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Controllability for Fractional Evolution Equations with Infinite Time-Delay and Non-Local Conditions in Compact and Noncompact Cases

Ahmed Salem [1,*] and Kholoud N. Alharbi [2]

[1] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[2] Department of Mathematics, College of Science and Arts in Uglat Asugour, Qassim University, Buraydah 51411, Saudi Arabia
* Correspondence: asaalshreef@kau.edu.sa

Abstract: The goal of this dissertation is to explore a system of fractional evolution equations with infinitesimal generator operators and an infinite time delay with non-local conditions. It turns out that there are two ways to regulate the solution. To demonstrate the presence of the controllability of mild solutions, it is usual practice to apply Krasnoselskii's theorem in the compactness case and the Sadvskii and Kuratowski measure of noncompactness. A fractional Caputo approach of order between 1 and 2 was used to construct our model. The families of linear operators cosine and sine, which are strongly continuous and uniformly bounded, are used to achieve the mild solution. To make our results seem to be applicable, a numerical example is provided.

Keywords: Caputo fractional derivative; evolution equation; infinite time-delay; mild solution; countability, Kuratowski measure of noncompactness

MSC: 34A08; 34A12; 34G99; 34K99; 34A60

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1. Introduction

Fractional calculus is a branch of mathematics that studies derivatives and integrals of arbitrary order, which are known as fractional derivatives and fractional integrals [1–3]. It is a generalization of classical calculus, which studies derivatives and integrals of integer order. Fractional calculus can be used to model various physical phenomena, such as diffusion and wave propagation, and can also be used to solve certain types of differential equations. It has applications in many fields, such as engineering, physics, chemistry, economics, and finance. Fractional studies based on the economic and financial systems have been investigated by [4,5].

Calculating the targets to which one can influence the state of a dynamical system using a control parameter that appears in the equation is the mathematical problem of controllability. It is the ability to control the evolution of a system by manipulating its parameters. This concept is used in many areas, such as control theory, dynamic systems, and engineering. Controllability is a key factor in the analysis and design of systems and can help to ensure that the system behaves as desired. Understanding the controllability of evolution equations can help us to better understand and control the behavior of complex systems [6,7]. Controllability results for impulsive neutral differential evolution inclusions with infinite delay have been discussed in [8].

Non-local conditions are also used to incorporate the effect of external influences, such as boundary conditions, on the system. By combining fractional derivatives and non-local conditions, we can gain a better understanding of the behavior of the system (see [9–13]).

Therefore, fractional evolution equations with infinite delay are a type of differential equation that can be used to model a variety of physical phenomena. These equations involve a fractional derivative of a certain order, which is a generalization of the standard derivative. The infinite delay term in the equation allows for the consideration of memory effects, which can be important in many real-world systems. Solving these equations can be challenging, but they can provide valuable insights into the behavior of complex systems [14–17]. In more detail, the existence and uniqueness of mild solutions for impulsive fractional equations with non-local conditions and infinite delay have been concerned in [14]. The existence of solutions for neutral fractional differential equations with indefinite delay is examined using the Banach fixed point theorem and the nonlinear alternative of the Leray-Schauder type [15]. In [16], Santra et al. have discovered a few necessary and sufficient criteria for the oscillation of the solutions to a second-order neutral differential equation. Local estimates, fixed point arguments, and a novel Halanay-type inequality are used to address the dissipativity, stability, and weak stability of solutions for non-local differential equations involving infinite delays [17].

In 2021, Bedi et al. [18] introduced a study about controllability and stability results for fractional evolution equations involving generalized Hilfer fractional derivatives such as

$$\begin{cases} \mathbb{D}_{0^+}^{\mathfrak{r},\mathfrak{y};\mathfrak{x}}\mathscr{E}\mathfrak{U}(t) = \mathbb{A}\mathfrak{U}(t) + \mathscr{E}\mathfrak{H}(t,\mathfrak{U}(t)) + \mathscr{E}(\mathfrak{K}\mathfrak{V}(t)), & t \in \mathfrak{J} = [0,a], \\ \mathscr{E}I_{0^+}^{(1-\mathfrak{r})(1-\mathfrak{y});\mathfrak{x}}\mathfrak{U}(0) = \mathscr{E}\mathfrak{U}_0, & \mathfrak{U}_0 \in D(\mathscr{E}). \end{cases} \tag{1}$$

Such that $\mathbb{D}_{0^+}^{\mathfrak{r},\mathfrak{y};\mathfrak{x}}$ portray the Hilfer fractional derivative of order $0 < \mathfrak{r} < 1$ and type $0 \leq \mathfrak{y} \leq 1$. The control function $\mathfrak{V}(\cdot)$ is defined in the Banach space of admissible control functions $\mathbb{L}^\infty(\mathfrak{J},\mathbb{U})$ and the state $\mathfrak{U}(\cdot)$ takes value in Banach space Ω. Furthermore, $\mathfrak{K}\colon \mathbb{U}) \to D(\mathscr{E})$ is bounded linear operator and $\mathfrak{H}\colon \mathfrak{J}\times\Omega \to D(\mathscr{E}) \subset \Omega$. Therefore, $(\mathbb{A},\mathfrak{E})$ is closed linear operator generates an exponentially bounded propagation family $\{T(t), t \leq 0\}$ from $D(\mathscr{E})$ to Ω. $I_{0^+}^{(1-\mathfrak{r})(1-\mathfrak{y});\mathfrak{x}}$ is the Riemann–Liouville fractional integral of order $(1-\mathfrak{r})(1-\mathfrak{y})$.

In [19], the researchers examined the existence of solutions and the approximate controllability of the Atangana–Baleanu fractional neutral stochastic inclusion with an infinite delay of the form

$$\begin{cases} {}^{ABC}D_{0+}^{\mathfrak{v}}[p(\xi) - N(\xi,p_t)] \in \mathfrak{A}[p(\xi) - N(\xi,p_t)] + Bu(\xi) & \\ +\mathcal{F}(\xi,p_\xi) + G(\xi,p_\xi)\frac{dW(\xi)}{d\xi}, & \xi \in J = [0,c], \\ p(\xi) = \phi(\xi) \in \mathbb{L}^\infty(\Omega,\mathfrak{P}_j U), & \xi \in (-\infty,0]. \end{cases}$$

As above, ${}^{ABC}D^{\mathfrak{v}}$ is the ABC fractional derivative of order $\mathfrak{v} \in (0,1)$, $\mathfrak{A}\colon D(\mathfrak{A}) \subset H \to H$ is infinitesimal generator of an q-resolvent operator $\{S_q(\xi)\}_{\xi\geq 0}$, $\{T_\rho(\xi)\}_{\xi\geq 0}$ is a solution on separable Hilbert space $(H, \|\cdot\|)$.

We are inspired by these masterpieces and hope to establish controllability of mild solution with infinite delay and non-local conditions of the evolution equation

$$\begin{cases} {}_c\mathfrak{D}_0^{\mathfrak{v}}\mathscr{U}(\xi) = \mathbb{A}\mathscr{U}(\xi) + \mathscr{F}(\xi,\mathscr{U}(\xi),\mathscr{U}_\xi) + \mathfrak{B}y(\xi), & \xi \in J = [0,a], \\ \mathscr{U}(\xi) = \phi(\xi), & \xi \in (-\infty,0], \\ \mathscr{U}'(0) + \eta(\mathscr{U}) = \xi_0, & \xi \in \mathfrak{X} \end{cases} \tag{2}$$

where ${}_c\mathfrak{D}_0^{\mathfrak{v}}(\cdot)$ is the Caputo fractional derivative of order $1 < \mathfrak{v} \leq 2$, $\mathscr{F} : [0,a]\times\mathfrak{X}\times\mathcal{P}_{\mathfrak{h}} \to \mathfrak{X}$ is a continuous function, $\phi(\xi) \in \mathscr{P}_{\mathscr{H}}$ ($\mathscr{P}_{\mathscr{H}}$ later judgment will be made over the phase space that is acceptable), a is a finite positive number, the state $\mathscr{U}(\cdot)$ takes values in a Banach space $\mathfrak{X}$, the control function $y(\cdot)$ is given in a Banach space $\mathbb{L}^2(J,\mathbb{U})$ and $\eta(\cdot)$ is a continuous function on $\mathfrak{X}$. Furthermore, $\mathscr{U}_\xi$ represents the state function's history up to the present time ξ, i.e., $\mathscr{U}_\xi(\mathfrak{K}) = \mathscr{U}(\xi+\mathfrak{K})$ for all $\mathfrak{K} \in (-\infty,0]$.

Let $\mathbb{A}$ be an infinitesimal generator of a strongly continuous cosine family $\{\mathscr{K}(\xi)\}_{\xi\geq 0}$ of uniformly bounded linear operators defined on a Banach space $\mathfrak{X}$. The Banach space of continuous and bounded functions from $(-\infty, a]$ into $\mathfrak{X}$ provided with the topology of uniform convergence is denoted by $\mathcal{C} = \mathcal{C}_a((-\infty, a], \mathfrak{X})$ with the norm

$$\|\mathscr{U}\|_{\mathcal{C}} = \sup_{\xi \in (-\infty, a]} |\mathscr{U}(\xi)|$$

and let $(\mathcal{B}(\mathfrak{X}), \|.\|_{(\mathcal{B}(\mathfrak{X}))})$ be the Banach space of all linear and bounded operators from $\mathfrak{X}$ to $\mathfrak{X}$. As $\{\mathscr{K}(\xi)\}_{\xi \geq 0}$ is cosine family on $\mathfrak{X}$, then there exists $\mathfrak{M} \geq 1$ where

$$\|\mathscr{K}(\xi)\| \leq \mathfrak{M}. \tag{3}$$

The fractional derivatives have many different types of definitions, among them Riemann–Liouville, Caputo, Hadamard, Conformable, Katugampola, Hilfer, etc. Riemann–Liouville and Caputo fractional derivatives are the most important ones in the applications of fractional calculus. A close relationship exists between the Riemann–Liouville fractional derivative and the Caputo fractional derivative. The Riemann–Liouville fractional derivative can be converted to the Caputo fractional derivative under some regularity assumptions of the function. However, the Caputo derivative is the most appropriate fractional operator to be used in modeling real-world problems. The Caputo derivative is of use in modeling phenomena that take account of interactions within the past and also problems with non-local properties. Furthermore, the initial conditions take the same form as that for integer-order differential equations, namely, the initial values of integer-order derivatives of functions at starting point [20]. However, the Riemann–Liouville approach needs initial conditions containing the limit values of the Riemann–Liouville fractional derivative at the starting point, whose physical meanings are not very clear.

Partial differential equations with time t as one of the independent variables, or nonlinear evolution equations, can be found in many areas of mathematics as well as in other scientific disciplines including physics, mechanics, and material science. Nonlinear evolution equations include, among others, the Navier–Stokes and Euler equations from fluid mechanics, the nonlinear reaction-diffusion equations from heat transfers and biological sciences, the nonlinear Klein-Gordon equations and nonlinear Schrodinger equations from quantum mechanics, and the Cahn-Hilliard equations from material science (see [21–23] and references cited therein).

Functional evolution equations with infinite-time delay arise often in mathematical modeling of a wide range of real-world issues, and as a result, research into these equations has gotten a lot of interest in recent years (see [24–28]. The time delay in the robot teleoperation system occurs when the system operator and the remote robot are far apart [29]. Zhang et al. [30] used the principle of compressed mapping to discuss the existence and uniqueness of the fractional diffusion equation with time delay. Anilkumar and Jose [31] analyzed a discrete-time queueing inventory model with service time and back-order in inventory. Some results of the existence and uniqueness of fixed points for a C-class of mappings satisfying an inequality of rational type in b-metric spaces have been studied by Asadi and Afsha [32].

The remainder of the text is organized as follows. We introduce some basic ideas and lemmas in Section 2. In Section 3, we formulate the mild solution of (2) by assuming that $\mathbb{A}$ is an infinitesimal generator of a strongly continuous cosine family $\{\mathscr{K}(\xi)\}_{\xi \geq 0}$. In Section 4, we handle the infinite delay by phase space. Section 5 provides the results of our analysis using two cases first in a compact case and second by the measure of the non-compactness technique. Section 6 offers an example that can be used as an application.

2. Preliminaries

In this section, a few concepts and terms related to the components of the research report are offered.

Definition 1 ([33]). *The expression of the Caputo derivative of fractional order* $\mathfrak{q}$ *for at least nth continuously differentiable function* $g\colon [0,\infty) \to \mathbb{R}$ *is*

$$ {}_c\mathfrak{D}^{\mathfrak{q}} g(t) = \frac{1}{\Gamma(n-\mathfrak{q})} \int_0^t (t-s)^{n-\mathfrak{q}-1} g^{(n)}(s) ds, \quad n-1 < \mathfrak{q} < n, n = [\mathfrak{q}] + 1, $$

where $[\mathfrak{q}]$ *denote the integer part of the real number* $\mathfrak{q}$.

Definition 2 ([33]). *Given below is the Laplace transform for the Caputo derivative of order* $\mathfrak{q} \in (1,2]$

$$ \mathcal{L}\{{}_c\mathfrak{D}_t^{\mathfrak{q}} g(t)\} = \lambda^{\mathfrak{q}} G(\lambda) - \lambda^{\mathfrak{q}-1} G(0) + \lambda^{\mathfrak{q}-2} G'(0), $$

where $G(\lambda) = \int_0^\infty e^{-\lambda t} g(t) dt$.

Definition 3 ([33]). *The left fractional integrals of the function* f *is*

$$ \mathcal{I}_a^{\mathfrak{q}} f(t) = \frac{1}{\Gamma(\mathfrak{q})} \int_a^t (t^\rho - s^\rho)^{\mathfrak{q}-1} f(s) ds, \quad t > a, \ \mathfrak{q} > 0. $$

Lemma 1 ([34]). *Let* $n \in \mathbb{N}$, $n-1 < \mathfrak{q} \le n$ *and* $x(t) \in C^n[0,1]$. *Then,*

$$ I_c^{\mathfrak{q}} \mathfrak{D}^{\mathfrak{q}} x(t) = x(t) + a_0 + a_1 t + \cdots + a_{n-1} t^{n-1}. $$

Definition 4 ([35]). *The Kuratowski measure of noncompactness* $\mu(\cdot)$ *is defined on bounded set* S *of Banach space* $\mathfrak{X}$ *as*

$$ \mu(S) \colon = \inf\left\{ \delta > 0 \colon S \subset \bigcup_{i=1}^m S_i, S_i \subset \mathfrak{X}, diam(S_i) < \delta \quad for \quad i = 1,2,\ldots,m; \ m \in \mathbb{N} \right\} $$

where

$$ diam(S_i) = \sup\{\|x_1 - x_2\| : x_1, x_2 \in S_i\}. $$

The following properties of the Kuratowski measure of noncompactness are well-known.

Lemma 2 ([35]). *Let* $\mathscr{T}, \mathscr{R}$ *be bounded in Banach space* $\mathfrak{X}$. *The following properties are satisfied:*

(i) $\mu(\mathscr{T}) = 0$, *if and only if* $\overline{\mathscr{T}}$ *is compact, where* $\overline{\mathscr{T}}$ *means the closure hull of* $\mathscr{T}$;

(ii) $\mu(\mathscr{T}) = \mu(\overline{\mathscr{T}}) = \mu(conv\mathscr{T})$, *where* $conv\mathscr{T}$ *means the convex hull of* $\mathscr{T}$;

(iii) $\mu(k\mathscr{T}) = |k|\mu(\mathscr{T})$ *for any* $k \in \mathbb{R}$;

(iv) $\mathscr{T} \subset \mathscr{R}$ *implies* $\mu(\mathscr{T}) \le \mu(\mathscr{R})$;

(v) $\mu(\mathscr{T} + \mathscr{R}) \le \mu(\mathscr{T}) + \mu(\mathscr{R})$, *where* $\mathscr{T} + \mathscr{R} = \{x | x = y + z, y \in \mathscr{T}, z \in \mathscr{R}\}$;

(vi) $\mu(\mathscr{T} \cup \mathscr{R}) = \max\{\mu\mathscr{T}, \mu\mathscr{R}\}$;

(vii) *If the map* $H : D(H) \subset \mathfrak{X} \to \mathfrak{Y}$ *is Lipschitz continuous with constant* c, *then* $\mu(H(U)) \le c\mu(U)$ *for any bounded subset* $U \in D(H)$, *where* $\mathfrak{Y}$ *is another Banach space.*

Lemma 3 (Sadovskii fixed point theorem [35]). *Let* Ψ *be bounded closed and convex subset in Banach space* $\mathfrak{X}$. *If the operator* $\mathscr{Q} : \Psi \to \Psi$ *is continuous* μ*-condensing, which means that* $\mu(\mathscr{Q}(\Psi)) < \mu(\Psi)$. *Then,* $\mathscr{Q}$ *has at least one fixed point in* Ψ.

Definition 5 ([36]). *Claim that the family of bounded linear operators* $\{\mathscr{K}(t)\}_{t \in \mathbb{R}_+}$, *namely maps the Banach space* $\mathfrak{X} \to \mathfrak{X}$, *has just one parameter, is referred to as a strongly continuous cosine family if and only if*

(i) $\mathscr{K}(0) = I$;

(ii) $\mathscr{K}(s+t) + \mathscr{K}(s-t) = 2\mathscr{L}(s)\mathscr{K}(t)$ *for all* $s, t \in \mathbb{R}_+$;

(iii) $\mathscr{K}(t)x$ *is a continuous on* $\mathbb{R}_+$ *for any* $x \in \mathfrak{X}$.

The substantially continuous cosine family $\{\mathscr{K}(t)\}_{t\in\mathbb{R}_+}$, *which is connected to the sine family* $\{\mathscr{L}(t)\}_{t\in\mathbb{R}_+}$, *is defined by*

$$\mathscr{L}(t)x = \int_0^t \mathscr{K}(s)xds, \quad x \in \mathfrak{X}, t \in \mathbb{R}_+.$$

Lemma 4 ([36]). *Unless* $\mathbb{A}$ *is an infinitesimal generator of a strongly continuous cosine family* $\{\mathscr{K}(t)\}_{t\in\mathbb{R}_+}$ *on a Banach space* $\mathfrak{X}$, *then* $\|\mathscr{K}(t)\|_{\mathcal{B}(\mathfrak{X})} \leq Me^{\xi t}, t \in \mathbb{R}_+$ *will be obtained. Then, given the value of* $\lambda > \xi$ *and* $(\xi^2, \infty) \subset \varrho(\mathbb{A})$ *(the resolvent set of the operator* $\mathbb{A}$*), we obtain*

$$\lambda R(\lambda^2; \mathbb{A})x = \int_0^\infty e^{-\lambda t}\mathscr{K}(t)xdt, \qquad R(\lambda^2; \mathbb{A})x = \int_0^\infty e^{-\lambda t}\mathscr{L}(t)xdt, \quad x \in \mathfrak{X}$$

where the operator $R(\lambda; \mathbb{A}) = (\lambda I - \mathbb{A})^{-1}$ *is the resolvent of the operator* $\mathbb{A}$ *and* $\lambda \in \varrho(\mathbb{A})$.

The operator $\mathbb{A}$ is characterized by

$$\mathbb{A}x = \frac{d^2}{dt^2}\mathscr{K}(0)x, \;\; \forall \;\; x \in \mathcal{D}(\mathbb{A})$$

where $\mathcal{D}(\mathbb{A}) = \{x \in \mathfrak{X} : \mathscr{K}(t)x \in \mathcal{C}^2(\mathbb{R}, \mathfrak{X})\}$. Clearly, the infinitesimal generator $\mathbb{A}$ is a densely defined operator in $\mathfrak{X}$ and closed.

Definition 6. *The Mainardi–Wright-type function when* $t > 0$ *is*

$$M_\rho(t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!\Gamma(1-\rho(n+1))}, \;\; \rho \in (0,1), \;\; t \in \mathbb{C}$$

and achieves

$$M_\rho(t) \geq 0, \qquad \int_0^\infty \theta^\xi M_\rho(\theta)d\theta = \frac{\Gamma(1+\xi)}{\Gamma(1+\rho\xi)}, \qquad \xi > -1.$$

3. Setting of Mild Solution

We first illustrate the following lemma before giving a formulation of the moderate solution of (2).

Lemma 5. *Allow* (2) *to hold. Then, there is*

$$\mathscr{U}(\xi) = \begin{cases} \mathscr{K}_q(\xi)\phi(0) + \int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\mathscr{U}))dt + \int_0^\xi (\xi-t)^{q-1}\mathscr{L}_q(\xi,t)\mathscr{F}(t)dt & \\ + \int_0^\xi (\xi-t)^{q-1}\mathscr{L}_q(\xi,t)\mathfrak{B}y(t)dt, & \xi \in [0,a], \\ \phi(\xi), & \xi \in (-\infty, 0], \end{cases}$$

where $1/2 < q = \frac{\mathfrak{v}}{2} < 1$,

$$\mathscr{K}_q(\xi) = \int_0^\infty M_q(\theta)\mathscr{K}(\xi^q\theta)d\theta,$$
$$\mathscr{L}_q(\xi, s) = q\int_0^\infty \theta M_q(\theta)\mathscr{L}((\xi-s)^q\theta)d\theta,$$

and M_q *is a probability density function defined by Definition 6.*

Proof. Presume that $\lambda > 0$

$$U(\lambda) = \int_0^\infty e^{-\lambda\xi}\mathscr{U}(\xi)d\xi, \qquad F(\lambda) + \mathfrak{B}Y(\lambda) = \int_0^\infty e^{-\lambda\xi}(\mathscr{F}(\xi) + \mathfrak{B}y(\xi))d\xi.$$

Let $\lambda^{\mathfrak{v}} \in \varrho(\mathbb{A})$. Now, that (2) has been transformed using Laplace and Lemma 4, we attain

$$\begin{aligned} U(\lambda) &= (\lambda^{\mathfrak{v}} - \mathbb{A})^{-1}\Big[F(\lambda) + \mathfrak{B}Y(\lambda) + \lambda^{-1}\phi(0) + \lambda^{-2}(\xi_0 - \eta(\mathscr{U}))\Big] \\ &= \lambda^{q-1}\int_0^\infty e^{-\lambda^q s}\mathscr{K}(s)\phi(0)ds + \lambda^{q-2}\int_0^\infty e^{-\lambda^q s}\mathscr{K}(t)(\xi_0 - \eta(\mathscr{U}))ds \\ &+ \int_0^\infty e^{-\lambda^q s}\mathscr{L}(s)[F(\lambda) + \mathfrak{B}Y(\lambda)]ds. \end{aligned}$$

Let $\theta \in (0,\infty)$, $q \in (\frac{1}{2},1)$ and $\Psi_q(\theta) = \frac{q}{\theta^{q+1}}M_q(\theta^{-q})$. Then,

$$\int_0^\infty e^{-\lambda\theta}\Psi_q(\theta)d\theta = e^{-\lambda^q}, for \ \ q \in (\frac{1}{2},1).$$

If we take $\rho \to 0$, we will still have the same answer for the first term in Lemma 5 in [37]. Afterward, we can write:

$$\lambda^{q-1}\int_0^\infty e^{-\lambda^q s}\mathscr{K}(s)\phi(0)ds = \int_0^\infty e^{-\lambda\xi}\mathscr{K}_q(\xi)\phi(0)d\xi.$$

In addition, since $\mathcal{L}[1](\lambda) = \lambda^{-1}$, we obtain

$$\lambda^{-1}\lambda^{q-1}\int_0^\infty e^{-\lambda^q s}\mathscr{K}(s)(\xi_0 - \eta(\mathscr{U}))ds = \int_0^\infty e^{-\lambda\xi}\left\{\int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\mathscr{U}))dt\right\}d\xi.$$

The last term, $\int_0^\infty e^{-\lambda^q s}\mathscr{L}(s)[F(\lambda) + \mathfrak{B}Y(\lambda)]ds$, is identical to the final term in [37] if we set $\rho \to 0$ and set $f(p) = F(\lambda) + \mathfrak{B}Y(\lambda)$, we get

$$\int_0^\infty e^{-\lambda^q s}\mathscr{L}(s)[F(\lambda) + \mathfrak{B}Y(\lambda)]ds = \int_0^\infty e^{-\lambda\xi}\left\{\int_0^\xi (\xi - t)^{q-1}\mathscr{L}_q(\xi,t)[\mathscr{F}(t) + \mathfrak{B}y(t)]dt\right\}d\xi.$$

To sum up, we can obtain

$$\begin{aligned} \int_0^\infty e^{-\lambda\xi}\mathscr{U}(\xi)d\xi = \int_0^\infty e^{-\lambda\xi}\bigg\{&\mathscr{K}_q(\xi)\phi(0) + \int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\mathscr{U}))dt \\ &+ \int_0^\xi (\xi - t)^{q-1}\mathscr{L}_q(\xi,t)[\mathscr{F}(t) + \mathfrak{B}y(t)]dt\bigg\}d\xi. \end{aligned}$$

The intended outcome is attained by using the inverse Laplace transform. □

Definition 7. *A function $\mathscr{U}(\xi) \in (\mathcal{C}(-\infty,a];\mathfrak{X})$ is considered to be the mild solution of (2) if it fulfills*

$$\mathscr{U}(\xi) = \begin{cases} \mathscr{K}_q(\xi)\phi(0) + \int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\mathscr{U}))dt \\ + \int_0^\xi (\xi - t)^{q-1}\mathscr{L}_q(\xi,t)[\mathscr{F}(t,\mathscr{U},\mathscr{U}_t) + \mathfrak{B}y(t)]dt, & \xi \in [0,a], \\ \phi(\xi), & \xi \in (-\infty,0]. \end{cases}$$

Remark 1 ([37]). *It is obvious to infer from the linearity of $\mathscr{K}(\xi)$ and $\mathscr{L}(\xi)$ for any $\xi \geq 0$ that $\mathscr{K}_q(\xi)$ and $\mathscr{L}_q(\xi,s)$ are also linear operators where $0 < s < \xi$.*

As a corollary, when ρ approaches 1, the proofs of all subsequent Lemmas are identical.

Lemma 6 ([37]). *The following estimates for $\mathscr{K}_q(\xi)$ and $\mathscr{L}_q(\xi,s)$ are verified for any fixed $\xi \geq 0$ and $0 < s < \xi$*

$$\left|\mathscr{K}_q(\xi)x\right| \leq \mathfrak{M}|x| \qquad \textit{and} \qquad \left|\mathscr{L}_q(\xi,s)x\right| \leq \frac{\mathfrak{M}a^q}{\Gamma(2q)}|x|.$$

Lemma 7 ([37]). *For any $0 < s < \xi$ and $\xi > 0$, the operators $\mathscr{K}_q(\xi)$ and $\mathscr{L}_q(s,\xi)$ are strongly continuous.*

Lemma 8 ([37]). *Pretend that $\mathscr{K}(\xi)$ and $\mathscr{L}(\xi,s)$ are compact for every $0 < s < t$. In that case, for any $0 < s < \xi$, the operators $\mathscr{K}_q(\xi)$ and $\mathscr{L}_q(s,\xi)$ are compact.*

4. Abstract Phases Space $\mathscr{P}_{\mathscr{H}}$ and Infinite Delay

By using the handy method of [14,15], we demonstrate the abstract phase $\mathscr{P}_{\mathscr{H}}$. Let us say that $\mathscr{H} = C((-\infty,0],[0,\infty))$ with $\int_{-\infty}^{0}\mathscr{H}(t)dt < \infty$ are used. Finally, we have stated that for every $c > 0$

$$\mathscr{P} = \{\mathfrak{A} : [-c,0] \to \mathfrak{X}\ ,\quad \mathfrak{A} \text{ is bounded and measurable}\}$$

identically, create the space $\mathscr{P}$ with

$$\|\mathfrak{A}\|_{\mathscr{P}} = \sup_{s\in[-c,0]} |\mathfrak{A}(s)|, \quad \text{for all} \quad \mathfrak{A} \in \mathscr{P}.$$

Let us specify the space

$$\mathscr{P}_{\mathscr{H}} = \left\{\mathfrak{A} : (-\infty,0] \to \mathfrak{X} \text{ such that for any } \ c > 0, \mathfrak{A}|_{[-c,0]} \in \mathscr{P} \text{ and } \int_{-\infty}^{0} \mathscr{H}(t) \sup_{t\leq s\leq 0} \mathfrak{A}(s)dt < \infty \right\}.$$

If $\mathscr{P}_{\mathscr{H}}$ are configured as

$$\|\mathfrak{A}\|_{\mathscr{P}_{\mathscr{H}}} = \int_{-\infty}^{0} \mathscr{H}(t) \sup_{t\leq s\leq 0} \|\mathfrak{A}(s)\|dt, \ \ \forall\, \mathfrak{A} \in \mathscr{P}_{\mathscr{H}},$$

then $\left(\mathscr{P}_{\mathscr{H}}, \|\cdot\|_{\mathscr{P}_{\mathscr{H}}}\right)$ is a Banach space.
The space is the first thing we consider

$$\overline{\mathscr{P}}_{\mathscr{H}} = \left\{v : (-\infty,a] \to \mathfrak{X} \ \text{ such that } v|_{[0,a]} \text{is continuous, } \ v|_{(-\infty,0]} = \phi \in \mathscr{P}_{\mathscr{H}} \right\}$$

which has the norm

$$\|x\|_{\overline{\mathscr{P}}_{\mathscr{H}}} = \sup_{s\in[0,a]} \|v(s)\| + \|\phi\|_{\mathscr{P}_{\mathscr{H}}}.$$

Definition 8 ([38]). *The prerequisites are true $\forall t \in [0,a]$. If $v : (-\infty,a] \to \mathfrak{X}$, $a > 0$, such that $\phi \in \mathscr{P}_{\mathscr{H}}$:*

1. $v_\tau \in \mathscr{P}_{\mathscr{H}}$;
2. *There are two function $\beta_1(t), \beta_2(t)$ such that $\beta_1(t)\colon [0,\infty) \to [0,\infty)$ is a continuous function and $\beta_2(t)\colon [0,\infty) \to [0,\infty)$ is a locally bounded function which are independent to $v(\cdot)$ whereas*
$$\|v_t\|_{\mathscr{P}_{\mathscr{H}}} \leq \beta_1(t) \sup_{0<s<t} \|v(s)\| + \beta_2(t)\|\phi\|_{\mathscr{P}_{\mathscr{H}}};$$
3. $\|v(t)\| \leq H\|v_t\|_{\mathscr{P}_{\mathscr{H}}}$, *where $H > 0$ is constant.*

Currently, the operator is defined $\mathscr{H} : \overline{\mathscr{P}}_{\mathscr{H}} \to \overline{\mathscr{P}}_{\mathscr{H}}$ as follows

$$\mathscr{H}(\mathscr{U})(\xi) = \begin{cases} \mathscr{K}_q(\xi)\phi(0) + \int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\mathscr{U}))dt \\ + \int_0^\xi (\xi - t)^{q-1}\mathscr{L}_q(\xi, t)[\mathscr{F}(t) + \mathfrak{B}y(t)]dt, & \xi \in [0, a], \\ \phi(\xi), & \xi \in (-\infty, 0]. \end{cases}$$

The function represented by $\varkappa(\cdot) : (-\infty, a] \to \mathfrak{X}$ should be considered as

$$\varkappa(\xi) = \begin{cases} 0, & \xi \in (0, a], \\ \phi(\xi), & \xi \in (-\infty, 0]. \end{cases}$$

After that, $\varkappa(0) = \phi(0)$. We indicate the function defined by κ for each $\mathscr{Z} \in C([0, a], \mathfrak{X})$ with $\mathscr{Z}(0) = 0$ and

$$\kappa(\xi) = \begin{cases} \mathscr{Z}(\xi), & \xi \in [0, a], \\ 0, & \xi \in (-\infty, 0]. \end{cases}$$

If $\mathscr{U}(\cdot)$ satisfies that $\mathscr{U}(\xi) = \mathscr{H}(\mathscr{U})(\xi)$ for all $\xi \in (-\infty, a]$, we can decompose that $\mathscr{U}(\xi) = \kappa(\xi) + \varkappa(\xi)$, $\xi \in (-\infty, a]$, it denotes $\mathscr{U}_\xi = \kappa_\xi + \varkappa_\xi$ for every $\xi \in (-\infty, a]$ and the function $\mathscr{Z}(\cdot)$ satisfies

$$\mathscr{Z}(\xi) = \mathscr{K}_q(\xi)\phi(0) + \int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\kappa + \varkappa))dt \\ + \int_0^\xi (\xi - t)^{q-1}\mathscr{L}_q(\xi, t)[\mathscr{F}(t, \kappa + \varkappa, \kappa_t + \varkappa_t) + \mathfrak{B}y(t)]dt.$$

Set the space $\Theta = \{\mathscr{Z} \in C([0, a], \mathfrak{X}), \mathscr{Z}(0) = 0\}$ equipped the norm

$$\|\mathscr{Z}\|_\Theta = \sup_{\xi \in [0, a]} \|\mathscr{Z}(\xi)\|.$$

Therefore, $(\Theta, \|\cdot\|_\Theta)$ is a Banach space. Assume that the operator $\mathfrak{G}$ is defined as follows: Let the operator $\mathfrak{G} : \Theta \to \Theta$ be formulated as follows:

$$\mathfrak{G}(\mathscr{Z})(\xi) = \mathscr{K}_q(\xi)\phi(0) + \int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\kappa + \varkappa))dt \\ + \int_0^\xi (\xi - t)^{q-1}\mathscr{L}_q(\xi, t)[\mathscr{F}(t, \kappa + \varkappa, \kappa_t + \varkappa_t) + \mathfrak{B}y(t)]dt.$$

The argument that the operator $\mathscr{H}$ appears to have a fixed point is similar to the claim that $\mathfrak{G}$ has a fixed point. Therefore, we continue to demonstrate this.

The subsequent assumptions, we make:

$(\mathcal{I}_1)$ The function $\mathscr{F} : J \times \mathfrak{X} \times \mathscr{P}_{\mathscr{H}} \to \mathfrak{X}$ is a continuous and there exist $d_{1f}, d_{2f} \geq 0$ such that for all $(\xi, \mathscr{U}, \mathscr{U}_\xi), (\xi, \mathscr{V}, \mathscr{V}_\xi) \in J \times \mathfrak{X} \times \mathscr{P}_{\mathscr{H}}$,

$$\|\mathscr{F}(\xi, \mathscr{U}, \mathscr{U}_\xi) - \mathscr{F}(\xi, \mathscr{V}, \mathscr{V}_\xi)\| \leq d_{1f}\|\mathscr{U} - \mathscr{V}\|_{\mathfrak{X}} + d_{2f}\|\mathscr{U}_\xi - \mathscr{V}_\xi\|_{\mathscr{P}_{\mathscr{H}}}.$$

$(\mathcal{I}_2)$ The linear operator $\mathscr{B} : \mathbb{U} \to \mathfrak{X}$ is bounded, and let $\mathbb{W} : \mathbb{L}^2(J, \mathbb{U}) \to \mathfrak{X}$ be the linear operator defined by

$$\mathbb{W}y = \int_0^a (a - t)^{q-1}\mathscr{L}_q(a, t)\mathfrak{B}y(t)dt,$$

has an invertible operator $\mathbb{W}^{-1}$ which takes value in $\mathbb{L}^2(J, \mathbb{U})/ker\mathbb{W}$, and there exist two positive constant $\mathfrak{O}_1$ and $\mathfrak{O}_1$ such that

$$\|\mathfrak{B}\| \leq \mathfrak{O}_1, \qquad \|\mathbb{W}^{-1}\| \leq \mathfrak{O}_2.$$

$(\mathcal{I}_3)$ The function $\eta\colon \mathfrak{X} \to \mathfrak{X}$ is continuous and there exist there exist a positive constant L_η such that

$$\|\eta(\mathscr{U}) - \eta(\mathscr{V})\| \leq L_\eta \|\mathscr{U} - \mathscr{V}\|.$$

Lemma 9. *Let $\beta_1^* = \sup_{\xi\in[0,a]} \beta_1(\xi)$ and $\beta_2^* = \sup_{\xi\in[0,a]} \beta_2(\xi)$ where $\beta_1(\cdot)$ and $\beta_2(\cdot)$ be defined in Definition (8). Assume that the assumptions $(\mathcal{I}_1)$ and $(\mathcal{I}_3)$ are satisfied with $\mathfrak{c} = \max\limits_{\xi\in[0,a]} |\mathscr{F}(\xi,0,0)|$ and $\gamma_\eta = |\eta(0)|$. Then,*

$$\begin{aligned}\|\mathscr{F}(\xi,\kappa+\varkappa,\kappa_\xi+\varkappa_\xi)\| &\leq \Big(d_{1f}H + d_{2f}\Big)\big(\beta_1(\xi)\|\mathscr{Z}\|_\Theta + \beta_2(\xi)\|\phi\|_{\mathscr{P}_\mathscr{H}}\big) + \mathfrak{c}\\ &\leq \Big(d_{1f}H + d_{2f}\Big)\big(\beta_1^*\|\mathscr{Z}\|_\Theta + \beta_2^*\|\phi\|_{\mathscr{P}_\mathscr{H}}\big) + \mathfrak{c} \triangleq \ell\end{aligned}$$

and

$$\eta(\mathscr{U})\| \leq L_\eta\|\mathscr{U}\| + \gamma_\eta.$$

Proof. By the same way in Lemma 9 in [37], we can easily reach the desired result. □

5. Controllability Results

Definition 9 ([39])**.** *The system (2) is said to be controllable on the interval J if for any $\phi(0) \in \mathscr{P}_\mathscr{H}$ and $\xi_0, y_a \in \mathfrak{X}$, there exists a control $y \in \mathbb{L}^2(J,\mathbb{U})$ such that a mild solution $\mathscr{Z}(\cdot)$ of system (2) satisfies $\mathscr{Z}(a) = y_a$.*

Lemma 10. *If the assumptions $(\mathcal{I}_1)$ and $(\mathcal{I}_3)$ hold, and $y_a \in \mathfrak{X}$ is target point. Then the control function*

$$\begin{aligned}y(\xi) = \mathbb{W}^{-1}\Bigg[&y_a - \mathscr{K}_q(a)\phi(0) + \int_0^a \mathscr{K}_q(t)(\xi_0 - \eta(\kappa+\varkappa))dt\\ &+ \int_0^a (a-t)^{q-1}\mathscr{L}_q(a,t)\mathscr{F}(t,\kappa+\varkappa,\kappa_t+\varkappa_t)dt\Bigg].\end{aligned}$$

steers the state $\mathscr{Z}(\xi)$ of the system (2) from initial points $\phi(0)$ and ξ_0 to target point y_a at time a. Furthermore, the control function $y(\xi)$ has an estimate $\|y(\xi\| \leq \Pi$ where

$$\Pi = \mathfrak{O}_2[\|y_a\| + \mathscr{T}_0 + \mathscr{M}_0\ell], \quad \mathscr{T}_0 = \mathfrak{M}\big(\|\phi(0)\| + a(\|\xi_0\| + \gamma_\eta)\big), \quad \text{and} \quad \mathscr{M}_0 = \frac{a^{2q}\mathfrak{M}}{q\Gamma(2q)}.$$

Proof. Consider the solution $\mathscr{Z}(\xi)$ of (2) defined by (7). For $\xi = a$, we get

$$\begin{aligned}\mathscr{Z}(a) &= \mathscr{K}_q(a)\phi(0) + \int_0^a \mathscr{K}_q(t)(\xi_0 - \eta(\kappa+\varkappa))dt + \int_0^a (a-t)^{q-1}\mathscr{L}_q(a,t)\mathscr{F}(\tau,\kappa+\varkappa,\kappa_\tau+\varkappa_\tau)d\tau dt\\ &+ \int_0^a (a-t)^{q-1}\mathscr{L}_q(a,t)\mathfrak{B}\mathbb{W}^{-1}\Bigg[y_a - \mathscr{K}_q(a)\phi(0) + \int_0^a \mathscr{K}_q(\tau)(\xi_0 - \eta(\kappa+\varkappa))d\tau\\ &+ \int_0^a (a-\tau)^{q-1}\mathscr{L}_q(a,t)\mathscr{F}(\tau,\kappa+\varkappa,\kappa_\tau+\varkappa_\tau)d\tau\Bigg]dt\\ &= \mathscr{K}_q(a)\phi(0) + \int_0^a \mathscr{K}_q(t)(\xi_0 - \eta(\kappa+\varkappa))dt + \int_0^a (a-t)^{q-1}\mathscr{L}_q(a,t)\mathscr{F}(\tau,\kappa+\varkappa,\kappa_\tau+\varkappa_\tau)d\tau dt\\ &+ \mathbb{W}\mathbb{W}^{-1}\Bigg[y_a - \mathscr{K}_q(a)\phi(0) + \int_0^a \mathscr{K}_q(\tau)(\xi_0 - \eta(\kappa+\varkappa))d\tau\\ &+ \int_0^a (a-\tau)^{q-1}\mathscr{L}_q(a,\tau)\mathscr{F}(\tau,\kappa+\varkappa,\kappa_\tau+\varkappa_\tau)d\tau\Bigg] = y_a.\end{aligned}$$

Furthermore, by using Lemma 9 the control function estimate

$$\begin{aligned}\|y(\xi)\| &\leq \left\|\mathbb{W}^{-1}\right\|\bigg[\|y_a\| + \|\mathscr{K}_q(a)\phi(0)\| + \int_0^a \|\mathscr{K}_q(t)\|(\|\xi_0\| + \|\eta(\kappa+\varkappa)\|)dt \\ &\quad + \int_0^a (a-t)^{q-1}\|\mathscr{L}_q(a,t)\|\|\mathscr{F}(t,\kappa+\varkappa,\kappa_t+\varkappa_t)\|dt\bigg] \\ &\leq \mathfrak{O}_2\left[\|y_a\| + \mathfrak{M}(\|\phi(0)\| + a(\|\xi_0\| + \gamma_\eta)) + \frac{a^{2q}\mathfrak{M}\ell}{q\Gamma(2q)}\right] = \Pi\end{aligned}$$

which ends the proof. □

5.1. Compactness Case

In this subsection, we assume the compactness of controllability of mild solution and investigate its existence of it by employing Krasnoselskii's fixed point theorem to deduce the first result about the existence of the solution of the problem (2).

Theorem 1. *Assume that $(\mathcal{I}_1)$, $(\mathcal{I}_2)$ and $(\mathcal{I}_3)$ are satisfied. Then the problem (2) is controllable on J if*

$$\mathcal{L}_{\mathfrak{v}} = \mathscr{M}_1\left[a\mathfrak{M}L_\eta + \mathscr{M}_0\beta_1^*(d_{1f}H + d_{2f})\right] < 1$$

where $\mathscr{M}_1 = \mathfrak{O}_1\mathfrak{O}_2\mathscr{M}_0$.

Proof. Designate

$$\Upsilon_\rho = \{\mathscr{Z} \in \theta : \|\mathscr{Z}\|_\theta \leq \rho\}$$

where

$$\rho \geq \frac{(1+\mathscr{M}_1)\left\{\mathscr{T}_0 + \mathscr{M}_0[\left(d_{1f}H + d_{2f}\right)\beta_2^*\|\phi\|_{\mathscr{P}_{\mathscr{H}}} + \mathfrak{c}]\right\} + \mathscr{M}_1\|y_a\|}{1-\mathcal{L}_{\mathfrak{v}}}.$$

The operator $\mathfrak{G}$ can be divided as a sum of two operators $\mathfrak{G}_1$ and $\mathfrak{G}_2$ which can be defined as

$$\begin{aligned}(\mathfrak{G}_1\mathscr{Z})(\xi) &= \mathscr{K}_q(\xi)\phi(0) + \int_0^\xi \mathscr{K}_q(t)(\xi_0 - \eta(\kappa+\varkappa))dt \\ &\quad + \int_0^\xi (\xi-t)^{q-1}\mathscr{L}_q(\xi,t)\left[\mathscr{F}(t,\kappa+\varkappa,\kappa_t+\varkappa_t) + \mathfrak{B}\mathbb{W}^{-1}(y_a - \mathscr{K}_q(a)\phi(0))\right]dt, \\ (\mathfrak{G}_2\mathscr{Z})(\xi) &= \mathfrak{B}\mathbb{W}^{-1}\int_0^\xi (\xi-t)^{q-1}\mathscr{L}_q(\xi,t)\bigg[\int_0^a \mathscr{K}_q(\tau)(\xi_0 - \eta(\kappa+\varkappa))d\tau \\ &\quad + \int_0^a (a-\tau)^{q-1}\mathscr{L}_q(a,\tau)\mathscr{F}(\tau,\kappa+\varkappa,\kappa_\tau+\varkappa_\tau)d\tau\bigg]dt.\end{aligned}$$

Then, for $u, v \in \Upsilon_\rho$, it follows that $\|\mathfrak{G}_1(\mathscr{Z})u + \mathfrak{G}_2(\mathscr{Z})v\| \leq \rho$, which concludes that $\mathfrak{G}_1(u) + \mathfrak{G}_2(v) \in \Upsilon_\rho$. Now, we want to show that $\mathfrak{G}$ maps bounded sets into the bounded set. For any $\rho \geq 0$ and for any $\mathscr{Z} \in \Upsilon_\rho$ and in light of Lemma 9, we have

$$
\begin{aligned}
\|(\mathfrak{G}\mathscr{Z})(\xi)\| \le{}& \mathfrak{M}\big(\|\phi(0)\| + a(\|\xi_0\| + \gamma_\eta))\big) \\
&+ \mathscr{M}_0\Big[\Big(d_{1f}H + d_{2f}\Big)(\beta_1^*\|\mathscr{Z}\|_\Theta + \beta_2^*\|\phi\|_{\mathscr{P}_\mathscr{H}}) + \mathfrak{c}\Big] \\
&+ \mathscr{M}_1\big[\|y_a\| + \mathfrak{M}\big(\|\phi(0)\| + a(\|\xi_0\| + \gamma_\eta)\big) \\
&+ \mathscr{M}_0\Big[(d_{1f}H + d_{2f})(\beta_1^*\|\mathscr{Z}\|_\Theta + \beta_2^*\|\phi\|_{\mathscr{P}_\mathscr{H}}) + \mathfrak{c}\Big]\Big] \\
&= (1 + \mathscr{M}_1)\Big\{\mathscr{T}_0 + \mathscr{M}_0\big[\Big(d_{1f}H + d_{2f}\Big)\beta_2^*\|\phi\|_{\mathscr{P}_\mathscr{H}} + \mathfrak{c}\big]\Big\} + \mathscr{M}_1\|y_a\| \\
&+ \rho\mathscr{M}_0\beta_1^*(1 + \mathscr{M}_1)(d_{1f}H + d_{2f})\rho \le \rho.
\end{aligned}
$$

The following step is to confirm that the operator $\mathfrak{G}_1$ is equicontinuous. In the light of the situations $(\mathcal{I}_1)$ and $(\mathcal{I}_3)$, $\mathfrak{G}_1$ is continuous. Let $v_1, v_2 \in J$ such that $0 \le v_1 < v_2 \le a$, then the following scenarios are therefore possible.

$$
\begin{aligned}
&\|(\mathfrak{G}_1\mathscr{Z})(v_2) - (\mathfrak{G}_1\mathscr{Z})(v_1)\| \le \|\mathscr{K}_q(v_2) - \mathscr{K}_q(v_1)\|\|\phi(0)\| + \mathfrak{M}\big(\|\xi_0\| + \gamma_\eta\big)(v_2 - v_1) \\
&\left[\frac{\mathfrak{M}\ell}{q\Gamma(2q)} + \frac{\mathfrak{M}\mathfrak{O}_1\mathfrak{O}_2}{q\Gamma(2q)}(\|y_a\| + \mathfrak{M}\|\phi(0)\|)\right](v_2 - v_1)^q \\
&+ (\ell + \mathfrak{O}_1\mathfrak{O}_2(\|y_a\| + \mathfrak{M}\|\phi(0)\|)) \int_0^{v_1} \Big\|(v_2 - t)^{q-1}\mathscr{L}_q(v_2, t) - (v_1 - t)^{q-1}\mathscr{L}_q(v_1, t)\Big\| dt.
\end{aligned}
$$

To evaluate the last term, we can follow the steps

$$
\begin{aligned}
I &= \int_0^{v_1} \Big\|(v_2 - t)^{q-1}\mathscr{L}_q(v_2, t) - (v_1 - t)^{q-1}\mathscr{L}_q(v_2, t) + (v_1 - t)^{q-1}\mathscr{L}_q(v_2, t) - (v_1 - t)^{q-1}\mathscr{L}_q(v_1, t)\Big\| dt \\
&\le \int_0^{v_1} [(v_1 - t)^{q-1} - (v_2 - t)^{q-1}]\|\mathscr{L}_q(v_2, t)\| dt + \int_0^{v_1} (v_1 - t)^{q-1}\|\mathscr{L}_q(v_2, t) - \mathscr{L}_q(v_1, t)\| dt \\
&= \frac{\mathfrak{M}}{q\Gamma(2q)}\Big[(v_2 - v_1)^q + (v_1^q - v_2^q)\Big] + \int_0^{v_1} (v_1 - t)^{q-1}\|\mathscr{L}_q(v_2, t) - \mathscr{L}_q(v_1, t)\| dt
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|(\mathfrak{G}_1\mathscr{Z})(v_2) - (\mathfrak{G}_1\mathscr{Z})(v_1)\| \le \|\mathscr{K}_q(v_2) - \mathscr{K}_q(v_1)\|\|\phi(0)\| + \mathfrak{M}\big(\|\xi_0\| + \gamma_\eta\big)(v_2 - v_1) \\
&+ \left[\frac{\mathfrak{M}\ell}{q\Gamma(2q)} + \frac{\mathfrak{M}\mathfrak{O}_1\mathfrak{O}_2}{q\Gamma(2q)}(\|y_a\| + \mathfrak{M}\|\phi(0)\|)\right](v_2 - v_1)^q \\
&+ (\ell + \mathfrak{O}_1\mathfrak{O}_2(\|y_a\| + \mathfrak{M}\|\phi(0)\|))\frac{\mathfrak{M}}{q\Gamma(2q)}\Big[(v_2 - v_1)^q + (v_1^q - v_2^q)\Big] \\
&+ (\ell + \mathfrak{O}_1\mathfrak{O}_2(\|y_a\| + \mathfrak{M}\|\phi(0)\|)) \int_0^{v_1} \Big\|(v_2 - t)^{q-1}\mathscr{L}_q(v_2, t) - (v_1 - t)^{q-1}\mathscr{L}_q(v_1, t)\Big\| dt.
\end{aligned}
$$

Due to compactness of operator $\mathscr{K}_q(y)$ and $\mathscr{L}_q(t, y)$ (see Lemma 8), we infer that $\|\mathfrak{G}_1(z)(v_1) - \mathfrak{G}_1(z)(v_2)\| \to 0$ as $v_2 \to v_1$. Thus, $\mathfrak{G}_1$ is a relatively compact on Υ_ρ. By Arezela Ascoli Theorem the operator $\mathfrak{G}_1$ is completely continuous on Υ_ρ. The only thing left to do is provide evidence that $\mathfrak{G}_2$ is a contraction mapping. Consider $\mathscr{Z}, \mathscr{Z}^* \in \Upsilon$. Then, for any $\xi \in [0, a]$,

$$
\begin{aligned}
&\|(\mathfrak{G}_2\mathscr{Z})(\xi)-(\mathfrak{G}_2\mathscr{Z}^*)(\xi)\|_{\Upsilon}\\
&\leq \frac{\mathfrak{O}_1\mathfrak{O}_2\mathfrak{M}}{\Gamma(2q)}\int_0^{\xi}(\xi-t)^{q-1}\Big[\mathfrak{M}\int_0^a\|\eta(\kappa+\varkappa)-\eta(\kappa^*+\varkappa)\|d\tau\\
&+\frac{\mathfrak{M}}{\Gamma(2q)}\int_0^a(a-\tau)^{q-1}\|\mathscr{F}(\tau,\kappa+\varkappa,\kappa_\tau+\varkappa_\tau)-\mathscr{F}(\tau,\kappa^*+\varkappa,\kappa_\tau+\varkappa_\tau)\|d\tau\Big]\\
&\leq \mathscr{M}_1\Big[a\mathfrak{M}L_\eta\|\kappa-\kappa^*\|_{\Upsilon}+\frac{\mathfrak{M}}{\Gamma(2q)}\int_0^a(a-\tau)^{q-1}(d_{1f}\|\kappa-\kappa^*\|_{\Upsilon}+d_{2f}\|\kappa_\tau-\kappa_\tau^*\|_{\mathscr{P}_{\mathscr{H}}})d\tau\Big]\\
&\leq \mathscr{M}_1\Big[a\mathfrak{M}L_\eta\|\kappa-\kappa^*\|_{\Upsilon}+\frac{\mathfrak{M}}{\Gamma(2q)}\int_0^a(a-\tau)^{q-1}(d_{1f}H+d_{2f})\|\kappa_\tau-\kappa_\tau^*\|_{\mathscr{P}_{\mathscr{H}}}d\tau\Big]\\
&\leq \mathscr{M}_1\Big[a\mathfrak{M}L_\eta+\mathscr{M}_0\beta_1^*(d_{1f}H+d_{2f})\Big]\|\kappa-\kappa^*\|_{\Upsilon}\\
&=\mathcal{L}_{\mathfrak{v}}\|\kappa-\kappa^*\|_{\Upsilon}.
\end{aligned}
$$

In a sense, the fractional evolution equation with non-instantaneous impulsive (2) has at least one mild solution on Υ, according to the Krasnoselskii Theorem. In view of the results in Lemma 10 and our results here, the evolution system (2) is controllable on J. The evidence is now complete. □

5.2. Noncompactness Case

The existence of a solution in the case of noncompactness of controllability of mild solution can be further explored by utilizing Kuratowski's measure of noncompactness through applying Sadovskii's fixed point Theorem 3. This matter can be addressed by considering the next existence result.

Theorem 2. *Assume that $(\mathcal{I}_1)$, $(\mathcal{I}_2)$ and $(\mathcal{I}_3)$ are satisfied. Furthermore, suppose that the following inequality holds*

$$
\mathfrak{P}_{\mathfrak{v}}=(1+\mathscr{M}_1)\Big[a\mathfrak{M}L_\eta+\mathscr{M}_0\beta_1^*(d_{1f}H+d_{2f})\Big]<1.
$$

Then, the evolution system (2) is controllable on J.

Proof. Firstly, we show that $\mathfrak{G}\colon\Upsilon_\rho\to\Upsilon_\rho$ is continuous where $\Upsilon_\rho\subset\theta$ is defined in the proof of Theorem 1. Plainly, the subset Υ_ρ is a closed, bounded, and convex nonempty subset of the Banach space θ. Let the sequence $\{\mathscr{Z}^n\}_{n\in\mathbb{N}}$ of a Banach space θ such that $\mathscr{Z}^n\to\mathscr{Z}$ as $n\to\infty$. For $0\leq\xi\leq a$, by the strongly continuity of $\mathscr{K}_q(\xi)$ and $\mathscr{L}_q(\xi,t)$ and Lemma 9, we get

$$
\begin{aligned}
\|(\mathfrak{G}\mathscr{Z}^n)(\xi)-(\mathfrak{G}\mathscr{Z})(\xi)\|&\leq\mathfrak{M}\int_0^{\xi}\|\eta(\kappa^n+\varkappa)-\eta(\kappa+\varkappa)\|dt\\
&+\frac{\mathfrak{M}}{\Gamma(2q)}\int_0^{\xi}(\xi-t)^{q-1}\|\mathscr{F}(t,\kappa^n+\varkappa,\kappa_t^n+\varkappa_t)-\mathscr{F}(t,\kappa+\varkappa,\kappa_t+\varkappa_t)\|dt\\
&\leq\mathfrak{M}L_\eta\int_0^{\xi}\|\kappa^n-\kappa\|_{\Upsilon}dt+\frac{\mathfrak{M}}{\Gamma(2q)}\int_0^{\xi}(\xi-t)^{q-1}\Big(d_{1f}\|\kappa^n-\kappa\|_{\Upsilon}+d_{2f}\|\kappa_t^n-\kappa_t\|_{\mathscr{P}_{\mathscr{H}}}\Big)dt\\
&\leq\mathfrak{M}L_\eta\int_0^{\xi}\|\kappa^n-\kappa\|_{\Upsilon}dt+\frac{\mathfrak{M}}{\Gamma(2q)}\int_0^{\xi}(\xi-t)^{q-1}\Big(d_{1f}H+d_{2f}\Big)\|\kappa_t^n-\kappa_t\|_{\mathscr{P}_{\mathscr{H}}}dt\\
&\leq\Big[a\mathfrak{M}L_\eta+\mathscr{M}_0\beta_1^*\Big(d_{1f}H+d_{2f}\Big)\Big]\|\mathscr{Z}^n-\mathscr{Z}\|_{\Upsilon}\to 0
\end{aligned}
$$

as $n\to\infty$ which implies that $\mathfrak{G}\colon\Upsilon_\rho\to\Upsilon_\rho$ is continuous.

Next, we show $\mathfrak{G}$ maps Υ_ρ into itself. It is verified as in Theorem 1. The operator $\mathfrak{G}$ must be shown to satisfy the inequality of the Kuratowski measure of noncompactness in

Lemma 3 as the last phase of this argument. Indeed, consider $\mathscr{X}, \mathscr{X}^* \in \mathrm{Y}_r$. Then, for any $\xi \in [0, a]$, with using the assumptions $(\mathcal{I}_1)$-$(\mathcal{I}_3)$, we get

$$\begin{aligned}
\|(\mathfrak{G}_1\mathscr{X})(\xi) - (\mathfrak{G}_1\mathscr{X}^*)(\xi)\|_{\mathrm{Y}} \leq{}& \mathfrak{M}\int_0^{\xi} \|\eta(\kappa + \varkappa) - \eta(\kappa^* + \varkappa)\| dt \\
&+ \frac{\mathfrak{M}a^q}{\Gamma(2q)}\int_0^{\xi} (\xi - t)^{q-1}\|\mathscr{F}(t, \kappa + \varkappa, \kappa_t + \varkappa_t) - \mathscr{F}(t, \kappa^* + \varkappa, \kappa_t^* + \varkappa_t)\| dt \\
&\leq \Big[a\mathfrak{M}L_\eta + \mathscr{M}_0\beta_1^*\Big(d_{1f}H + d_{2f}\Big)\Big]\|\mathscr{X} - \mathscr{X}^*\|_{\mathrm{Y}}.
\end{aligned}$$

By exploiting the results obtained in the previous theorem, we find that

$$\begin{aligned}
\|(\mathfrak{G}\mathscr{X})(\xi) - (\mathfrak{G}\mathscr{X}^*)(\xi)\|_{\mathrm{Y}} &\leq \|(\mathfrak{G}_1\mathscr{X})(\xi) - (\mathfrak{G}_1\mathscr{X}^*)(\xi)\|_{\mathrm{Y}} + \|(\mathfrak{G}_2\mathscr{X})(\xi) - (\mathfrak{G}_2\mathscr{X}^*)(\xi)\|_{\mathrm{Y}} \\
&\leq (1 + \mathscr{M}_1)\Big[a\mathfrak{M}L_\eta + \mathscr{M}_0\beta_1^*(d_{1f}H + d_{2f})\Big]\|\mathscr{X} - \mathscr{X}^*\|_{\mathrm{Y}}
\end{aligned}$$

which implies that

$$\|(\mathfrak{G}\mathscr{X})(\xi) - (\mathfrak{G}\mathscr{X}^*)(\xi)\|_{\mathrm{Y}} \leq \mathfrak{P}_\rho\|\mathscr{X} - \mathscr{X}^*\|_{\mathrm{Y}}.$$

Let $U \subset \mathrm{Y}_\rho$ be closed such that there are $U_i,\ i = 1, 2, \dots, n;\ n \in \mathbb{N}$ and $U \subseteq \bigcup_{i=1}^n U_i$. Then, according to the definitions of diameter and Kuratowski measure of noncompactness, we conclude that

$$\begin{aligned}
\mu(\mathfrak{G}U) &= \inf\left\{r \colon diam(\mathfrak{G}U_i) \leq r,\ U \subseteq \bigcup_{i=1}^n U_i\right\} \\
&= \inf\left\{r \colon \sup\{\|(\mathfrak{G}\mathscr{X})(\xi) - (\mathfrak{G}\mathscr{X}^*)(\xi)\|_{\mathrm{Y}}\} \leq r, \mathscr{X}, \mathscr{X}^* \in U_i, U \subseteq \bigcup_{i=1}^n U_i\right\} \\
&\leq \mathfrak{P}_\rho \inf\left\{r \colon \sup\{\|\mathscr{X}(\xi) - \mathscr{X}^*(\xi)\|_{\mathrm{Y}}\} \leq r, \mathscr{X}, \mathscr{X}^* \in U_i, U \subseteq \bigcup_{i=1}^n U_i\right\} \\
&= \mathfrak{P}_\rho \inf\left\{r \colon diam(U_i) \leq r,\ U \subseteq \bigcup_{i=1}^n U_i\right\} \\
&= \mathfrak{P}_\rho \mu(U).
\end{aligned}$$

By Lemma 2 (vii), we know that for any bounded $U \in \mathrm{Y}_\rho$

$$\mu(\mathfrak{G}(U)) \leq \mathfrak{P}_{\mathfrak{v}}\mu(U).$$

This means that the operator $\mathfrak{G} \colon \mathrm{Y}_\rho \to \mathrm{Y}_\rho$ is μ-condensing. It follows from Sadovskii fixed point theorem the operator $\mathfrak{G}$ has at least one fixed point $\mathscr{X} \in \mathrm{Y}_\rho$, which is just a mild solution to problem (2). This with Lemma 10 completes the proof. □

6. An Application

Consider the following fractional evolution with infinite delay

$$\begin{cases}
{}_cD_0^{\frac{5}{3}}\mathscr{U}(\xi, x) = \mathbb{A}\mathscr{U}(\xi, x) + \mathscr{F}(\xi, \mathscr{U}(\xi, x), \mathscr{U}_\xi(\xi, x)) + \mathfrak{B}y(\xi, x), & \xi \in [0, 1], x \in [0, \pi] \\
\mathscr{U}(\xi, x) = \frac{1}{5}e^{-0.5\xi}, & \xi \in (-\infty, 0], x \in [0, \pi] \\
\mathscr{U}'(0, x) + \frac{1}{13}\sin\mathscr{U}(\xi, x) = \frac{1}{2}, & \xi \in [0, 1], x \in [0, \pi] \\
\mathscr{U}(\xi, 0) = \mathscr{U}(\xi, 1) = 0, & \xi \in [0, 1].
\end{cases}$$

Let the space $\mathfrak{X} = C([0, 1] \times [0, \pi], \mathbb{R})$ and $\mathbb{U} = L^2[0, 1]$ the space of a square-integrable function equipped with the norm

$$\|\mathscr{U}\|_{L^2[0,1]} = \left(\int_0^1 |\mathscr{U}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

Furthermore, the operator $\mathbb{A}\colon D(\mathbb{A}) \subset \mathfrak{X} \to \mathfrak{X}$ is defined as $\mathbb{A} = \frac{\partial^2}{\partial x^2}$ with a domain

$$D(\mathbb{A}) = \left\{ \mathscr{U} \in \mathfrak{X} | \frac{\partial}{\partial x}\mathscr{U}, \frac{\partial^2}{\partial x^2}\mathscr{U} \in \mathfrak{X} \right\}$$

Apparently, the operator $\mathbb{A}$ is densely defined in $\mathfrak{X}$ and is the infinitesimal generator of a resolvent cosine family $\mathscr{K}(\xi), \xi > 0$ on $\mathfrak{X}$. Here, we take $\mathfrak{v} = \frac{5}{3}$ which implies $q = \frac{5}{6}$ and $\mathbb{A} = \frac{\partial^2}{\partial x^2}, x \in [0,\pi]$, we take $H = \frac{1}{16}$, $\beta_1(\xi) = \frac{\xi^2+1}{5} \to \beta_1^* = \frac{2}{5}$, $\beta_2(\xi) = \frac{1}{\sqrt{\xi+1}}$, $\beta_2^* = \frac{1}{\sqrt{2}}$, $\|\mathscr{K}_q(\xi)\| \leq 1, \|\mathscr{L}_q(\xi,t)\| \leq 0.36 \ \forall\, 0 < s < \xi \leq 1$.

The non-local function given by $\eta(\mathscr{U}(\xi,\cdot)) = \frac{1}{13}\sin\mathscr{U}(\xi,\cdot)$, so we have

$$\left\| \frac{1}{13}\sin\mathscr{U} - \frac{1}{13}\sin\mathscr{V} \right\| \leq \frac{1}{13}\|\mathscr{U} - \mathscr{V}\|$$

then, $L_\eta = \frac{1}{13}$.

Let $h(s) = e^{7s}, s < 0$, then $\int_{-\infty}^{0} h(s)ds = \frac{1}{7}$, we define

$$\|\phi\|_{\mathscr{P}_{\mathscr{H}}} = \int_{-\infty}^{0} e^{7s} \sup_{s \leq \xi \leq 0} \|\phi(\xi)\| ds.$$

Then, we can say

$$\|\phi\|_{\mathscr{P}_{\mathscr{H}}} = \|\frac{1}{5}e^{-0.5\xi}\|_{\mathscr{P}_{\mathscr{H}}} = \frac{1}{35}.$$

Assume that the operator $\mathfrak{B} = \mathfrak{O}_1 I$ where I is the identity operator. For $x \in [0,\pi]$, we also assume the operator $\mathbb{W}\colon (\mathbb{U},\mathbb{R}) \to \mathfrak{X}$ is defined as

$$\mathbb{W}y = \mathfrak{O}_1 \int_0^1 (1-\xi)^{\frac{-1}{6}} \mathscr{L}_q(1,\xi) I y(\xi,x) d\xi$$

and its norm can be given easily by

$$\|\mathbb{W}y\| = \left\| \int_0^1 (1-\xi)^{\frac{-1}{6}} \mathscr{L}_q(1,\xi) \mathfrak{B} y(\xi,x) d\xi \right\| \leq \frac{6\mathfrak{O}_1}{5\Gamma(\frac{5}{3})}\|y\|.$$

Plainly, $\mathbb{W}$ is linear and bounded operator with $\mathbb{W} \leq \frac{6\mathfrak{O}_1}{5\Gamma(\frac{5}{3})}$. Therefore Assumption 2 holds for a suitable constant $\mathfrak{O}_2 > 0$.

Finally, suppose that

$$\mathscr{F}(\xi, \mathscr{U}(\xi), \mathscr{U}_\xi) = \frac{1}{15}\xi^{\frac{1}{3}}\sin\mathscr{U} + \frac{\mathscr{U}_\xi}{5 + \xi^{\frac{3}{2}}}$$

Clearly $\mathscr{F} : [0,1] \times \mathfrak{X} \times \mathscr{P}_{\mathscr{H}} \to \mathfrak{X}$ is continuous and satisfies

$$\|\mathscr{F}(\xi,\mathscr{U}(\xi),\mathscr{U}_\xi) - \mathscr{F}(\xi,\mathscr{V}(\xi),\mathscr{V}_\xi)\| \leq \frac{1}{15}\xi^{\frac{1}{3}}\|\sin\mathscr{U} - \sin\mathscr{V}\|_{\mathscr{X}} + \frac{1}{5+\xi^{\frac{3}{2}}}\|\mathscr{U}_\xi - \mathscr{V}_\xi\|_{\mathscr{P}_{\mathscr{H}}}.$$

Then, we have $d_{1f} = \frac{1}{15}$ and $d_{2f} = \frac{1}{6}$ and

$$a\mathfrak{M}L_\eta + \mathscr{M}_0\beta_1^*(d_{1f}H + d_{2f}) \sim 0.167757.$$

- Case I: Krasnoselskii fixed point theorem:
  To check the presumption of Theorem 1, we have $\mathcal{L}_{\mathfrak{v}} \sim 0.167757\mathscr{M}_1 < 1$ which is true for all $0 < \mathfrak{O}_1 < 4.48439/\mathfrak{O}_2$. Thus, all assumptions of this theorem are satisfied. Therefore, the problem (2) has a unique mild solution and is controllable on $(-\infty, 1]$.

- Case II: Sadovskii fixed point theorem:
  To check the presumption of Theorem 2, we have $\mathfrak{P}_\rho \sim 0.167757(1+\mathscr{M}_1) < 1$ which is true for all $0 < \mathfrak{O}_1 < 3.7321/\mathfrak{O}_2$. Thus, all assumptions of this theorem are satisfied. Therefore, the problem (2) has a unique mild solution and is controllable on $(-\infty, 1]$.

7. Conclusions

In the current study, we analyzed an infinitely delaying system of fractional evolution equations. The foundation for our observations is furnished by current functional analysis approaches. In order to provide a reasonable remedy, we employ the unbounded operator $\mathbb{A}$ as the generator of the strongly continuous Cosine family. In the case of the problem (2), we had to examine a moderate controllability solution by two different arguments, the first of which used compactness technology and the second, noncompactness. By using the Sadovskii fixed point theorem and the measure of non-compactness, we present a new approach to analyzing the controllability of mild solutions. The first argument is based on Krasnoselskii's theorem, which allows $\mathscr{F}(\xi, \mathscr{U}, \mathscr{U}_\xi)$ to behave as

$$\|\mathscr{F}(\xi, \mathscr{U}, \mathscr{U}_\xi) - \mathscr{F}(\xi, \mathscr{V}, \mathscr{V}_\xi)\| \leq d_{1f}\|\mathscr{U} - \mathscr{V}\|_{\mathfrak{X}} + d_{2f}\|\mathscr{U}_\xi - \mathscr{V}_\xi\|_{\mathscr{P}_{\mathscr{H}}}.$$

The tools of fixed point theory in the case of simple assumptions are simple to install and enhance the range of results offered to meet our demands. The second result, which is rooted in the Kuratowski measure of noncompactness and the Sadovskii fixed point theorem, establishes a stipulation to utilize the operator of the solution is a condensing map in order to comply with the Lipschitz continuance, ensuring that the problem at hand has no prior solutions. Our conclusion is then illustrated with a numerical example that looks at a function that meets all the requirements.

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