axioms

MDPI

Article

# Fixed Point Theorems for Mann's Iteration Scheme in Convex $G_b$-Metric Spaces with an Application

**Dong Ji [1,2,*,†], Chaobo Li [2,*,†] and Yunan Cui [2,†]**

[1] Department of Mathematics, Harbin University, Harbin 150086, China
[2] Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, China
* Correspondence: jidong@hrbu.edu.cn (D.J.); lichaobo6898@163.com (C.L.)
† These authors contributed equally to this work.

**Abstract:** In this paper, we present a series of fixed point results for Mann's iteration scheme in the framework of $G_b$-metric spaces. First, we introduce the concept of convex $G_b$-metric space by means of a convex structure and Mann's iteration algorithm is extended to this space. Furthermore, using Mann's iteration scheme, we prove some fixed point results for several mappings satisfying various suitable conditions on complete convex $G_b$-metric spaces. Some examples supporting our main results are also presented. We also discuss the well-posedness of the fixed point problems and the $P$ property for given mappings. Moreover, as an application, we apply our main result to prove the existence of the solutions to integral equations.

**Keywords:** $G_b$-metric spaces; convex structure; Mann's iteration; fixed point; integral equation

**MSC:** 47H10; 54H25

check for updates

**Citation:** Ji, D.; Li, C.; Cui, Y. Fixed Point Theorems for Mann's Iteration Scheme in Convex $G_b$-Metric Spaces with an Application. *Axioms* **2023**, *12*, 108. https://doi.org/10.3390/axioms12020108

Academic Editors: Behzad Djafari-Rouhani and Adrian Petrusel

Received: 23 November 2022
Revised: 13 January 2023
Accepted: 16 January 2023
Published: 19 January 2023



## 1. Introduction and Preliminaries

It is well know the fixed point theory in metric spaces plays an important role in nonlinear analysis. In 1922, Banach [1] proved a well-known fixed point theorem called the Banach fixed point theorem, which various applications in different branches of science. Since then, many researchers have extended these results by considering classes of nonlinear mappings and in other important spaces. In particular, generalizations of metric spaces were reported by Gahler [2] and Dhage [3] to aim to solve the more complex nonlinear analysis problems. Later, in 1993, Czerwik [4] proposed the concepts of $b$-metric spaces and generalized the classical Banach fixed point principle to these spaces.

**Definition 1** ([4])**.** *Let $X$ be a nonempty set and assume that a mapping $d : X \times X \to [0, +\infty)$ satisfies for all $x, y, u \in X$,*

*(1) $d(x, y) = 0$ if and only if $x = y$;*
*(2) $d(x, y) = d(y, x)$;*
*(3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, u) + d(u, y)]$.*

*Then $d$ is called a $b$-metric on $X$ and the pair $(X, d)$ is called a $b$-metric space with coefficient $s \geq 1$.*

Obviously, the class of $b$-metric is considerably larger than the class of metric spaces since a metric is a $b$-metric with $s = 1$. Note that a $b$-metric function is not necessarily continuous in each of its arguments [5].

In 2006, the concept of $G$-metric spaces was introduced by Mustafa and Sims [6]. Then, Aghajani et al. [7] introduced the notion of $G_b$-metric spaces which can be viewed as a generalization of $G$-metric spaces and $b$-metric spaces.

**Definition 2** ([7]). *Let $X$ be a nonempty set. Suppose that a mapping $G : X \times X \times X \to [0, \infty)$ satisfies the following conditions:*

(1) $G(x, y, z) = 0$ *if* $x = y = z$;
(2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$;
(3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $y \neq z$;
(4) $G(x, y, z) = G(x, z, y) = G(z, x, y) = \ldots$, *(symmetry in all three variables);*
(5) *there exists a real number* $s \geq 1$ *such that* $G(x, y, z) \leq s[G(x, u, u) + G(u, y, z)]$ *for all* $x, y, z, u \in X$.

*Then $G$ is called a $G_b$-metric on $X$ and the pair $(X, G)$ is called a $G_b$-metric space.*

**Example 1** ([7]). *Let $X = \mathbb{R}$ and $(X, d)$ be a b-metric space with $s \geq 1$. Let*

$$G_1(x, y, z) = d(x, y) + d(x, z) + d(y, z).$$

*Then $(X, G_1)$ is not a $G_b$-metric space. However, let*

$$G_2(x, y, z) = \max\{d(x, y), d(x, z), d(y, z)\}.$$

*Then $(X, G_2)$ is a $G_b$-metric space with $s$.*

**Remark 1.** *It is worth mentioning that $G_b$-metric spaces and b-metric spaces are topologically equivalent [7]. This allows us to readily transport many concepts and results from b-metric spaces into $G_b$-metric spaces.*

**Proposition 1** ([7]). *Let $(X, G)$ be a $G_b$-metric space. Then for any $x, y, z, u \in X$, we have:*

(1) *if* $G(x, y, z) = 0$, *then* $x = y = z$ ;
(2) $G(x, y, z) \leq s(G(x, x, y) + G(x, x, z))$;
(3) $G(x, y, y) \leq 2sG(y, x, x)$;
(4) $G(x, y, z) \leq s(G(x, u, z) + G(u, y, z))$.

**Definition 3** ([7]). *Let $(X, G)$ be a $G_b$-metric space. A sequence $\{x_n\}$ in $X$ is said to be convergent in $X$ if there exists $x^* \in X$ such that $\lim\limits_{n,m \to \infty} G(x_n, x_m, x^*) = 0$. $(X, G)$ is said to be a complete $G_b$-metric space if every Cauchy sequence in $X$ is convergent.*

**Proposition 2** ([7]). *Let $(X, G)$ be a $G_b$-metric space. Then, the following are equivalent:*

(1) *the sequence $\{x_n\}$ is a Cauchy sequence;*
(2) *for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq n_0$.*

**Definition 4** ([7]). *A $G_b$-metric $G$ is said to be symmetric if $G(x_n, x_m, x_m) = G(x_m, x_n, x_n)$ for all $x_n, x_m \in X$.*

**Definition 5** ([8]). *Let $(X_1, G_1)$ and $(X_2, G_2)$ be two $G_b$-metric spaces. Let $f : (X_1, G_1) \to (X_2, G_2)$ be a mapping; then $f$ is said to be $G$-continuous at a point $x^* \in X$; for any $y, z \in X$ and $\varepsilon > 0$, there exists $\delta > 0$, such that $G_1(x^*, y, z) < \delta$ implies $G_2(fx^*, fy, fz) < \varepsilon$.*

**Proposition 3** ([8]). *Let $(X_1, G_1)$ and $(X_2, G_2)$ be two $G_b$-metric spaces. Then a mapping $f : (X_1, G_1) \to (X_2, G_2)$ is $G$-continuous at a point $x^* \in X$ if and only if $f(x_n)$ is $G$-convergent $f(x^*)$ whenever $\{x_n\}$ is $G$-convergent to $x^*$.*

On the other hand, the concepts of a convex structure and a convex metric space were introduced by Takahashi [9].

**Definition 6** ([9]). *Let $(X, d)$ be a metric space and $I = [0, 1]$. A continuous function $w : X \times X \times [0, 1] \to X$ is said to be a convex structure on $X$ if for each $x, y, u \in E$ and $\alpha \in I$,*

$$d(u, w(x, y; \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$$

*holds. A metric space $(X, d)$ with a convex structure $w$ is called a convex metric space.*

Norouzian et al. [10] introduced convex structure in $G$-metric spaces.

**Definition 7** ([10])**.** *Let $(X, G)$ be a $G$-metric space. A mapping $w : X \times X \times X \times [0,1] \times [0,1] \times [0,1] \to X$ is said to be a convex structure on $X$ if for each $(x, y, z; \lambda_1, \lambda_2, \lambda_3) \in X \times X \times X \times [0,1] \times [0,1] \times [0,1]$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ then $w(x, y, z; \lambda_1, \lambda_2, \lambda_3) \in X$, where $w(x, y, z; \lambda_1, \lambda_2, \lambda_3) = \lambda_1 x + \lambda_2 y + \lambda_3 z$. If $w$ is a convex structure on $X$, then the triplet $(X, G, w)$ is called a convex $G$-metric space.*

Howeve, iterative methods have received vast investigation for finding fixed points of nonexpansive mappings—see [11–18]. Particularly, in the research on some approximations of the fixed points problem using the iteration scheme, one of the most famous fixed point methods is the Mann iteration [19] as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \alpha_n \in [0,1].$$

For example, Reich [20] proved that if $\{\alpha_n\}$ is chosen such that $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$, then the Mann sequence $\{x_n\}$ converges weakly to a fixed point of $T$ in a uniformly convex Banach space with a Fréchet differentiable norm.

In this article, we first give the notion of convex $G_b$-metric spaces by means of convex structure in the sense of Takahashi. Then, we generalize the Mann iterative algorithm to $G_b$-metric spaces and present the existence and uniqueness theorem for contraction mapping. Moreover, we show concrete examples supporting our main results. The results greatly generalize the previous results from [16]. Furthermore, we consider the well-posedness of the fixed problems and the $P$ property for a given mapping. Finally, we apply our main result to approximating the solutions of integral equations.

In the following, we always denote by $\mathbb{N}_0$ the set of nonnegative integers.

## 2. Main Results

We begin with the following definition which generalizes the notion of $G_b$-metric spaces and convex structure in the sense of Takahashi.

**Definition 8.** *Let $(X, G)$ be a $G_b$-metric space with coefficient $s \geq 1$ and $I = [0,1]$. A mapping $w : X \times X \times [0,1] \to X$ is said to be a convex structure on $X$ if for each $x, y, u, v \in X$ and $\alpha \in I$,*

$$G(u, v, w(x, y; \alpha)) \leq \alpha G(u, v, x) + (1 - \alpha) G(u, v, y) \tag{1}$$

*holds. Then the triplet $(X, G, w)$ is called a convex $G_b$-metric space with coefficient $s \geq 1$.*

**Remark 2.** *A convex $G_b$-metric space reduces a convex $G$-metric space for $s = 1$.*

**Definition 9.** *Let $(X, G)$ be a $G_b$-metric space and $T : X \to X$ be a mapping. We say the sequence $\{x_n\}$ is a Mann sequence if*

$$x_{n+1} = w(x_n, T x_n; \alpha_n), n \in \mathbb{N}_0,$$

*where $x_0 \in X$ and $\alpha_n \in [0,1]$.*

We present now some specific examples of convex $G_b$-metric spaces.

**Example 2.** *Let $X = \mathbb{R}$ and the metric $G : X \times X \times X \to [0, \infty)$ be defined by*

$$G(x, y, z) = \left[\frac{1}{3}(|x - y| + |y - z| + |x - z|)\right]^2, \; for \; all \; x, y, z \in X,$$

*as well as the mapping $w : X \times X \times [0,1] \to X$ defined by the formula*

$$w(x, y; \alpha) = \alpha x + (1 - \alpha) y.$$

*Then $(X,G,w)$ is a convex $G_b$-metric space with $s=2$. Indeed, it is clear that that $(X,G)$ is a $G_b$-metric space with $s=2$ (see [21], Example 4). For any $x,y,u,v\in X$, we get*

$$\begin{aligned}
G(x,y,w(u,v;\alpha)) =& \frac{1}{9}\times(|x-y|+|y-\alpha u-(1-\alpha)v|+|x-\alpha u-(1-\alpha)v|)^2\\
\leq& \frac{1}{9}\times[\alpha|x-y|+(1-\alpha)|x-y|+\alpha|y-u|+(1-\alpha)|y-v|\\
&+\alpha|x-u|+(1-\alpha)|x-v|]\\
=& \frac{1}{9}\times[\alpha(|x-y|+|y-u|+|x-u|)+(1-\alpha)(|x-y|+|y-v|+|x-v|)]^2\\
\leq& \frac{1}{9}\times\Big[\alpha^2(|x-y|+|y-u|+|x-u|)^2+(1-\alpha)^2(|x-y|+|y-v|+|x-v|)^2\\
&+2\alpha(1-\alpha)\,(|x-y|+|y-u|+|x-u|)^2\Big]\\
\leq& \frac{1}{9}\times\Big[\alpha(|x-y|+|y-u|+|x-u|)^2+(1-\alpha)(|x-y|+|y-v|+|x-v|)^2\Big]\\
=& \alpha G(x,y,u)+(1-\alpha)G(x,y,v).
\end{aligned}$$

*Hence, $(X,G,w)$ is a convex $G_b$-metric space with $s=2$.*

**Example 3.** *Let $X=\mathbb{R}$, and for $x,y\in X$, let us define the metric $d: X\times X\to[0,+\infty)$ by the formula*

$$d(x,y)=\sum_{i=1}^{n}(x_i-y_i)^2,$$

*for all $x=(x_1,x_2,\cdots,x_n)\in X$ and $y=(y_1,y_2,\cdots,y_n)\in X$, and define the mapping $w: X\times X\times[0,1]\to X$ by the formula*

$$w(x,y;\alpha)=\frac{x+y}{2}.$$

*We can know that $(X,d)$ is a convex b-metric space with $s=2$ (see [16], Example 4). The metric $G: X\times X\times X\to[0,\infty)$ is defined by*

$$G(x,y,z)=\max\{d(x,y),d(x,z),d(y,z)\}.$$

*For any $x,y,u,v\in X$, we have*

$$\begin{aligned}
G(x,y,w(u,v;\alpha)) &= \max\{d(x,y),d(x,w(u,v;\alpha)),d(y,w(u,v;\alpha))\}\\
&\leq \max\{d(x,y),\alpha d(x,u)+(1-\alpha)d(x,v),\alpha d(y,u)+(1-\alpha)d(y,v)\}\\
&\leq \alpha\max\{d(x,y),d(x,u),d(y,u)\}+(1-\alpha)\max\{d(x,y),d(x,v),d(y,v)\}\\
&= \alpha G(x,y,u)+(1-\alpha)G(x,y,v).
\end{aligned}$$

*Hence, $(X,G,w)$ is a convex $G_b$-metric space with $s=2^{p-1}$.*

The next example shows that the mapping $w$ defined in the above examples sometimes may not be a convex structure on some $G_b$-metric spaces.

**Example 4.** *Let $L(X,m,\vartheta)$ be a measure space and suppose that $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. We define the $L^p_{\mathbb{K}}(\vartheta)$ space as follows:*

$$L^p_{\mathbb{K}}(\vartheta)=\left\{x: X\to\mathbb{K}|\ x \text{ is measureable and } \int_X|x|^p<\infty\right\}, 0<p<1.$$

*We define $G: L^p_{\mathbb{K}}(\vartheta)\times L^p_{\mathbb{K}}(\vartheta)\times L^p_{\mathbb{K}}(\vartheta)\to[0,\infty)$ by the formula*

$$G(x,y,z)=\left(\int_X|x-y|^p d\nu+\int_X|y-z|^p d\nu+\int_X|x-z|^p d\nu\right)^{1/p},$$

*where $x, y, z \in L^p_{\mathbb{K}}(\vartheta)$. It is not hard to see that $(X, G)$ is a $G_b$-metric space with $s = 2^{1/p}$. Let $w(x, y; \alpha) = \alpha x + (1-\alpha)y$ for all $x, y \in L^p_{\mathbb{K}}(\vartheta)$. Then, for all $\alpha \in (0,1)$, we get*

$$
\begin{aligned}
G(x,y,w(u,v;\alpha)) =& \left(\int_X |x-y|^p d\vartheta + \int_X |y-w(u,v;\alpha)|^p d\vartheta + \int_X |x-w(u,v;\alpha)|^p d\vartheta\right)^{1/p} \\
\geq& \left(\int_X |x-y|^p d\vartheta\right)^{\frac{1}{p}} + \left(\int_X (\alpha|y-u| + (1-\alpha)|y-v|)^p d\vartheta\right)^{\frac{1}{p}} \\
&+ \left(\int_X (\alpha|x-u| + (1-\alpha)|x-v|)^p d\vartheta\right)^{\frac{1}{p}} \\
>& \left(\int_X |x-y|^p d\vartheta\right)^{\frac{1}{p}} + \left(\int_X \alpha^p|y-u|^p d\vartheta\right)^{\frac{1}{p}} + \left(\int_X (1-\alpha)^p|y-v|^p d\vartheta\right)^{\frac{1}{p}} \\
&+ \left(\int_X \alpha^p|x-u|^p d\vartheta\right)^{\frac{1}{p}} + \left(\int_X (1-\alpha)^p|x-v|^p d\vartheta\right)^{\frac{1}{p}} \\
=& \alpha G(x,y,u) + (1-\alpha)G(x,y,v),
\end{aligned}
$$

*which implies that $w$ is not a convex structure on $X$.*

The following properties are consequences of Definition 8.

**Proposition 4.** *Let $(X, G, w)$ be a convex $G_b$-metric space. If $\alpha \in (0,1)$, then $G_b$-metric $G$ is symmetric.*

**Proof.** If $x = y$, then obviously $G(x,x,y) = G(x,y,y)$ holds. Suppose that $x \neq y$. Due to $\alpha < 1$, it is not difficult to see that $x \neq w(x,y;\alpha)$ and $y \neq w(x,y;\alpha)$. Indeed, if $x = w(x,y;\alpha)$, we have

$$G(x,y,y) = G(w(x,y;\alpha),y,y) \leq \alpha G(x,y,y),$$

a contradiction. Therefore, $x \neq w(x,y;\alpha)$. Using similar arguments, we get $y \neq w(x,y;\alpha)$. Now consider

$$G(x,y,y) \leq G(x,y,w(x,y;\alpha)) \leq \alpha G(x,y,x) + (1-\alpha)G(x,y,y).$$

This implies that $G(x,y,y) \leq G(x,x,y)$. In addition

$$G(x,x,y) \leq G(x,w(x,y;\alpha),y) \leq \alpha G(x,x,y) + (1-\alpha)G(x,y,y).$$

This implies that $G(x,x,y) \leq G(x,y,y)$. By induction, we have $G(x,x,y) = G(x,y,y)$. □

Now we generalize Banach's contraction principle for convex $G_b$-metric space as follows:

**Theorem 1.** *Let $(X, G, w)$ be a complete convex $G_b$-metric space with constant $s \geq 1$ and $T : X \to X$ be a mapping such that*

$$G(Tx, Ty, Tz) \leq \lambda G(x,y,z)$$

*for all $x, y, z \in X$ and $\lambda \in [0,1)$. Suppose that the sequence $\{x_n\}$ is generated by the Mann iterative process and $x_0 \in X$. If the sequence $\{\alpha_n\} \in (0,1)$ converges to $\alpha < \frac{1-s^2\lambda}{s^2-s^2\lambda}$ and $\lambda < \frac{1}{s^2}$, then $T$ has a unique fixed point $x^*$ in $X$. Moreover $T$ is $G$-continuous at $x^*$.*

**Proof.** For any $n \in \mathbb{N}_0$, we have

$$G(x_n, x_n, x_{n+1}) = G(x_n, x_n, w(x_n, Tx_n; \alpha_n)) \leq (1-\alpha_n)G(x_n, x_n, Tx_n).$$

Thanks to Definition 8 and Proposition 4, we obtain

$$\begin{aligned}
G(x_n, x_n, Tx_n) &= G(x_n, Tx_n, Tx_n) \\
&\leq s[G(x_n, Tx_{n-1}, Tx_{n-1}) + G(Tx_{n-1}, Tx_n, Tx_n)] \\
&\leq s[G(w(x_{n-1}, Tx_{n-1}; \alpha_{n-1}), Tx_{n-1}, Tx_{n-1}) + \lambda G(x_{n-1}, x_n, x_n)] \\
&\leq s[\alpha_{n-1} G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + \lambda(1-\alpha_{n-1}) G(x_{n-1}, x_{n-1}, Tx_{n-1})] \\
&= s[\alpha_{n-1} + \lambda(1-\alpha_{n-1})] G(x_{n-1}, x_{n-1}, Tx_{n-1}).
\end{aligned}$$

Set $k_n = s[\alpha_n + \lambda(1-\alpha_n)]$. By the hypothesis, we get $\lim_{n\to\infty} k_n = s\alpha + s\lambda(1-\alpha) < \frac{1}{s}$. Thus we have

$$G(x_n, x_n, Tx_n) \leq k_{n-1} G(x_{n-1}, x_{n-1}, Tx_{n-1}) \leq \ldots \leq \prod_{i=0}^{n-1} k_i G(x_0, x_0, Tx_0).$$

Furthermore, we get that

$$G(x_n, x_n, x_{n+1}) \leq (1-\alpha_n) G(x_n, x_n, Tx_n) \leq \prod_{i=0}^{n-1} k_i G(x_0, x_0, Tx_0).$$

For any $p \in \mathbb{N}$, we have

$$\begin{aligned}
G(x_n, x_n, x_{n+p}) =& G(x_n, x_{n+p}, x_{n+p}) \\
\leq& sG(x_n, x_{n+1}, x_{n+1}) + sG(x_{n+1}, x_{n+p}, x_{n+p}) \\
\leq& sG(x_n, x_{n+1}, x_{n+1}) + s^2 G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + s^p G(x_{n+p-1}, x_{n+p}, x_{n+p}) \\
\leq& s\prod_{i=0}^{n} k_i G(x_0, x_0, Tx_0) G(x_0, x_0, Tx_0) + s^2 \prod_{i=0}^{n+1} k_i G(x_0, x_0, Tx_0) + \ldots \\
&+ s^p \prod_{i=0}^{n+p-1} k_i G(x_0, x_0, Tx_0) \\
=& \left[ s\prod_{i=0}^{n} k_i + s^2 \prod_{i=0}^{n+1} k_i + \ldots + s^p \prod_{i=0}^{n+p-1} k_i \right] G(x_0, x_0, Tx_0).
\end{aligned}$$

Let $Z_{n+i} = s^{i+1} \prod_{i=0}^{n+i} k_i, i = 0, 1, 2, \ldots, p-1$. Then we deduce that

$$G(x_n, x_n, x_{n+p}) \leq \left( Z_n + Z_{n+1} + \ldots + Z_{n+p-1} \right) G(x_0, x_0, Tx_0).$$

Notice that

$$\limsup_{i\to\infty} \frac{Z_{n+i+1}}{Z_{n+i}} = \limsup_{i\to\infty} \frac{s^{i+2} \prod_{i=0}^{n+i+1} k_i}{s^{i+1} \prod_{i=0}^{n+i} k_i} = \limsup_{i\to\infty} sk_{n+i+1} < 1.$$

By D'Alembert's test, we can deduce that $\sum_{i=0}^{\infty} Z_i$ is convergent which yields $\lim_{n\to\infty} \sum_{i=n}^{\infty} Z_n = 0$. Hence, we get $\lim_{n\to\infty} G(x_n, x_n, x_{n+p}) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence in $X$. Since $(X, G, w)$ is a complete convex $G_b$-metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} G(x_n, x_n, x^*) = 0$. Note that

$$\begin{aligned}
G(x^*, Tx^*, Tx^*) &\leq s[G(x^*, x_n, x_n) + G(x_n, Tx^*, Tx^*)] \\
&\leq sG(x^*, x_n, x_n) + s^2[G(x_n, Tx_n, Tx_n) + G(Tx_n, Tx^*, Tx^*)] \\
&\leq sG(x^*, x_n, x_n) + s^2 G(x_n, Tx_n, Tx_n) + s^2 \lambda G(x_n, x^*, x^*);
\end{aligned}$$

letting $n \to \infty$, we deduce $G(x^*, Tx^*, Tx^*) = 0$ which implies $Tx^* = x^*$. Thus $x^*$ is a fixed point of $T$. Suppose that $x^*, y^* \in X$ are two distinct fixed points of $T$. Then

$$0 < G(x^*, x^*, y^*) = G(Tx^*, Tx^*, Ty^*) \leq \lambda G(x^*, x^*, y^*),$$

which is a contradiction. Therefore, we must have $G(x^*, x^*, y^*) = 0$. To see that $T$ is $G$-continuous at a fixed point $x^*$, let $\{y_n\}$ be a sequence such that $\lim_{n\to\infty} y_n = x^*$. Then

$$G(x^*, Ty_n, Ty_n) = G(Tx^*, Ty_n, Ty_n) \leq \lambda G(x^*, y_n, y_n).$$

Taking the limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} G(x^*, Ty_n, Ty_n) = 0$ which implies $\lim_{n\to\infty} Ty_n = x^* = Tx^*$. Combining this with Proposition 4, we have that $T$ is $G$-continuous at $x^*$. $\square$

Let us give an example illustrating the above theorem.

**Example 5.** *Let $X = \mathbb{R}^+ \cup \{0\}$ and $Tx = \frac{x}{3}$ for all $x \in x$. For any $x, y, z \in X$, we define $G : X \times X \times X \to [0, \infty)$ with the formula*

$$G(x, y, z) = \left[\frac{1}{3}(|x - y| + |y - z| + |x - z|)\right]^2,$$

*while the mapping is defined by*

$$w(x, y; \alpha) = \alpha x + (1 - \alpha)y.$$

*Then $(X, G, w)$ is a convex $G_b$-metric space with $s = 2$. Set $x_{n+1} = w(x_n, Tx_n; \alpha_n)$ and $\alpha_n = \frac{1}{8}$. For any $u, v, x, y \in X$, it is not difficult to see that $T$ satisfies*

$$G(Tx, Ty, Tz) \leq \frac{1}{9} G(x, y, z) \leq \lambda G(x, y, z),$$

*for $\lambda \in [0, \frac{1}{4})$. We choose $x_0 \in X\backslash\{0\}$; according to $x_{n+1} = w(x_n, Tx_n; \alpha_n)$, we have $x_n = \frac{1}{8}x_{n-1} + \frac{7}{8}Tx_{n-1}$. Combining with $Tx = \frac{x}{3}$, we obtain $x_n = \frac{1}{8}x_{n-1} + \frac{7}{24}x_{n-1} = \frac{5}{12}x_{n-1}$, that is, $x_n = \frac{5}{12}x_{n-1}$. Then we have $x_n = (\frac{5}{12})^n x_0$ and $Tx_n = \frac{1}{3} \times (\frac{5}{12})^n x_0$. Let $n \to \infty$; we get $x_n \to 0 \in X$ and $Tx_n \to 0 \in X$. Hence, 0 is a fixed point of $T$ in $X$. Suppose $x^*, y^* \in X$ are two distinct fixed points of $T$. Thus we have*

$$G(x^*, x^*, y^*) = G(Tx^*, Tx^*, Ty^*) \leq \frac{1}{8} G(x^*, x^*, y^*),$$

*which shows that $G(x^*, x^*, y^*) = 0$, that is, $x^* = y^*$. Thus 0 is a unique fixed point of $T$.*

We denote the set of all fixed points of $T$ by $F(T)$, that is, $F(T) = \{x \in X : Tx = x\}$.

**Theorem 2.** *Let $(X, G, w)$ be a complete convex $G_b$-metric space with constant $s \geq 1$ and $T : X \to X$ be a mapping such that for all $x, y, z \in X$ and $\beta > 0$.*

$$\begin{aligned} G(Tx, Ty, Tz) \leq & \lambda_1 \frac{G(x, x, y)G(y, y, x)}{M(x, y)} + \lambda_2 \frac{G(x, x, Ty)G(y, y, Tx)}{M(x, y)} \\ & + \lambda_3 \frac{G(y, y, z)G(z, z, y)}{M(y, z)} + \lambda_4 \frac{G(y, y, Tz)G(z, z, Ty)}{M(y, z)} \\ & + \lambda_5 \frac{G(x, x, z)G(z, z, x)}{M(x, z)} + \lambda_6 \frac{G(x, x, Tz)G(z, z, Tx)}{M(x, z)}, \end{aligned}$$

*where*

$$\begin{aligned} M(x, y) &= \max\{\beta, G(x, x, Tx), G(y, y, Ty)\} \\ M(x, z) &= \max\{\beta, G(x, x, Tx), G(z, z, Tz)\} \\ M(y, z) &= \max\{\beta, G(y, y, Ty), G(z, z, Tz)\}, \end{aligned}$$

*and $\lambda_1 + \lambda_3 + \lambda_5 \leq \frac{1}{3s}, \lambda_2 + \lambda_4 + \lambda_6 \leq \frac{1}{3s}$. Suppose that the sequence $\{x_n\}$ is generated by the Mann iterative process and $x_0 \in X$. If the sequence $\{\alpha_n\} \in [0, \frac{1}{2s^2}]$, then $T$ has a fixed point, that is, $F(T) \neq \emptyset$.*

**Proof.** For any $n \in \mathbb{N}_0$, we have

$$G(x_n, x_n, x_{n+1}) = G(x_n, x_n, w(x_n, Tx_n; \alpha_n)) \leq (1 - \alpha_n)G(x_n, x_n, Tx_n).$$

If $x_n = x_{n+1}$, then $G(x_n, Tx_n, Tx_n) = G(x_{n+1}, Tx_n, Tx_n) \leq \alpha_n G(x_n, Tx_n, Tx_n)$, which implies that $x_n = Tx_n$ and $x_n$ is a fixed point of $T$. So, assume that $x_n \neq x_{n+1}$ and $x_n \neq Tx_n$. From Definition 8 and Proposition 4, it follows that

$$\begin{aligned} G(x_n, x_n, Tx_n) &= G(x_n, Tx_n, Tx_n) \\ &\leq s[G(x_n, Tx_{n-1}, Tx_{n-1}) + G(Tx_{n-1}, Tx_n, Tx_n)] \\ &\leq s[\alpha_{n-1} G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(Tx_{n-1}, Tx_n, Tx_n)]. \end{aligned}$$

On the one hand, we can consider $\{G(Tx_{n-1}, Tx_n, Tx_n)\}$ in the following cases.
Case 1 For any $n \in \mathbb{N}_0$, we have

$$\begin{aligned} G(Tx_{n-1}, Tx_n, Tx_n) \leq& \lambda_1 \frac{G(x_{n-1}, x_{n-1}, x_n)G(x_n, x_n, x_{n-1})}{M(x_{n-1}, x_n)} + \lambda_2 \frac{G(x_{n-1}, x_{n-1}, Tx_n)G(x_n, x_n, Tx_{n-1})}{M(x_{n-1}, x_n)} \\ &+ \lambda_3 \frac{G(x_{n-1}, x_{n-1}, x_n)G(x_n, x_n, x_{n-1})}{M(x_{n-1}, x_n)} + \lambda_4 \frac{G(x_{n-1}, x_{n-1}, Tx_n)G(x_n, x_n, Tx_{n-1})}{M(x_{n-1}, x_n)} \\ &+ \lambda_5 \frac{G(x_n, x_n, x_n)G(x_n, x_n, x_n)}{M(x_n, x_n)} + \lambda_6 \frac{G(x_n, x_n, Tx_n)G(x_n, x_n, Tx_n)}{M(x_n, x_n)} \\ &\leq (\lambda_1 + \lambda_3) \frac{((1 - \alpha_{n-1})G(x_{n-1}, x_{n-1}, Tx_{n-1}))^2}{M(x_{n-1}, x_n)} \\ &+ (\lambda_2 + \lambda_4) \frac{\alpha_{n-1} G(x_{n-1}, x_{n-1}, Tx_n)G(x_{n-1}, x_{n-1}, Tx_{n-1})}{M(x_n, x_{n-1})} + \lambda_6 G(x_n, x_n, Tx_n) \\ \leq& (1 - \alpha_{n-1})^2 (\lambda_1 + \lambda_3) G(x_{n-1}, x_{n-1}, Tx_{n-1}) \\ &+ \alpha_{n-1}(\lambda_2 + \lambda_4) s[G(Tx_n, x_n, x_n) + G(x_n, x_{n-1}, x_{n-1})] + \lambda_6 G(x_n, x_n, Tx_n) \\ \leq& [(1 - \alpha_{n-1})^2 (\lambda_1 + \lambda_3) + \alpha_{n-1}(1 - \alpha_{n-1}) s(\lambda_2 + \lambda_4)] G(x_{n-1}, x_{n-1}, Tx_{n-1}) \\ &+ [\alpha_n s(\lambda_2 + \lambda_4) + \lambda_6] G(x_n, x_n, Tx_n). \end{aligned}$$

Case 2 For any $n \in \mathbb{N}_0$, we have

$$\begin{aligned} G(Tx_{n-1}, Tx_n, Tx_n) =& G(Tx_n, Tx_{n-1}, Tx_n) \\ \leq& \lambda_1 \frac{G(x_n, x_n, x_{n-1})G(x_{n-1}, x_{n-1}, x_n)}{M(x_n, x_{n-1})} + \lambda_2 \frac{G(x_n, x_n, Tx_{n-1})G(x_{n-1}, x_{n-1}, Tx_n)}{M(x_n, x_{n-1})} \\ &+ \lambda_3 \frac{G(x_n, x_n, x_n)G(x_n, x_n, x_n)}{M(x_n, x_n)} + \lambda_4 \frac{G(x_n, x_n, Tx_n)G(x_{n-1}, x_n, Tx_n)}{M(x_n, x_n)} \\ &+ \lambda_5 \frac{G(x_{n-1}, x_{n-1}, x_n)G(x_n, x_n, x_{n-1})}{M(x_{n-1}, x_n)} + \lambda_6 \frac{G(x_{n-1}, x_{n-1}, Tx_n)G(x_n, x_n, Tx_{n-1})}{M(x_{n-1}, x_n)} \\ \leq& (\lambda_1 + \lambda_5) \frac{((1 - \alpha_{n-1})G(x_{n-1}, x_{n-1}, Tx_{n-1}))^2}{M(x_{n-1}, x_n)} \\ &+ (\lambda_2 + \lambda_6) \frac{\alpha_{n-1} G(x_{n-1}, x_{n-1}, Tx_n)G(x_{n-1}, x_{n-1}, Tx_{n-1})}{M(x_n, x_{n-1})} + \lambda_4 G(x_n, x_n, Tx_n) \\ \leq& (1 - \alpha_{n-1})^2 (\lambda_1 + \lambda_5) G(x_{n-1}, x_{n-1}, Tx_{n-1}) + \alpha_{n-1}(\lambda_2 + \lambda_6) s[G(Tx_n, x_n, x_n) \\ &+ G(x_n, x_{n-1}, x_{n-1})] + \lambda_4 G(x_n, x_n, Tx_n) \\ \leq& [(1 - \alpha_{n-1})^2 (\lambda_1 + \lambda_5) + \alpha_{n-1}(1 - \alpha_{n-1}) s(\lambda_2 + \lambda_6)] G(x_{n-1}, x_{n-1}, Tx_{n-1}) \\ &+ [\alpha_{n-1} s(\lambda_2 + \lambda_6) + \lambda_4] G(x_n, x_n, Tx_n). \end{aligned}$$

Case 3 For any $n \in \mathbb{N}_0$, we have

$$
\begin{aligned}
G(Tx_{n-1},Tx_n,Tx_n) =& G(Tx_n,Tx_n,Tx_{n-1}) \\
\leq& \lambda_1 \frac{G(x_n,x_n,x_n)G(x_n,x_n,x_n)}{M(x_n,x_n)} + \lambda_2 \frac{G(x_n,x_n,Tx_n)G(x_n,x_n,Tx_n)}{M(x_n,x_n)} \\
&+ \lambda_3 \frac{G(x_n,x_n,x_{n-1})G(x_{n-1},x_{n-1},x_n)}{M(x_n,x_{n-1})} + \lambda_4 \frac{G(x_n,x_n,Tx_{n-1})G(x_{n-1},x_{n-1},Tx_n)}{M(x_n,x_{n-1})} \\
&+ \lambda_5 \frac{G(x_n,x_n,x_{n-1})G(x_{n-1},x_{n-1},x_n)}{M(x_n,x_{n-1})} + \lambda_6 \frac{G(x_n,x_n,Tx_{n-1})G(x_{n-1},x_{n-1},Tx_n)}{M(x_n,x_{n-1})} \\
\leq& \lambda_2 G(x_{n-1},x_{n-1},Tx_{n-1}) + (\lambda_3+\lambda_5)\frac{((1-\alpha_{n-1})G(x_{n-1},x_{n-1},Tx_{n-1}))^2}{M(x_{n-1},x_n)} \\
&+ (\lambda_4+\lambda_6)\frac{\alpha_{n-1}G(x_{n-1},x_{n-1},Tx_n)G(x_{n-1},x_{n-1},Tx_{n-1})}{M(x_n,x_{n-1})} \\
\leq& \lambda_2 G(x_n,x_n,Tx_n) + (1-\alpha_{n-1})^2(\lambda_3+\lambda_5)G(x_{n-1},x_{n-1},Tx_{n-1}) \\
&+ \alpha_{n-1}(\lambda_4+\lambda_6)s[G(Tx_n,x_n,x_n)+G(x_n,x_{n-1},x_{n-1})] \\
\leq& [(1-\alpha_{n-1})^2(\lambda_3+\lambda_5)+\alpha_{n-1}(1-\alpha_{n-1})s(\lambda_4+\lambda_6)]G(x_{n-1},x_{n-1},Tx_{n-1}) \\
&+ [\alpha_{n-1}(\lambda_4+\lambda_6)+\lambda_2]G(x_n,x_n,Tx_n).
\end{aligned}
$$

Since $G(Tx_{n-1},Tx_n,Tx_n) = G(Tx_n,Tx_{n-1},Tx_{n-1})$, on the other hand, similar to the procedure of the above cases, we can deduce that

Case 4 For any $n \in \mathbb{N}_0$, we have

$$
\begin{aligned}
G(Tx_n,Tx_{n-1},Tx_{n-1}) \leq& [(1-\alpha_{n-1})^2(\lambda_1+\lambda_3)+\alpha_{n-1}(1-\alpha_{n-1})s(\lambda_2+\lambda_4)+\lambda_6]G(x_{n-1},x_{n-1},Tx_{n-1}) \\
&+ \alpha_{n-1}s(\lambda_4+\lambda_6)G(x_n,x_n,Tx_n).
\end{aligned}
$$

Case 5 For any $n \in \mathbb{N}_0$, we have

$$
\begin{aligned}
G(Tx_{n-1},Tx_n,Tx_{n-1}) \leq& [(1-\alpha_{n-1})^2(\lambda_1+\lambda_5)+\alpha_{n-1}(1-\alpha_{n-1})s(\lambda_2+\lambda_6)+\lambda_4]G(x_{n-1},x_{n-1},Tx_{n-1}) \\
&+ \alpha_{n-1}s(\lambda_2+\lambda_6)G(x_n,x_n,Tx_n).
\end{aligned}
$$

Case 6 For any $n \in \mathbb{N}_0$, we have

$$
\begin{aligned}
G(Tx_{n-1},Tx_{n-1},Tx_n) \leq& [(1-\alpha_{n-1})^2(\lambda_3+\lambda_5)+\alpha_{n-1}(1-\alpha_{n-1})s(\lambda_4+\lambda_6)+\lambda_2]G(x_{n-1},x_{n-1},Tx_{n-1}) \\
&+ \alpha_{n-1}s(\lambda_4+\lambda_6)G(x_n,x_n,Tx_n).
\end{aligned}
$$

In view of all the above cases, we deduce that

$$
\begin{aligned}
6G(Tx_{n-1},Tx_n,Tx_n) \leq& [4(1-\alpha_{n-1})^2(\lambda_1+\lambda_3+\lambda_5)+(4\alpha_{n-1}(1-\alpha_{n-1})s+1)(\lambda_2+\lambda_4+\lambda_6)] \\
&\times G(x_{n-1},x_{n-1},Tx_{n-1}) + (4\alpha_{n-1}s+1)(\lambda_2+\lambda_4+\lambda_6)G(x_n,x_n,Tx_n).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
G(x_n,x_n,Tx_n) \leq& s[\alpha_{n-1} + \frac{4(1-\alpha_{n-1})^2(\lambda_1+\lambda_3+\lambda_5)+(4\alpha_{n-1}(1-\alpha_{n-1})s+1)(\lambda_2+\lambda_4+\lambda_6)}{6}] \\
&\times G(x_{n-1},x_{n-1},Tx_{n-1}) + \frac{(4\alpha_{n-1}s+1)(\lambda_2+\lambda_4+\lambda_6)}{6}G(x_n,x_n,Tx_n) \\
\leq& \frac{1}{s}[\frac{1}{2} + \frac{5+\frac{2}{s}\times(1-\frac{1}{2s^2})}{18}]G(x_{n-1},x_{n-1},Tx_{n-1}) + \frac{1}{6}G(x_n,x_n,Tx_n) \\
\leq& \frac{1}{s}(\frac{1}{2} + \frac{5+\frac{28}{27}}{18})G(x_{n-1},x_{n-1},Tx_{n-1}) + \frac{1}{6}G(x_n,x_n,Tx_n),
\end{aligned}
$$

that is,

$$G(x_n, x_n, Tx_n) \leq \frac{1}{s} \frac{\left(\frac{1}{2} + \frac{5\times 27+28}{27\times 18}\right)}{\frac{5}{6}} G(x_{n-1}, x_{n-1}, Tx_{n-1}).$$

Let $\eta = \frac{1}{s} \frac{\left(\frac{1}{2} + \frac{5\times 27+28}{27\times 18}\right)}{\frac{5}{6}}$; we also note that $\eta < \frac{1}{s}$. It follows from the above inequality that

$$G(x_n, x_n, Tx_n) \leq \eta G(x_{n-1}, x_{n-1}, Tx_{n-1}).$$

Moreover,

$$G(x_n, x_n, x_{n+1}) \leq \eta G(x_n, x_n, Tx_n).$$

Thus, for any $p \in \mathbb{N}$, we get

$$\begin{aligned} G(x_n, x_n, x_{n+p}) =& G(x_n, x_{n+p}, x_{n+p}) \\ \leq& sG(x_n, x_{n+1}, x_{n+1}) + sG(x_{n+1}, x_{n+p}, x_{n+p}) \\ \leq& sG(x_n, x_{n+1}, x_{n+1}) + s^2 G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots + s^p G(x_{n+p-1}, x_{n+p}, x_{n+p}) \\ \leq& s\eta^n G(x_0, x_0, Tx_0) + s^2 \eta^{n+1} G(x_0, x_0, Tx_0) + \ldots + s^p \eta^{n+p-1} G(x_0, x_0, Tx_0) \\ \leq& \eta^n (s + s^2 \eta + s^3 \eta^2 + \ldots) G(x_0, x_0, Tx_0) \\ \leq& \frac{1}{1 - s\eta} s\eta^n G(x_0, x_0, Tx_0), \end{aligned}$$

and, letting $n \to \infty$, we deduce that $\lim_{n\to\infty} G(x_n, x_n, x_{n+p}) = 0$, which shows that $\{x_n\}$ is a Cauchy sequence in $X$. Since $(X, G, w)$ is a complete convex $G_b$-metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} G(x_n, x_n, x^*) = 0$. Note that

$$\begin{aligned} G(Tx^*, x^*, x^*) \leq& s[G(Tx^*, Tx_n, Tx_n) + sG(Tx_n, x_n, x_n) + sG(x_n, x^*, x^*)] \\ \leq& s[(\lambda_1 + \lambda_3) \frac{G(x^*, x^*, x_n) G(x_n, x_n, x^*)}{M(x^*, x_n)} + (\lambda_2 + \lambda_4) \frac{G(x^*, x^*, Tx_n) G(x_n, x_n, Tx^*)}{M(x^*, x_n)} \\ &+ \lambda_5 \frac{G(x_n, x_n, Tx_n) G(x_n, x_n, Tx_n)}{M(x_n, x_n)}] + s^2 G(Tx_n, x_n, x_n) + s^2 G(x_n, x^*, x^*) \\ \leq& s[(\lambda_1 + \lambda_3) G(x^*, x^*, x_n) G(x_n, x_n, x^*) \\ &+ (\lambda_2 + \lambda_4) s[G(x^*, x_n, x_n) + G(x_n, Tx_n, Tx_n)] G(x_n, x_n, Tx^*) \\ &+ \lambda_5 G(x_n, x_n, Tx_n)] + s^2 G(Tx_n, x_n, x_n) + s^2 G(x_n, x^*, x^*). \end{aligned}$$

Letting $n \to \infty$, we deduce $G(x^*, x^*, Tx^*) = 0$, which implies $x^* = Tx^*$. Thus $x^*$ is a fixed point of $T$. $\square$

**Remark 3.** *The condition in Theorem 2 does not guarantee the uniqueness of the fixed point. The following example illustrates this fact.*

**Example 6.** *Let $X = \{1, 2, 3\}$ and $G : X \times X \times X \to [0, \infty)$ be a mapping for any $x, y, z \in X$ such that $G(x, y, z) = G(y, x, z) = G(z, y, x) = \ldots$ and $G(1, 1, 1) = G(2, 2, 2) = G(3, 3, 3) = 0, G(1, 1, 2) = G(2, 2, 1) = 3, G(1, 1, 3) = G(3, 3, 1) = 4, G(2, 2, 3) = G(3, 3, 2) = 5, G(1, 2, 3) = 6$. Then $(X, d)$ is a complete $G_b$-metric space with $s = 1$. Let $T$ be a mapping defined by $Tx = x$ for any $x \in X$. Set*

$$\begin{aligned} F(x, y, z) =& \lambda_1 \frac{G(x, x, y) G(y, y, x)}{M(x, y)} + \lambda_2 \frac{G(x, x, Ty) G(y, y, Tx)}{M(x, y)} \\ &+ \lambda_3 \frac{G(y, y, z) G(z, z, y)}{M(y, z)} + \lambda_4 \frac{G(y, y, Tz) G(z, z, Ty)}{M(y, z)} \\ &+ \lambda_5 \frac{G(x, x, z) G(z, z, x)}{M(x, z)} + \lambda_6 \frac{G(x, x, Tz) G(z, z, Tx)}{M(x, z)} \\ =& (\lambda_1 + \lambda_2) G(x, x, y) + (\lambda_3 + \lambda_4) G(y, y, z) + (\lambda_5 + \lambda_6) G(x, x, z) \end{aligned}$$

*For any $x, y, z \in X$, we have*

$$F(x,y,z) =(\lambda_1+\lambda_2)(G(x,x,y))^2+(\lambda_3+\lambda_4)(G(y,y,z))^2+(\lambda_5+\lambda_6)(G(x,x,z))^2$$
$$F(z,x,y) =(\lambda_3+\lambda_4)(G(x,x,y))^2+(\lambda_5+\lambda_6)(G(y,y,z))^2+(\lambda_1+\lambda_2)(G(x,x,z))^2$$
$$F(y,z,x) =(\lambda_5+\lambda_6)(G(x,x,y))^2+(\lambda_1+\lambda_2)(G(y,y,z))^2+(\lambda_3+\lambda_4)(G(x,x,z))^2.$$

*Therefore,*

$$3F(x,y,z) = (\sum_{i=1}^{6}\lambda_i)(G(x,x,y))^2+(\sum_{i=1}^{6}\lambda_i)(G(y,y,z))^2+(\sum_{i=1}^{6}\lambda_i)(G(x,x,z))^2.$$

*Now, we consider the following cases:*

*Case 1 If $x = 1, y = 2, z = 3$, then*

$$\begin{aligned}3F(1,2,3) &=(\sum_{i=1}^{6}\lambda_i)(G(1,1,2))^2+(\sum_{i=1}^{6}\lambda_i)(G(2,2,3))^2+(\sum_{i=1}^{6}\lambda_i)(G(1,1,3))^2\\ &=(\sum_{i=1}^{6}\lambda_i)(9+25+16)=\frac{100}{3},\end{aligned}$$

*which implies*

$$F(1,2,3)=\frac{100}{9}>G(T1,T2,T3)=G(1,2,3)=6;$$

*Case 2 If $x = y = 1, z = 2$, then*

$$\begin{aligned}3F(1,1,2) &=(\sum_{i=1}^{6}\lambda_i)(G(1,1,1))^2+(\sum_{i=1}^{6}\lambda_i)(G(1,1,2))^2+(\sum_{i=1}^{6}\lambda_i)(G(1,1,2))^2\\ &=(\sum_{i=1}^{6}\lambda_i)\times(8+8)=\frac{32}{3},\end{aligned}$$

*which implies*

$$F(1,1,2)=\frac{32}{9}>G(T1,T1,T2)=G(1,1,2)=3;$$

*Case 3 If $x = y = 1, z = 3$, then*

$$\begin{aligned}3F(1,1,3) &=(\sum_{i=1}^{6}\lambda_i)(G(1,1,1))^2+(\sum_{i=1}^{6}\lambda_i)(G(1,1,3))^2+(\sum_{i=1}^{6}\lambda_i)(G(1,1,3))^2\\ &=(\sum_{i=1}^{6}\lambda_i)\times(9+9)=\frac{36}{3},\end{aligned}$$

*which implies*

$$F(1,1,2)=\frac{36}{9}\geq G(T1,T1,T2)=G(1,1,2)=4;$$

*Case 4 If $x = y = 2, z = 3$, then*

$$\begin{aligned}3F(2,2,3) &=(\sum_{i=1}^{6}\lambda_i)(G(2,2,2))^2+(\sum_{i=1}^{6}\lambda_i)(G(2,2,3))^2+(\sum_{i=1}^{6}\lambda_i)(G(2,2,3))^2\\ &=(\sum_{i=1}^{6}\lambda_i)\times(25+25)=\frac{100}{3},\end{aligned}$$

*which implies*

$$F(2,2,3)=\frac{100}{9}>G(T2,T2,T3)=G(2,2,3)=5.$$

*Therefore, we obtain that $G(Tx, Ty, Tz) \leq F(x, y, z)$ for any $x, y, z \in X$. Hence, all conditions of Theorem 2 are satisfied and $F(T) = \{1, 2, 3\}$.*

The well-posedness of a fixed point problem has evoked much interest to many authors (see [22–26]).

**Definition 10** ([22,23])**.** *Let $(X,d)$ be a metric space and $T : X \to X$ be a mapping. The fixed point problem of $T$ is said to be well-posed if*

*(1) $T$ has a unique fixed point $x^* \in X$;*

*(2) For any sequence $\{x_n\}$ in $X$ with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $\lim_{n\to\infty} d(x_n, x^*) = 0$.*

We introduce the concept of well-posedness in $G_b$-metric space.

**Definition 11.** *Let $(X,G)$ be a $G_b$-metric space and $T : X \to X$ be a mapping. The fixed point problem of $T$ is said to be well-posed if*

*(1) $T$ has a unique fixed point $x^* \in X$;*

*(2) For any sequence $\{x_n\}$ in $X$, if $\lim_{n\to\infty} G(x_n, x_n, Tx_n) = 0$, then $\lim_{n\to\infty} G(x_n, x_n, x^*) = 0$, or, if $\lim_{n\to\infty} G(x_n, Tx_n, Tx_n) = 0$, then $\lim_{n\to\infty} G(x_n, x^*, x^*) = 0$.*

**Theorem 3.** *Under the conditions of Theorem 2, if*

$$\sum_{i=1}^{6} \lambda_i \leq \max\{\lambda_1 + \lambda_2, \lambda_3 + \lambda_4, \lambda_5 + \lambda_6\},$$

*then the fixed point problem for $T$ is well-posed.*

**Proof.** Taking advantage of Theorem 2, we get that $T$ has a fixed point $x^* \in X$. We shall prove that $x^*$ is a unique fixed point of $T$. Assume the contrary, that $y^*$ is another fixed point of $T$. By virtue of the hypotheses, let $\sum_{i=1}^{6} \lambda_i \leq \lambda_1 + \lambda_2$, which is only true if $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$. Then we get

$$G(x^*, x^*, y^*) \leq \lambda_1 \frac{G(x^*, x^*, x^*)G(x^*, x^*, x^*)}{\beta + M(x^*, x^*)} + \lambda_2 \frac{G(x^*, x^*, Tx^*)G(x^*, x^*, Tx^*)}{\beta + M(x^*, x^*)} = 0,$$

that is, $G(x^*, x^*, y^*) = 0$, a contradiction. In the other cases, it is easy to get that $x^* = y^*$. Therefore, $x^*$ is a unique fixed point. Suppose that $\{y_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} G(y_n, y_n, Ty_n) = 0$. Next, we discuss following cases.

Case 1 If $\sum_{i=1}^{6} \lambda_i \leq \lambda_1 + \lambda_2$, which implies that $\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$, we have

$$\begin{aligned} G(Ty_n, x^*, x^*) =& G(Ty_n, Ty_n, Tx^*) \\ \leq& \lambda_1 \frac{G(y_n, y_n, y_n)G(y_n, y_n, y_n)}{\beta + M(y_n, y_n)} + \lambda_2 \frac{G(y_n, y_n, Ty_n)G(y_n, y_n, Ty_n)}{\beta + M(y_n, y_n)} \\ =& \lambda_2 G(y_n, y_n, Ty_n). \end{aligned}$$

Letting $n \to \infty$, we conclude that $\lim_{n\to\infty} G(Ty_n, x^*, x^*) = 0$.

Case 2 If $\sum_{i=1}^{6} \lambda_i \leq \lambda_3 + \lambda_4$, which implies that $\lambda_1 = \lambda_2 = \lambda_5 = \lambda_6 = 0$, we have

$$\begin{aligned} G(Ty_n, x^*, x^*) =& G(Ty_n, Tx^*, Ty_n) \\ \leq& \lambda_3 \frac{G(y_n, y_n, y_n)G(y_n, y_n, y_n)}{M(y_n, y_n)} + \lambda_4 \frac{G(y_n, y_n, Ty_n)G(y_n, y_n, Ty_n)}{M(y_n, y_n)} \\ =& \lambda_4 G(y_n, y_n, Ty_n). \end{aligned}$$

Letting $n \to \infty$, we conclude that $\lim_{n\to\infty} G(Ty_n, x^*, x^*) = 0$.

Case 3 If $\sum_{i=1}^{6} \lambda_i \leq \lambda_5 + \lambda_6$, which implies that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, we have

$$\begin{aligned} G(Ty_n, x^*, x^*) =& G(Tx^*, Ty_n, Ty_n) \\ &\leq \lambda_5 \frac{G(y_n, y_n, y_n)G(y_n, y_n, y_n)}{M(y_n, y_n)} + \lambda_6 \frac{G(y_n, y_n, Ty_n)G(y_n, y_n, Ty_n)}{M(y_n, y_n)} \\ &= \lambda_6 G(y_n, y_n, Ty_n). \end{aligned}$$

Letting $n \to \infty$, we conclude that $\lim_{n\to\infty} G(Ty_n, x^*, x^*) = 0$.

By the above cases, we have $G(Ty_n, x^*, x^*) \leq \frac{1}{3}(\lambda_2 + \lambda_4 + \lambda_6)G(y_n, y_n, Ty_n)$. Then

$$\begin{aligned} G(y_n, x^*, x^*) \leq& s[G(y_n, Ty_n, Ty_n) + G(Ty_n, x^*, x^*)] \\ \leq& s(1 + \lambda_2 + \lambda_4 + \lambda_6)G(y_n, y_n, Ty_n). \end{aligned}$$

Letting $n \to \infty$, we conclude that $\lim_{n\to\infty} G(y_n, x^*, x^*) = 0$, hence $\lim_{n\to\infty} G(y_n, y_n, x^*) = 0$. □

If a map $T$ satisfies $F(T) = F(T^n)$, $n \in \mathbb{N}_0$, then $T$ is said to have the $P$ property [27,28]. Note that if $T$ has a fixed point $x^*$, then $x^*$ is also a fixed point of $T^n$, but it is well-known that the converse is not true.

**Theorem 4.** *Let $(X, G)$ be a $G_b$-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping with $F(T) \neq \emptyset$ satisfying*

$$G(Tx, Tx, T^2x) \leq \eta G(x, x, Tx) \tag{2}$$

*for any $x \in X$, where $\eta \in [0, 1)$. Then $T$ has the $P$ property.*

**Proof.** Obviously, we can assume that $n > 1$. Let $z = T^n z$ for all $n > 1$. We have

$$\begin{aligned} G(z, z, Tz) &= G(TT^{n-1}z, TT^{n-1}z, T^2T^{n-1}z) \\ &\leq \eta G(T^{n-1}z, T^{n-1}z, TT^n z) \\ &= \eta G(TT^{n-1}z, TT^{n-2}z, T^2T^n z) \\ &\leq \eta^2 G(T^{n-1}z, T^{n-2}z, TT^n z) \\ &\leq \dots \\ &\leq \eta^n G(z, z, Tz). \end{aligned}$$

Letting $n \to \infty$, we get $G(z, z, Tz) = 0$, which implies that $z = Tz$. □

**Theorem 5.** *Under the conditions of Theorem 2, $T$ has the $P$ property.*

**Proof.** For any $x \in X$, we have

$$\begin{aligned} G(Tx, Tx, T^2x) =& G(Tx, Tx, TTx) \\ \leq& \lambda_1 \frac{G(x, x, x)G(x, x, x)}{1 + M(x, x)} + \lambda_2 \frac{G(x, x, Tx)G(x, x, Tx)}{1 + M(x, x)} \\ &+ (\lambda_3 + \lambda_5)\frac{G(x, x, Tx)G(Tx, Tx, x)}{1 + M(x, Tx)} + (\lambda_4 + \lambda_6)\frac{G(x, x, T^2x)G(Tx, Tx, Tx)}{1 + M(x, Tx)} \\ \leq& (\lambda_2 + \lambda_3 + \lambda_5)G(x, x, Tx), \end{aligned}$$

which implies that

$$G(Tx, Tx, T^2x) \leq (\lambda_2 + \lambda_3 + \lambda_5)G(x, x, Tx).$$

Note that $\lambda_2 + \lambda_3 + \lambda_5 < 1$; accordingly, (2) is satisfied. Thus, $T$ has the $P$ property. □

## 3. Application

In this section, we apply Theorem 1 to guarantee the existence of a solution to the following integral equation:

$$x(t) = f(t) + \gamma \int_a^b u(t,\tau)K_1(\tau, x(\tau))d\tau \int_a^b u(t,\tau)K_2(\tau, x(\tau))d\tau \quad (3)$$

for $t \in [a,b]$, where $f : [a,b] \to R$, $u : [a,b] \times [a,b] \to R$ and $K_1, K_2 : [a,b] \times R \to R$ are continuous functions. Let $X = C([a,b], \mathbb{R})$ denote the space of all continuous functions on $[a,b]$. We endow with the $G_b$-metric mapping

$$G(x,y,z) = \left( \sup_{t\in[a,b]} |x(t) - y(t)| + \sup_{t\in[a,b]} |y(t) - z(t)| + \sup_{t\in[a,b]} |x(t) - z(t)| \right)^2,$$

while the function $w : X \times X \times (0,1) \to X$ is defined as $w(x,y;\alpha) = \alpha x + (1-\alpha)y$. It is clear that $(X, G, w)$ is a complete convex $G_b$-metric space with $s = 2$. Define $T : X \to X$ by

$$Tx(t) = f(t) + \gamma \int_a^b u(t,\tau)K_1(\tau, x(\tau))d\tau \int_a^b u(t,\tau)K_2(\tau, x(\tau))d\tau. \quad (4)$$

Obviously, $T$ is well-defined. In order to find a solution for integral Equation (3), it is sufficient to find a fixed point of the operator $T$.

Now, we state the following consequence.

**Theorem 6.** *Assume that the following conditions are satisfied:*

*(1)* $\gamma \leq 1$;

*(2)* $\int_a^b u(t,\tau)d(\tau) \leq 1$;

*(3)* $|K_i(\tau, x(\tau)) - K_i(\tau, y(\tau))| \leq \frac{\sqrt{5}}{5}|x-y|, i = 1,2,$ *and*

$$\int_a^b u(t,\tau)|K_1(\tau, y(\tau)) + K_2(\tau, x(\tau))|d\tau \leq 1.$$

*Then, the integral Equation (3) has a unique solution in X.*

**Proof.** It is clear that any fixed point of (4) is a solution of (3). Using the condtions of (1)–(3), we obtain

$$
\begin{aligned}
G(Tx,Ty,Ty) =&\left(2\sup_{t\in[a,b]}|Tx(t)-Ty(t)|\right)^2\\
\leq&\gamma^2\left(2\sup_{t\in[a,b]}\left|\int_a^b u(t,\tau)K_1(\tau,x(\tau))d\tau\int_a^b u(t,\tau)K_2(\tau,x(\tau))d\tau\right.\right.\\
&\left.\left.-\int_a^b u(t,\tau)K_1(\tau,y(\tau))d\tau\int_a^b u(t,\tau)K_2(\tau,y(\tau))d\tau\right|\right)^2\\
\leq&\gamma^2\left(2\sup_{t\in[a,b]}\left|\int_a^b u(t,\tau)|K_1(\tau,x(\tau))-K_1(\tau,y(\tau))|d\tau\int_a^b u(t,\tau)K_2(\tau,x(\tau))d\tau\right.\right.\\
&\left.\left.+\int_a^b u(t,\tau)K_1(\tau,y(\tau))d\tau\int_a^b u(t,\tau)|K_2(\tau,x(\tau))-K_2(\tau,y(\tau))|d\tau\right|\right)^2\\
\leq&4\gamma^2\left(\sup_{t\in[a,b]}\sup_{\tau\in[a,b]}|K_1(\tau,x(\tau))-K_1(\tau,y(\tau))|\left|\sup_{t\in[a,b]}\int_a^b u(t,\tau)d\tau\int_a^b u(t,\tau)K_2(\tau,x(\tau))d\tau\right|\right.\\
&\left.+\sup_{t\in[a,b]}\sup_{\tau\in[a,b]}|K_2(\tau,x(\tau))-K_2(\tau,y(\tau))|\left|\int_a^b u(t,\tau)K_1(\tau,y(\tau))d\tau\int_a^b u(t,\tau)d\tau\right|\right)^2\\
\leq&4\gamma^2\left(\frac{\sqrt{5}}{5}\sup_{t\in[a,b]}|x-y|\sup_{t\in[a,b]}\left|\int_a^b u(t,\tau)d\tau\int_a^b u(t,\tau)K_2(\tau,x(\tau))d\tau\right.\right.\\
&\left.\left.+\int_a^b u(t,\tau)K_1(\tau,y(\tau))d\tau\int_a^b u(t,\tau)d\tau\right|\right)^2\\
\leq&\frac{4}{5}\gamma^2\sup_{t\in[a,b]}\left(\int_a^b u(t,\tau)d\tau\right)^2\left(\sup_{t\in[a,b]}|x-y|\sup_{t\in[a,b]}\left|\int_a^b u(t,\tau)K_2(\tau,x(\tau))d\tau\right.\right.\\
&\left.\left.+\int_a^b u(t,\tau)K_1(\tau,y(\tau))d\tau\right|\right)^2\\
\leq&\frac{4}{5}\gamma^2\left(2\sup_{t\in[a,b]}|x-y|\sup_{t\in[a,b]}\left|\int_a^b u(t,\tau)|K_1(\tau,y(\tau))+K_2(\tau,x(\tau))|d\tau\right|\right)^2\\
\leq&\frac{1}{5}\left(2\sup_{t\in[a,b]}|x-y|\right)^2\\
=&\frac{1}{5}G(x,y,y),
\end{aligned}
$$

which satisfies all conditions of Theorem 1 with $y=z$. Hence, we can get that the integral Equation (3) has a unique solution $x(t)$ satisfying $\lim_{n\to\infty}x_n(t)=x(t)$ where the sequence $\{x_n\}$ is defined by $x_n=\alpha_{n-1}x_{n-1}+(1-\alpha_{n-1})Tx_{n-1}$, $\alpha_n\in(0,\frac{1}{4})$. □

**Author Contributions:** D.J., C.L. and Y.C. contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was supported by the Natural Science Foundation of Heilongjiang Province of China (Grant No. YQ2021C025).

**Data Availability Statement:** Not applicable.

**Acknowledgments:** We would like to express our thanks to the anonymous referees and the editor for their constructive comments and suggestions, which greatly improved this article.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References


1. Banach, S. Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Gahler, S. 2-metrische Raume und ihre topologische Strüktür. *Math. Nachr.* **1963**, *26*, 115–148. [CrossRef]
3. Dhage, B.C. Generalized metric space and mapping with fixed point. Bull. *Calcutta Math. Soc.* **1992**, *84*, 329–336.
4. Czerwik, S. Contraction mappings in metric spaces. Acta Math. Inform. *Univ. Ostrav.* **1993**, *1*, 5–11.
5. Huang H.; Deng, G.; Radenović, S. Fixed point theorems in $b$-metric spaces with applications to differential equations. *J. Fixed Point Theory Appl.* **2018**, *20*, 1–24. [CrossRef]
6. Mustafa, Z.; Sims, B. A new approach to generalized metric spaces. *J. Nonlinear Convex Anal.* **2006**, *7*, 289–297.
7. Aghajani, A.; Abbas, M.; Roshan, J.R. Common fixed point of generallized weak contractive mappings in partially ordered $G_b$-metric spaces. *Filomat* **2014**, *28*, 1087–1101. [CrossRef]
8. Mustafa, Z.; Sims, B. Fixed point theorems for contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2009**, *2009*, 1–10. [CrossRef]
9. Takahashi, W. A convexity in metric space and nonexpansive mappings I. *Kodai Math. J.* **1970**, *22*, 142–149. [CrossRef]
10. Norouzian, M.; Abkar, A. Tripartite coincidence-best proximity points and convexity in generalized metric spaces. *Bull. Braz. Math. Soc.* **2019**, *50*, 999–1028. [CrossRef]
11. Goebel, K.; Kirk, W.A. Iteration processes for nonexpansive mappings. *Contemp. Math.* **1983**, *21*, 115–123.
12. Ding, X.P. Iteration processes for nonlinear mappings in convex metric spaces. *Contemp. Math.* **1983**, *21*, 115–123. [CrossRef]
13. Reich, S.; Shafrir, I. Nonexpansive iterations in hyperbolic spaces. *Nonlinear Anal.* **1990**, *19*, 537–558. [CrossRef]
14. Chen, L.; Gao, L.; Zhao, Y. A new iterative scheme for finding attractive points of generalized hybrid set-valued mappings. *J. Nonlinear Sci. Appl.* **2017**, *10*, 1228–1237. [CrossRef]
15. Chen, L.; Zou, J.; Zhao, Y.; Zhang, M. Iterative approximation of common attractive points of generalized hybrid set-valued mappings. *J. Fixed Point Theory Appl.* **2019**, *21*, 58. [CrossRef]
16. Chen, L.L.; Li, C.B.; Kaczmarek, R.; Zhao, Y.F. Several fixed point theorems in convex $b$-metric spaces and applications. *Mathematics* **2020**, *8*, 242. [CrossRef]
17. Li C.; Cui Y.; Chen L. Fixed point results on closed ball in convex rectangular $b$-metric spaces and applications. *J. Funct. Spaces* **2022**, *2022*, 8840964. [CrossRef]
18. Asif, A.; Alansari, M.; Hussain, N.; Ali, A. Iterating fixed point via generalized Mann's iteration in convex $b$-metric spaces with application. *Complexity* **2021**, *2021*, 8534239. [CrossRef]
19. Mann, W.R.: Mean value methods in iteration. *Proc. Am. Math. Soc.* **1953**, *4*, 506–510. [CrossRef]
20. Reich S. Weak convergence theorems for nonexpansive mappings in Banach spaces. *J. Math. Anal. Appl.* **1979**, *67*, 274–276. [CrossRef]
21. Aydi, H.; Rakíc, D.; Aghajani, A.; Došenović, T.; Noorani, M.S.M.; Qawaqneh, H. On fixed point results in $G_b$-metric spaces. *Mathematics* **2019**, *7*, 617. [CrossRef]
22. Reich, S.; Zaslavski, A.J. Well-posedness of fixed point problems. *Far East J. Math. Sci.* **2001**, *3*, 393–401. [CrossRef]
23. Reich, S.; Zaslavski, A.J. Generic well-posedness of fixed point problems. *Vietnam. J. Math.* **2018**, *46*, 5–13. [CrossRef]
24. Chen, L.; Huang, S.; Li, C.; Zhao, Y. Several fixed point theorems for $F-$contractions in complete Branciari metric spaces and applications. *J. Funct. Spaces* **2020**, *2020*, 7963242.
25. Chen, L.; Yang, N.; Zhao, Y.; Ma, Z. Fixed point theorems for set-valued $G-$contractions in a graphical convex metric space with applications. *J. Fixed Point Theory Appl.* **2020**, *22*, 88. [CrossRef]
26. Aydi, H.; Jellali, M.; Karapınar, E. On fixed point results for alpha-implicit contractions in quasi-metric spaces and consequences. *Nonlinear Anal.-Model.* **2016**, *21*, 40–56. [CrossRef]
27. Huang, H.; Xu, S.; Liu, H.; Radenović, S. Fixed point theorems and $T-$stability of Picard iteration for generalized Lipschitz mappings in cone metric spaces over Banach algebras. *J. Comput. Anal. Appl.* **2016**, *20*, 869–888.
28. Jeong, G.S.; Rhoades, B.E. Maps for which $F(T) = F(T^n)$. *Fixed Point Theory Appl.* **2005**, *6*, 71–105.