(JX)\eta(JY)], \;\; \forall\, X, Y \in \Gamma(TM), \tag{13}$$

or, equivalently,

$$QX = \alpha X + \beta JX + \gamma[\eta(X)\xi + \eta(JX)J\xi], \;\; \forall\, X \in \Gamma(TM),$$

where ξ is the dual vector field of η with respect to g, i.e., $g(\xi, X) = \eta(X)$.

Remark 6. *On an n-dimensional holomorphic η-Einstein B-manifold, the scalar and holomorphic scalar curvatures are given by (3).*

Proposition 5. *Let (M^n, J, g) be a B-manifold. If M is holomorphic η-Einstein, then M is Bochner flat if and only if*

$$\begin{aligned} R(X,Y,Z,W) = {} & U(X,Y,Z,W) - U(X,Y,W,Z) \\ & - U(JX,JY,Z,W) + U(JX,JY,W,Z), \;\; \forall\, X,Y,Z,W \in \Gamma(TM), \end{aligned} \tag{14}$$

where

$$\begin{aligned} U(X,Y,Z,W) = {} & \frac{1}{n(n-2)}[rg(X,W)g(Y,Z) - r^* g(X,W)g(JY,Z)] \\ & + \frac{\gamma}{n-4}\Big[\Big(\eta(X)\eta(W) + \eta(JX)\eta(JW)\Big)g(Y,Z) \\ & + \Big(\eta(Y)\eta(Z) + \eta(JY)\eta(JZ)\Big)g(X,W)\Big]. \end{aligned}$$

Proof. From (13), we have

$$
\begin{aligned}
B(X,Y,Z,W) = R(X,Y,Z,W)&\\
&-\frac{1}{n(n-2)}\{r[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\\
&-g(JX,W)g(JY,Z)+g(JX,Z)g(JY,W)]\\
&-r^*[g(X,W)g(JY,Z)-g(X,Z)g(JY,W)\\
&+g(JX,W)g(Y,Z)-g(JX,Z)g(Y,W)]\}\\
&-\frac{\gamma}{(n-4)}[\eta(Y)\eta(Z)g(X,W)+\eta(JY)\eta(JZ)g(X,W)\\
&-\eta(Y)\eta(W)g(X,Z)-\eta(JY)\eta(JW)g(X,Z)\\
&+\eta(X)\eta(W)g(Y,Z)+\eta(JX)\eta(JW)g(Y,Z)\\
&-\eta(X)\eta(Z)g(Y,W)-\eta(JX)\eta(JZ)g(Y,W)\\
&-\eta(JY)\eta(Z)g(JX,W)+\eta(Y)\eta(JZ)g(JX,W)\\
&+\eta(JY)\eta(W)g(JX,Z)-\eta(Y)\eta(JW)g(JX,Z)\\
&-\eta(JX)\eta(W)g(JY,Z)+\eta(X)\eta(JW)g(JY,Z)\\
&+\eta(JX)\eta(Z)g(JY,W)-\eta(X)\eta(JZ)g(JY,W)],
\end{aligned}
$$

for any $X,Y,Z,W \in \Gamma(TM)$, which, by a straightforward computation, yields (14). □

We recall that on a semi-Riemannian manifold (N,g), a vector field ξ is called spacelike (respectively, timelike) if

$$g(\xi,\xi) = 1 \tag{15}$$

(respectively, $g(\xi,\xi) = -1$), [18].

In our context, we introduce the following notion.

Definition 6. *Let (M,J,g) be a B-manifold endowed with a unit spacelike vector field ξ. We say that M is of quasi-constant totally real sectional curvatures if, for any 2-plane $\pi \subset T_pM$, $p \in M$, making the angle θ with ξ and the angle ψ with $J\xi$, its sectional curvatures depend only on $p \in M$, θ, and ψ.*

Remark 7. *All the totally real planes have the same sectional curvatures.*

The geometric properties of the Riemannian manifolds with constant sectional curvature are characterized in terms of the Riemannian curvature. In [2], Ganchev and Mihova characterized, in terms of their Riemannian curvature, the Riemannian manifolds of quasi-constant sectional curvature, meaning that all planes having the same angle with a fixed non-zero direction ξ have the same curvature. Corresponding to the Riemannian case, in the Kähler case, the manifolds of holomorphic quasi-constant sectional curvature have been characterized independently by Bejan and Benyounes in [3] and by Ganchev and Mihova in [4]. Then, in the context of para-Kähler geometry, a corresponding characterization for para-Kähler manifolds of quasi-constant P-sectional curvature was carried out by Bejan and Ferrara in [5]. Now, we give a corresponding result in the context of B-manifolds.

Following similar straightforward but very long computations as in [2–5], we obtain the next characterization theorem:

Theorem 3. *Let (M,J,g) be a B-manifold endowed with a unit spacelike vector field ξ. Then, M is of quasi-constant totally real sectional curvatures if and only if there exist three real constants a,b,c such that*

$$\begin{aligned}
R(X,Y,Z,W) = {} & a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\
& - g(JY,Z)g(JX,W) + g(JX,Z)g(JY,W)] \\
& + b[g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\
& + g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z) \\
& + g(Y,Z)g(X,JW) - g(X,Z)g(Y,JW) \\
& + g(Y,JZ)g(X,W) - g(X,JZ)g(Y,W) \\
& + g(Y,Z)\eta(JX)\eta(JW) - g(X,Z)\eta(JY)\eta(JW) \\
& + g(X,W)\eta(JY)\eta(JZ) - g(Y,W)\eta(JX)\eta(JZ)] \\
& + c[g(JY,Z)\eta(X)\eta(JW) - g(JX,Z)\eta(Y)\eta(JW) \\
& + g(JX,W)\eta(Y)\eta(JZ) - g(JY,W)\eta(X)\eta(JZ) \\
& - g(JY,Z)\eta(JX)\eta(W) + g(JX,Z)\eta(JY)\eta(W) \\
& - g(JX,W)\eta(JY)\eta(Z) + g(JY,W)\eta(JX)\eta(Z)]. \qquad (16)
\end{aligned}$$

Definition 7 ([19]). *On a (semi-)Riemannian manifold (N,g), a vector field ξ is called semi-torse-forming if $R(X,\xi)\xi = 0$ for any $X \in \Gamma(TN)$.*

Remark 8. *If ξ is a unit vector field on a B-manifold (M^n, J, g), then it is semi-torse-forming if and only if*

$$\begin{aligned}
B(X,\xi)\xi = {} & -\frac{1}{n-4}[\eta(Q\xi)X - \eta(X)Q\xi + QX - \eta(QX)\xi \\
& - \eta(QJ\xi)JX + \eta(JX)QJ\xi - \eta(J\xi)QJX + \eta(QJX)J\xi] \\
& + \frac{1}{(n-2)(n-4)}\{r[X - \eta(X)\xi - \eta(J\xi)JX + \eta(JX)J\xi] \\
& - r^*[JX - \eta(JX)\xi + \eta(J\xi)X - \eta(X)J\xi]\}, \ \ \forall\, X \in \Gamma(TM).
\end{aligned}$$

Proposition 6. *Let (M^n, J, g) be a B-manifold endowed with a unit spacelike vector field ξ. If M is of quasi-constant totally real sectional curvatures, then M is holomorphic η-Einstein.*

Proof. If in (16) we take the trace, by using (13) and (15), we obtain

$$\begin{aligned}
S(X,W) = {} & \sum_{i=1}^{n} \epsilon_i R(X,e_i,e_i,W) \\
= {} & a[\sum_{i=1}^{n} \epsilon_i g(e_i,e_i)g(X,W) - \sum_{i=1}^{n} \epsilon_i g(X,e_i)g(e_i,W) \\
& - \sum_{i=1}^{n} \epsilon_i g(Je_i,e_i)g(JX,W) + \sum_{i=1}^{n} \epsilon_i g(JX,e_i)g(Je_i,W)] \\
& + b[\sum_{i=1}^{n} \epsilon_i g(e_i,e_i)\eta(X)\eta(W) - \sum_{i=1}^{n} \epsilon_i g(X,e_i)\eta(e_i)\eta(W) \\
& + \sum_{i=1}^{n} \epsilon_i g(X,W)\eta(e_i)\eta(e_i) - \sum_{i=1}^{n} \epsilon_i g(e_i,W)\eta(X)\eta(e_i) \\
& + \sum_{i=1}^{n} \epsilon_i g(e_i,e_i)g(X,JW) - \sum_{i=1}^{n} \epsilon_i g(X,e_i)g(e_i,JW)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\epsilon_i g(e_i,Je_i)g(X,W)-\sum_{i=1}^{n}\epsilon_i g(X,Je_i)g(e_i,W)\\
&+\sum_{i=1}^{n}\epsilon_i g(e_i,e_i)\eta(JX)\eta(JW)-\sum_{i=1}^{n}\epsilon_i g(X,e_i)\eta(Je_i)\eta(JW)\\
&+\sum_{i=1}^{n}\epsilon_i g(X,W)\eta(Je_i)\eta(Je_i)-\sum_{i=1}^{n}\epsilon_i g(e_i,W)\eta(JX)\eta(Je_i)]\\
&+c[\sum_{i=1}^{n}\epsilon_i g(Je_i,e_i)\eta(X)\eta(JW)-\sum_{i=1}^{n}\epsilon_i g(JX,e_i)\eta(e_i)\eta(JW)\\
&+\sum_{i=1}^{n}\epsilon_i g(JX,W)\eta(e_i)\eta(Je_i)-\sum_{i=1}^{n}\epsilon_i g(Je_i,W)\eta(X)\eta(Je_i)\\
&-\sum_{i=1}^{n}\epsilon_i g(Je_i,e_i)\eta(JX)\eta(W)+\sum_{i=1}^{n}\epsilon_i g(JX,e_i)\eta(Je_i)\eta(W)\\
&-\sum_{i=1}^{n}\epsilon_i g(JX,W)\eta(Je_i)\eta(e_i)+\sum_{i=1}^{n}\epsilon_i g(Je_i,W)\eta(JX)\eta(e_i)]\\
&=a[ng(X,W)-g(X,W)+g(JX,JW)]\\
&+b[n\eta(X)\eta(W)-\eta(X)\eta(W)+g(X,W)-\eta(X)\eta(W)\\
&+ng(X,JW)-g(X,JW)-g(X,JW)\\
&+n\eta(JX)\eta(JW)-\eta(JX)\eta(JW)+g(JX,JW)-\eta(JX)\eta(JW)\\
&+c[-\eta(JX)\eta(JW)+g(JX,W)\eta(J\xi)-\eta(X)\eta(J^2W)\\
&+\eta(J^2X)\eta(W)-g(JX,W)\eta(J\xi)+\eta(JX)\eta(JW)]\\
&=(n-2)\{ag(X,W)+b[\eta(X)\eta(W)+\eta(JX)\eta(JW)+g(JX,W)]\},
\end{aligned}
$$

for any $X,W\in\Gamma(TM)$, where $\{e_i\}_{1\le i\le n}$ is an orthonormal basis on M. □

Remark 9. *Under the conditions of Proposition 6, we obtain (6).*

By direct computations, in which we use Theorem 3 and Proposition 6, we obtain the following:

Theorem 4. *Let (M,J,g) be a B-manifold endowed with a unit spacelike vector field ξ. Then, the following statements are equivalent:*

(i) *M is of quasi-constant totally real sectional curvatures;*
(ii) *M is a holomorphic η-Einstein, Bochner flat manifold.*

Remark 10. *The statements of Theorem 3, Proposition 6, and Theorem 4 do not change if ξ is timelike instead of spacelike.*

5. Conclusions

The Bochner curvature tensor plays a similar role, in Kähler geometry, to the Weyl curvature tensor on Riemannian manifolds. Even if initially it was just formally defined, its geometrical meaning was pointed out by Blair in [10]. Different from the class of Bochner flat Kählerian manifolds, the Bochner flat B-manifolds (i.e., Kähler manifolds with Norden metrics) have some special behavior. Our results here relate the notion of Bochner flatness to certain generalized Einstein equations (holomorphic Einstein, Ricci soliton, and holomorphic η-Einstein) and to the notion of constant sectional curvature (totally real sectional curvatures and quasi-totally real sectional curvatures). Precisely, in the context of B-manifolds, we characterize the manifolds of constant and of quasi-constant totally real sectional curvatures and provide some necessary and sufficient conditions for a gradient Ricci soliton or a holomorphic η-Einstein Kähler B-manifold to be Bochner flat.

The results of our work may be of interest for both researchers in Differential Geometry, PDE, and Theoretical Physics.

Author Contributions: Conceptualization, C.-L.B.; validation, C.-L.B., G.N. and A.M.B.; formal analysis, C.-L.B., G.N. and A.M.B.; investigation, C.-L.B., G.N. and A.M.B.; writing—original draft preparation, C.-L.B.; writing—review and editing, C.-L.B., G.N. and A.M.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No data was used.

Conflicts of Interest: The authors declare that there are no conflicts of interest.

References

1. Boju, V.; Popescu, M. Espaces à courbure quasi-constante. *J. Differ. Geom.* **1978**, *13*, 373–383. [CrossRef]
2. Ganchev, G.; Mihova, V. Riemannian manifolds of quasi-constant sectional curvatures. *J. Reine Angew. Math.* **2000**, *522*, 119–141. [CrossRef]
3. Bejan, C.L.; Benyounes, M. Kähler manifolds of quasi-constant holomorphic sectional curvature. *J. Geom.* **2008**, *88*, 1–14. [CrossRef]
4. Ganchev, G.; Mihova, V. Kähler manifolds of quasi-constant holomorphic sectional curvatures. *Centr. Eur. J. Math.* **2008**, *6*, 43–75. [CrossRef]
5. Bejan, C.L.; Ferrara, M. Para-Kähler manifolds of quasi-constant P-sectional curvature. In *Workshop Contemporary Geometry and Related Topics*; Čigoja Publishing Company: Belgrade, Serbia, 2006; pp. 29–36.
6. Bochner, S. Curvature and Betti numbers II. *Ann. Math.* **1949**, *50*, 77–93. [CrossRef]
7. Yano, K.; Bochner, S. Volume 32 in the series Annals of Mathematics Studies. In *Curvature and Betti Numbers*; Princeton University Press: Princeton, NJ, USA, 1953. [CrossRef]
8. Tachibana, S.-I. On the Bochner curvature tensor. *Nat. Sci. Rep. Ochanomizu Univ.* **1967**, *18*, 15–19.
9. Chen, B.-Y.; Yano, K. Manifolds with vanishing Weyl or Bochner curvature tensor. *J. Math. Soc. Jpn.* **1975** *27*, 106–112. [CrossRef]
10. Blair, D.E. On the geometric meaning of the Bochner tensor. *Geom. Dedicata* **1975**, *4*, 33–38. [CrossRef]
11. Norden, A.P. On a certain class of four-dimensional A-spaces. *Izv. Vuzov Mat.* **1960**, *4*, 145–157.
12. Olszak, K. On the Bochner conformal curvature of Kähler-Norden manifolds. *Cent. Eur. J. Math.* **2005**, *3*, 309–317. [CrossRef]
13. Olszak, K.; Olszak, Z. Generalized Einstein conditions on holomorphic Riemannian manifolds. *Acta Math. Hung.* **2006**, *113*, 345–358. [CrossRef]
14. Dokuzova, I.; Razpopov, D. Four-dimensional almost Einstein manifolds with skew-circular structures. *J. Geom.* **2020**, *111*, 9. [CrossRef]
15. Dokuzova, I. Curvature properties of Riemannian manifolds with skew-circulant structures. *Balk. J. Geom. Appl.* **2022**, *27*, 50–62.
16. Borisov, A.; Ganchev, G. Curvature properties of Kählerian manifolds with B-metric. In Proceedings of the XIV Spring Conference UBM, Sunny Beach, Bulgaria, 6–9 April, 1985; pp. 220–226.
17. Hamilton, R. The Ricci flow on surfaces. Mathematics and General Relativity. *Contemp. Math.* **1998**, *71*, 237–261. [CrossRef]
18. Duggal, K.L.; Bejancu, A. Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications. In *Series: Mathematics and Its Applications (MAIA)*; Springer: Dordrecht, The Netherlands, 1996; Volume 364. [CrossRef]
19. Rachunek, L.; Mikes, J. On semi-torse-forming vector fields. In Proceedings of the 3rd International Conference APLIMAT, Slovak University of Technology, Bratislava, Slovakia, 4–6 February 2004; pp. 835–840.