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A Generalization of Gegenbauer Polynomials and Bi-Univalent Functions

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Abstract: Three subclasses of analytic and bi-univalent functions are introduced through the use of $q-$Gegenbauer polynomials, which are a generalization of Gegenbauer polynomials. For functions falling within these subclasses, coefficient bounds $|a_2|$ and $|a_3|$ as well as Fekete–Szegö inequalities are derived. Specializing the parameters used in our main results leads to a number of new results.

Keywords: Fekete–Szegö problem; q–Gegenbauer polynomials; bi-univalent functions; q–calculus; analytic functions

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1. Introduction

Legendre first made the discovery of orthogonal polynomials in 1784 [1]. Under specific model restrictions, orthogonal polynomials are frequently employed to solve ordinary differential equations. Furthermore, a crucial function in the approximation theory is performed by orthogonal polynomials [2].

P_m and P_n are two polynomials of order m and n, respectively, and are orthogonal if

$$\int_a^b P_n(x)P_m(x)s(x)dx = 0, \qquad \text{for} \qquad m \neq n,$$

where $s(x)$ is a suitably specified function in the interval (a,b); therefore, all finite order polynomials $P_n(x)$ have a well-defined integral.

Gegenbauer polynomials are orthogonal polynomials of a specified type. As found in [3], when traditional algebraic formulations are used, the generating function of Gegenbauer polynomials and the integral representation of typically real functions T_R are related to each other in a symbolic way T_R. This undoubtedly caused a number of helpful inequalities to emerge from the world of Gegenbauer polynomials.

$q-$orthogonal polynomials are now of particular relevance in both physics and mathematics due to the development of quantum groups. The $q-$deformed harmonic oscillator, for instance, has a group-theoretic setting for the $q-$Laguerre and $q-$Hermite polynomials. Jackson's $q-$exponential plays a crucial role in the mathematical framework required to characterize the properties of these $q-$polynomials, such as the recurrence relations, generating functions, and orthogonality relations. Jackson's $q-$exponential has recently been expressed by Quesne [4] as a closed-form multiplicative series of regular exponentials with known coefficients. In this case, it is crucial to look into how this discovery might affect the theory of $q-$orthogonal polynomials. An effort in this regard was made in the current work. To obtain novel nonlinear connection equations for $q-$Gegenbauer polynomials in terms of their respective classical equivalents, we used the aforementioned result in particular.

This study analyzed various features of the class under consideration after associating some bi-univalent functions with $q-$Gegenbauer polynomials. The following part lays the foundation for mathematical notations and definitions.

2. Preliminaries

Let $\mathcal{A}$ denote the class of all analytical functions f that are defined on the open unit disk $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ and normalized by the formula $f(0) = f'(0) - 1 = 0$. As a result, each $f \in \mathcal{A}$ has the following Taylor–Maclaurin series expansion:

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \quad (\xi \in \mathbb{U}). \tag{1}$$

In addition, let $\mathcal{S}$ denote the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{U}$.

Let the functions $g(\xi)$ and $f(\xi)$ be analytic in $\mathbb{U}$. We say that the function $f(\xi)$ is subordinate to $g(\xi)$, written as $f(\xi) \prec g(\xi)$, if there exists a Schwarz function ω that is analytic in $\mathbb{U}$ with

$$|\omega(\xi)| < 1 \text{ and } \omega(0) = 0 \qquad (\xi \in \mathbb{U})$$

such that

$$g(\omega(\xi)) = f(\xi).$$

Beside that, if the function g is univalent in $\mathbb{U}$, then the following equivalence holds:

$$f(\xi) \prec g(\xi) \quad \text{if} \quad g(0) = f(0)$$

and

$$f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$\xi = f^{-1}(f(\xi)) \qquad (\xi \in \mathbb{U})$$

and

$$f^{-1}(f(w)) = w \qquad \left(r_0(f) \geq \frac{1}{4};\ |w| < r_0(f)\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 - w^3(a_3 - 2a_2^2) + w^4(5a_2 a_3 - a_4 - 5a_2^3) + \cdots. \tag{2}$$

If both $f^{-1}(\xi)$ and $f(\xi)$ are univalent in $\mathbb{U}$, then a function is said to be bi-univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Examples of functions in the class Σ are $\frac{\xi}{1-\xi}$, $\log\sqrt{\frac{1+\xi}{1-\xi}}$.

Fekete and Szegö achieved a sharp bound of the functional $\eta a_2^2 - a_3$, with real η $(0 \leq \eta \leq 1)$ for a univalent function f in 1933 [5]. Since that time, it has been known as the classical Fekete and Szegö problem of establishing the sharp bounds for this functional of any compact family of functions $f \in \mathcal{A}$ with any complex η.

In 1983, Askey and Ismail [6] found a class of polynomials that can be interpreted as q–analogues of the Gegenbauer polynomials. These are essentially the polynomials $\mathfrak{B}_q^{(\lambda)}(\xi, z)$

$$\mathfrak{G}_q^{(\lambda)}(x, \xi) = \sum_{n=0}^{\infty} \mathcal{C}_n^{(\lambda)}(x; q)\xi^n, \tag{3}$$

where $x \in [-1, 1]$ and $\xi \in \mathbb{U}$.

In 2006, Chakrabarti et al. [7] found a class of polynomials that can be interpreted as q–analogues of the Gegenbauer polynomials by the following recurrence relations:

$$\mathcal{C}_0^{(\lambda)}(x;q) = 1,\ \mathcal{C}_1^{(\lambda)}(x;q) = [\lambda]_q C_1^1(x) = 2[\lambda]_q x, \tag{4}$$

$$\mathcal{C}_2^{(\lambda)}(x;q) = [\lambda]_{q^2} C_2^1(x) - \frac{1}{2}\Big([\lambda]_{q^2} - [\lambda]_q^2\Big) C_1^2(x) = 2\Big([\lambda]_{q^2} + [\lambda]_q^2\Big)x^2 - [\lambda]_{q^2}. \tag{5}$$

where $0 < q < 1$ and $\lambda \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

In 2021, Amourah et al. [8,9] considered the classical Gegenbauer polynomials $\mathfrak{G}^{(\lambda)}(x, \xi)$, where $\xi \in \mathbb{U}$ and $x \in [-1, 1]$. For fixed x, the function $\mathfrak{G}^{(\lambda)}$ is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as

$$\mathfrak{G}^{(\lambda)}(x, \xi) = \sum_{n=0}^{\infty} C_n^{\alpha}(x)\xi^n,$$

where $C_n^{\alpha}(x)$ is the classical Gegenbauer polynomial of degree n.

Recently, several authors have begun examining bi-univalent functions connected to orthogonal polynomials (such as [10–28]).

As far as we are aware, there is no published work on bi-univalent functions for $q-$Gegenbauer polynomials. The major objective of this work is to start an investigation of the characteristics of bi-univalent functions related to $q-$Gegenbauer polynomials. To perform this, we consider the following definitions in the next section.

3. Coefficient Bounds of the Class $\mathfrak{B}_\Sigma(x, \alpha; q)$

Here, we introduce some new bi-univalent function subclasses that are subordinate to the $q-$Gegenbauer polynomial.

Definition 1. *For $x \in (\frac{1}{2}, 1]$ and $0 < q < 1$, if the following subordinations are satisfied, a function f belonging to Σ is said to be in the class $\mathfrak{B}_\Sigma(x, \alpha; q)$ given by (1):*

$$\partial_q f(\xi) \prec \mathfrak{G}_q^{(\lambda)}(x, \xi) \tag{6}$$

and

$$\partial_q g(w) \prec \mathfrak{G}_q^{(\lambda)}(x, w), \tag{7}$$

where $\lambda \in \mathbb{N} = \{1, 2, 3, \cdots\}$, α is a nonzero real constant, the function $f^{-1}(w) = g(w)$ is defined by (2), and $\mathfrak{G}_q^{(\lambda)}$ is the generating function of q–analogues of the Gegenbauer polynomials given by (3).

We start by providing the coefficient estimates for the class $\mathfrak{B}_\Sigma(x, \alpha; q)$ specified in Definition 1.

Theorem 1. *Let $f \in \Sigma$ given by (1) be in the class $\mathfrak{B}_\Sigma(x, \alpha; q)$. Then,*

$$|a_2| \leq \frac{2x\big|[\lambda]_q\big|\sqrt{2[\lambda]_q x}}{[2]_q\sqrt{\left[\left(\frac{4[3]_q}{[2]_q^2} - 2\right)[\lambda]_q^2 - 2[\lambda]_{q^2}\right]x^2 + [\lambda]_q}},$$

and

$$|a_3| \leq \frac{4[\lambda]_q^2 x^2}{[2]_q^2} + \frac{2\big|[\lambda]_q\big|x}{[3]_q}.$$

Proof. Let $f \in \mathfrak{B}_\Sigma(x, \alpha; q)$. From Definition 1, for some analytic functions w and v such that $w(0) = 0 = v(0)$ and $|w(\xi)| < 1$, $|v(w)| < 1$ for all $w, \xi \in \mathbb{U}$; then, we can write

$$\partial_q f(\xi) = \mathfrak{G}_q^{(\lambda)}(x, w(\xi)) \tag{8}$$

and

$$\partial_q g(w) = \mathfrak{G}_q^{(\lambda)}(x, v(w)), \tag{9}$$

From the Equations (8) and (9), we obtain that

$$\partial_q f(\xi) = 1 + C_1^{(\lambda)}(x;q)c_1\xi + \left[C_1^{(\lambda)}(x;q)c_2 + C_2^{(\lambda)}(x;q)c_1^2\right]\xi^2 + \cdots \tag{10}$$

and

$$\partial_q g(w) = 1 + C_1^{(\lambda)}(x;q)d_1 w + \left[C_1^{(\lambda)}(x;q)d_2 + C_2^{(\lambda)}(x;q)d_1^2\right])w^2 + \cdots. \tag{11}$$

It is generally understood that if

$$|w(\xi)| = \left|c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots\right| < 1, \ \ (\xi \in \mathbb{U})$$

and

$$|v(w)| = \left|d_1 w + d_2 w^2 + d_3 w^3 + \cdots\right| < 1, \ \ (w \in \mathbb{U}),$$

then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{12}$$

As a result, we have the following after comparing the relevant coefficients in (10) and (11):

$$[2]_q a_2 = C_1^{(\lambda)}(x;q)c_1, \tag{13}$$

$$[3]_q a_3 = C_1^{(\lambda)}(x;q)c_2 + C_2^{(\lambda)}(x;q)c_1^2, \tag{14}$$

$$-[2]_q a_2 = C_1^{(\lambda)}(x;q)d_1, \tag{15}$$

and

$$[3]_q\left(2a_2^2 - a_3\right) = C_1^{(\lambda)}(x;q)d_2 + C_2^{(\lambda)}(x;q)d_1^2. \tag{16}$$

From the Equations (13) and (15), we have

$$c_1 = -d_1 \tag{17}$$

and

$$2[2]_q^2 a_2^2 = \left[C_1^{(\lambda)}(x;q)\right]^2\left(c_1^2 + d_1^2\right). \tag{18}$$

By adding (14) to (16), yields

$$2[3]_q a_2^2 = C_1^{(\lambda)}(x;q)(c_2 + d_2) + C_2^{(\lambda)}(x;q)\left(c_1^2 + d_1^2\right). \tag{19}$$

We determine that, by replacing the value of $\left(c_1^2 + d_1^2\right)$ from (18) on the right side of (19),

$$\left(2[3]_q - \frac{2C_2^{(\lambda)}(x;q)[2]_q^2}{\left[C_1^{(\lambda)}(x;q)\right]^2}\right)a_2^2 = C_1^{(\lambda)}(x;q)(c_2 + d_2). \tag{20}$$

Through computations using (11), (5), and (20), we find that

$$|a_2| \leq \frac{2\left|[\lambda]_q\right|x\sqrt{2[\lambda]_q x}}{[2]_q\sqrt{\left[\left(\frac{4[3]_q}{[2]_q^2} - 2\right)[\lambda]_q^2 - 2[\lambda]_{q^2}\right]x^2 + [\lambda]_q}}.$$

In addition, if we subtract (16) from (14), we obtain

$$2[3]_q\left(a_3 - a_2^2\right) = C_1^{(\lambda)}(x;q)(c_2 - d_2) + C_2^{(\lambda)}(x;q)\left(c_1^2 - d_1^2\right). \tag{21}$$

Then, in view of (18) and (21), we obtain

$$a_3 = \frac{\left[C_1^{(\lambda)}(x;q)\right]^2}{2[2]_q^2}\left(c_1^2 + d_1^2\right) + \frac{C_1^{(\lambda)}(x;q)}{2[3]_q}(c_2 - d_2).$$

By applying (4), we conclude that

$$|a_3| \leq \frac{4[\lambda]_q^2 x^2}{[2]_q^2} + \frac{2\left|[\lambda]_q\right|x}{[3]_q}.$$

The proof of the theorem is now complete. □

Using the values of a_2^2 and a_3, we prove the following Fekete–Szegö inequality for functions in the class $\mathfrak{B}_\Sigma(x,\alpha;q)$.

Theorem 2. *Let $f \in \Sigma$ given by (1) be in the class $\mathfrak{B}_\Sigma(x,\alpha;q)$. Then,*

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{2x\left|[\lambda]_q\right|}{[3]_q}, & |\sigma - 1| \leq \left|\frac{\left(\left(2[3]_q - [2]_q^2\right)[\lambda]_q^2 - [2]_q^2[\lambda]_{q^2}\right)x^2 + 2[2]_q^2[\lambda]_{q^2}}{2[3]_q[\lambda]_q^2 x^2}\right|, \\ 2\left|2[\lambda]_q x\right||h(\eta)|, & |\sigma - 1| \geq \left|\frac{\left(\left(2[3]_q - [2]_q^2\right)[\lambda]_q^2 - [2]_q^2[\lambda]_{q^2}\right)x^2 + 2[2]_q^2[\lambda]_{q^2}}{2[3]_q[\lambda]_q^2 x^2}\right|. \end{cases}$$

Proof. From (20) and (21),

$$\begin{aligned} a_3 - \sigma a_2^2 &= (1-\sigma)\frac{\left[C_1^{(\lambda)}(x;q)\right]^3(c_2 + d_2)}{\left[2[3]_q\left[C_1^{(\lambda)}(x;q)\right]^2 - 2[2]_q^2 C_2^{(\lambda)}(x;q)\right]} + \frac{C_1^{(\lambda)}(x;q)}{2[3]_q}(c_2 - d_2) \\ &= C_1^{\alpha}(x)\left[\left[h(\eta) + \frac{1}{2[3]_q}\right]c_2 + \left[h(\eta) - \frac{1}{2[3]_q}\right]d_2\right], \end{aligned}$$

where

$$\mathcal{K}(\sigma) = \frac{\left[C_1^{(\lambda)}(x;q)\right]^2(1-\sigma)}{2[3]_q\left[C_1^{(\lambda)}(x;q)\right]^2 - 2[2]_q^2 C_2^{(\lambda)}(x;q)},$$

In view of (4) and (5), we conclude that

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{\left|C_1^{(\lambda)}(x;q)\right|}{[3]_q}, & |\mathcal{K}(\sigma)| \leq \frac{1}{2[3]_q}, \\ 2\left|C_1^{(\lambda)}(x;q)\right||\mathcal{K}(\sigma)|, & |\mathcal{K}(\sigma)| \geq \frac{1}{2[3]_q}. \end{cases}$$

The proof of the theorem is now complete. □

Corollary 1. *Let $f \in \Sigma$ given by (1) belong to the class $\mathfrak{B}_\Sigma(x,\alpha;1)$. Then,*

$$|a_2| \leq \frac{|\alpha|x\sqrt{2|\alpha|x}}{\sqrt{|[(\alpha-2)x^2+1]\alpha|}}.$$

$$|a_3| \leq \lambda^2 x^2 + \frac{2|\lambda|x}{3},$$

and $$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2|\alpha|x}{3}, & |\eta - 1| \leq \left|\frac{(\alpha-2)x^2+1}{3\alpha x^2}\right| \\ \frac{2|\alpha x|^3|1-\eta|}{\left|[(\alpha-2)x^2+1]\alpha\right|}, & |\eta - 1| \geq \left|\frac{(\alpha-2)x^2+1}{3\alpha x^2}\right|. \end{cases}$$

Corollary 2. *Let $f \in \Sigma$ given by (1) belong to the class $\mathfrak{B}_\Sigma(x,1;1)$. Then,*

$$|a_2| \leq \frac{x\sqrt{2x}}{\sqrt{|1-x^2|}},$$

$$|a_3| \leq x^2 + \frac{2x}{3},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2x}{3}, & |\eta - 1| \leq \left|\frac{1-x^2}{3x^2}\right| \\ \frac{2x^3|1-\eta|}{\left|1-x^2\right|}, & |\eta - 1| \geq \left|\frac{1-x^2}{3x^2}\right|. \end{cases}$$

4. Coefficient Bounds of the Class $\mathcal{S}^*_\Sigma(x,\alpha;q)$

Definition 2. *For $x \in (\frac{1}{2}, 1]$ and $0 < q < 1$, if the following subordinations are satisfied, a function f belonging to Σ is said to be in the class $\mathcal{S}^*_\Sigma(x,\alpha;q)$ given by (1) :*

$$\frac{\xi \partial_q f(\xi)}{f(\xi)} \prec \mathfrak{G}_q^{(\lambda)}(x,\xi), \tag{22}$$

and

$$\frac{w \partial_q g(w)}{g(w)} \prec \mathfrak{G}_q^{(\lambda)}(x,w), \tag{23}$$

where $\lambda \in \mathbb{N} = \{1,2,3,\cdots\}$, α is a nonzero real constant, the function $g(w) = f^{-1}(w)$ is defined by (2), and $\mathfrak{G}_q^{(\lambda)}$ is the generating function of the q–analogues of Gegenbauer polynomials given by (3).

Theorem 3. *Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{S}^*_\Sigma(x,\alpha;q)$. Then, we have*

$$|a_2| \leq \frac{2|[\lambda]_q|x\sqrt{2[\lambda]_q x}}{q\sqrt{2\left([\lambda]_q^2 - [\lambda]_{q^2}\right)x^2 + [\lambda]_{q^2}}},$$

and

$$|a_3| \leq \frac{4[\lambda]_q^2 x^2}{q^2} + \frac{2[\lambda]_q x}{q(1+q)}.$$

Proof. Let $f \in \mathcal{S}^*_\Sigma(x,\alpha;q)$. From Definition 2, for some analytic functions w and v such that $w(0) = 0 = v(0)$ and $|w(\xi)| < 1$, $|v(w)| < 1$ for all $\xi, w \in \mathbb{U}$,

$$\frac{\xi \partial_q f(\xi)}{f(\xi)} = \mathfrak{G}_q^{(\lambda)}(x, w(\xi)), \tag{24}$$

and

$$\frac{\xi \partial_q g(w)}{g(w)} = \mathfrak{G}_q^{(\lambda)}(x, v(w)). \tag{25}$$

From the equalities (24) and (25), we obtain that

$$\frac{\xi\partial_q f(\xi)}{f(\xi)} = 1 + C_1^{(\lambda)}(x;q)\xi + \left[C_1^{(\lambda)}(x;q)c_2 + C_2^{(\lambda)}(x;q)c_1^2\right]\xi^2 + \cdots \tag{26}$$

and

$$\frac{\xi\partial_q g(w)}{g(w)} = 1 + C_1^{(\lambda)}(x;q)d_1 w + \left[C_1^{(\lambda)}(x;q)d_2 + C_2^{(\lambda)}(x;q)d_1^2\right])w^2 + \cdots. \tag{27}$$

Thus, upon comparing the corresponding coefficients in (26) and (27), we have

$$qa_2 = C_1^{(\lambda)}(x;q)c_1, \tag{28}$$

$$q(1+q)a_3 - qa_2^2 = C_1^{(\lambda)}(x;q)c_2 + C_2^{(\lambda)}(x;q)c_1^2, \tag{29}$$

$$-qa_2 = C_1^{(\lambda)}(x;q)d_1, \tag{30}$$

and

$$q(1+2q)a_2^2 - q(1+q)a_3 = C_1^{(\lambda)}(x;q)d_2 + C_2^{(\lambda)}(x;q)d_1^2. \tag{31}$$

From the Equations (28) and (30), it follows that

$$c_1 = -d_1 \tag{32}$$

and

$$2q^2a_2^2 = \left[C_1^{(\lambda)}(x;q)\right]^2\left(c_1^2 + d_1^2\right). \tag{33}$$

By adding (29) to (31), yields

$$2q^2a_2^2 = C_1^{(\lambda)}(x;q)(c_2+d_2) + C_2^{(\lambda)}(x;q)\left(c_1^2 + d_1^2\right). \tag{34}$$

We determine that, by replacing the value of $\left(c_1^2 + d_1^2\right)$ from (33) on the right side of (34),

$$2q^2\left(1 - \frac{C_2^{(\lambda)}(x;q)}{\left[C_1^{(\lambda)}(x;q)\right]^2}\right)a_2^2 = C_1^{(\lambda)}(x;q)(c_2+d_2). \tag{35}$$

Moreover, through computations using (5) and (35), we find that

$$|a_2| \leq \frac{2|[\lambda]_q|x\sqrt{2[\lambda]_q x}}{q\sqrt{2\left([\lambda]_q^2 - [\lambda]_{q^2}\right)x^2 + [\lambda]_{q^2}}}.$$

Now, if we subtract (31) from (29), we obtain

$$2q(1+q)\left(a_3 - a_2^2\right) = C_1^{(\lambda)}(x;q)(c_2-d_2) + C_2^{(\lambda)}(x;q)\left(c_1^2 - d_1^2\right). \tag{36}$$

By viewing of (33) and (36), we conclude that

$$a_3 = \frac{\left[C_1^{(\lambda)}(x;q)\right]^2}{2q^2}\left(c_1^2 + d_1^2\right). + \frac{C_1^{(\lambda)}(x;q)}{2q(1+q)}(c_2 - d_2).$$

By applying (4) and (5), we have

$$|a_3| \leq \frac{4[\lambda]_q^2 x^2}{q^2} + \frac{2[\lambda]_q x}{q(1+q)}.$$

This completes the proof of the Theorem 3. □

Theorem 4. *Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{S}^*_\Sigma(x,\alpha;q)$. Then,*

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{\left|[\lambda]_q\right| x}{q(1+q)}, & |1-\sigma| \leq \frac{q^2\left[\left(2\left([\lambda]_q^2 - [\lambda]_{q^2}\right)x^2\right) + [\lambda]_{q^2}\right]}{8(1+q)\left|[\lambda]_q x\right|^3}, \\ \frac{8\left|[\lambda]_q x\right|^3 |1-\sigma|}{q^2\left[\left(2\left([\lambda]_q^2 - [\lambda]_{q^2}\right)x^2\right) + [\lambda]_{q^2}\right]}, & |1-\sigma| \geq \frac{q^2\left[\left(2\left([\lambda]_q^2 - [\lambda]_{q^2}\right)x^2\right) + [\lambda]_{q^2}\right]}{8(1+q)\left|[\lambda]_q x\right|^3}. \end{cases}$$

Proof. From (35) and (36),

$$\begin{aligned} a_3 - \sigma a_2^2 &= \frac{(1-\sigma)\left(C_1^{(\lambda)}(x;q)\right)^3}{2q^2\left(\left[C_1^{(\lambda)}(x;q)\right]^2 - C_2^{(\lambda)}(x;q)\right)}(c_2+d_2) + \frac{C_1^{(\lambda)}(x;q)}{2q(1+q)}(c_2-d_2) \\ &= \frac{C_1^{(\lambda)}(x;q)}{2q}\left[\left[\mathfrak{K}(\sigma) + \frac{1}{1+q}\right]c_2 + \left[\mathfrak{K}(\sigma) - \frac{1}{1+q}\right]d_2\right], \end{aligned}$$

where

$$\mathfrak{K}(\sigma) = \frac{\left[C_1^{(\lambda)}(x;q)\right]^2(1-\sigma)}{q\left[\left[C_1^{(\lambda)}(x;q)\right]^2 - C_2^{(\lambda)}(x;q)\right]},$$

$$|1-\sigma| \geq \frac{q^2\left[\left(2\left([\lambda]_q^2 - [\lambda]_{q^2}\right)x^2\right) + [\lambda]_{q^2}\right]}{8(1+q)\left|[\lambda]_q x\right|^3},$$

then, in view of (4) and (5), we conclude that

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{\left|C_1^{(\lambda)}(x;q)\right|}{2q(1+q)}, & |\mathfrak{K}(\sigma)| \leq \frac{1}{1+q}, \\ \frac{1}{q}\left|C_1^{(\lambda)}(x;q)\right||\mathfrak{K}(\sigma)|, & |\mathfrak{K}(\sigma)| \geq \frac{1}{1+q}. \end{cases}$$

This completes the proof of the Theorem 4. □

Corollary 3. *Let $f \in \Sigma$ given by (1) belong to the class $\mathcal{S}^*_\Sigma(x,\alpha;1)$. Then, we have*

$$|a_2| \leq \frac{2|\lambda| x\sqrt{2x}}{\sqrt{2(\lambda-1)x^2 - 1}}, \ |a_3| \leq \lambda(4\lambda x + 1),$$

and

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{|\lambda| x}{2}, & |1-\sigma| \leq \frac{\left(2(\lambda-1)x^2\right)+1}{16|\lambda|^2 x^3}, \\ \frac{8\lambda^2 x^3 |1-\sigma|}{\left(2(\lambda-1)x^2\right)+1}, & |1-\sigma| \geq \frac{\left(2(\lambda-1)x^2\right)+1}{16|\lambda|^2 x^3}. \end{cases}$$

5. Coefficient Bounds of the Class $\mathfrak{C}_\Sigma(x,\alpha;q)$

Definition 3. *For $x \in (\frac{1}{2},1]$ and $0<q<1$, if the following subordinations are satisfied, a function f belonging to Σ is said to be in the class $\mathfrak{C}_\Sigma(x,\alpha;q)$ given by (1):*

$$1+\frac{\xi\partial_q^2 f(\xi)}{\partial_q f(\xi)} \prec \mathfrak{G}_q^{(\lambda)}(x,\xi) \tag{37}$$

and

$$1+\frac{\xi\partial_q^2 g(w)}{\partial_q g(w)} \prec \mathfrak{G}_q^{(\lambda)}(x,w), \tag{38}$$

where $\lambda \in \mathbb{N} = \{1,2,3,\cdots\}$, α is a nonzero real constant, the function $g(w) = f^{-1}(w)$ is defined by (2), and $\mathfrak{G}_q^{(\lambda)}$ is the generating function of the q–analogues of Gegenbauer polynomials given by (3).

Theorem 5. *Let $f \in \Sigma$ given by (1) belong to the class $\mathfrak{C}_\Sigma(x,\alpha;q)$. Then,*

$$|a_2| \le \frac{2[\lambda]_q x\sqrt{2|[\lambda]_q|x}}{\sqrt{[2]_q\Big(\big((2[3]_q-3[2]_q)[\lambda]_q^2-2[2]_q[\lambda]_{q^2}\big)x^2+[2]_q[\lambda]_{q^2}\Big)}},$$

and

$$|a_3| \le \frac{4[\lambda]_q^2x^2}{[2]_q^2}+\frac{2[\lambda]_q x}{[2]_q[3]_q}.$$

Proof. Let $f \in \mathfrak{C}_\Sigma(x,\alpha,\mu;q)$. From Definition 3, for some analytic functions w, v such that $w(0)=v(0)=0$ and $|w(\xi)|<1$, $|v(w)|<1$ for all $\xi, w \in \mathbb{U}$,

$$1+\frac{\xi\partial_q^2 f(\xi)}{\partial_q f(\xi)} = \mathfrak{G}_q^{(\lambda)}(x,w(\xi)), \tag{39}$$

and

$$1+\frac{\xi\partial_q^2 g(w)}{\partial_q g(w)} = \mathfrak{G}_q^{(\lambda)}(x,v(w)). \tag{40}$$

By expanding the Equations (39) and (40), we obtain that

$$1+\frac{\xi\partial_q^2 f(\xi)}{\partial_q f(\xi)} = 1+C_1^{(\lambda)}(x;q)c_1\xi+\Big[C_1^{(\lambda)}(x;q)c_2+C_2^{(\lambda)}(x;q)c_1^2\Big]\xi^2+\cdots \tag{41}$$

and

$$1+\frac{\xi\partial_q^2 g(w)}{\partial_q g(w)} = 1+C_1^{(\lambda)}(x;q)d_1w+\Big[C_1^{(\lambda)}(x;q)d_2+C_2^{(\lambda)}(x;q)d_1^2\Big])w^2+\cdots. \tag{42}$$

Upon comparing the corresponding coefficients in (41) and (42), we have

$$[2]_q a_2 = C_1^{(\lambda)}(x;q)c_1, \tag{43}$$

$$[2]_q[3]_q a_3 - [2]_q^2 a_2^2 = C_1^{(\lambda)}(x;q)c_2+C_2^{(\lambda)}(x;q)c_1^2, \tag{44}$$

$$-[2]_q a_2 = C_1^{(\lambda)}(x;q)d_1, \tag{45}$$

and
$$[2]_q\left(2[3]_q-[2]_q\right)a_2^2-[2]_q[3]_q a_3=C_1^{(\lambda)}(x;q)d_2+C_2^{(\lambda)}(x;q)d_1^2. \tag{46}$$

We get from (43) and (45) that
$$c_1=-d_1 \tag{47}$$
and
$$2\left([2]_q\right)^2 a_2^2=\left[C_1^{(\lambda)}(x;q)\right]^2\left(c_1^2+d_1^2\right). \tag{48}$$

By adding (44) to (46), we obtain
$$2[2]_q\left([3]_q-[2]_q\right)a_2^2=C_1^{(\lambda)}(x;q)(c_2+d_2)+C_2^{(\lambda)}(x;q)\left(c_1^2+d_1^2\right). \tag{49}$$

We determine that, by replacing the value of $\left(c_1^2+d_1^2\right)$ from (48) on the right side of (49),
$$2[2]_q\left(\left([3]_q-[2]_q\right)-[2]_q\frac{C_2^{(\lambda)}(x;q)}{\left[C_1^{(\lambda)}(x;q)\right]^2}\right)a_2^2=C_1^{(\lambda)}(x;q)(c_2+d_2). \tag{50}$$
Moreover, by doing computations along (12) and (50), we find that
$$|a_2|\le\frac{2[\lambda]_q x\sqrt{2|[\lambda]_q|x}}{\sqrt{[2]_q\left(\left(\left(2[3]_q-3[2]_q\right)[\lambda]_q^2-[2]_q[\lambda]_{q^2}\right)x^2+[2]_q[\lambda]_{q^2}\right)}}.$$

By subtracting (44) from (46), we obtain
$$2[2]_q[3]_q\left(a_3-a_2^2\right)=C_1^{(\lambda)}(x;q)(c_2-d_2)+C_2^{(\lambda)}(x;q)\left(c_1^2-d_1^2\right). \tag{51}$$

In view of (48) and (51), we obtain
$$a_3=\frac{\left[C_1^{(\lambda)}(x;q)\right]^2}{2\left([2]_q\right)^2}\left(c_1^2+d_1^2\right)+\frac{C_1^{(\lambda)}(x;q)}{2[2]_q[3]_q}(c_2-d_2).$$

By applying (5), we conclude that
$$|a_3|\le\frac{4[\lambda]_q^2x^2}{[2]_q^2}+\frac{2[\lambda]_q x}{[2]_q[3]_q}.$$
This completes the proof of the Theorem 5. □

Theorem 6. *Let $f\in\Sigma$ given by (1) belong to the class $\mathfrak{C}_\Sigma(x,\alpha;q)$. Then,*

$$\left|a_3-\sigma a_2^2\right|\le$$
$$\begin{cases}\frac{2|\lambda|x}{[2]_q[3]_q}, & |1-\sigma|\le F,\\[2ex] \frac{16(1-\sigma)\left|[\lambda]_q\right|^3x^3}{[2]_q\left(\left(2\left(\left(2[3]_q2-3[2]_q\right)[\lambda]_q^2-2[2]_q[\lambda]_{q^2}\right)\right)x^2+[2]_q[\lambda]_{q^2}\right)}, & |1-\sigma|\ge F,\end{cases}$$
where
$$F=\frac{\left(2\left(\left(2[3]_q2-3[2]_q\right)[\lambda]_q^2-2[2]_q[\lambda]_{q^2}\right)\right)x^2+[2]_q[\lambda]_{q^2}}{16[2]_q[3]_q[\lambda]_q^2x^2}.$$

Proof. From (50) and (51),

$$a_3 - \sigma a_2^2 = (1-\sigma)\frac{\left[C_1^{(\lambda)}(x;q)\right]^3}{2[2]_q\left(q^2\left[C_1^{(\lambda)}(x;q)\right]^2 - [2]_q C_2^{(\lambda)}(x;q)\right)}(c_2+d_2)$$
$$+\frac{C_1^{(\lambda)}(x;q)}{2[2]_q[3]_q}(c_2-d_2)$$
$$= C_1^{(\lambda)}(x;q)\left[\left[\mathcal{K}(\sigma)+\frac{1}{2[2]_q[3]_q}\right]c_2 + \left[\mathcal{K}(\sigma)-\frac{1}{2[2]_q[3]_q}\right]d_2\right],$$

where

$$\mathcal{K}(\sigma) = \frac{(1-\sigma)\left[C_1^{(\lambda)}(x;q)\right]^2}{2[2]_q\left(q^2\left[C_1^{(\lambda)}(x;q)\right]^2 - [2]_q C_2^{(\lambda)}(x;q)\right)},$$

$$|1-\sigma| \leq \frac{\left(4q^2[\lambda]_q^2 x^2 - [2]_q C_2^{(\lambda)}(x;q)\right)}{4[3]_q[\lambda]_q^2 x}$$

Then, in view of (5), we conclude that

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{\left|C_1^{(\lambda)}(x;q)\right|}{[2]_q[3]_q}, & |\mathcal{K}(\sigma)| \leq \frac{1}{2[2]_q[3]_q}, \\ \\ \frac{1}{[2]_q}\left|C_1^{(\lambda)}(x;q)\right||\mathcal{K}(\sigma)|, & |\mathcal{K}(\sigma)| \geq \frac{1}{2[2]_q[3]_q}. \end{cases}$$

This completes the proof of the last theorem. □

Corollary 4. *Let $f \in \Sigma$ given by (1) belong to the class $\mathfrak{C}_\Sigma(x,\alpha;1)$. Then,*

$$|a_2| \leq \frac{\lambda x\sqrt{2x}}{\sqrt{1-2x^2}},$$

$$|a_3| \leq \lambda^2 x^2 + \frac{\lambda x}{3}.$$

and

$$\left|a_3 - \sigma a_2^2\right| \leq \begin{cases} \frac{2|[\lambda]_q|x}{[2]_q[3]_q}, & |\sigma - 1| \leq \left|\frac{1-2\lambda x^2}{24\lambda x^2}\right|, \\ \frac{1}{[2]_q}\left|C_1^{(\lambda)}(x;q)\right||\mathcal{K}(\sigma)|, & |\sigma - 1| \geq \left|\frac{1-2\lambda x^2}{24\lambda x^2}\right|. \end{cases}$$

6. Conclusions

In the current study, we introduced and examined the coefficient issues related to each of the three new subclasses of the class of bi-univalent functions in the open unit disk $\mathbb{U}$: $\mathfrak{B}_\Sigma(x,\alpha;q)$, $\mathcal{S}^*_\Sigma(x,\alpha;q)$, and $\mathfrak{C}_\Sigma(x,\alpha;q)$. These bi-univalent function classes are described, accordingly, in Definitions 1 to 3. We calculated the estimates of the Fekete–Szegö functional problems and the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in each of these three bi-univalent function classes. Several more fresh outcomes are revealed to follow following specializing the parameters involved in our main results. Obtaining estimates on the bound of $|a_n|$ for $n \geq 4; n \in \mathbb{N}$ for the classes that have been introduced here is still a problem.

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