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Strong and Δ-Convergence Fixed-Point Theorems Using Noor Iterations

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Abstract: A wide range of new research articles in artificial intelligence, logic programming, and other applied sciences are based on fixed-point theorems. The aim of this article is to present an approximation method for finding the fixed point of generalized Suzuki nonexpansive mappings on hyperbolic spaces. Strong and Δ-convergence theorems are proved using the Noor iterative process for generalized Suzuki nonexpansive mappings (GSNM) on uniform convex hyperbolic spaces. Due to the richness of uniform convex hyperbolic spaces, the results of this paper can be used as an extension and generalization of many famous results in Banach spaces together with $CAT(0)$ spaces.

Keywords: mappings; convergence; hyperbolic spaces; iteration process

MSC: 46S40; 54H25; 47H09; 47H10

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1. Introduction and Motivation

Metric fixed-point theory has emerged as a powerful tool to represent the virtual space as a digital environment [1] and explore web topology [2]. New research on fixed-point theory also emphasizes the significance of solving real-world issues. Functional equations and iterative procedures are applicable to the solution of routing problems in artificial intelligence. The capacitated vehicle routing problem [3] describes a technique for selecting the optimum strategy to distribute comprehensible things from a pickup location using a huge number of carriages with a convinced adaptive volume. It certainly meets the claim of a network of clients spread across the globe. Conversely, communication engineering utilizes fixed-point theory as a tool for problem-solving. The resolution of chemical equations, genetics, algorithm testing, and control theory are additional real-world applications. These findings present pleasant opportunities for approximating the solutions of differential and integral equations that are both linear and nonlinear in nature [4,5]. The theory of fixed points has become a potent and essential tool for the study of nonlinear problems [6–9] due to its novel emergence as a confluence of analysis [10–13] and geometry [14–17]. More specifically, the fixed-point approximation for SKC mappings in hyperbolic spaces has remained a focal point of recent and past research [18]. In this research, we will prove strong and Δ-convergence theorems by using Noor iterative process for generalized Suzuki nonexpansive mappings (GSNM) on uniform convex hyperbolic spaces. To achieve this purpose, we intend to start with the basic definitions and preliminaries in the next section.

2. Basic Definitions and Preliminaries

A family of single-valued mappings introduced by Suzuki [19] is defined as

Definition 1. *Let us consider a Banach space $\mathbb{B}$ and a mapping $\mathfrak{F}$ on the subset $\mathbb{S}$ of $\mathbb{B}$ satisfying the following condition:*

$$\frac{1}{2}\|u - \mathfrak{F}v\| \leq \|u - v\| \implies \|\mathfrak{F}u - \mathfrak{F}v\| \leq \|u - v\|, \tag{1}$$

$\forall\, u, v \in \mathbb{S}$.

This mapping works as an intermediate class of mapping between nonexpansiveness and quasi-nonexpansiveness as given below:

Definition 2. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $\mathbb{B}$ is a Banach Space. Then $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ is nonexpansive if $\|\mathfrak{F}u - \mathfrak{F}v\| \leq \|u - v\|\ \forall\, u, v \in \mathbb{S}$.*

Definition 3. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $\mathbb{B}$ is a Banach Space. Then $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ is quasi-nonexpansive if $\|\mathfrak{F}x - \rho\| \leq \|x - \rho\|$ for every $\rho \in FP(\mathfrak{F})$ and $\forall\, x \in \mathbb{S}$. Here $FP(\mathfrak{F})$ represents fixed point set of $\mathfrak{F}$.*

Example 1. *Let $\mathfrak{F}$ on [0, 5] be defined by;*

$$\mathfrak{F}x = \begin{cases} 0, & x \neq 5; \\ 1, & x = 5. \end{cases}$$

Then clearly $\mathfrak{F}$ is not nonexpansive but it satisfies condition (1).

Example 2. *Let $\mathfrak{F}$ on [0, 5] be defined by*

$$\mathfrak{F}x = \begin{cases} 0, & x \neq 5; \\ 2, & x = 5. \end{cases}$$

Then $\mathfrak{F}$ fails to fullfill condition (1), however $\mathfrak{F}$ is quasi-nonexpansive and $FP(\mathfrak{F}) = \{0\} \neq \phi$.

Suzuki [19] conducted significant work in showing the presence of the fixed point and convergence theorem in Banach spaces equipped with mapping a satisfying condition (1).

In [20] Dhompongsa et al. enhanced the conclusions of Suzuki [19] with different conditions on Banach spaces and obtained a fixed point result in these spaces equipped with mapping satisfying condition C.

Nanjaras et al. [21] rendered sundry characterization of existing fixed point results equipped with mappings satisfying condition C in the skeleton of CAT(0) spaces. Abbas et al. [22] also analyzed such spaces whereas the asymptotic regularity is discussed in [23]. Other related work can be found in [24–26]. There is need to generalize the result of Suzuki-type nonexpansive mappings which were efficiently conducted by Karapınar et al. [20] in 2011 as given below.

Definition 4. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space, equipped with mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ and if*

$$\frac{1}{2}\rho(\mathfrak{F}u, \mathfrak{F}v) \leq \rho(u, v) \Rightarrow \rho(\mathfrak{F}u, \mathfrak{F}v) \leq \theta(u, v),$$

where $\theta(u, v) = max\{\rho(u, v), \rho(u, \mathfrak{F}u), \rho(v, \mathfrak{F}v), \rho(u, \mathfrak{F}v), d(v, \mathfrak{F}u)\}\ \forall\, u, v \in \mathbb{S}$. Then $\mathfrak{F}$ is considered to be a Suzuki–Ciric mapping (SCC) [27].

Definition 5. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space, equipped with mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ and if*

$$\frac{1}{2}\rho(\mathfrak{F}u, \mathfrak{F}v) \leq \rho(u, v) \Rightarrow \rho(\mathfrak{F}u, \mathfrak{F}v) \leq v(u, v),$$

where $\nu(u,v) = \max\left\{\rho(u,v), \frac{\rho(u,\mathfrak{F}u)+\rho(v,\mathfrak{F}v)}{2}, \frac{\rho(u,\mathfrak{F}v)+\rho(v,\mathfrak{F}u)}{2}\right\} \ \forall\, u,v \in \mathbb{S}$.

Then $\mathfrak{F}$ *is considered to be a Suzuki–KC mapping (SKC).*

Definition 6. *Assume* $\mathbb{S} \subset \mathbb{B}$, *where* $\mathbb{S}$ *is nonempty and* $(\mathbb{B},\rho)$ *represents metric space, equipped with mapping* $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ *and if*

$$\frac{1}{2}\rho(\mathfrak{F}u,\mathfrak{F}v) \leq \rho(u,v) \Rightarrow \rho(\mathfrak{F}u,\mathfrak{F}v) \leq \frac{\rho(u,\mathfrak{F}u)+\rho(v,\mathfrak{F}v)}{2},$$

$\forall\, u,v. \in \mathbb{S}$.

Then $\mathfrak{F}$ *is considered to be a Kannan–Suzuki mapping (KSC).*

Definition 7. *Assume* $\mathbb{S} \subset \mathbb{B}$, *where* $\mathbb{S}$ *is nonempty and* $(\mathbb{B},\rho)$ *represents metric space, equipped with mapping* $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ *and if*

$$\frac{1}{2}\rho(\mathfrak{F}u,\mathfrak{F}v) \leq \rho(u,v) \Rightarrow \rho(\mathfrak{F}u,\mathfrak{F}v) \leq \frac{\rho(v,\mathfrak{F}u)+\rho(u,\mathfrak{F}v)}{2},$$

$\forall\, x,y \in \mathbb{S}$.

Then $\mathfrak{F}$ *is considered to be a Chatterjea–Suzuki mapping (CSC).*

Clearly every nonexpansive mapping is SKC, but the converse may not true [27].

Example 3. *Set* $\mathfrak{F}$ *on [0, 6] by:*

$$\mathfrak{F}x = \begin{cases} 0, & x \neq 6; \\ 1, & x = 6. \end{cases}$$

Clearly $\mathfrak{F}$ *is not nonexpansive but* $\mathfrak{F}$ *fullfill both the SCC and SKC conditions.*

Example 4. *Set R on [0, 6] by:*

$$Rx = \begin{cases} 0, & x \neq 6; \\ 3, & x = 6. \end{cases}$$

Clearly R does not fulfill the SKC condition, moreover R is quasi-nonexpansive and $FP(R) \neq \phi$.

Example 5. *Let the space* $\mathbb{B} = \{(0,0),(0,1),(1,1),(1,2)\}$ *with metric:*

$$\rho((u_1,v_1),(u_2,v_2)) = \max\{|u_1-u_2|,|v_1-v_2|\}.$$

Set $\mathfrak{F}$ *on* $\mathbb{B}$ *by:*

$$\mathfrak{F}(u,v) = \begin{cases} (1,1), & if(u,v) \neq (0,0); \\ (0,1), & if(u,v) = (0,0). \end{cases}$$

Clearly $\mathfrak{F}$ *fullfill SKC's condition. Assume that* $(u,v) = (0,0)$ *and* $(u,v) = (1,1)$, *then*

$$\frac{1}{2}\rho(\mathfrak{F}(0,0),\mathfrak{F}(0,0)) \leq \rho((0,0),(1,1))$$

and

$$\begin{aligned} \nu((0,0),(1,1)) = \max\{&\rho((0,0),(1,1)), \frac{1}{2}[\rho(\mathfrak{F}(0,0),\mathfrak{F}(0,0)), \rho(\mathfrak{F}(1,1),\mathfrak{F}(1,1))], \\ &\frac{1}{2}[\rho(\mathfrak{F}(1,1),\mathfrak{F}(0,0)), \rho(\mathfrak{F}(0,0),\mathfrak{F}(1,1))]\} \\ =\ &1, \end{aligned}$$

thus

$$\rho(\mathfrak{F}(0,0),\mathfrak{F}(1,1)) = 1 \leq \rho((0,0),(1,1)) = 1.$$

Clearly SKC condition is fulfilled by other points in $\mathbb{B}$.
Moreover $FP(\mathfrak{F}) = \{(1,1)\} \neq \phi$, *and* $FP(\mathfrak{F})$ *is convex and closed.*

This is significant to understand the different iterative process [28] adapted by several writers [29,30] in locating fixed points of the space equipped with nonlinear mappings, moreover solution of their operator equations.

The iteration process manufacture by Mann (see [31,32]) is explained below:

Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is convex and $\mathbb{B}$ is Banach Space, and let $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be a nonlinear mapping, for every point $u_0 \in \mathbb{S}$, the sequence $\{u_n\}$ in $\mathbb{S}$ is defined by

$$u_{n+1} = (1 - \gamma_n)u_n + \gamma_n \mathfrak{F} u_n = M(u_n, \gamma_n, \mathfrak{F}), \quad n \in N,$$

called Mann iterative process.

It should be noted that $\{\gamma_n\}$ represents a real sequence in [0, 1] which fulfills the conditions given below:
(M_1): $0 \leq \gamma_n < 1$,
(M_2): $\lim_{n\to\infty} \gamma_n = 0$,
(M_3): $\sum_{n=1}^{\infty} \gamma_n = \infty$.
One can replace M_3 by $\sum_{n=1}^{\infty} \gamma_n(1 - \gamma_n) = \infty$ in other applications.

The Ishikawa manufacture iteration process improves the Mann iteration process (see [33–35]) as follows:

Setting $\mathbb{S}, \mathbb{B}$, and $\mathfrak{F}$ as in (M), for every point $u_0 \in \mathbb{S}$, the sequence $\{u_n\}$ in $\mathbb{S}$ is defined by:

$$u_{n+1} = (1 - \gamma_n)u_n + \gamma_n \mathfrak{F}((1 - \alpha_n)u_n + \alpha_n \mathfrak{F} u_n), \; n \in N,$$

called the Ishikawa iterative process, where $\{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in [0, 1] which satisfy the following conditions:
(I_1): $0 \leq \gamma_n \leq \alpha_n < 1$,
(I_2): $\lim_{n\to\infty} \alpha_n = 0$,
(I_3): $\sum_{n=1}^{\infty} \gamma_n \alpha_n = \infty$.

Some authors switch condition $(I_1) : \; 0 \leq \gamma_n \leq \alpha_n < 1$, with the general condition $(I_1') : \; 0 < \gamma_n, \alpha_n < 1$, and notice that, with this switching, the iterative process defined by Ishikawa (I) is a spontaneous generalization of the iterative process given by Mann (M). It is perceived that, if the iterative process defined by Mann (M) is convergent, then the iterative process defined by Ishikawa (I) through condition (I_1') is also convergent, with appropriate conditions on γ_n and α_n.

Recently, Agarwal et al. [36] broached the S-iteration process which is independent of the above two iterative processes as follows:

For $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is convex and $\mathbb{B}$ is linear space, and let $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be a mapping, for every point $u_0 \in \mathbb{S}$, the iterative sequence $\{u_n\}$ in $\mathbb{S}$ is defined by the S-iteration process is given below:

$$\begin{cases} u_{n+1} = (1 - \gamma_n)u_n + \gamma_n \mathfrak{F} u_n \\ v_n = (1 - \alpha_n)u_n + \alpha_n \mathfrak{F} u_n, \qquad n \in N, \end{cases}$$

where $\{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in (0, 1) filling the condition:

$$\sum_{n=0}^{\infty} \gamma_n \alpha_n (1 - \alpha_n) = \infty.$$

It is perceived that both the S-iteration process and the Picard has the same rate of convergence, which is more rapid than the iteration process defined by Mann which is equipped with contraction mapping (see [31,36,37]).

We use the definition of a hyperbolic space given in [38–40], because the definition given by Reich and Shafrir [41] is a bit more repressive. The hyperbolic spaces in the Reich and Shafrir sense [41] is unbounded by taking family of metric lines M instead of metric segments. Further related research can be seen in [42,43]. Moreover, every subset of hyperbolic space is hyperbolic itself by definition, which we consider, and it gives convergence results too.

Definition 8. *Consider the metric space* $(\mathbb{B}, \rho)$ *equipped with convex mapping* $\Omega : \mathbb{B}^2 \times [0,1]$ *then the triplet* $(\mathbb{B}, \rho, \Omega)$ *is said to be hyperbolic space if it fulfills the conditions given below:*
$(\Omega_1) : \rho(x, \Omega(u, v, \gamma)) \leq \gamma\rho(x, u) + (1-\gamma)d(x, v)$;
$(\Omega_2) : \rho(\Omega(u, v, \gamma), \Omega(u, v, \alpha)) = \gamma - \alpha|\rho(u, v)$;
$(\Omega_3) : \Omega(u, v, \gamma) = \Omega(v, u, 1-\gamma)$;
$(\Omega_4) : \rho(\Omega(u, w, \gamma), \Omega(v, y, \gamma)) \leq (1-\gamma)\rho(u, v) + \gamma\rho(w, y)$,
$\forall\, u, v, x$ *and* $y \in \mathbb{B}$ *and* $\gamma, \alpha \in [0,1]$.

Takahashi established the convex metric space [44], in which the triplet $(\mathbb{B}, \rho, \Omega)$ *fulfills* Ω_1*. Goebel and Kirk in [45] gave their own definition of above space, where triplet* $(\mathbb{B}, \rho, \Omega)$ *fill conditions* (Ω_1)*–*(Ω_3)*.*

Reich and Shafrir [41] and Kirk [46] manufactured their definition of hyperbolic space by using 'condition III' of Itoh [47] which is equivalent to Ω_4*.*

The class of hyperbolic spaces is rich in nature and contains different spaces, manifold of the Hadamard type and convex subsets thereof. For more see [48], and the CAT(0) spaces along with Ω *as the unique geodesic path between any two points in* $\mathbb{B}$*. Bruhat and Tits [49] show that hyperbolic space is a* $CAT(0)$*-space if and only if it fulfills the so called CN-inequality.*

Wataru Takahashi [44] introduce the notion of a convex set $\mathbb{S}$ *of hyperbolic spaces* $\mathbb{B}$ *if it satisfies the following condition* $\Omega(u, v, \gamma) \in \mathbb{S}\ \forall\, u, vs. \in \mathbb{S}$ *and* $\gamma \in [0,1]$*. We often use the notion* $(1-\mu)u \oplus \mu v$ *for* $\Omega(u, v, \mu)$*,*$\forall\, u, v \in \mathbb{B}$ *and* $\mu \in [0,1]$*.*

Assume $\forall\, u, v \in \mathbb{B}$*, and* $\mu \in [0,1]$*, and setting*

$$\rho(u, (1-\mu)u \oplus \mu v) = \mu\rho(u, v)$$

and

$$\rho(v, (1-\mu)u \oplus \mu v) = (1-\mu)\rho(u, v)$$

which is considered to be a more general setting of a convex metric space [44,50].

A hyperbolic space $(\mathbb{B}, \rho, \Omega)$ *is uniformly convex in the sense of [37] if, for any* $q > 0$ *and* $\epsilon \in (0,2]$*, there exists* $\delta \in (0,1]$ *such that,* $\forall\, c, u, v \in \mathbb{B}$*,*

$$\rho(\frac{1}{2}u \oplus \frac{1}{2}v, b) \leq (1-\delta)q,$$

provided $\rho(u, c) \leq q, \rho(v, c) \leq r$*, and* $\rho(u, v) \geq \epsilon q$*.*

Setting $\eta : (0, \infty) \times (0,2] \to (0,1]$ *equipped with* $\delta = \eta(q, \varepsilon)$ *such that* $q > 0$ *and* $\varepsilon \in (0,2]$ *then* η *is said to be modulus of uniform convexity. Clearly with this setting if* q *decreases for stationary* ε *then* η *is monotone.*

The aim of this article is to prove strong convergence and Δ-*convergence* of Noor iterative process for GSNM in uniform convex hyperbolic spaces. First, we recall the notion of Δ-convergence and a few of its primary characteristics.

Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $(\mathbb{B}, \rho)$ represents metric space and let $\{u_n\}$ be any sequence in $\mathbb{S}$. Moreover, diam$(\mathbb{S})$ signify the diameter of $\mathbb{S}$. Set a continuous functional $r_b(., u_n) : \mathbb{B} \to \mathbb{R}^+$ as

$$r_b(u, \{u_n\}) = \lim_{n\to\infty} \sup \rho(u_n, u), u \in \mathbb{B}.$$

The asymptotic radius of $\{u_n\}$ is signified by $r_b(\mathbb{S}, \{u_n\})$ in connection with $\mathbb{S}$ and is defined to be the infimum of $r_b(., u_n)$ over $\mathbb{S}$.

Furthermore, if

$$r_b(w, \{u_n\}) = \inf\{(u, \{u_n\}) : u \in \mathbb{S}\},$$

then the point $w \in \mathbb{S}$ signifies as an asymptotic center of the sequence $\{u_n\}$ in connection with $\mathbb{S}$.

AC($\mathbb{S}$, $\{u_n\}$) signifies the set of all asymptotic centers of $\{u_n\}$ in connection with $\mathbb{S}$, which is the set of minimizers of the functional $r(., \{u_n\})$ and it can be empty or a singleton or contain infinite points.

The notions $r_b(\mathbb{B}, \{u_n\}) = r_b(\{u_n\})$ and AC($\mathbb{B}$, $\{u_n\}$) = AC($\{u_n\}$), respectively, signify the asymptotic radius and asymptotic center taken in connection with $\mathbb{B}$.

Clearly, for $u \in \mathbb{B}, r_b(u, \{u_n\}) = 0$ if and only if $\lim_{n\to\infty} u_n = u$.

Moreover, every sequence which is bounded has a unique asymptotic center in connection with each closed convex subset in uniformly convex Banach spaces and even CAT(0) spaces.

The following lemma is due to Leuştean [51] and we know that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1 ([51]). *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty. Moreover, $\mathbb{S}$ is also closed and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space, which is complete and has η as a monotone modulus of uniform convexity. Then every sequence $\{u_n\}$ in $\mathbb{B}$, which is bounded, has a unique asymptotic center referring to $\mathbb{S}$ as defined above.*

Definition 9. *Let $\mathbb{B}$ be hyperbolic space and $\{u_n\}$ in any sequence in $\mathbb{B}$. If u is the unique asymptotic center of every subsequence $\{t_n\}$ of $\{u_n\}$ then $\{u_n\}$ is considered to be Δ-convergent to $u \in \mathbb{B}$. In such a case, we set Δ-$lim_n u_n = u$ and we refer u the Δ-limit of u_n.*

Lemma 2 ([42]). *The triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space having η as a monotone modulus of uniform convexity. Moreover, assume $u \in \mathbb{B}$ and $\{s_n\}$ be a sequence in $[c, d]$ with $0 < c, d < 1$. If $\{u_n\}$ and $\{v_n\}$ are any two sequences in $\mathbb{B}$ so that*
$\limsup_{n\to\infty} \rho(u_n, u) \le e$,
$\limsup_{n\to\infty} \rho(v_n, p) \le e$,
$\lim_{n\to\infty} \rho(\Omega(u_n, v_n, s_n), u) = e$,
for some $e \ge 0$, then $\lim_{n\to\infty} \rho(u_n, v_n) = 0$.

3. Main Results

First, we will give the definition of $Fej\acute{e}r$ monotone sequences.

Definition 10. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $\mathbb{B}$ is a hyperbolic space. Moreover, suppose that $\{u_n\}$ be a sequence in $\mathbb{B}$. Then the sequence $\{u_n\}$ is said to be Fejér monotone in connection with $\mathbb{S}$ if $\forall\ u \in \mathbb{S}$ and $n \in N$,*

$$\rho(u_{n+1}, u) \le \rho(u_n, u).$$

Proposition 1 ([40]). *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $\mathbb{B}$ is a hyperbolic space. Moreover, suppose that $\{u_n\}$ be a Fejér monotone sequence in connection with $\mathbb{S}$. Then the following conditions hold:*
(1) $\{u_n\}$ is bounded;
(2) the sequence $\{\rho(u_n, t)\}$ is decreasing and convergent $\forall\ t \in FP(\mathfrak{F})$.

We are now able to present the iterative process defined by Noor in hyperbolic spaces (see [40]):

Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty, moreover $\mathbb{S}$ is closed and convex, and $\mathbb{B}$ is hyperbolic space. Furthermore, $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ is a mapping. For any $u_1 \in \mathbb{S}$, the sequence $\{u_n\}$ of the Noor iteration process is defined by:

$$\begin{cases} u_{n+1} = \Omega(u_n, \mathfrak{F}v_n, \gamma_n), \\ v_n = \Omega(u_n, \mathfrak{F}w_n, \alpha_n), \\ w_n = \Omega(u_n, \mathfrak{F}u_n, \beta_n), \qquad n \in N, \end{cases} \tag{2}$$

where $\{\gamma_n\}$ and $\{\alpha_n\}$ are real sequences such that $0 < a \le \gamma_n, \alpha_n$ and $\beta_n \le b < 1$.

We are able to manufacture the proof of the following lemma from the definition of SKC mapping.

Lemma 3. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty and $\mathbb{B}$ is a hyperbolic space. Moreover, suppose that $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an SKC mapping. If $\{u_n\}$ is a sequence given by (2), then $\{u_n\}$ is Fejḱer monotone sequence in connection with $FP(\mathfrak{F})$.*

Proof. Let $q \in FP(\mathfrak{F})$. Then by (2), we have

$$\begin{aligned} \rho(w_n, q) &= \rho(\Omega(u_n, \mathfrak{F}u_n, \beta_n), q) \\ &\leq (1 - \beta_n\rho(u_n, q) + \beta_n\rho(\mathfrak{F}u_n, q) \\ &\leq (1 - \beta_n\rho(u_n, q) + \beta_n[5\rho(q, \mathfrak{F}_q) + (u_n, q)] \\ &\leq (1 - \beta_n\rho(u_n, q) + \beta_n\rho(u_n, q) \\ &\leq \rho(u_n, q). \end{aligned} \tag{3}$$

$$\begin{aligned} \rho(v_n, q) &= \rho(\Omega(u_n, \mathfrak{F}w_n, \alpha_n), q) \\ &\leq (1 - \alpha_n\rho(u_n, q) + \alpha_n\rho(\mathfrak{F}w_n, q) \\ &\leq (1 - \alpha_n\rho(u_n, q) + \alpha_n[5\rho(q, \mathfrak{F}_q) + (w_n, q)] \\ &\leq (1 - \alpha_n\rho(u_n, q) + \alpha_n\rho(w_n, q) \\ &\leq (1 - \alpha_n\rho(u_n, q) + \alpha_n\rho(u_n, q) \\ &\leq \rho(u_n, q). \end{aligned} \tag{4}$$

$$\begin{aligned} \rho(u_{n+1}, q) &= \rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), q) \\ &\leq (1 - \gamma_n\rho(u_n, q) + \gamma_n\rho(\mathfrak{F}v_n, q) \\ &\leq (1 - \gamma_n\rho(u_n, q) + \gamma_n[5\rho(q, \mathfrak{F}_q) + (v_n, q)] \\ &\leq (1 - \gamma_n\rho(u_n, q) + \gamma_n\rho(v_n, q) \\ &\leq (1 - \gamma_n\rho(u_n, q) + \gamma_n\rho(u_n, q) \\ &\leq \rho(u_n, q). \end{aligned} \tag{5}$$

$\forall\, q \in FP(\mathfrak{F})$, which completes the proof. □

Lemma 4. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty, closed, and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space, which is complete, having η as a monotone modulus of uniform convexity, and let $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an SKC mapping. If the sequence $\{u_n\}$ is defined by (2), then $FP(\mathfrak{F})$ is nonempty if and only if $\{u_n\}$ is bounded and $\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0$.*

Proof. Suppose $FP(\mathfrak{F})$ is nonempty and $q \in FP(\mathfrak{F})$. Then, the sequence $\{u_n\}$ is *Fejḱer* monotone with respect to $FP(\mathfrak{F})$ by using by Lemma 3. Furthermore, $\{u_n\}$ is bounded and $\lim_{n\to\infty} \rho(u_n, q)$ by using Proposition 1.

Set $\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = e \geq 0$. If $e = 0$, then clearly we have

$$\begin{aligned} \rho(u_n, \mathfrak{F}u_n) &\leq \rho(u_n, q) + \rho(\mathfrak{F}u_n, q) \\ &\leq \rho(u_n, q) + 5\rho(q, \mathfrak{F}_q) + \rho(u_n, q) \\ &\leq 2\rho(u_n, q) \end{aligned}$$

Applying the limit supremum, we have

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0.$$

Set $e > 0$. Moreover, $\mathfrak{F}$ is an SKC mapping, then

$$\rho(\mathfrak{F}_q, \mathfrak{F}v_n) \leq \rho(q, v_n)$$

and

$$\rho(\mathfrak{F}_q, \mathfrak{F}u_n) \leq \rho(q, u_n).$$

Therefore,

$$\begin{aligned} \rho(\mathfrak{F}_n, q) &\leq \rho(\mathfrak{F}_n, \mathfrak{F}q). \\ &\leq \rho(u_n, q) \end{aligned}$$

for every $n \in N$. Applying the limit supremum, we get

$$\limsup_{n\to\infty} \rho(\mathfrak{F}_n, q) \leq e,$$

for $e > 0$. Further, we have

$$\limsup_{n\to\infty} \rho(\mathfrak{F}_v, q) \leq e.$$

Applying the limit supremum, we get

$$\limsup_{n\to\infty} \rho(v_n, q) \leq e.$$

Since

$$\begin{aligned} e &= \limsup_{n\to\infty} \rho(u_{n+1}, q) \\ &\leq \limsup_{n\to\infty} \{\rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), q)\} \\ &\leq \limsup_{n\to\infty} \{(1-\gamma_n)\rho(u_n, q) + \gamma_n \rho(\mathfrak{F}v_n, q)\} \\ &\leq (1-\gamma_n) \limsup_{n\to\infty} \rho(u_n, q) + \gamma_n \limsup_{n\to\infty} \rho(\mathfrak{F}v_n, q) \end{aligned}$$

we have

$$e \leq ((1-\alpha_n)e + \alpha_n e) = e.$$

Thus,

$$\lim_{n\to\infty} \{\rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), q)\} = e,$$

for $e > 0$. Consequently it occurs from the Lemma 3 that

$$\lim_{n\to\infty} \rho(\mathfrak{F}u_n, \mathfrak{F}v_n) = 0.$$

Next,

$$\begin{aligned} \rho(u_{n+1}, \mathfrak{F}u_n) &= \rho(\Omega(u_n, \mathfrak{F}v_n, \gamma_n), \mathfrak{F}u_n) \\ &\leq d\rho(\mathfrak{F}v_n, \mathfrak{F}u_n) \\ &\to 0 \qquad\qquad as : n \to \infty. \end{aligned}$$

Hence, we have

$$\begin{aligned} \rho(u_{n+1}, \mathfrak{F}v_n) &= \rho(u_{n+1}, \mathfrak{F}u_n) + (\mathfrak{F}u_n), \mathfrak{F}v_n) \\ &\to 0 \qquad\qquad as : n \to \infty. \end{aligned}$$

Notice that

$$\begin{aligned}\rho(u_{n+1},q) &= \rho(u_{n+1},\mathfrak{F}v_n)+(\mathfrak{F}v_n),q)\\ &\leq \rho(u_{n+1},\mathfrak{F}v_n)+(\mathfrak{F}_n),q)\end{aligned}$$

which produces

$$c \leq \liminf_{n\to\infty} \rho(v_n,q).$$

From the above inequalities, we get

$$\lim_{n\to\infty} \rho(v_n,q) = e.$$

Thus, we get

$$\lim_{n\to\infty}\{\rho(\Omega(u_n,\mathfrak{F}u_n,\alpha_n),q)\} = e,$$

which implies

$$\lim_{n\to\infty} \rho(u_n,\mathfrak{F}u_n) = 0.$$

Conversely, assume that the sequence $\{u_n\}$ is bounded and $\lim_{n\to\infty}\rho(u_n,\mathfrak{F}u_n) = 0$. Set $AC(\mathbb{S},\{u_n\}) = u$ be a singleton. Then $u \in \mathbb{S}$. Further $\mathfrak{F}$ is an $SKCmapping$

$$d(u_n,\mathfrak{F}u) \leq 5\rho(u_n,\mathfrak{F}u_n)+\rho(u_n,u),$$

which implies that

$$\begin{aligned}r_b(\mathfrak{F}u,u_n) &= \limsup_{n\to\infty} \rho(u_n,\mathfrak{F}u)\\ &\leq \limsup_{n\to\infty}[5\rho(u_n,\mathfrak{F}u_n)+\rho(u_n,u)]\\ &\leq \limsup_{n\to\infty} \rho(u_n,u)\\ &= r_b(u,u_n).\end{aligned}$$

By utilizing the uniqueness of the asymptotic center, $\mathfrak{F}u = u$, so u is a fixed point of $\mathfrak{F}$. Consequently, $FP(\mathfrak{F})$ is nonempty. □

Now, we are able to prove the Δ-*convergence* theorem.

Theorem 1. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty, closed, and convex. Furthermore, the triplet $(\mathbb{B},\rho,\Omega)$ represents uniformly convex hyperbolic space, which is complete and has η as monotone modulus of uniform convexity, and let $\mathfrak{F}:\mathbb{S}\to\mathbb{S}$ be a mapping such that $FP(\mathfrak{F}) \neq \phi$. If the sequence $\{u_n\}$ is defined by (2), then the sequence $\{u_n\}$ is Δ-convergent to a fixed point of $\mathfrak{F}$.*

Proof. Suppose $\mathfrak{F}$ is an SKC mapping. We observe that $\{u_n\}$ be a bounded sequence. Therefore, $\{u_n\}$ has a Δ-*convergent* subsequence. We have to show that every Δ-*convergent* subsequence of $\{u_n\}$ has a unique Δ-*limit* in $FP(\mathfrak{F})$. To prove this claim, suppose s and t be Δ-*limits* of the subsequences $\{s_n\}$ and $\{t_n\}$ of $\{u_n\}$, respectively. Since $AC(\mathbb{S},s_n) = s$ and $AC(\mathbb{S},t_n) = t$ by using Lemma 1. Now by Lemma 3, $\{s_n\}$ is a bounded sequence and $\lim_{n\to\infty}\rho(s_n,\mathfrak{F}s_n) = 0$.

We have to show that s is a fixed point of $\mathfrak{F}$.

$$\rho(s_n,\mathfrak{F}s) \leq 5\rho(s_n,\mathfrak{F}s_n)+\rho(s_n,s).$$

Applying the limit supremum, we get

$$\begin{aligned} r_b(s_n, \mathfrak{F}s) &= \limsup_{n\to\infty} \rho(s_n, \mathfrak{F}s) \\ &\leq \limsup_{n\to\infty} [5\rho(s_n, \mathfrak{F}s_n) + \rho(s_n, s)] \\ &\leq \limsup_{n\to\infty} \rho(s_n, s) \\ &= r_b(s_n, s). \end{aligned}$$

Hence, we have

$$r_b(s_n, \mathfrak{F}s) \leq r_b(s_n, s).$$

By uniqueness of the asymptotic center, $\mathfrak{F}s = s$.

By using same argument, we can show that $\mathfrak{F}t = t$. Consequently, s and t are fixed points of $\mathfrak{F}$. Now, we show that $s = t$. Suppose on contrary that $s \neq t$, moreover by the uniqueness of the asymptotic center,

$$\begin{aligned} \limsup_{n\to\infty} \rho(u_n, s) &= \limsup_{n\to\infty} \rho(s_n, s) \\ &< \limsup_{n\to\infty} \rho(s_n, t) \\ &= \limsup_{n\to\infty} \rho(u_n, t) \\ &= \limsup_{n\to\infty} \rho(t_n, t) \\ &< \limsup_{n\to\infty} \rho(t_n, s) \\ &= \limsup_{n\to\infty} \rho(u_n, s) \end{aligned}$$

which is a contradiction. Therefore s = t. □

Now, we will introduce the strong convergence theorems in hyperbolic spaces.

Theorem 2. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty, closed, and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space, which is complete and has η as the monotone modulus of uniform convexity and let $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an SKC mapping. If the sequence $\{u_n\}$ is defined by (2), then the sequence $\{u_n\}$ converges strongly to some fixed point of $\mathfrak{F}$ if and only if*

$$\liminf_{n\to\infty} D(u_n, FP(\mathfrak{F})) = 0,$$

where $D(u_n, FP(\mathfrak{F})) = \inf_{u\in FP(\mathfrak{F})} \rho(u_n, u)$.

Proof. Clearly the necessary condition is trivial. The proof completes only by showing the sufficient condition. So we show that $FP(\mathfrak{F})$ is closed. Assume that $\mathfrak{F}$ is SKC mapping, moreover let $\{u_n\}$ be any sequence in $FP(\mathfrak{F})$ which converges to some point $u \in \mathbb{S}$.

$$\rho(u_n, \mathfrak{F}u) \leq 5\rho(\mathfrak{F}u_n, \mathfrak{F}u) + \rho(u_n, u) \leq \rho(u_n, u).$$

Applying the limit, we get

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u) \leq \lim_{n\to\infty} \rho(u_n, u) = 0.$$

Since the limit is unique, we get $u = \mathfrak{F}u$, which shows that $FP(\mathfrak{F})$ is closed.

Assume that

$$\liminf_{n\to\infty} D(u_n, FP(\mathfrak{F})) = 0.$$

Moreover, we obtain

$$D(u_{n+1}, FP(\mathfrak{F})) \leq D(u_n, FP(\mathfrak{F}))$$

Thus, $\lim_{n\to\infty} \rho(u_n, FP(\mathfrak{F}))$ exists by applying Lemma 3 and using Proposition 1. Consequently, we know that

$$\lim_{n\to\infty} D(u_n, FP(\mathfrak{F})) = 0.$$

Consequently, we can set a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ so that

$$\rho(u_{n_k}, q_k) < \frac{1}{2^k},$$

for every $k \geq 1$, where $q_k \in FP(\mathfrak{F})$.

Applying Lemma 3, we get

$$\rho(u_{n_{k+1}}, q_k) \leq \rho(u_{n_k}, q_k) < \frac{1}{2^k}$$

from which we can deduce that

$$\begin{aligned}\rho(q_{k+1}, q_k) &\leq \rho(q_{k+1}, u_{n_{k+1}}) + \rho(u_{n_{k+1}}, q_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}.\end{aligned}$$

Thus, $\{q_k\}$ is a Cauchy sequence. Whereas $FP(\mathfrak{F})$ is closed. Then $\{q_k\}$ is a convergent sequence.

Suppose $\lim_{k\to\infty} q_k = q$. Then, the proof completes by showing that $\{u_n\}$ converges to q. In fact,

$$\rho(u_{n_k}, q) \leq \rho(u_{n_k}, q_k) + \rho(q_k, q) \to 0$$

as $k \to \infty$.

We have

$$\lim_{k\to\infty} \rho(u_{n_k}, q) = 0.$$

Since $\lim_{k\to\infty} \rho(u_n, q)$ exists, the sequence $\{u_n\}$ converges to q. □

Next, we will give one more strong convergence theorem by usingTheorem 2. We call up the definition of condition (I) broached by Senter and Doston [52].

Assume $(\mathbb{B}, \rho)$ be a metric space and $\mathbb{S} \subset \mathbb{B}$ which is nonempty, be equipped with a mapping $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$. Then $\mathfrak{F}$ is claimed to fulfill condition (I), if $\exists$ a nondecreasing function $f[0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0 \ \forall \ t \in (0,\infty)$ so that

$$\rho(u, \mathfrak{F}u) \geq f(D(u, FP(\mathfrak{F}))),$$

$\forall \ u \in \mathbb{S}$, where $D(u, FP(\mathfrak{F})) = \inf d(u, q) : q \in FP(\mathfrak{F})$.

Theorem 3. *Assume $\mathbb{S} \subset \mathbb{B}$, where $\mathbb{S}$ is nonempty, closed, and convex. Furthermore, the triplet $(\mathbb{B}, \rho, \Omega)$ represents uniformly convex hyperbolic space, which is complete and has η as monotone modulus of uniform convexity, and let $\mathfrak{F} : \mathbb{S} \to \mathbb{S}$ be an SKC mapping with condition (I) and $FP(\mathfrak{F}) \neq \phi$. Then, the sequence $\{u_n\}$, which is defined by (2) converges strongly to some fixed point of $\mathfrak{F}$.*

Proof. From Theorem 2, and applying Lemma 4, we have

$$\lim_{n\to\infty} \rho(u_n, \mathfrak{F}u_n) = 0.$$

The condition (I) gives us

$$\lim_{n\to\infty}\rho(u_n,\mathfrak{F}u_n)\geq\lim_{n\to\infty}f(D(u_n,FP(\mathfrak{F}))),$$

for $f[0,\infty)\to[0,\infty)$,which is nondecreasing with $f(0)=0$, $f(t)>0$ for t, such that $0<t<\infty$.

Consequently, we get

$$\lim_{n\to\infty}f(D(u_n,FP(\mathfrak{F})))=0.$$

Whereas f is a nondecreasing mapping filling $f(0)=0$ for every t, such that $0<t<\infty$, we get

$$\lim_{n\to\infty}D(u_n,FP(\mathfrak{F}))=0.$$

Which completes the proof from Theorem. □

Example 6. *Consider the real line R with usual metric ρ defined as $\rho(u,v)=|u-v|$, moreover suppose $\mathbb{S}=[0,4]\subset R$. Set*

$$\Omega(u,v,\gamma)=\gamma u+(1-\gamma v),$$

for every $u,vs.\in\mathbb{S}$.

Then (R,ρ,Ω) is a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity and clearly $\mathbb{S}\subset R$, which is nonempty closed and convex. Set a mapping $\mathfrak{F}$ as defined in Example 2.

Clearly $\mathfrak{F}$ fulfills the SKC condition with $0\in\mathbb{S}$ as a fixed point of $\mathfrak{F}$. Moreover, it is noticed that it fulfills all conditions in Theorem 2. Suppose $\{\gamma_n\}$ and $\{\alpha_n\}$ be constant sequences such that $\gamma_n=\alpha_n=\beta_n=\frac{1}{2}$ for every $n\geq 0$. We encounter the following cases for $\mathfrak{F}$.

Case 1: Set $u\neq 4$; for the sake of simplicity, we suppose that $u_0=1$. Moreover, by the iterative process defined in (Definition 10) and the definition of Ω, we get

$$\begin{aligned}w_0&=\Omega(u_0,\mathfrak{F}u_0,\frac{1}{2})\\&=\frac{1}{2}(u_0)+(1-\frac{1}{2})(\mathfrak{F}u_0)\\&=\frac{1}{2}(u_0+\mathfrak{F}u_0)\end{aligned}$$

and

$$\begin{aligned}v_0&=\Omega(u_0,\mathfrak{F}w_0,\frac{1}{2})\\&=\frac{1}{2}(u_0)+(1-\frac{1}{2})(\mathfrak{F}w_0)\\&=\frac{1}{2}(u_0+\mathfrak{F}w_0)\end{aligned}$$

and

$$\begin{aligned}u_1&=\Omega(u_0,\mathfrak{F}v_0,\frac{1}{2})\\&=\frac{1}{2}(u_0)+(1-\frac{1}{2})(\mathfrak{F}v_0)\\&=\frac{1}{2}(u_0+\mathfrak{F}v_0).\end{aligned}$$

Case 2: Set $u = 4$, for the sake of simplicity, we suppose that $u_0 = 4$. Moreover, by the iterative process manufactured in (Definition 10) and the definition of Ω, we get

$$\begin{aligned} w_0 &= \Omega(u_0, \mathfrak{F}u_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}u_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}u_0) \end{aligned}$$

and

$$\begin{aligned} v_0 &= \Omega(u_0, \mathfrak{F}w_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}w_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}w_0) \end{aligned}$$

and

$$\begin{aligned} u_1 &= \Omega(u_0, \mathfrak{F}v_0, \frac{1}{2}) \\ &= \frac{1}{2}(u_0) + (1 - \frac{1}{2})(\mathfrak{F}v_0) \\ &= \frac{1}{2}(u_0 + \mathfrak{F}v_0), \end{aligned}$$

where $w_1 = \Omega(u_1, \mathfrak{F}u_1, \frac{1}{2})$; $v_1 = \Omega(u_1, \mathfrak{F}w_1, \frac{1}{2})$ and $u_2 = \Omega(u_1, \mathfrak{F}v_1, \frac{1}{2})$.

Consequently, by simple calculations, it can be seen that the sequence $\{u_n\}$ converges to $0 \in FP(\mathfrak{F})$.

4. Discussion

Fixed-point theorems are the foundation of numerous recent research publications in applied sciences, logic programming, and artificial intelligence. This article's aim is to provide an approximate technique for determining the fixed point of generalized Suzuki nonexpansive mappings on hyperbolic spaces. For generalized Suzuki nonexpansive mappings (GSNM) on uniform convex hyperbolic spaces, the theorems of strong and Δ-convergence are demonstrated using the Noor iterative method. The findings of this study can be used as an extension and generalization of numerous well-known conclusions in Banach spaces along with CAT(0) spaces due to the richness of uniform convex hyperbolic spaces.

5. Conclusions

Fixed-point theory is a tool for problem-solving in communication engineering. Additional real-world applications include genetics, testing of algorithms, control theory, and the solving of chemical equations. These results offer interesting possibilities for approximate solutions of linear and nonlinear differential and integral equations. We conclude our results with some open questions and future directions:

1. Whether the condition of boundedness of sequence $\{u_n\}$ in Lemma 4 can be relaxed?
2. Whether in Lemma 4 a convergent sequence is not enough to prove it?
3. In Theorem 1, is it possible that the conditions on $\mathbb{S}$ will be removed or replaced with less strong conditions?
4. What about the proof of Theorem 2 if $\mathfrak{F}$ is KSC mapping instead of SKC mapping?
5. What about the proof of Theorem 3 if $\mathfrak{F}$ is CSC mapping instead of SKC mapping?

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References

1. Lippert, K.J.; Cloutier, R. Cyberspace: A Digital Ecosystem. *Systems* **2021**, *9*, 48. [CrossRef]
2. Camelo, M.; Papadimitriou, D.; Fàbrega, L.; Vilà, P. Geometric Routing with Word-Metric Spaces. *IEEE Commun. Lett.* **2014**, *18*, 2125–2128. [CrossRef]
3. Khachay, M.Y.; Ogorodnikov, Y.Y. Efficient Approximation of the Capacitated Vehicle Routing Problem in a Metric Space of an Arbitrary Fixed Doubling Dimension. *Dokl. Math.* **2020**, *102*, 324–329. [CrossRef]
4. Panda, S.K.; Tassaddiq, A.; Agarwal, R.P. A new approach to the solution of non-linear integral equations via various FBe-contractions. *Symmetry* **2019**, *11*, 206. [CrossRef]
5. Khunpanuk, C.; Pakkaranang, N.; Pholasa, N. On solving pseudomonotone equilibrium problems via two new extragradient-type methods under convex constraints. *Demonstr. Math.* **2022**, *55*, 297–314. [CrossRef]
6. Tassaddiq, A.; Shabbir, M.S.; Din, Q.; Naaz, H. Discretization, bifurcation, and control for a class of predator-prey interactions. *Fractal Fract.* **2022**, *6*, 31. [CrossRef]
7. Tassaddiq, A.; Shabbir, M.S.; Din, Q.; Ahmad, K.; Kazi, S. A ratio-dependent nonlinear predator-prey model with certain dynamical results. *IEEE Access* **2020**, *8*, 195074–195088. [CrossRef]
8. Shabbir, M.S.; Din, Q.; Ahmad, K.; Tassaddiq, A.; Soori, A.H.; Khan, M.A. Stability, bifurcation, and chaos control of a novel discrete-time model involving Allee effect and cannibalism. *Adv. Differ. Equ.* **2020**, *2020*, 379. [CrossRef]
9. Shabbir, M.S.; Din, Q.; Alabdan, R.; Tassaddiq, A.; Ahmad, K. Dynamical complexity in a class of novel discrete-time predator-prey interaction with cannibalism. *IEEE Access* **2020**, *8*, 100226–100240. [CrossRef]
10. Tassaddiq, A.; Kanwal, S.; Perveen, S.; Srivastava, R. Fixed points of single-valued and multi-valued mappings in sb-metric spaces. *J. Inequal. Appl.* **2022**, *2022*, 85. [CrossRef]
11. Shoaib, A.; Kazi, S.; Tassaddiq, A.; Alshoraify, S.S.; Rasham, T. Double controlled quasi-metric type spaces and some results. *Complexity* **2020**, *2020*, 3460938. [CrossRef]
12. Olatinwo, M.O.; Omidire, O.J. Some new convergence and stability results for Jungck generalized pseudo-contractive and Lipschitzian type operators using hybrid iterative techniques in the Hilbert space. *Rend. Circ. Mat. Palermo II* **2022**. [CrossRef]
13. Kwelegano, K.M.T.; Zegeye, H.; Boikanyo, O.A. An Iterative method for split equality variational inequality problems for non-Lipschitz pseudomonotone mappings. *Rend. Circ. Mat. Palermo II* **2022**, *71*, 325–348. [CrossRef]
14. Tassaddiq, A. General escape criteria for the generation of fractals in extended Jungck–Noor orbit. *Math. Comput. Simul.* **2022**, *196*, 1–14. [CrossRef]
15. Li, D.; Shahid, A.A.; Tassaddiq, A.; Khan, A.; Guo, X.; Ahmad, M. CR iteration in generation of antifractals with s-convexity. *IEEE Access* **2020**, *8*, 61621–61630. [CrossRef]
16. Zou, C.; Shahid, A.A.; Tassaddiq, A.; Khan, A.; Ahmad, M. Mandelbrot sets and Julia sets in Picard–Mann orbit. *IEEE Access* **2020**, *8*, 64411–64421. [CrossRef]
17. Ogwo, G.N.; Alakoya, T.O.; Mewomo, O.T. Inertial iterative method with self-adaptive step size for finite family of split monotone variational inclusion and fixed point problems in Banach spaces. *Demonstr. Math.* **2022**, *55*, 193–216. [CrossRef]
18. Kim, J.K. Samir Dashutre: Fixed point approximation for $SKC - mappings$ in hyperbolic spaces. *J. Inequal. Appl.* **2015**, *2015*, 341. [CrossRef]
19. Suzuki, T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2008**, *340*, 1088–1095. [CrossRef]
20. Dhompongsa, S.; Inthakon, W.; Kaewkhao, A. Edelsteins method and fixed point theorems for some generalized nonexpansive mappings. *J. Math. Anal. Appl.* **2009**, *350*, 12–17. [CrossRef]
21. Nanjaras, B.; Panyanak, B.; Phuengrattana, W. Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces. *Nonlinear Anal. Hybrid Syst.* **2010**, *4*, 25–31. [CrossRef]
22. Abbas, M.; Khan, S.H.; Postolache, M. Existence and approximation results for SKC mappings in $CAT(0)$ spaces. *J. Inequal. Appl.* **2014**, *2014*, 212. [CrossRef]

23. Leuştean, L. A quadratic rate of asymptotic regularity for CAT(0) spaces. *J. Math. Anal. Appl.* **2007**, *325*, 386–399. [CrossRef]
24. Dhompongsa, S.; Panyanak, B. On Δ-*convergence* theorems in CAT(0) spaces. *Comput. Math. Appl.* **2008**, *56*, 2572–2579. [CrossRef]
25. Khan, S.H.; Abbas, M. Strong and Δ-*convergence* of some iterative schemes in CAT(0) spaces. *Comput. Math. Appl.* **2011**, *61*, 109–116. [CrossRef]
26. Laokul, T.; Panyanak, B. Approximating fixed points of nonexpansive mappings in CAT(0) spaces. *Int. J. Math. Anal.* **2009**, *3*, 1305–1315.
27. Karapinar, E.; Tas, K. Generalized (C)-conditions and related fixed point theorems. *Comput. Math. Appl.* **2011**, *61*, 3370–3380. [CrossRef]
28. Chidume, C.E. Global iteration schemes for strongly pseudocontractive maps. *Proc. Am. Math. Soc.* **1998**, *126*, 2641–2649. [CrossRef]
29. Zhou, H.Y.; Jia, Y.T. Approximation of fixed points of strongly pseudocontractive maps without Lipschitz assumption. *Proc. Am. Math. Soc.* **1997**, *125*, 1705–1709.
30. Gromov, M. *Metric Structure of Riemannian and Non-Riemannian Spaces*; Birkhäuser: Boston, MA, USA, 1984.
31. Kim, J.K.; Dashputre, S.; Hyun, H.G. Convergence theorems of S-iteration process for quasi-contractive mappings in Banach spaces. *Commun. Appl. Nonlinear Anal.* **2014**, *21*, 89–99.
32. Mann, W.R. Mean value methods in iteration. *Proc. Am. Math. Soc.* **1953**, *4*, 506–610. [CrossRef]
33. Ishikawa, S. Fixed points by a new iteration method. *Proc. Am. Math. Soc.* **1974**, *44*, 147–150. [CrossRef]
34. Deng, L. Convergence of the Ishikawa iteration process for nonexpansive mappings. *J. Math. Anal. Appl.* **1996**, *199*, 769–775. [CrossRef]
35. Zeng, L.C. A note on approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. *J. Math. Anal. Appl.* **1998**, *178*, 245–250. [CrossRef]
36. Agarwal, R.P.; O'Regan, D.; Sahu, D.R. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *J. Convex Anal.* **2007**, *8*, 61–79.
37. Saluja, G.S.; Kim, J.K. On the convergence of modified S-iteration process for asymptotically quasi-nonexpansive type mappings in a CAT(0) space. *Nonlinear Funct. Anal. Appl.* **2014**, *19*, 329–339. [CrossRef]
38. Kohlenbach, U. Some logical metatheorems with applications in functional analysis. *Trans. Am. Math. Soc.* **2005**, *357*, 89–128. [CrossRef]
39. Zhao, L.C.; Chang, S.S.; Kim, J.K. Mixed type iteration for total asymptotically nonexpansive mappings in hyperbolic spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 363. [CrossRef]
40. Kim, J.K.; Pathak, R.P.; Dashputre, S.; Diwan, S.D.; Gupta, R. Fixed point approximation of generalized nonexpansive mappings in hyperbolic spaces. *Int. J. Math. Math. Sci.* **2015**, *2015*, 368204. [CrossRef]
41. Reich, S.; Shafrir, I. Nonexpansive iterations in hyperbolic spaces. *Nonlinear Anal.* **1990**, *15*, 537–558. [CrossRef]
42. Chang, S.S.; Agarwal, R.P.; Wang, L. Existence and convergence theorems of fixed points for multi-valued SCC-, SKC-, KSC-, SCS- and C-type mappings in hyperbolic spaces. *Fixed Point Theory Appl.* **2015**, *2015*, 83. [CrossRef]
43. Khan, A.R.; Fukhar-ud-din, H.; Khan, M.A. An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. *Fixed Point Theory Appl.* **2012**, *2012*, 54. [CrossRef]
44. Takahashi, W.A. A convexity in metric space and nonexpansive mappings, I. *Kodai Math. Semin. Rep.* **1970**, *22*, 142–149. [CrossRef]
45. Goebel, K.; Kirk, W.A. Iteration processes for nonexpansive mappings. *Contemp. Math.* **1983**, *21*, 115–123.
46. Kirk, W.A. Iteration process in hyperbolic spaces. *Numer. Funct. Anal. Optim.* **1982**, *4*, 371–381. [CrossRef]
47. Itoh, S. Some fixed point theorems in metric spaces. *Fundam. Math.* **1979**, *102*, 109–117. [CrossRef]
48. Goebel, K.; Reich, S. *Uniformly Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*; Dekker: New York, NY, USA, 1984.
49. Tan, K.K.; Xu, H.K. Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. *J. Math. Anal. Appl.* **1993**, *178*, 301–308. [CrossRef]
50. Kim, J.K.; Kim, K.S.; Nam, Y.M. Convergence of stability of iterative processes for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in convex metric spaces. *J. Comput. Anal. Appl.* **2007**, *9*, 159–172.
51. Leuştean, L. Nonexpansive iteration in uniformly convex W-hyperbolic spaces. *Contemp. Math.* **2010**, *513*, 193–210.
52. Senter, H.F.; Doston, W.G., Jr. Approximating fixed points of nonexpansive mappings. *Proc. Am. Math. Soc.* **1974**, *44*, 375–380. [CrossRef]