axioms

Article

Coinciding Mean of the Two Symmetries on the Set of Mean Functions

Lenka Mihoković

Faculty of Electrical Engineering and Computing, University of Zagreb, Unska 3, 10000 Zagreb, Croatia; lenka.mihokovic@fer.hr

Abstract: On the set $\mathcal{M}$ of mean functions, the symmetric mean of M with respect to mean M_0 can be defined in several ways. The first one is related to the group structure on $\mathcal{M}$, and the second one is defined trough Gauss' functional equation. In this paper, we provide an answer to the open question formulated by B. Farhi about the matching of these two different mappings called symmetries on the set of mean functions. Using techniques of asymptotic expansions developed by T. Burić, N. Elezović, and L. Mihoković (Vukšić), we discuss some properties of such symmetries trough connection with asymptotic expansions of means involved. As a result of coefficient comparison, a new class of means was discovered, which interpolates between harmonic, geometric, and arithmetic mean.

Keywords: mean; asymptotic expansion; symmetry; Catalan numbers

MSC: 26E60; 41A60; 26E40; 39B22

check for updates

Citation: Mihoković, L. Coinciding Mean of the Two Symmetries on the Set of Mean Functions. *Axioms* **2023**, *12*, 238. https://doi.org/10.3390/axioms12030238

Academic Editor: Inna Kalchuk

Received: 1 February 2023
Revised: 21 February 2023
Accepted: 23 February 2023
Published: 25 February 2023

1. Introduction

Function $M\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}$ is called a mean if for all $s,t \in \mathbf{R}^+$

$$\min(s,t) \le M(s,t) \le \max(s,t). \tag{1}$$

Mean M is symmetric if for all $s,t \in \mathbf{R}^+$

$$M(s,t) = M(t,s)$$

and homogeneous (of degree 1) if for all $\lambda, s, t \in \mathbf{R}^+$

$$M(\lambda s, \lambda t) = \lambda M(s,t).$$

This paper was motivated by the problem of matching two different mappings on the set of mean functions formulated in paper [1] in which author introduced algebraic and topological structures on the set $\mathcal{M}_\mathcal{D}$ of symmetric means on a symmetric domain $\mathcal{D}$ with additional property

$$M(s,t) = s \Rightarrow s = t, \quad \forall (s,t) \in \mathcal{D}.$$

The first mapping is related to the group structure and the second one is defined trough Gauss' functional equation. It was found that those mappings coincide for arithmetic, geometric, and harmonic mean, but the question of the existence of other solutions remained open. We shall take $\mathcal{D} = \mathbf{R}^+ \times \mathbf{R}^+$.

First, let $\mathcal{A}_\mathcal{D}$ be set of all functions $f\colon \mathcal{D} \to \mathbf{R}$ such that

$$(\forall (x,y) \in \mathcal{D})\ f(x,y) = -f(y,x).$$

$(\mathcal{A}_\mathcal{D}, +)$ is an abelian group with the neutral element 0. Function $\varphi\colon \mathcal{M}_\mathcal{D} \to \mathcal{A}^\mathcal{D}$ defined by

$$\varphi(M)(x,y) := \begin{cases} \log\left(-\frac{M(x,y)-x}{M(x,y)-y}\right), & x \neq y, \\ 0, & x = y, \end{cases}$$

is a bijection. The composition law $*\colon \mathcal{M}_{\mathcal{D}} \times \mathcal{M}_{\mathcal{D}} \to \mathcal{M}_{\mathcal{D}}$ is defined by

$$M_1 * M_2 = \varphi^{-1}(\varphi(M_1) + \varphi(M_2)).$$

Thus $(\mathcal{M}_{\mathcal{D}}, *)$ is an abelian group with the neutral element $\varphi^{-1}(0) = A$. It can also easily be shown that the explicit formula for the composition law $*$ holds:

$$(M_1 * M_2)(x,y) = \begin{cases} \frac{x(M_1-y)(M_2-y)+y(M_1-x)(M_2-x)}{(M_1-x)(M_2-x)+(M_1-y)(M_2-y)}, & x \neq y, \\ x, & x = y. \end{cases} \tag{2}$$

For the sake of simplicity, variables (x,y) were omitted. By sum and difference of means, we assume usual pointwise addition and subtraction. More on the topological structures on set of bivariate means can also be found in [2].

Based on the operation $*$ defined in (2), the first type of the symmetry was defined.

Definition 1 ([1])**.** *The symmetric mean M_2 to a mean M_1 with respect to mean M_0 via the group structure $(\mathcal{M}_{\mathcal{D}}, *)$ is defined with the expression*

$$S_{M_0}(M_1) = M_2 \Leftrightarrow M_1 * M_2 = M_0 * M_0. \tag{3}$$

Combining (3) with (2), the explicit formula for symmetric mean of mean M_1 with respect to M_0 can easily be calculated:

$$S_{M_0}(M_1) = \frac{x(M_1-x)(M_0-y)^2 - y(M_0-x)^2(M_1-y)}{(M_1-x)(M_0-y)^2 - (M_0-x)^2(M_1-y)}. \tag{4}$$

We shall see the behavior of S_{M_0} for some basic well known means M_0. For $(s,t) \in \mathcal{D} = \mathbf{R}^+ \times \mathbf{R}^+$ let

$$A(s,t) = \frac{s+t}{2}, \quad G(s,t) = \sqrt{st}, \quad H(s,t) = \frac{2st}{s+t},$$

be the arithmetic, geometric, and harmonic means, respectively.

Example 1 ([1])**.** *For any mean $M \in \mathcal{M}_{\mathcal{D}}$, we have:*

1. $S_A(M) = 2A - M$,
2. $S_G(M) = \frac{G^2}{M}$,
3. $S_H(M) = \frac{HM}{2M-H}$.

Notice that the denominator in $S_H(M)$ from Example 1 cannot be equal to 0, since $M = \frac{1}{2}H$ does not satisfy the left hand side inequality in (1) and, hence, it is not a mean.

Another type of symmetry, independent of the group structure $(\mathcal{M}_{\mathcal{D}}, *)$, can also be defined.

Definition 2 ([1])**.** *Mean M_2 is said to be functional symmetric mean of M_1 with respect to M_0 if the following functional equation is satisfied:*

$$\sigma_{M_0}(M_1) = M_2 \Leftrightarrow M_0(M_1, M_2) = M_0. \tag{5}$$

We can also say that mean M_0 is the functional middle of M_1 and M_2. Defining equation on the right side of the equivalence relation (5) is known as the Gauss functional equation. Some authors refer to means M_1 and M_2 as a pair of M_0-complementary means.

Mean M_0 is also said to be (M_1, M_2)-invariant. For recent related results, see [3–6] and also survey article on invariance of means [7] and references therein. Furthermore, if functional symmetric mean exists, then it is unique.

With respect to the same means as in the latter exmple, we may calculate the symmetric means. For instance, when $M_0 = H$, we have

$$H(M, \sigma_H(M)) = H \Leftrightarrow \frac{2M\sigma_H(M)}{M+\sigma_H(M)} = H \Leftrightarrow 2M\sigma_H(M) = H(M + \sigma_H(M)) \Leftrightarrow \sigma_H(M) = \frac{HM}{2M-H}.$$

Other symmetric pairs, with respect to A and G, are obtained in similar manner.

Example 2 ([1])**.** *For any mean $M \in \mathcal{M}_{\mathcal{D}}$, we have:*

1. $\sigma_A(M) = 2A - M$,
2. $\sigma_G(M) = \frac{G^2}{M}$,
3. $\sigma_H(M) = \frac{HM}{2M-H}$.

Taking into account Examples 1 and 2, in which the same mappings appear with respect to arithmetic, geometric, and harmonic mean appear, the author in [1] states the following.

Open question. For which mean functions M_0 on $\mathcal{D} = \mathbf{R}^+ \times \mathbf{R}^+$ do the two symmetries, S and σ, with respect to M_0, coincide?

The goal of this paper is to analyze the open question and offer the answer in the setting of symmetric homogeneous means, which possess the asymptotic expansion. Techniques of asymptotic expansions were developed in [8–10] and appeared to be very useful in comparison and finding inequalities for bivariate means ([11,12]), comparison of bivariate parameter means ([10]), finding optimal parameters in convex combinations of means ([12,13]), and solving the functional equations of the form $B(A(x)) = C(x)$, where asymptotic expansions of B and C are known ([14]). In the latter example, A, B, and C are functions of a real variable, which possess asymptotic expansion as $x \to \infty$ with respect to asymptotic sequences $(x^{w-n})_{n\in\mathbf{N}_0}$, $(x^{u-n})_{n\in\mathbf{N}_0}$, and $(x^{v-n})_{n\in\mathbf{N}_0}$, respectively, where w, u, and v are real numbers. When used with $B(x) = f(x)$ and $C(x) = \frac{1}{t-s}\int_s^t f(x+u)\,du$, finding $A(x)$ is then equivalent to determining integral f-mean $I_f(x+s, x+t) = f^{-1}\left(\frac{1}{t-s}\int_{x+s}^{x+t} f(u)\,du\right)$ for a given function f as it was described in detail in above mentioned paper. We may perceive the significance of this approach when explicit formula for the inverse function is not known, which is case for the digamma function.

Techniques and results applyed in this paper were described in Section 2. In the next step, we obtained the algorithm for calculating the coefficients in the asymptotic expansions of means $M_2^S = S_{M_0}(M_1)$ and $M_2^\sigma = \sigma_{M_0}(M_1)$. Comparing the first few obtained coefficents, we anticipated the general form of the coefficients in the asymptotic expansion of mean M_0 for which symmetries S_{M_0} and σ_{M_0} coincide, i.e., such that $M_2^S = M_2^\sigma$.

At the beginning of Section 3, we found closed formula and explored some properties, such as limit behavior and monotonocity with respect to the parameter. We proved that proposed function represents the well defined one parameter class of means. We have shown that it also covers, as the special cases, means from Examples 1 and 2.

Lastly, in Section 4, we have proved that this class of means answered the open question and stated the hypothesis that there were not any other solutions in the context of homogeneous symmetric means, which possess asymptotic power series expansions.

In addition, methods presented in this paper may be useful with similar problems regarding functional equations, especially in case when the explicit formula for included function was not known.

2. Asymptotic Expansions

Recall the definition of an asymptotic power series expansion as $x \to \infty$.

Definition 3. *The series $\sum_{n=0}^{\infty} c_n x^{-n}$ is said to be an asymptotic expansion of a function $f(x)$ as $x \to \infty$ if for each $N \in \mathbf{N}$*

$$f(x) = \sum_{n=0}^{N} c_n x^{-n} + o(x^{-N}).$$

Main properties of asymptotic series and asymptotic expansions can be found in [15]. Taylor series expansion can also be seen as an asymptotic expansion, but the converse is not generally true, and the asymptotic series may also be divergent. The main characteristic of asymptotic expansion is that it provides good approximation using a finite number of terms while letting $x \to \infty$.

Beacause of the intrinsity (1), mean M would possess the asymptotic power series as $x \to \infty$ of the form

$$M(x+s, x+t) = \sum_{n=0}^{\infty} c_n(s,t) x^{-n+1}$$

with $c_0(s,t) = 1$. For a homogeneous symmetric mean, the coefficients $c_n(s,t)$ are also homogeneous symmetric polynomials of degree n in variables s and t, and for $s = -t$, they have a simpler form. Let the means included possess the asymptotic expansions as $x \to \infty$ of the form

$$\begin{aligned} M_0(x-t, x+t) &= \sum_{n=0}^{\infty} c_n t^{2n} x^{-2n+1}, \qquad (6) \\ M_1(x-t, x+t) &= \sum_{n=0}^{\infty} a_n t^{2n} x^{-2n+1}, \\ M_2(x-t, x+t) &= \sum_{n=0}^{\infty} b_n t^{2n} x^{-2n+1}. \end{aligned}$$

Conversely, it can also be shown that the expansion in variables $(x-t, x+t)$ is sufficent to obtain the so-called two variable expansion, i.e., the expansion in variables $(x+s, x+t)$. Furthermore, note that

$$a_0 = b_0 = c_0 = 1. \qquad (7)$$

In this section, we will find the asymptotic expansions of means $M_2^S = S_{M_0}(M_1)$ and $M_2^\sigma = \sigma_{M_0}(M_1)$.

2.1. Symmetry S_{M_0}

Recall the recently developed results for tansformations of asymptotic series, i.e., the complete asymptotic expansions of the quotient and the power of asymptotic series.

Lemma 1 ([10], Lemma 1.1.). *Let function $f(x)$ and $g(x)$ have the following asymptotic expansions ($a_0 \neq 0, b_0 \neq 0$) as $x \to \infty$:*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \qquad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}.$$

Then, asymptotic expansion of their quotient $f(x)/g(x)$ reads as

$$\frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} c_n x^{-n},$$

where coefficients c_n are defined by

$$c_n = \frac{1}{b_0}\left(a_n - \sum_{k=0}^{n-1} b_{n-k} c_k \right).$$

Lemma 2 ([8,16]). *Let $m(x)$ be a function with asymptotic expansion ($c_0 \neq 0$):*

$$m(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad (x \to \infty).$$

Then, for all real r, it holds

$$[m(x)]^r \sim \sum_{n=0}^{\infty} P[n, r, (c_j)_{j\in\mathbf{N}_0}] x^{-n}$$

where

$$\begin{aligned} &P[0, r, (c_j)_{j\in\mathbf{N}_0}] = c_0^r, \\ &P[n, r, (c_j)_{j\in\mathbf{N}_0}] = \frac{1}{nc_0} \sum_{k=1}^{n} [k(1+r) - n] c_k P[n-k, r, (c_j)_{j\in\mathbf{N}_0}]. \end{aligned} \tag{8}$$

Symmetric mean with respect to mean M_0 of mean M_1 via the group structure $(\mathcal{M}_{\mathcal{D}}, *)$ as a consequence of (4) can be expressed as:

$$\begin{aligned} M_2^S(x-t, x+t) &= S_{M_0}(M_1)(x-t, x+t) \\ &= \frac{(x-t)(M_1 - x + t)(M_0 - x - t)^2 - (x+t)(M_0 - x + t)^2(M_1 - x - t)}{(M_1 - x + t)(M_0 - x - t)^2 - (M_0 - x + t)^2(M_1 - x - t)} \\ &= \frac{(x-t)(\overline{M}_1 + t)(\overline{M}_0 - t)^2 - (x+t)(\overline{M}_0 + t)^2(\overline{M}_1 - t)}{(\overline{M}_1 + t)(\overline{M}_0 - t)^2 - (\overline{M}_0 + t)^2(\overline{M}_1 - t)} \\ &= x + \frac{2t^2\overline{M}_0 - t^2\overline{M}_1 - \overline{M}_0^2\overline{M}_1}{t^2 + \overline{M}_0^2 - 2\overline{M}_0\overline{M}_1}, \end{aligned}$$

where $\overline{M}_i$, $i = 1, 2, 3$, stands for $M_i - x$. The variables $(x - t, x + t)$ were omitted for the sake of symplicity. Further calculations reveal that:

$$\begin{aligned} M_2^S(x-t, x+t) = x + t^2x^{-1}\Big[&(2c_1 - a_1) + \\ &+ \sum_{n=0}^{\infty}\Big(2c_{n+2} - a_{n+2} + \sum_{k=0}^{n}\Big(\sum_{j=0}^{k}\Big(c_{j+1}c_{k-j+1}\Big)a_{n+1-k}\Big)\Big)t^{2n+2}x^{-2n-2}\Big] \times \\ &\times \Big[1 + \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_{k+1}(c_{n-k+1} - 2a_{n-k+1})t^{2n+2}x^{-2n-2}\Big]^{-1}. \end{aligned}$$

Coefficients b_n^S for $n \geq 1$ are obtained using Lemma 1 for the division of asymptotic series. Hence, we have the following:

$$\begin{aligned} &b_0^S = 1, \\ &b_n^S = num_n - \sum_{k=0}^{n-2} den_{n-1-k} b_{k+1}^S, \quad n \geq 1, \end{aligned}$$

where $(num_n)_{n\in\mathbf{N}_0}$ and $(den_n)_{n\in\mathbf{N}_0}$ dentote auxiliary sequences, which appear in the numerator and the denominator:

$$\begin{aligned} &num_0 = 2c_1 - a_1, \\ &num_n = 2c_{n+1} - a_{n+1} + \sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}(c_{j+1}c_{k-j+1})a_{n-k}\right), \quad n \geq 1, \end{aligned}$$

and

$$den_0 = 1,$$
$$den_n = \sum_{k=0}^{n-1} c_{k+1}(c_{n-k} - 2a_{n-k}), \quad n \geq 1.$$

We shall calculate the first few coefficients:

$$\begin{aligned}
b_0^S &= 1,\\
b_1^S &= 2c_1 - a_1,\\
b_2^S &= 2c_2 - a_2 - 2c_1(a_1 - c_1)^2,\\
b_3^S &= 2c_3 - a_3 - 2(a_1 - c_1)(2a_2c_1 + c_1^2(2a_1^2 - 3a_1c_1 + c_1^2) + (a_1 - 3c_1)c_2),\\
b_4^S &= 2c_4 - a_4 - 2(a_2^2c_1 + 4a_1^4c_1^3 + 4a_1^3c_1(-3c_1^3 + c_2)\\
&\quad + 2a_2((3a_1 - 2c_1)(a_1 - c_1)c_1^2 + (a_1 - 2c_1)c_2)\\
&\quad + a_1^2(13c_1^5 - 15c_1^2c_2 + c_3) + 2a_1(a_3c_1 - 3c_1^6 + 8c_1^3c_2 - c_2^2 - 2c_1c_3)\\
&\quad + c_1(-2a_3c_1 + c_1^6 - 5c_1^3c_2 + 3c_2^2 + 3c_1c_3)),\\
b_5^S &= 2c_5 - a_5 - 2(-2a_4c_1^2 + 8a_1^5c_1^4 + 4a_3c_1^4 - c_1^9 - 4a_3c_1c_2 + 7c_1^6c_2 - 10c_1^3c_2^2\\
&\quad + c_2^3 + a_2^2(6a_1c_1^2 - 5c_1^3 + c_2) + 4a_1^4c_1^2(-7c_1^3 + 3c_2) - 5c_1^4c_3 + 6c_1c_2c_3\\
&\quad + 2a_1^3(19c_1^6 - 24c_1^3c_2 + c_2^2 + 2c_1c_3) + 2a_2(a_3c_1 + 8a_1^3c_1^3 - 3c_1^6 + 8c_1^3c_2\\
&\quad - c_2^2 + 6a_1^2c_1(-3c_1^3 + c_2) - 2c_1c_3 + a_1(13c_1^5 - 15c_1^2c_2 + c_3)) + 3c_1^2c_4\\
&\quad + a_1^2(6a_3c_1^2 - 5c_1(5c_1^6 - 13c_1^3c_2 + 3c_2^2 + 3c_1c_3) + c_4) + 2a_1(a_4c_1\\
&\quad + a_3(-5c_1^3 + c_2) + 2(2c_1^8 - 9c_1^5c_2 + 6c_1^2c_2^2 + 4c_1^3c_3 - c_2c_3 - c_1c_4))).
\end{aligned}$$

2.2. Symmetry σ_{M_0}

The problem of functional symmetic mean corresponds the functional equation

$$M_0(x-t, x+t) = M_0(M_1(x-t, x+t), M_2(x-t, x+t))$$

which we will solve in terms of asymptotic series. To this end, we shall use the following result from Burić and Elezović about the asymptotic expansion of the composition of means.

Theorem 1 ([17], Theorem 2.2.). *Let M and N be given homogeneous symmetric means with asymptotic expansions*

$$M(x-t, x+t) = \sum_{k=0}^{\infty} a_k t^{2k} x^{-2k+1}, \quad N(x-t, x+t) = \sum_{k=0}^{\infty} b_k t^{2k} x^{-2k+1},$$

and let F be homogeneous symmetric mean with expansion

$$F(x-t, x+t) = \sum_{k=0}^{\infty} \gamma_k t^{2k} x^{-2k+1}.$$

Then, the composition $H = F(M, N)$ has asymptotic expansion

$$H(x-t, x+t) = \sum_{k=0}^{\infty} h_n t^{2n} x^{-2n+1},$$

where coefficients (h_n) are calculated by

$$h_n = \sum_{k=0}^{\lfloor \frac{n}{2z} \rfloor} \gamma_k \sum_{j=0}^{n-2zk} P[j, 2k, (d_m)_{m\in\mathbf{N}_0}] P[n-2zk-j, -2k+1, (c_m)_{m\in\mathbf{N}_0}].$$

Sequences (c_n) and (d_n) are defined by

$$c_n = \frac{1}{2}(a_n + b_n), \;\; d_n = \frac{1}{2}(a_{n+z} - b_{n+z}), \quad n \geq 0,$$

where z is the smallest number such that $d_n \neq 0$.

Applying Theorem 1 on $M = M_1$, $N = M_2$ (or equivalently $M = M_2$, $N = M_1$) and $F = M_0$, we obtain the asymptotic expansion of the composition $M_0(M_1, M_2)$. Since the equation $M_0 = M_0(M_1, M_2)$ holds, on the other side, in Theorem 1, we also have $H = M_0$. The coeficients in the asymptotic expansion of the composition $M_0(M_1, M_2)$ equal the coefficients c_n in the asymptotic expansion of mean M_0. In the end, we obtain the recursive algorithm for coefficients c_n:

$$c_0 = 1;$$
$$c_n = \sum_{k=0}^{\lfloor \frac{n}{2z} \rfloor} c_k \sum_{j=0}^{n-2zk} P[j, 2k, (\tfrac{1}{2}(a_m - b_m^\sigma))_{m \geq z}] P[n - 2zk - j, -2k + 1, (\tfrac{1}{2}(a_m + b_m^\sigma))_{m \in \mathbf{N}_0}], \quad (9)$$

where $P[n, r, (c_m)_{m \in \mathbf{N}_0}]$, $n \in \mathbf{N}_0$ denotes the n-th coefficient in the asymptotic expansion of r-th power of the asymptotic seires with coefficients $(c_m)_{m \in \mathbf{N}_0}$, as it was defined in (8). Because of (7), z is always greater or equal to 1.

For $z = 1$ we calculate the first few coefficients:

$$\begin{aligned}
c_0 &= 1, \\
c_1 &= \frac{1}{2}(a_1 + b_1^\sigma), \\
c_2 &= \frac{1}{2}(a_2 + b_2^\sigma) + \frac{1}{4}(a_1 - b_1^\sigma)^2 c_1, \\
c_3 &= \frac{1}{2}(a_3 + b_3^\sigma) - \frac{1}{8}(a_1 - b_1^\sigma)(a_1^2 - 4a_2 - (b_1^\sigma)^2 + 4b_2^\sigma)c_1, \\
c_4 &= \frac{1}{2}(a_4 + b_4^\sigma) + \frac{1}{16}((a_1^4 + 4a_2^2 - 8a_3 b_1^\sigma + (b_1^\sigma)^4 + 2a_2((b_1^\sigma)^2 - 4b_2^\sigma) \\
&\quad - 2a_1^2(3a_2 + (b_1^\sigma)^2 - b_2^\sigma) - 6(b_1^\sigma)^2 b_2^\sigma + 4(b_2^\sigma)^2 \\
&\quad + 4a_1(2a_3 + b_1^\sigma(a_2 + b_2^\sigma) - 2b_3^\sigma) + 8b_1^\sigma b_3^\sigma)c_1 + (a_1 - b_1^\sigma)^4 c_2), \\
c_5 &= \frac{1}{2}(a_5 + b_5^\sigma) - \frac{1}{32}((a_1^5 + a_1^4 b_1^\sigma - 4a_2^2 b_1^\sigma + 16a_4 b_1^\sigma - 4a_3(b_1^\sigma)^2 + (b_1^\sigma)^5 \\
&\quad - 2a_1^3(4a_2 + (b_1^\sigma)^2) + 16a_3 b_2^\sigma - 8(b_1^\sigma)^3 b_2^\sigma + 12b_1^\sigma(b_2^\sigma)^2 \\
&\quad - 8a_2(2a_3 + b_1^\sigma b_2^\sigma - 2b_3^\sigma) + 2a_1^2(6a_3 - (b_1^\sigma)^3 + 4b_1^\sigma b_2^\sigma - 2b_3^\sigma) + 12(b_1^\sigma)^2 b_3^\sigma \\
&\quad - 16b_2^\sigma b_3^\sigma - 16b_1^\sigma b_4^\sigma + a_1(12a_2^2 - 16a_4 - 8a_3 b_1^\sigma + (b_1^\sigma)^4 + 8a_2((b_1^\sigma)^2 - b_2^\sigma) \\
&\quad - 4(b_2^\sigma)^2 - 8b_1^\sigma b_3^\sigma + 16b_4^\sigma))c_1 - (a_1 - b_1^\sigma)^3(3a_1^2 - 8a_2 - 3(b_1^\sigma)^2 + 8b_2^\sigma)c_2).
\end{aligned}$$

The connetcion between b_n^σ and c_n with the highest index n in each equation is linear. In the expression (9), b_n^σ appears ony in the second part

$$P[n - 2zk - j, -2k + 1, (\tfrac{1}{2}(a_m + b_m^\sigma))_{m \in \mathbf{N}_0}], \quad (10)$$

when $k = j = 0$. Then, (10) becomes $P[n, 1, (\frac{1}{2}(a_m + b_m^\sigma))_{m \in \mathbf{N}_0}]$, which represents the n-th coefficient in the $\sum_{n=0}^{\infty} \frac{1}{2}(a_n + b_n^\sigma) t^{2n} x^{-2n+1}$ to the power of 1, which equals $\frac{1}{2}(a_n + b_n^\sigma)$. So, we can easily extract b_n^σ. The first few coefficients b_n^σ are:

$$\begin{aligned}
b_0^\sigma &= 1, \\
b_1^\sigma &= 2c_1 - a_1,
\end{aligned}$$

$$
\begin{aligned}
b_2^\sigma &= 2c_2 - a_2 - \frac{1}{2}c_1(a_1 - b_1^\sigma),\\
b_3^\sigma &= 2c_3 - a_3 + \frac{1}{4}(a_1 - b_1^\sigma)(a_1^2 - 4a_2 - (b_1^\sigma)^2 + 4b_2^\sigma)c_1,\\
b_4^\sigma &= 2c_4 - a_4 - \frac{1}{8}((a_1^4 + 4a_2^2 - 8a_3b_1^\sigma + (b_1^\sigma)^4 + 2a_2((b_1^\sigma)^2 - 4b_2^\sigma)\\
&\quad - 2a_1^2(3a_2 + (b_1^\sigma)^2 - b_2^\sigma) - 6(b_1^\sigma)^2 b_2^\sigma + 4(b_2^\sigma)^2\\
&\quad + 4a_1(2a_3 + b_1^\sigma(a_2 + b_2^\sigma) - 2b_3^\sigma) + 8b_1^\sigma b_3^\sigma)c_1 + (a_1 - b_1^\sigma)^4 c_2),\\
b_5^\sigma &= 2c_5 - a_5 + \frac{1}{16}((a_1^5 + a_1^4 b_1^\sigma - 4a_2^2 b_1^\sigma + 16a_4 b_1^\sigma - 4a_3(b_1^\sigma)^2 + (b_1^\sigma)^5\\
&\quad - 2a_1^3(4a_2 + (b_1^\sigma)^2) + 16a_3b_2^\sigma - 8(b_1^\sigma)^3 b_2^\sigma + 12b_1^\sigma(b_2^\sigma)^2\\
&\quad - 8a_2(2a_3 + b_1^\sigma b_2^\sigma - 2b_3^\sigma) + 2a_1^2(6a_3 - (b_1^\sigma)^3 + 4b_1^\sigma b_2^\sigma - 2b_3^\sigma) + 12(b_1^\sigma)^2 b_3^\sigma\\
&\quad - 16b_2^\sigma b_3^\sigma - 16b_1^\sigma b_4^\sigma + a_1(12a_2^2 - 16a_4 - 8a_3b_1^\sigma + (b_1^\sigma)^4 + 8a_2((b_1^\sigma)^2 - b_2^\sigma)\\
&\quad - 4(b_2^\sigma)^2 - 8b_1^\sigma b_3^\sigma + 16b_4^\sigma))c_1 - (a_1 - b_1^\sigma)^3(3a_1^2 - 8a_2 - 3(b_1^\sigma)^2 + 8b_2^\sigma)c_2).
\end{aligned}
$$

For beter understanding the role of the parameter z, we shall recall the idea behind the proof of Theorem 1. The composition $F(M, N)$ has the asymptotic expansion

$$
\begin{aligned}
&F(M(x-t, x+t), N(x-t, x+t)) =\\
&\qquad = F\left(\frac{M+N}{2} - \frac{N-M}{2}, \frac{M+N}{2} + \frac{N-M}{2}\right)\\
&\qquad = \sum_{k=0}^{\infty} \gamma_k \left(\frac{N-M}{2}\right)^{2k} \left(\frac{M+N}{2}\right)^{-2k+1}.
\end{aligned}
$$

Larger z corresponds with the equating a_i and b_i^σ and some parts of the coefficients c_n reduce. Observation of the cases with $z > 1$ in sequel did not provide any new information about the ceofficients c_n.

2.3. Comparison of Coefficients

Sequences $(b_n^S)_{n\in\mathbf{N}_0}$ and $(b_n^\sigma)_{n\in\mathbf{N}_0}$ represent the coefficients in asymptotic expansions of means, which are results of mappings $S_{M_0}(M_1)$ and $\sigma_{M_0}(M_1)$, respectively. Since we are looking for a mean M_0 such those mappings coincide, b_n^S and b_n^σ need to be equal. Since the equality must hold for any mean M_1, we may suppose that $z = 1$, which is equivalent with $a_1 \neq c_1$. Equating b_0^S with b_0^σ and b_1^S with b_1^σ does not provide any new information, except

$$b_0 = b_0^S = b_0^\sigma = 1 \text{ and } b_1 = b_1^S = b_1^\sigma = 2c_1 - a_1.$$

With such b_1^σ we may express b_2^σ as

$$b_2^\sigma = 2c_2 - a_2 - 2c_1(a_1 - c_1)^2,$$

which is already equal to b_2^S. Now, we can substitute

$$b_2 = b_2^S = b_2^\sigma = 2c_2 - a_2 - 2c_1(a_1 - c_1)^2,$$

in b_3^σ to obtain

$$b_3^\sigma = 2c_3 - a_3 - 2c_1(a_1 - c_1)(2a_2 + 2c_1(a_1 - c_1)^2 + c_1^2 - a_1c_1 - 2c_2),$$

which, after equating with b_3^S, gives the following condition

$$(a_1 - c_1)^2(c_1^2 + c_1^3 + c_2) = 0.$$

Since we assumed that a_1 and c_1 are not equal, it is necessarily

$$c_2 = -c_1^2(1 + c_1).$$

Now, we have

$$b_3 = b_3^S = b_3^\sigma = 2c_3 - a_3 - 2c_1(a_1 - c_1)\Big((3 - 4a_1)c_1^2 + a_1(2a_1 - 1)c_1 + 2a_2 + 4c_1^3\Big).$$

After substitutions, we observe the next coefficient

$$\begin{aligned}
b_4^\sigma &= 2c_4 - a_4 - 2c_1\big(2a_2c_1\big(-a_1(6c_1 + 1) + 3a_1^2 + 2c_1(2c_1 + 1)\big) \\
&\quad + c_1\big(c_1\big(-4a_1^3(4c_1 + 1) + a_1^2(2c_1 + 1)(15c_1 + 2) - 2a_1c_1(14c_1(c_1 + 1) + 3) \\
&\quad + 4a_1^4 + c_1^2(c_1(11c_1 + 15) + 5)\big) - 2a_3\big) + 2c_3(c_1 - a_1) + a_2^2 + 2a_1a_3\big)
\end{aligned}$$

which, after equating with b_4^S, gives the following condition:

$$(a_1 - c_1)^2\Big(2c_1^3(c_1 + 1)^2 - c_3\Big) = 0,$$

and we conclude that it must be

$$c_3 = 2c_1^3(1 + c_1)^2.$$

We continue with this procedure as it was described above. Further calculations reveal that the first few coefficients c_n have the following form:

$$\begin{aligned}
c_0 &= 1, \\
c_1 &= c, \\
c_2 &= -c^2(1 + c), \\
c_3 &= 2c^3(1 + c)^2, \\
c_4 &= -5c^4(1 + c)^3, \\
c_5 &= 14c^5(1 + c)^4, \\
c_6 &= -42c^6(1 + c)^5.
\end{aligned}$$

After these first steps, it is natural to state the following hypothesis about the general formula for the coefficients in the asymptotic expansion of mean M_0:

$$\begin{aligned}
&c_0 = 1, \\
&c_n = (-1)^{n-1}C_{n-1}c^n(1 + c)^{n-1},\ n \geq 1,
\end{aligned} \tag{11}$$

where C_n denotes the n-th Catalan number. Catalan numbers appear in many occasions, and their behavior has been widely explored. Here, we mention only a few properties, which we will use in sequel. Catalan numbers are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \in \mathbf{N}_0$$

and they satisfy the recursive relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad n \in \mathbf{N}_0.$$

Based on this recursive relation, the generating function for Catalan numbers can be obtained ([18]):

$$\sum_{n=0}^{\infty} C_n y^n = \frac{1-\sqrt{1-4y}}{2y}, \tag{12}$$

which is convergent for $|y| < \frac{1}{4}$.

3. New Mean Function

In this section, we shall find closed a form for a mean whose coefficients are given in (11). We start from asymptotic expansion (6):

$$\begin{aligned} M_0(x-t, x+t) &= x + \sum_{n=1}^{\infty} (-1)^{n-1} C_{n-1} c^n (1+c)^{n-1} t^{2n} x^{-2n+1} \\ &= x + \sum_{n=0}^{\infty} (-1)^n C_n c^{n+1} (1+c)^n t^{2n+2} x^{-2n-1} \\ &= x + ct^2 x^{-1} \sum_{n=0}^{\infty} C_n \left[-\frac{c(1+c)t^2}{x^2} \right]^n . \end{aligned} \tag{13}$$

Introducing the substitution $y = -\frac{c(1+c)t^2}{x^2}$, as $x \to \infty$ and thereby $y \to 0$, yields

$$M_0(x-t, x+t) = x + ct^2 x^{-1} \sum_{n=0}^{\infty} C_n y^n,$$

and, then, according to the formula (12), for $c + 1 \neq 0$, we obtain

$$\begin{aligned} M_0(x-t, x+t) &= x + ct^2 x^{-1} \frac{1-\sqrt{1-4y}}{2y} \\ &= \frac{1+2c}{2(1+c)} x + \frac{1}{2(1+c)} \sqrt{x^2 + 4c(1+c)t^2}. \end{aligned} \tag{14}$$

Abandoning series expansion in this moment, from the Equation (14) with substitution

$$x = \frac{a+b}{2}, \quad t = \frac{b-a}{2},$$

we obtain the expression for M_0 in terms of variables a and b. For $c \in \mathbf{R} \setminus \{-1\}$ and $a, b > 0$ we define function $L_c \colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$

$$L_c(a,b) = \frac{a+b}{2} \frac{1+2c}{2(1+c)} + \frac{1}{2(1+c)} \sqrt{\left(\frac{a+b}{2}\right)^2 + 4c(1+c)\left(\frac{b-a}{2}\right)^2}. \tag{15}$$

Remark 1. *Function L_c is well defined for all $(a,b) \in \mathbf{R}^+ \times \mathbf{R}^+$ as we can rearrange terms under the square root:*

$$\begin{aligned} \left(\frac{a+b}{2}\right)^2 + 4c(1+c)\left(\frac{b-a}{2}\right)^2 &= \frac{1}{4}\Big((a+b)^2 + 4c(1+c)(b-a)^2\Big) \\ &= \frac{1}{4}\Big((1+2c)^2(a-b)^2 + 4ab\Big) > 0. \end{aligned}$$

Remark 2. *For $c = -1$ function L_c corresponds to the harmonic mean which will be proved in sequel. Therefore, definition (15) can be considered for all $c \in \mathbf{R}$.*

Remark 3. *Formula for L_c can also be written in a following way:*

$$L_c(a,b) = A(a,b)\frac{1+2c}{2(1+c)} + \frac{1}{2(1+c)} \sqrt{A(a,b)^2 + 4c(1+c)(A(a,b)^2 - G(a,b)^2)}. \tag{16}$$

3.1. Limit Cases and Monotonicity

In this subsection, we study properties of L_c with respect to parameter c. First, we state the following proposition, which can be proved using basic methods of mathematical analysis.

Proposition 1. *For a fixed pair $(a,b) \in \mathbf{R}^+ \times \mathbf{R}^+$, function L_c holds*

1. $\lim_{c \to -\infty} L_c(a,b) = \min(a,b)$,
2. $\lim_{c \to -1-} L_c(a,b) = \lim_{c \to -1+} L_c(a,b) = \dfrac{2ab}{a+b} = H(a,b)$
3. $\lim_{c \to +\infty} L_c(a,b) = \max(a,b)$,

It is well known that the following double inequality holds

$$H < A < G.$$

Also, $H = L_c$ for $c \to -1$, $G = L_c$ for $c = -\frac{1}{2}$, and $A = L_c$ for $c = 0$. In the next Theorem, we explore the ordering of means L_c with respect to parameter c.

Theorem 2. *For a fixed pair $(a,b) \in \mathbf{R}^+ \times \mathbf{R}^+$, $a \neq b$, function $f\colon \mathbf{R} \setminus \{-1\} \to \mathbf{R}$,*

$$f(c) = L_c(a,b)$$

is strictly increasing.

Proof. Starting form the (16), with $A = A(a,b)$ and $G = G(a,b)$, we have

$$f(c) = A\frac{1+2c}{2(1+c)} + \frac{1}{2(1+c)}\sqrt{g(c)},$$

where

$$g(c) = A^2 + 4c(1+c)\left(A^2 - G^2\right) > 0$$

according to Remarks 1 and 3. The first derivative of function f equals

$$\begin{aligned} f'(c) &= \frac{1}{2(1+c)^2}\left(A + 2(1+2c)(1+c)(A^2-G^2)g(c)^{-\frac{1}{2}} - g(c)^{\frac{1}{2}}\right) \\ &= \frac{1}{2(1+c)^2 g(c)^{\frac{1}{2}}}\left(Ag(c)^{\frac{1}{2}} + 2(1+2c)(1+c)(A^2-G^2) - g(c)\right) \\ &= \frac{1}{2(1+c)^2 g(c)^{\frac{1}{2}}}\left(Ag(c)^{\frac{1}{2}} + 2(1+c)(A^2-G^2) - A^2\right). \end{aligned}$$

So, the condition $f'(c) > 0$ is equivalent to

$$Ag(c)^{\frac{1}{2}} > A^2 - 2(1+c)(A^2-G^2).$$

If the right-hand side is negative, than the inequality obviusly holds. If it is positive, then we may observe the squared inequality:

$$A^2 g(c) > A^4 - 4(1+c)(A^2-G^2)A^2 + 4(1+c)^2(A^2-G^2)$$

which is equivalent to

$$4c(1+c)A^2(A^2-G^2) > -4(1+c)(A^2-G^2)A^2 + 4(1+c)^2(A^2-G^2)$$

and

$$(A^2 - G^2)(1+c)^2 G^2 > 0$$

which is true for $a \neq b$ and $c \neq 1$. □

Since L_c assumes values between minimum and maximum of a and b, we may conclude the following.

Corollary 1. *For $c \in \mathbf{R}$ funcion L_c is a mean.*

Remark 4. *Notice that we proved that L_c is a strict mean, i.e., for $a \neq b$, strict inequalities hold:*

$$\min(a,b) < M(a,b) < \max(a,b).$$

3.2. Special Cases

Before we continue further, let us see what happens with some of the special cases of parameter c. We shall also connect results form this paper with the previously obtained asymptotic expansions of classical means.

Example 3. *(a)* $c = -1$. *Then mean has two non-zero coefficients:*

$$c_0 = 1, \quad c_1 = c, \quad c_n = 0,\ n \geq 2.$$

Corresponding asymptotic expansion is finite. From (13), we obtain

$$L_c(x-t, x+t) = x - t^2 x^{-1},$$

which, after substitution $x = \frac{a+b}{2}$, $t = \frac{b-a}{2}$ becomes

$$L_c(a,b) = \frac{a+b}{2} - \frac{(b-a)^2}{4} \cdot \frac{2}{a+b} = \frac{2ab}{a+b} = H(a,b).$$

(b) $c = 0$. *All coefficients except c_0 equal zero. Then, either from the (13) or (14), we obtain*

$$L_c(x-t, x+t) = x,$$

and after the substitution

$$L_c(a,b) = \frac{a+b}{2} = A(a,b).$$

(c) $c = -\frac{1}{2}$. *The coefficients are*

$$c_0 = 1, \quad c_n = -\frac{1}{2^{2n-1}} C_{n-1},\ n \geq 1. \tag{17}$$

Coefficients (17) correspond to the coefficients in asymptotic expansion of geometric mean obtained in [9] for $\alpha = 0$ and $\beta = t$, and also to coefficients of power mean M_p with $p = 0$ obtained in [10]. On the other side, from the formula (14), we obtain

$$L_c(x-t, x+t) = \sqrt{x^2 - t^2},$$

and, after substitution

$$L_c(a,b) = \sqrt{ab} = G(a,b).$$

From the example above, we see that we covered the cases of means for which in [1] was stated that symmetries S and σ coincide.

4. Answer to the Open Question

Theorem 3. *For mean L_c, $c \in \mathbf{R}$, defined in (15), symmetries S_{L_c} and σ_{L_c} coincide.*

Proof. Let us rewrite mean L_c in the following manner:

$$L_c(a,b) = \frac{1}{4(1+c)}\left[(1+2c)(a+b) + \sqrt{(a+b)^2 + 4c(1+c)(b-a)^2}\right].$$

For $M_0 = L_c$ and variable mean $M_1 = M$, there exists symmetric mean $\sigma = \sigma_{L_c}(M)$, i.e., the condition $L_c(M,\sigma) = L_c$ holds, which yields (for the sake of brevity, the variables will be ommited):

$$\frac{1}{4(1+c)}\left[(1+2c)(M+\sigma) + \sqrt{(M+\sigma)^2 + 4c(1+c)(M-\sigma)^2}\right] = L_c,$$

or equivalently

$$\sqrt{(M+\sigma)^2 + 4c(1+c)(M-\sigma)^2} = 4(1+c)L_c - (1+2c)(M+\sigma).$$

We rearrange the terms and, because of the existence of mean $\sigma = \sigma_{L_c}(M)$, we may square the latter expression:

$$\begin{aligned} &M^2(1+2c)^2 + 2M\sigma(1-4c-4c^2) + \sigma^2(1+2c)^2 \\ &\quad = [4(1+c)L_c - (1+2c)M]^2 - 2[4(1+c)L_c - (1+2c)M]^2 + \sigma^2(1-2c)^2. \end{aligned}$$

The terms $\sigma^2(1-2c)^2$ cancel from both sides. Further calculation gives

$$\begin{aligned} &2M(1-4c-4c^2)\sigma + 2\big(4(1+c)L_c - (1+2c)M\big)(1+2c)\sigma \\ &\qquad = -M^2(1+2c)^2 + \big(4(1+c)L_c - (1+2c)M\big)^2, \end{aligned}$$

and finally

$$\sigma = \frac{L_c\big((1+2c)M - 2(1+c)L_c\big)}{2cM - (1+2c)L_c}. \tag{18}$$

Thus, we obtained the explicit expression for mean $\sigma = \sigma_{L_c}(M)$ in terms of M and L_c.

On the other side, from (4), we know that

$$S_{L_c}(M) = \frac{a(M-a)(L_c-b)^2 - b(L_c-a)^2(M-b)}{(M-a)(L_c-b)^2 - (L_c-a)^2(M-b)},$$

which may be written as

$$S_{L_c}(M) = \frac{K_1 M - K_2}{K_0 M - K_1}, \tag{19}$$

where

$$\begin{aligned} K_0 &= (L_c-b)^2 - (L_c-a)^2, \\ K_1 &= a(L_c-b)^2 - b(L_c-a)^2, \\ K_2 &= a^2(L_c-b)^2 - b^2(L_c-a)^2. \end{aligned}$$

By equating the results of mappings σ and S with respect to mean L_c of a mean M and employing Formulas (18) and (19), we obtain

$$\frac{L_c\big((1+2c)M - 2(1+c)L_c\big)}{2cM - (1+2c)L_c} = \frac{K_1 M - K_2}{K_0 M - K_1}$$

which needs to be proved. We calculate

$$L_c[2(1+c)L_c - (1+2c)M](K_0M - K_1) = [(1+2c)L_c - 2cM](K_1M - K_2).$$

Grouping by the powers of M yields

$$\begin{aligned}[M_0(1+2c)K_0 - 2cK_1]M^2 + 2\Big[K_2c - (1+c)L_c^2K_0\Big]M& \\ + L_c[2(1+c)L_cK_1 - (1+2c)K_2] = 0.&\end{aligned} \tag{20}$$

Now, we simplify each coefficient by the powers of M. First,

$$\begin{aligned}&M_0(1+2c)K_0 - 2cK_1 = \\ &\qquad = M_0(1+2c)\Big[(L_c-b)^2 - (L_c-a)^2\Big] - 2c\Big[a(L_c-b)^2 - b(L_c-a)^2\Big] \\ &\qquad = (a-b)\Big[2(1+c)L_c^2 - (a+b)(1+2c)L_c + 2abc\Big],\end{aligned}$$

second,

$$\begin{aligned}&cK_2 - (1+c)L_c^2K_0 = \\ &\qquad = c\Big[a^2(L_c-b)^2 - b^2(L_c-a)^2\Big] - (1+c)L_c^2\Big[(L_c-b)^2 - (L_c-a)^2\Big] \\ &\qquad = -(a-b)L_c\Big[2(1+c)L_c^2 - (a+b)(1+2c)L_c + 2abc\Big],\end{aligned}$$

and third

$$\begin{aligned}&2(1+c)L_cK_1 - (1+2c)K_2 = \\ &\quad = 2(1+c)L_c\Big[a(L_c-b)^2 - b(L_c-a)^2\Big] - (1+2c)\Big[a^2(L_c-b)^2 - b^2(L_c-a)^2\Big] \\ &\quad = (a-b)L_c\Big[2(1+c)L_c^2 - (a+b)(1+2c)L_c + 2abc\Big].\end{aligned}$$

Hence, the equation (20) factorizes as

$$(a-b)\Big[2(1+c)L_c^2 - (a+b)(1+2c)L_c + 2abc\Big]\Big(M^2 - 2L_cM + L_c^2\Big) = 0. \tag{21}$$

Notice that the mean L_c defined in (15) is one of the solutions of quadratic equation

$$2(1+c)L_c^2 - (a+b)(1+2c)L_c + 2abc,$$

and the condition (21) is fulfilled, which proves the theorem. □

We will close this section with a conjecture. Based on the analysis in this paper, we may conclude the following.

Hypothesis 1. *Symmetric homogeneous mean, which has the asymptotic power series expansion and fulfills the requirements of the open question from [1] necessarily has the same coefficients as mean L_c, $c \in \mathbf{R}$.*

5. Concluding Remarks

Using techniques of asymptotic expansions, we were able to compare two symmetries of different origins on the set of mean functions. Finding asymptotic series expansion for both of them, in terms of recursive algorithm for their coefficients, enabled us to carry out the coefficient comparison, which resulted in obtaining a class of means, which interpolates between harmonic, geometric, and arithmetic mean.

Methods presented in this paper may be useful with various problems regarding bivariate means and further. For example, in case of dual means, generalized inverses of means and similar problems where some functional connection is given, and especially when the explicit formula for some of the means involved, was not known.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Farhi, B. Algebraic and topological structures on the set of mean functions and generalization of the AGM mean. *Colloq. Math.* **2013**, *132*, 139–149. [CrossRef]
2. Raissouli, M.; Chergui, M. On a metric topology on the set of bivariate means. *Acta Univ. Sapientiae Math.* **2022**, *14*, 147–159. [CrossRef]
3. Głazowska, D.; Matkowski, J. Generalized Classical Weighted Means, the Invariance, Complementarity and Convergence of Iterates of the Mean-Type Mappings. *Results Math.* **2022**, *77*, 72. [CrossRef]
4. Matkowski, J.; Nowicka, M.; Witkowski, A. Explicit Solutions of the Invariance Equation for Means. *Results Math.* **2017**, *71*, 397–410. [CrossRef]
5. Toader, G.; Costin, I.; Toader, S. Invariance in Some Families of Means. In *Functional Equations in Mathematical Analysis*; Springer: New York, NY, USA,
6. Toader, S.; Rassias, T.M.; Toader, G. A Gauss type functional equation. *Int. J. Math. Math. Sci.* **2001**, *25*, 856942. [CrossRef]
7. Jarczyk, J.; Jarczyk, W. Invariance of means. *Aequat. Math.* **2018**, *92*, 801–872. [CrossRef]
8. Chen, C.-P.; Elezović, N.; Vukšić, L. Asymptotic formulae associated with the Wallis power function and digamma function. *J. Class. Anal.* **2013**, *2*, 151–166. [CrossRef]
9. Elezović, N.; Vukšić, L. Asymptotic expansions of bivariate classical means and related inequalities. *J. Math. Inequal.* **2014**, *8*, 707–724. [CrossRef]
10. Elezović, N.; Vukšić, L. Asymptotic expansions and comparison of bivariate parameter means. *Math. Inequal. Appl.* **2014** *17*, 1225–1244. [CrossRef]
11. Elezović, N. Asymptotic inequalities and comparison of classical means. *J. Math. Inequal.* **2015**, *9*, 177–196. [CrossRef]
12. Elezović, N.; Mihoković, L. Inequalities between reciprocals of means. *Sarajevo J. Math.* **2015**, *11*, 171–180.
13. Vukšić, L. Seiffert means, asymptotic expansions and inequalities. *Rad Hrvat. Akad. Znan. Umjet. Mat. Znan.* **2015**, *19*, 129–142.
14. Elezović, N.; Vukšić, L. Asymptotic expansions of integral means and applications to the ratio of gamma functions. *Appl. Math. Comput.* **2014**, *235*, 187–200. [CrossRef]
15. Erdélyi, A. *Asymptotic Expansions*; Dover Publications: Mineola, NY, USA, 1956.
16. Gould, H.W. Coefficient identities for powers of Taylor and Dirichlet series. *Am. Math. Monthly* **1974**, *81*, 3–14. [CrossRef]
17. Burić, T.; Elezović, N. Computation and analysis of the asymptotic expansions of the compound means. *Appl. Math. Comput.* **2017**, *303*, 48–54. [CrossRef]
18. Graham, R.L.; Knuth, D.B.; Patashnik, O. *Concrete Mathematics*; Addison-Wesley: New York, NY, USA, 1989.