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A Fast Calculation Method for Sensitivity Analysis Using Matrix Decomposition Technique

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Abstract: The sensitivity reanalysis technique is an important tool for selecting the search direction in structural optimization design. Based on the decomposition perturbation of the flexibility matrix, a fast and exact structural displacement sensitivity reanalysis method is proposed in this work. For this purpose, the direct formulas for computing the first-order and second-order sensitivities of structural displacements are derived. The algorithm can be applied to a variety of the modifications in optimal design, including the low-rank modifications, high-rank modifications, small modifications and large modifications. Two numerical examples are given to verify the effectiveness of the proposed approach. The results show that the presented algorithm is exact and effective. Compared with the existing two reanalysis methods, this method has obvious advantages in calculation accuracy and efficiency. This new algorithm is very useful for calculating displacement sensitivity in engineering problems such as structure optimization, model correction and defect detection.

Keywords: sensitivity reanalysis; flexibility matrix; disassembly perturbation; structural displacement; exact method

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1. Introduction

Sensitivity analysis is often used in structural optimization design, vibration control, and damage identification. In general, sensitivity refers to the first derivative of structural response parameters to its physical parameters [1,2]. In engineering design, it is often necessary to modify the structure repeatedly. As a result, the computational cost for sensitivity analysis will be very expensive. To reduce the computational burden, reanalysis and sensitivity reanalysis techniques have been studied continuously in the past decades [3–8]. Sensitivity reanalysis uses the original response of the structure and its sensitivity to find the response sensitivity coefficients of the modified structure, whose calculation cost is far lower than the cost required for the complete analysis. For a structure under a given load vector y, the displacement vector x in the initial design can be computed by the static equilibrium equation as

$$K \cdot x = y \tag{1}$$

in which K is the structural stiffness matrix of $n \times n$ dimension in the initial finite element model (FEM). From Equation (1), the displacement x and its sensitivity $\frac{\partial x}{\partial p_i}$ of the initial design can be calculated from the complete analysis as

$$x = K^{-1} \cdot y = F \cdot y \tag{2}$$

$$\frac{\partial x}{\partial p_i} = -K^{-1}\frac{\partial K}{\partial p_i} \cdot x = -F\frac{\partial K}{\partial p_i}F \cdot y \tag{3}$$

where p_i is a design variable such as geometry size, elastic modulus, and so on. The matrix F is called the structural flexibility matrix, that is, $F = K^{-1}$. Correspondingly, the static balance equation of the modified structure can be expressed as

$$K_d \cdot x_d = y \tag{4}$$

$$K_d = K + \Delta K \tag{5}$$

in which K_d is the modified stiffness matrix, ΔK is the stiffness change caused by the optimal design, and x_d is the modified displacement vector. From Equation (4), x_d and its sensitivity $\frac{\partial x_d}{\partial p_i}$ can also be computed by the complete analysis as

$$x_d = K_d^{-1} \cdot y = F_d \cdot y \tag{6}$$

$$\frac{\partial x_d}{\partial p_i} = -K_d^{-1} \frac{\partial K_d}{\partial p_i} \cdot x_d = -F_d \frac{\partial K_d}{\partial p_i} F_d \cdot y \tag{7}$$

in which F_d is the modified flexibility matrix, i.e., $F_d = K_d^{-1}$. As mentioned earlier, when the half-bandwidth of the stiffness matrix is large, the complete analysis based on Equations (6) and (7) is very inefficient and time-consuming. For solving this problem, many reanalysis algorithms have been presented to calculate x_d and its sensitivity $\frac{\partial x_d}{\partial p_i}$ more effectively. The existing sensitivity reanalysis methods can be divided into two types: finite-difference method [9–12] and direct (analytic) method [13–16]. Most of the existing reanalysis methods can only obtain the approximate solution of displacement sensitivity. Moreover, these methods may be inefficient for large modifications or high-rank modifications. The high-rank modification refers to the design changes in many components of the structure. In view of this, an exact sensitivity reanalysis approach using flexibility disassembly perturbation (FDP) [17–19] is developed in this work for computing the displacement sensitivity. The presented algorithm is accurate and efficient, and it can be used for many types of modifications in design, such as the low-rank, high-rank, small and large modifications. Numerical examples show that the results obtained by the presented sensitivity reanalysis algorithm are the same as those obtained by the complete analysis. In addition, this approach has higher computing efficiency than the existing sensitivity reanalysis methods.

2. Sensitivity Reanalysis Using FDP

Reference [19] presented a static reanalysis method using the FDP technique for quickly and exactly calculating the displacement vector after structural modification. In addition to the displacement vector, the displacement sensitivity is another quantity that needs to be repeatedly calculated in structural optimization design, which indicates the direction of optimization design. So, in this work, FDP is used again to exactly compute the displacement sensitivity after structural modification. The research content of this work can be seen as an extension of reference [19]. From Equation (7), the modified displacement sensitivity $\frac{\partial x_d}{\partial p_i}$ can be easily calculated by the modified flexibility matrix F_d. Thus, the reanalysis problem of displacement sensitivity can be transformed into the reanalysis problem of structural flexibility matrix after modification. According to references [17–19], the modified flexibility matrix can be fast computed using FDP. The core idea of FDP is to decompose the flexibility matrix into a connected matrix reflecting the topological relationship between the degrees of freedom (DOFs) and the diagonal matrix reflecting the material and geometric information. The formulas of FDP are briefly derived as follows. According to the FEM theory, structural stiffness matrix K is the sum of all elementary stiffness matrices K_i ($i = 1 \sim N$), that is

$$K = \sum_{i=1}^{N} K_i \tag{8}$$

in which N is the number of all elements in FEM. Performing the spectral decomposition on K_i yields

$$K_i = [c_i^1, \cdots, c_i^r] \begin{bmatrix} p_i^1 & & \\ & \ddots & \\ & & p_i^r \end{bmatrix} [c_i^1, \cdots, c_i^r]^T \tag{9}$$

In Equation (9), the non-zero eigenvalues $p_i^1, \cdots, p_i^r$ are purely functions of the material and geometric properties such as elastic modulus E, cross-sectional area A and moment of inertia I. The eigenvectors $c_i^1, \cdots, c_i^r$ reflect the topological relationship between degrees of freedom. For instance, the spectral decomposition on a plane beam element gives [20]:

$$[p_i] = \begin{bmatrix} \frac{2EA}{L} & 0 & 0 \\ 0 & \frac{2EI}{L} & 0 \\ 0 & 0 & \frac{6EI(L^2+4)}{L^3} \end{bmatrix} \tag{10}$$

$$[c_i] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{\sqrt{L^2+4}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{L}{\sqrt{2}\sqrt{L^2+4}} \\ \frac{-1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{2}}{\sqrt{L^2+4}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{L}{\sqrt{2}\sqrt{L^2+4}} \end{bmatrix} \tag{11}$$

in which L denotes the beam element length. Thus, $p_i^1, \cdots, p_i^r$ are also called the elementary stiffness coefficients and $c_i^1, \cdots, c_i^r$ are called the topological connection vectors. From Equations (8) and (9), the stiffness disassembly formula can be obtained as

$$K = CPC^T \tag{12}$$

$$C = [C_1^1, \cdots, c_1^r, c_2^1, \cdots, c_2^r, \cdots, c_N^r] \tag{13}$$

$$P = \begin{bmatrix} p_1^1 & & & & \\ & \ddots & & & \\ & & p_1^r & & \\ & & & \ddots & \\ & & & & p_N^r \end{bmatrix} \tag{14}$$

in which C is a $n \times rN$ dimension matrix, and P is a $rN \times rN$ dimension matrix. C is a full-rank matrix with $rank(C_{n \times rN}) = n$ because of $rank(K_{n \times n}) = n$. For the statically determinate system, C is a square matrix of $n = rN$. For the statically indeterminate system, C is a rectangular matrix of $n < rN$. Commonly, structural modifications such as the section correction or material correction only lead to the change of stiffness coefficients $p_i^1, \cdots, p_i^r$. This means that only P is changed in the structural modifications. As a result, the disassembly of the stiffness matrix K_d after modification can be derived as

$$K_d = CP_dC^T \tag{15}$$

$$P_d = \begin{bmatrix} p_1^1(1+\alpha_1^1) & & & & \\ & \ddots & & & \\ & & p_1^r(1+\alpha_1^r) & & \\ & & & \ddots & \\ & & & & p_N^r(1+\alpha_N^r) \end{bmatrix} \tag{16}$$

where α_i^j ($i = 1 \sim N, j = 1 \sim r$) denotes the modification ratio of the stiffness parameter p_i^j. As stated before, C is a full-rank square matrix for the statically determinate system. Thus, the flexibility matrix F_d can be fast computed from Equation (15) by $F_d = K_d^{-1}$ as

$$F_d = DQ_dD^T \tag{17}$$

$$D = (C^{-1})^T \tag{18}$$

$$Q_d = P_d^{-1} = \begin{bmatrix} \frac{1}{p_1^1(1+\alpha_1^1)} & & & & \\ & \ddots & & & \\ & & \frac{1}{p_1^r(1+\alpha_1^r)} & & \\ & & & \ddots & \\ & & & & \frac{1}{p_N^r(1+\alpha_N^r)} \end{bmatrix} \tag{19}$$

It should be pointed out that the computational burden of the flexibility matrix reanalysis is only focused on the diagonal matrix Q_d, which only requires simple division operation when the modification ratios α_i^j are given. The computation of the matrix D should be attributed to the initial analysis, since D is unchanged in each modification. For the statically indeterminate structure, the flexibility disassembly as in Equation (17) is nonexistent, since C is a rectangular matrix with $n < rN$. In this case, the flexible disassembly can be realized by converting the statically indeterminate system into a statically determinate substructure and the redundant constraints. Correspondingly, the stiffness disassembly of the statically indeterminate system can be expressed from Equation (15) by

$$K_d = CP_dC^T = C'P'_d(C')^T + C''P''_d(C'')^T \tag{20}$$

where C' and P'_d are associated with the statically determinate substructure, while C'' and P''_d are associated with the redundant constraints. The dimensions of C' and P'_d are both $n \times n$. The dimensions of C'' and P''_d are $n \times (rN - n)$ and $(rN - n) \times (rN - n)$, respectively. From Equation (20), the flexibility disassembly can be derived by $F_d = K_d^{-1}$ with the help of Sherman–Morrison–Woodbury formulas [21,22] as

$$F_d = D'Q'_d(D')^T - D'Q'_d(D')^T C''P''_d\left[I_e + (C'')^T D'Q'_d(D')^T C''P''_d\right]^{-1}(C'')^T D'Q'_d(D')^T \tag{21}$$

$$D' = ((C')^{-1})^T, \tag{22}$$

$$Q'_d = (P'_d)^{-1} \tag{23}$$

where I_e is the identity matrix, while Q'_d and P''_d are the corrections corresponding to the statically determinate subsystem and the redundant constraints. Equation (21) is the flexibility reanalysis formula for the statically indeterminate system with the given Q'_d and P''_d.

According to the above theory and derivation, the modified displacement sensitivity $\frac{\partial x_d}{\partial p_i}$ can be fast computed using Equation (7) with F_d determined by Equation (17) or (21). It is clear that Equation (17) is an exceptional case of Equation (21). The step-by-step summary for the proposed sensitivity reanalysis approach is as follows. Step 1: Perform the stiffness disassembly of the initial structure using Equations (8)–(14) to obtain the matrices C, or C' and C''. Step 2: Compute the matrix D or D' by Equation (18) or (22). Step 3: Calculate the modified flexibility matrix F_d by Equation (17) or (21) with the given modifications Q_d, or Q'_d and P''_d. Step 4: Compute the displacement sensitivity $\frac{\partial x_d}{\partial p_i}$ of the modified structure using Equation (7). Note that the calculations in steps 1 and 2 should be attributed to the initial analysis. The computational burden of the sensitivity reanalysis algorithm is the focus of steps 3 and 4. Another virtue of this algorithm is that it can be readily extended to calculate the second-order sensitivity of static displacement. Differentiating Equation (4) with respect to p_i twice and rearranging gives the second-order sensitivity $\frac{\partial^2 x_d}{\partial p_i^2}$ as

$$\frac{\partial^2 x_d}{\partial p_i^2} = -F_d \frac{\partial^2 K_d}{\partial p_i^2} F_d y - 2F_d \frac{\partial K_d}{\partial p_i} \cdot \frac{\partial x_d}{\partial p_i} \tag{24}$$

Apparently, the second-order sensitivity of static displacement can also be fast calculated by Equation (24) using the proposed method for the modified structure.

3. Numerical Examples

3.1. Statically Determinate Structure

As presented in Figure 1, a statically determinate system of a 23-bar truss is used firstly to demonstrate the proposed approach. The values of the concentrated loads applied to the structure shown in Figure 1 are $f_1 = f_2 = f_3 = f_4 = f_5 = 10$ kN. Assuming the change rate of cross-sectional area is the correction factor α_i, Table 1 gives several modification cases including the low-rank, high-rank, small and large corrections. Tables 2 and 3 present the first-order sensitivity $\frac{\partial x_d}{\partial p_{10}}$ and second-order sensitivity $\frac{\partial^2 x_d}{\partial p_{10}^2}$ using the proposed method and complete analysis for these modification cases. It is found from Tables 2 and 3 that the reanalysis results of the presented algorithm are the same as the complete analysis results. This shows that the proposed method is an exact algorithm for displacement sensitivity reanalysis.

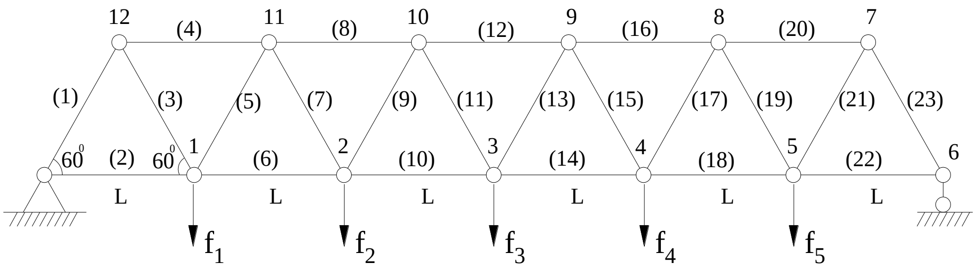

Figure 1. An initial structure of a 23-bar truss. Material parameters: Elastic modulus is 200 GPa, density is 7800 kg/m^3, L = 1 m, and initial cross-sectional area of each bar is 175.9 mm^2.

3.2. Statically Indeterminate Structure

As presented in Figure 2, a statically indeterminate system of a 275-bar truss is used to conduct the comparison study on the computation efficiency between this method and two existing sensitivity reanalysis approaches. The first existing technique is the combined approximate (CA) method proposed by Kirsch in reference [10]. The second existing technique is the method proposed by Zuo et al. in reference [16], which combines Taylor series expansion and the CA method. Table 4 gives three types of corrections for this example. As shown in Figure 2, the modified bars of the three types of corrections are: bars 1~10, bars 1~93 (the first story), and all bars (1~275) of the system, respectively. For each correction, 200 modifications are performed, and the total calculation times of displacement sensitivities $\frac{\partial x_d}{\partial p_8}$ using the complete analysis, the CA method, Zuo's method, and the proposed method are given in Table 5. Note that the correction coefficient α_i increases with the modification number z ($z = 1 \sim 150$). This means that the early stage corresponds to small modifications and the later stage corresponds to large modifications. Tables 6–11 show the displacement sensitivity data of some DOFs for each correction scenario with $z = 1$, $z = 2$, $z = 10$ and $z = 15$, respectively. From Table 5, one can see that the presented algorithm has the highest calculation efficiency among the four sensitivity reanalysis methods. For type 1 (10 bars are modified), the calculation times of the four methods are: t_1 = 0.262 s (the complete analysis), t_2 = 0.166 s (CA method), t_3 = 0.161 s (Zuo's method) and t_4 = 0.083 s (the presented algorithm), respectively. For type 2 (93 bars are modified), the calculation times of the four methods are: t_1 = 0.254 s (the complete analysis), t_2 = 0.191 s (CA method), t_3 = 0.174 s (Zuo's method) and t_4 = 0.097 s (the presented algorithm), respectively. For the third type (all bars are modified), the calculation times of the four methods are: t_1 = 0.292 s (the complete analysis), t_2 = 0.232 s (CA method), t_3 = 0.217 s (Zuo's method) and t_4 = 0.140 s (the presented algorithm), respectively. Overall, the calculation time of the presented algorithm is about 30~40% of that of the complete analysis method, and it is about 50~60% of that of CA or Zuo's method. This means that whether the number of correction bars is small or large, the presented algorithm always has the high computation efficiency. According to Tables 6–11, it can be seen that the results achieved by the presented approach and the complete analysis method are exactly the same. One can also find that the results obtained by CA and Zuo's methods have some errors compared with the exact results. These results show that the presented approach is an exact algorithm for displacement sensitivity reanalysis, and the CA and Zuo's methods are approximate methods.

Table 1. Different correction cases of a 23-bar truss.

The Correction Coefficient α_i	Scenario 1: Low-Rank Correction	Scenario 2: High-Rank Small Correction	Scenario 3: High-Rank Large Correction
α_1	0	0.15	4.87
α_2	0	0.17	4.07
α_3	0	−0.08	−4.22
α_4	0	0.15	3.32
α_5	0.21	0.19	−1.93
α_6	0	−0.09	−1.15
α_7	0	−0.10	−0.88
α_8	0	0.14	−0.53
α_9	0.44	−0.02	−1.40
α_{10}	0	0.19	−4.66
α_{11}	0	−0.18	0.32
α_{12}	0	0.12	1.81
α_{13}	0	0.06	−1.32
α_{14}	−0.32	−0.16	3.08
α_{15}	0	0.17	−0.87
α_{16}	0	−0.08	1.16
α_{17}	0	0.13	0.54
α_{18}	0	0.09	−1.76
α_{19}	0	−0.13	−0.03
α_{20}	0	−0.10	4.27
α_{21}	0	0.05	4.35
α_{22}	0	0.10	−3.18
α_{23}	0	0.08	4.06

Table 2. The first-order sensitivities of displacements for modified structures ($\times 10^{-3}$).

DOF Number	Scenario 1: Low-Rank Correction		Scenario 2: High-Rank Small Correction		Scenario 3: High-Rank Large Correction	
	The Complete Analysis	The Proposed Reanalysis Algorithm	The Complete Analysis	The Proposed Reanalysis Algorithm	The Complete Analysis	The Proposed Reanalysis Algorithm
1	0.000	0.000	0.000	0.000	0.000	0.000
2	0.940	0.940	0.664	0.664	0.070	0.070
3	0.000	0.000	0.000	0.000	0.000	0.000
4	1.879	1.879	1.327	1.327	0.140	0.140
5	−1.395	−1.395	−0.985	−0.985	−0.104	−0.104
6	2.013	2.013	1.422	1.422	0.150	0.150
7	−1.395	−1.395	−0.985	−0.985	−0.104	−0.104
8	1.342	1.342	0.948	0.948	0.100	0.100
9	−1.395	−1.395	−0.985	−0.985	−0.104	−0.104
10	0.671	0.671	0.474	0.474	0.050	0.050
11	−1.395	−1.395	−0.985	−0.985	−0.104	−0.104
12	−0.814	−0.814	−0.575	−0.575	−0.061	−0.061
13	0.336	0.336	0.237	0.237	0.025	0.025
14	−0.814	−0.814	−0.575	−0.575	−0.061	−0.061
15	1.007	1.007	0.711	0.711	0.075	0.075
16	−0.814	−0.814	−0.575	−0.575	−0.061	−0.061
17	1.678	1.678	1.185	1.185	0.125	0.125
18	−0.814	−0.814	−0.575	−0.575	−0.061	−0.061
19	2.349	2.349	1.659	1.659	0.175	0.175
20	−0.814	−0.814	−0.575	−0.575	−0.061	−0.061
21	1.409	1.409	0.995	0.995	0.105	0.105
22	−0.814	−0.814	−0.575	−0.575	−0.061	−0.061
23	0.470	0.470	0.332	0.332	0.035	0.035

Table 3. The second-order sensitivities of displacements for modified structures ($\times 10^{-3}$).

DOF Number	Scenario 1: Low-Rank Correction		Scenario 2: High-Rank Small Correction		Scenario 3: High-Rank Large Correction	
	The Complete Analysis	The Proposed Reanalysis Algorithm	The Complete Analysis	The Proposed Reanalysis Algorithm	The Complete Analysis	The Proposed Reanalysis Algorithm
1	0.000	0.000	0.000	0.000	0.000	0.000
2	−1.879	−1.879	−1.115	−1.115	0.038	0.038
3	0.000	0.000	0.000	0.000	0.000	0.000
4	−3.758	−3.758	−2.230	−2.230	0.077	0.077
5	2.790	2.790	1.656	1.656	−0.057	−0.057
6	−4.027	−4.027	−2.390	−2.390	0.082	0.082
7	2.790	2.790	1.656	1.656	−0.057	−0.057
8	−2.685	−2.685	−1.593	−1.593	0.055	0.055
9	2.790	2.790	1.656	1.656	−0.057	−0.057
10	−1.342	−1.342	−0.797	−0.797	0.027	0.027
11	2.790	2.790	1.656	1.656	−0.057	−0.057
12	1.627	1.627	0.966	0.966	−0.033	−0.033
13	−0.671	−0.671	−0.398	−0.398	0.014	0.014
14	1.627	1.627	0.966	0.966	−0.033	−0.033
15	−2.013	−2.013	−1.195	−1.195	0.041	0.041
16	1.627	1.627	0.966	0.966	−0.033	−0.033
17	−3.356	−3.356	−1.991	−1.991	0.068	0.068

Table 3. *Cont.*

DOF Number	Scenario 1: Low-Rank Correction		Scenario 2: High-Rank Small Correction		Scenario 3: High-Rank Large Correction	
	The Complete Analysis	The Proposed Reanalysis Algorithm	The Complete Analysis	The Proposed Reanalysis Algorithm	The Complete Analysis	The Proposed Reanalysis Algorithm
18	1.627	1.627	0.966	0.966	−0.033	−0.033
19	−4.698	−4.698	−2.788	−2.788	0.096	0.096
20	1.627	1.627	0.966	0.966	−0.033	−0.033
21	−2.819	−2.819	−1.673	−1.673	0.057	0.057
22	1.627	1.627	0.966	0.966	−0.033	−0.033
23	−0.940	−0.940	−0.558	−0.558	0.019	0.019

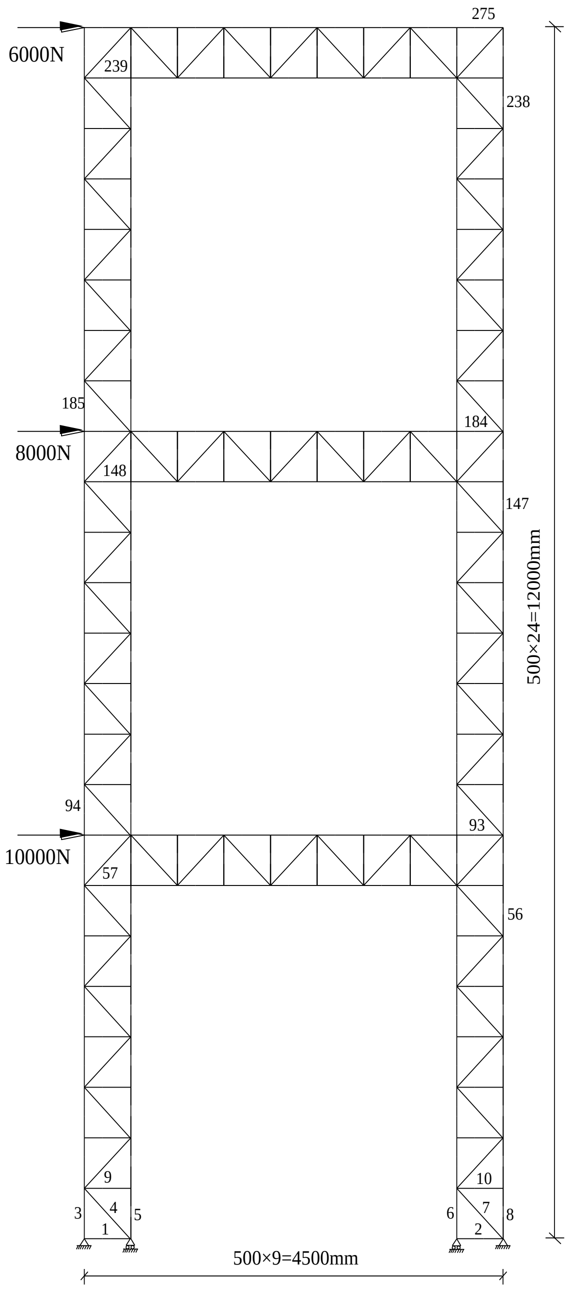

Figure 2. An initial structure of a 275-bar truss. Material parameters: Elastic modulus is 200 GPa, density is 7800 kg/m^3, L = 0.5 m, and initial cross-sectional area of each bar is 314 mm^2.

Table 4. Types of corrections in the 275-bar truss system.

Type of Correction	Modified Bars	Correction Coefficients α_i^z (i is the Bar Number, z is the Modification Number, z = 1~150)
Type 1	Bars 1~10 as shown in Figure 2	$\alpha_i^z = \frac{z}{20}, i = 1 \sim 10$
Type 2	Bars 1~93 of the first story as shown in Figure 2	$\alpha_i^z = \begin{cases} \frac{z}{40}, i = 1 \sim 56 \\ \frac{z}{50}, i = 57 \sim 93 \end{cases}$
Type 3	All bars (1~275) in Figure 2	First story: $\alpha_i^z = \begin{cases} \frac{z}{40}, i = 1 \sim 56 \\ \frac{z}{50}, i = 57 \sim 93 \end{cases}$ Second story: $\alpha_i^z = \begin{cases} \frac{z}{60}, i = 94 \sim 147 \\ \frac{z}{75}, i = 148 \sim 184 \end{cases}$ Third story: $\alpha_i^z = \begin{cases} \frac{z}{80}, i = 185 \sim 238 \\ \frac{z}{100}, i = 239 \sim 275 \end{cases}$

Table 5. Computation times of the four algorithms for the three types of modifications.

Type of Modification	The Complete Analysis t_1	CA Method t_2	Zuo's Method t_3	The Proposed Method t_4
Type 1 (10 elements are revised)	t_1 = 0.262 s	t_2 = 0.166 s	t_3 = 0.161 s	t_4 = 0.083 s
		$(t_1 - t_2)/t_1$ = 36.6%	$(t_1 - t_3)/t_1$ = 38.5%	$(t_1 - t_4)/t_1$ = 68.3%
			$(t_2 - t_3)/t_2$ = 3.0%	$(t_2 - t_4)/t_2$ = 50.0%
				$(t_3 - t_4)/t_3$ = 48.4%
Type 2 (93 elements are revised)	t_1 = 0.254 s	t_2 = 0.191 s	t_3 = 0.174 s	t_4 = 0.097 s
		$(t_1 - t_2)/t_1$ = 24.8%	$(t_1 - t_3)/t_1$ = 31.5%	$(t_1 - t_4)/t_1$ = 61.8%
			$(t_2 - t_3)/t_2$ = 8.9%	$(t_2 - t_4)/t_2$ = 49.2%
				$(t_3 - t_4)/t_3$ = 44.3%
Type 3 (all elements are revised)	t_1 = 0.292 s	t_2 = 0.232 s	t_3 = 0.217 s	t_4 = 0.140 s
		$(t_1 - t_2)/t_1$ = 20.5%	$(t_1 - t_3)/t_1$ = 25.7%	$(t_1 - t_4)/t_1$ = 52.1%
			$(t_2 - t_3)/t_2$ = 6.5%	$(t_2 - t_4)/t_2$ = 39.7%
				$(t_3 - t_4)/t_3$ = 35.5%

Table 6. Displacement sensitivities for modification type 1 when $z = 1$ and $z = 2$ ($\times 10^{-5}$).

DOF Number	The Complete Analysis		CA Method		Zuo's Method		The Proposed Method	
	z = 1	z = 2	z = 1	z = 2	z = 1	z = 2	z = 1	z = 2
10	1.659	1.518	1.657	1.516	1.663	1.531	1.659	1.518
11	−0.551	−0.496	−0.551	−0.495	−0.552	−0.500	−0.551	−0.496
12	−0.187	−0.169	−0.187	−0.169	−0.188	−0.171	−0.187	−0.169
13	−0.551	−0.496	−0.551	−0.495	−0.552	−0.500	−0.551	−0.496
14	0.169	0.154	0.168	0.154	0.169	0.155	0.169	0.154
15	−0.979	−0.908	−0.979	−0.909	−0.981	−0.915	−0.979	−0.908
16	0.344	0.310	0.344	0.309	0.345	0.313	0.344	0.310
17	−0.979	−0.908	−0.979	−0.909	−0.981	−0.915	−0.979	−0.908
18	1.492	1.364	1.491	1.362	1.496	1.376	1.492	1.364
19	−1.047	−0.949	−1.046	−0.948	−1.049	−0.957	−1.047	−0.949
20	−0.260	−0.237	−0.260	−0.237	−0.260	−0.239	−0.260	−0.237
21	−1.047	−0.949	−1.046	−0.948	−1.049	−0.957	−1.047	−0.949

Table 7. Displacement sensitivities for modification type 1 when $z = 10$ and $z = 15$ ($\times 10^{-5}$).

DOF Number	The Complete Analysis		CA Method		Zuo's Method		The Proposed Method	
	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$
10	0.837	0.622	0.835	0.619	0.971	0.813	0.837	0.622
11	−0.244	−0.171	−0.240	−0.165	−0.283	−0.224	−0.244	−0.171
12	−0.087	−0.062	−0.086	−0.061	−0.101	−0.081	−0.087	−0.062
13	−0.244	−0.171	−0.240	−0.165	−0.283	−0.224	−0.244	−0.171
14	0.083	0.060	0.084	0.062	0.096	0.079	0.083	0.060
15	−0.540	−0.413	−0.563	−0.447	−0.626	−0.540	−0.540	−0.413
16	0.155	0.110	0.143	0.093	0.180	0.144	0.155	0.110
17	−0.540	−0.413	−0.563	−0.447	−0.626	−0.540	−0.540	−0.413
18	0.747	0.553	0.742	0.547	0.866	0.723	0.747	0.553
19	−0.492	−0.355	−0.490	−0.352	−0.571	−0.464	−0.492	−0.355
20	−0.128	−0.094	−0.129	−0.095	−0.149	−0.123	−0.128	−0.094
21	−0.492	−0.355	−0.490	−0.352	−0.571	−0.464	−0.492	−0.355

Table 8. Displacement sensitivities for modification type 2 when $z = 1$ and $z = 2$ ($\times 10^{-5}$).

DOF Number	The Complete Analysis		CA Method		Zuo's Method		The Proposed Method	
	$z = 1$	$z = 2$	$z = 1$	$z = 2$	$z = 1$	$z = 2$	$z = 1$	$z = 2$
10	1.738	1.660	1.736	1.658	1.739	1.664	1.738	1.660
11	−0.587	−0.561	−0.587	−0.560	−0.588	−0.562	−0.587	−0.561
12	−0.198	−0.190	−0.198	−0.189	−0.198	−0.190	−0.198	−0.190
13	−0.587	−0.561	−0.587	−0.560	−0.588	−0.562	−0.587	−0.561
14	0.177	0.170	0.177	0.169	0.177	0.170	0.177	0.170
15	−1.012	−0.967	−1.010	−0.966	−1.012	−0.969	−1.012	−0.967
16	0.366	0.350	0.366	0.349	0.367	0.350	0.366	0.350
17	−1.012	−0.967	−1.010	−0.966	−1.012	−0.969	−1.012	−0.967
18	1.565	1.496	1.563	1.494	1.566	1.498	1.565	1.496
19	−1.107	−1.058	−1.106	−1.056	−1.108	−1.060	−1.107	−1.058
20	−0.273	−0.261	−0.273	−0.261	−0.273	−0.262	−0.273	−0.261
21	−1.107	−1.058	−1.106	−1.056	−1.108	−1.060	−1.107	−1.058

Table 9. Displacement sensitivities for modification type 2 when $z = 10$ and $z = 15$ ($\times 10^{-5}$).

DOF Number	The Complete Analysis		CA Method		Zuo's Method		The Proposed Method	
	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$
10	1.193	0.996	1.181	0.976	1.242	1.079	1.193	0.996
11	−0.403	−0.336	−0.399	−0.329	−0.420	−0.364	−0.403	−0.336
12	−0.137	−0.114	−0.136	−0.113	−0.142	−0.124	−0.137	−0.114
13	−0.403	−0.336	−0.399	−0.329	−0.420	−0.364	−0.403	−0.336
14	0.122	0.102	0.121	0.100	0.128	0.111	0.122	0.102
15	−0.698	−0.584	−0.692	−0.574	−0.727	−0.633	−0.698	−0.584

Table 9. *Cont.*

DOF Number	The Complete Analysis		CA Method		Zuo's Method		The Proposed Method	
	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$
16	0.250	0.208	0.247	0.203	0.260	0.225	0.250	0.208
17	−0.698	−0.584	−0.692	−0.574	−0.727	−0.633	−0.698	−0.584
18	1.075	0.898	1.065	0.880	1.120	0.973	1.075	0.898
19	−0.761	−0.635	−0.753	−0.623	−0.792	−0.688	−0.761	−0.635
20	−0.188	−0.157	−0.187	−0.155	−0.196	−0.171	−0.188	−0.157
21	−0.761	−0.635	−0.753	−0.623	−0.792	−0.688	−0.761	−0.635

Table 10. Displacement sensitivities for modification type 3 when $z = 1$ and $z = 2$ ($\times 10^{-5}$).

DOF Number	The Complete Analysis		CA Method		Zuo's Method		The Proposed Method	
	$z = 1$	$z = 2$	$z = 1$	$z = 2$	$z = 1$	$z = 2$	$z = 1$	$z = 2$
10	1.736	1.656	1.734	1.655	1.737	1.660	1.736	1.656
11	−0.587	−0.560	−0.586	−0.559	−0.587	−0.561	−0.587	−0.560
12	−0.198	−0.189	−0.198	−0.189	−0.198	−0.190	−0.198	−0.189
13	−0.587	−0.560	−0.586	−0.559	−0.587	−0.561	−0.587	−0.560
14	0.177	0.169	0.177	0.169	0.177	0.169	0.177	0.169
15	−1.010	−0.964	−1.009	−0.963	−1.011	−0.966	−1.010	−0.964
16	0.366	0.349	0.366	0.349	0.366	0.350	0.366	0.349
17	−1.010	−0.964	−1.009	−0.963	−1.011	−0.966	−1.010	−0.964
18	1.563	1.492	1.562	1.490	1.564	1.495	1.563	1.492
19	−1.106	−1.055	−1.105	−1.054	−1.106	−1.058	−1.106	−1.055
20	−0.273	−0.260	−0.272	−0.260	−0.273	−0.261	−0.273	−0.260
21	−1.106	−1.055	−1.105	−1.054	−1.106	−1.058	−1.106	−1.055

Table 11. Displacement sensitivities for modification type 3 when $z = 10$ and $z = 15$ ($\times 10^{-5}$).

DOF Number	The Complete Analysis		CA Method		Zuo's Method		The Proposed Method	
	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$	$z = 10$	$z = 15$
10	1.181	0.981	1.173	0.969	1.234	1.071	1.181	0.981
11	−0.399	−0.332	−0.397	−0.328	−0.417	−0.362	−0.399	−0.332
12	−0.135	−0.112	−0.134	−0.111	−0.141	−0.123	−0.135	−0.112
13	−0.399	−0.332	−0.397	−0.328	−0.417	−0.362	−0.399	−0.332
14	0.121	0.101	0.120	0.099	0.126	0.110	0.121	0.101
15	−0.689	−0.573	−0.685	−0.567	−0.720	−0.626	−0.689	−0.573
16	0.248	0.206	0.246	0.203	0.259	0.224	0.248	0.206
17	−0.689	−0.573	−0.685	−0.567	−0.720	−0.626	−0.689	−0.573
18	1.064	0.884	1.057	0.874	1.111	0.965	1.064	0.884
19	−0.753	−0.626	−0.749	−0.619	−0.787	−0.683	−0.753	−0.626
20	−0.186	−0.155	−0.185	−0.153	−0.194	−0.169	−0.186	−0.155
21	−0.753	−0.626	−0.749	−0.619	−0.787	−0.683	−0.753	−0.626

4. Conclusions

In this paper, an exact algorithm for the reanalysis of static displacement sensitivity based on flexibility disassembly perturbation is proposed. The presented algorithm is exact and efficient, and it can be used in many types of corrections in structural optimal design, including the low-rank, high-rank, small and large corrections. Numerical examples show that the presented approach can achieve the same results as the complete analysis method with less computational time. Compared with CA and Zuo's techniques, this algorithm has obvious advantages in computational efficiency and accuracy. It has been shown that the proposed algorithm has great application potential in structural optimization design based on gradient.

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