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Remarks on Conjectures in Block Theory of Finite Groups †

Manal H. Algreagri [1,2,*] and Ahmad M. Alghamdi [1]

[1] Mathematics Department, Faculty of Sciences, Umm Al-Qura University, P.O. Box 14035, Makkah 21955, Saudi Arabia; amghamdi@uqu.edu.sa
[2] Department of Mathematics, Jamoum University College, Umm Al-Qura University, Makkah 21955, Saudi Arabia
* Correspondence: s44277183@st.uqu.edu.sa or mhgreagri@uqu.edu.sa
† Dedicated to Professor Geoffrey R. Robinson on the occasion of his 70th birthday.

Abstract: In this paper, we focus on Brauer's height zero conjecture, Robinson's conjecture, and Olsson's conjecture regarding the direct product of finite groups and give relative versions of these conjectures by restricting them to the algebraic concept of the anchor group of an irreducible character. Consider G to be a finite simple group. We prove that the anchor group of the irreducible character of G with degree p is the trivial group, where p is an odd prime. Additionally, we introduce the relative version of the Green correspondence theorem with respect to this group. We then apply the relative versions of these conjectures to suitable examples of simple groups. Classical and standard theories on the direct product of finite groups, block theory, and character theory are used to achieve these results.

Keywords: finite group; group algebra; character; block; defect group; direct product

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1. Introduction

Let G be a finite group and p be a prime divisor of $|G|$. Let B be a p-block of group G with defect group D. We consider the triple $(k, \mathcal{R}, F)$ to be a p-modular system [1–3]. This system comprises a complete discrete valuation ring $\mathcal{R}$ with a field of fractions k of characteristic 0, where k contains all the primitive $|G|^{\text{th}}$ roots of unity. We denote v_p as a valuation on the field k such that $v_p(p) = 1$. Next, there is the residual field $F = \mathcal{R}/J(\mathcal{R})$ of characteristic p, where $J(\mathcal{R})$ is the Jacobson radical of ring $\mathcal{R}$. We can use the field k as a splitting field and F as an algebraically closed field. Let $Irr(G)$ be the set of all ordinary irreducible characters of G, which corresponds to the set of all simple kG-modules. Let $\mathcal{M}$ be a simple kG-module, affording the irreducible character ψ of G. Then, there exists an $\mathcal{R}G$-lattice L such that $k \otimes_{\mathcal{R}} L = \mathcal{M}$, but L is not uniquely determined up to isomorphism (see [1,4]). In this case, L is said to be a full $\mathcal{R}G$-lattice in $\mathcal{M}$, and, according to ([1], Chapter 2, Exercise 16.7), L is an indecomposable $\mathcal{R}G$-lattice. Recall that an $\mathcal{R}G$-lattice L is a left $\mathcal{R}G$-module that has a finite $\mathcal{R}$-basis. Let $K(B)$ be the number of ordinary irreducible characters of B and $IBr(G)$ be the set of all irreducible Brauer characters of G. We use ψ^0 to denote the restriction of the ordinary irreducible character ψ to the set of all p-regular elements (p does not divide the order of the elements) of G. Let $L(B)$ be the number of irreducible Brauer characters of B. We define $BL(G)$ as the set of all p-blocks of G. We use $=_G$ to refer to equivalence up to G-conjugacy.

Consider the order of the finite group G to be $|G| = p^{\alpha}m$ such that $g.c.d.(p, m) = 1$, $\alpha, m \in \mathbb{Z}^+$ for a fixed prime number p. Let $\psi \in Irr(G)$. As is well-known, the degree of ψ divides the order of G, as demonstrated in ([5], Theorem 2.4) and ([6], Theorem 3.11). If $p^n = \frac{|G|_p}{\psi(1)_p}$, where x_p denotes the p-part of a natural number x, then n is the highest power of p such that p^n divides $\frac{|G|}{\psi(1)}$. The non-negative number n is called the p-defect of ψ. We can also define the p-defect of ψ as $def(\psi) := v_p(\frac{|G|}{\psi(1)})$. Let $Irr(B)$ be the set of all ordinary irreducible characters of G that belong to a p-block B of G. The defect number of B refers to the maximum p-defect of irreducible characters belonging to the p-block B, and we write

$$def(B) := Max\{def(\psi); \psi \in Irr(B)\}.$$

The height of ψ can be written as $h(\psi) = def(B) - def(\psi)$. If $def(\psi) = \alpha$, we can say that ψ is of height zero or the full defect, and we write $K_0(B) = \{\psi \in Irr(B) | h(\psi) = 0\}$. On the other hand, if $def(\psi) = 0$, then we say that ψ is of defect zero, and we have $\psi(1)_p = |G|_p$ (see [3,5,6]). The work in this paper relies on these numerical invariants of the p-block B of the finite group G. Many questions and conjectures exist in this area of research. We are concerned with Brauer's height zero conjecture (BHZC), Robinson's conjecture (RC), and Olsson's conjecture (OC) (see Sections 1.1–1.3 below).

Consider $\mathcal{R}G$ to be an interior G-algebra over $\mathcal{R}$. Let e_B be a p-block idempotent of $\mathcal{R}G$; that is, $e_B^2 = e_B$, and e_B is in the center of $\mathcal{R}G$. Then, there exists a p-subgroup D of G in which D is a minimal p-subgroup of G, such that $e_B \in tr_D^G((\mathcal{R}G)^D)$. Here, tr_D^G is the relative trace map, and $(\mathcal{R}G)^D$ is the set of D-fixed elements of $\mathcal{R}G$ (see ([4], Chapter 2, Section 11)). A defect group of a p-block B is of order $p^{def(B)}$. We refer the reader to ([7], Definition 4j), ([8], p. 71), ([5], Chapter 5, Theorem 1.2), and ([9], Chapter 7, Definition (57.10)) for further theory on defect groups.

The remainder of this paper is organized as follows. This section contains five subsections: a literature review of BHZC, a literature review of RC, a literature review of OC, the anchor group of irreducible characters, and a description of our methods for solving and dealing with these problems. Section 2 provides preliminaries of classical and standard theories regarding the direct product of finite groups. We offer some of the characteristics of ordinary irreducible characters. In Section 3, we present the main results; in particular, we prove that RC holds for the direct product $H_1 \times H_2$ of two finite groups H_1 and H_2 if and only if it holds for each of them. We prove that the same conclusion holds for Brauer's height zero and Olsson's conjectures. In Section 3, we give the conjectures MARC, MHZC, and MAOC related to the algebraic concept of "the anchor group of an irreducible character". These conjectures are the relative versions of RC, BHZC, and OC, respectively. We prove the relative version of Robinson's conjecture MARC in some cases. Let G be a finite simple group that contains the irreducible character ψ of degree p, where p is an odd prime. We prove that the anchor group of ψ is the trivial group. We also introduce the relative version of the Green correspondence theorem for this group and give suitable examples of this type of theory. Finally, we include a discussion and conclusions that support our results and arguments.

1.1. Literature Review of Brauer's Height Zero Conjecture

In 1955, R. Brauer [10] conjectured that "the defect groups of a p-block B are abelian if and only if all irreducible characters in B have height zero." This conjecture is called Brauer's height zero conjecture (BHZC) and is considered to be one of the most challenging and fundamental conjectures in the representation theory of finite groups, having a significant impact on group theory research. Over the past few decades, several authors have contributed to proving the "only if" implication of BHZC. First, in 1961, P. Fong [11] proved the "only if" implication of BHZC for principle blocks. He also proved the "if" implication of BHZC for the p-solvable group. Later, in [12], he proved the "only if" implication of BHZC for the solvable groups, where the prime number is the largest divisor of the group order. Then, the proof of BHZC was completed for solvable groups in [13,14]. In 1984, D. Gluck and T. R. Wolf [15] proved the "only if" implication of BHZC for the p-solvable group. More recently, in 2012, G. Navarro and P. H. Tiep [16] proved the "only if" implication of BHZC for a 2-block B with a Sylow 2-subgroup as a defect group of B. In 2013, R. Kessar and G. Malle [17] proved the "if" implication of BHZC for all finite groups after decades of other contributions on the subject. The next year, B. Sambale [18] investigated BHZC in the case of p-blocks of finite groups with metacyclic defect groups. He proved that BHZC holds for all 2-blocks with defect groups of order 16 at most. Very recently, in 2021, G. Malle and G. Navarro [19] proved the "only if" implication of BHZC for the principle p-block for all prime numbers. After that, the proof of BHZC was completed by proving the "only if" implication of BHZC for any odd prime (see [20]).

1.2. Literature Review of Robinson's Conjecture

In 1996, G. Robinson [21] submitted a proposal for the expansion of BHZC, comparing the order of the center of a defect group of a p-block and the p-part of characters' degrees that belong to the p-block of a finite group G:

Robinson's conjecture. Suppose G is a finite group. Let $\chi \in Irr(G)$, which belongs to a p-block B of G with a defect group D. Then, $p^{def(\chi)} \geq |Z(D)|$. Moreover, the equality holds if and only if D is abelian.

The other form of RC comes from the relation between the p-defect of the irreducible character χ and the height of χ:

$$\begin{aligned} p^{h(\chi)} &= p^{def(B)-def(\chi)}, && (1)\\ &= \frac{p^{def(B)}}{p^{def(\chi)}}, && (2)\\ &\leq \frac{|D|}{|Z(D)|} = [D:Z(D)]. && (3) \end{aligned}$$

The equality in RC holds if and only if D is abelian. If D is abelian, then $D = Z(D)$ according to ([22], Section 2.2, Example (1)), which implies that all irreducible characters in B have height zero from (3). Then, we obtain the "if" implication of BHZC; hence, RC is an expansion for this implication of BHZC. In 1998, M. Murai [23] introduced a reduction of RC to p-blocks of the covering groups for all primes $p \geq 3$. In 2014, B. Sambale [18] investigated RC in the case of p-blocks of finite groups with metacyclic defect groups. He proved that RC holds for all 2-blocks with a defect group of order 16 at most. Recently, in 2018, Z. Feng, C. Li, Y. Liu, G. Malle, and J. Zhang [24] proved that RC holds for all primes $p \geq 3$ for all finite groups using Murai's reduction of RC. Later, they proved [25] that RC holds using Murai's reduction in the case $p = 2$ of finite quasi-simple classical groups. Thus, to complete the proof of RC, it only remains to investigate the so-called isolated 2-blocks of the covering groups of exceptional Lie type in the case of an odd characteristic.

1.3. Literature Review of Olsson's Conjecture

In [26], J. B. Olsson conjectured that "$K_0(B) \leq [D_B : \acute{D_B}]$", where D_B is the defect group of the p-block B of G and $\acute{D_B}$ denotes the commutator subgroup of D, called Olsson's conjecture (OC). The definition of the commutator subgroup can be found in [22,27,28]. This conjecture has been proven under certain conditions, but it remains open in general. For instance, in [29], B. Külshammer showed that OC for the p-block B can be derived from the Alperin–Mckay conjecture for B. The same result appeared in [30,31]. We remind the reader that the Alperin–Mckay conjecture states that $K_0(b) = K_0(B)$, where b is the Brauer correspondent of the p-block B in $\mathcal{R}N_G(D_B)$. The meaning of the Brauer correspondent of the p-block can be found in [2,5,32,33]. However, OC is satisfied for p-solvable, alternating, or symmetric groups in [34–36]. If D_B is the abelian group, then the commutator $\acute{D_B}$ is the trivial subgroup $\{1_D\}$. Thus, OC leads to Brauer's $K(B)$ conjecture. Recall that Brauer's $K(B)$ conjecture predicts that $K(B) \leq |D_B|$; see [37]. In particular, OC holds if D_B is metacyclic (see [38,39]) or if D_B is minimal non-abelian and $p = 2$ (see [40]). In [41,42], S. Hendren proved OC for some p-block with a defect group that is an extraspecial p-group of order p^3 and exponents p and p^2. Recently, the authors of [43] proved that OC is fulfilled for controlled blocks with certain defect groups. Furthermore, in the same paper [43], they used the classification of a finite simple group to verify OC for defect groups of p-rank 2 and cases where $p > 3$ for a minimal non-abelian defect group.

The following example appeared in [37]:

Example 1. *Let $G = S_4$ be the symmetric group of degree four. The number of irreducible characters $|Irr(S_4)| = 5$.*
***For the case** $p = 2$*
We have that the Klein four V_4 is a normal 2-subgroup of S_4 and the centralizer $C_{S_4}(V_4) = V_4$. From ([32], Chapter V, Corollary 3.11), there is only one 2-block B_0 of S_4 with $def(B_0) = 3$. For the defect group of B_0, $D(B_0) \cong D_8$, the dihedral group of order 8 is a non-abelian 2-group. Note that there exists $\chi_3 \in Irr(B_0)$ with non-zero height. The center $Z(D_8) \cong C_2$, which is the cyclic group of order 2. We have

$$p^{def(\chi)} > |Z(D)| = 2, \; for\ all\ \chi \in Irr(B_0).$$

The commutator of $D(B_0)$ is isomorphic to C_2. We have $K_0(B_0) = 4 = [D_8 : C_2]$.

***For the case** $p = 3$*

We have the principal 3-block $B_0 = \{\chi_1, \chi_2, \chi_3\}$, with $def(B_0) = 1$. For the defect group of B_0, $D(B_0) \cong C_3$, which is the cyclic group of order 3. Note that all $\chi \in Irr(B_0)$ are of height zero and satisfy $p^{def(\chi)} = |Z(D)| = 3$. As $D(B_0)$ is an abelian group, the commutator $\acute{D}(B_0) = \{1_{D(B_0)}\}$ and $K_0(B_0) = 3 = |C_3|$.

1.4. Anchor Group of Irreducible Characters

Let $\psi \in Irr(G)$. Then, ψ may be extended to an algebra map in a unique way with $\psi : kG \to k$. We consider the element

$$e_\psi = \frac{\psi(1)}{|G|} \sum_{x \in G} \psi(x^{-1})x;$$

which is the unique central primitive idempotent in kG such that $\psi(e_\psi) \neq 0$ (see ([44], Theorem 3.3.1)). As the center $Z(\mathcal{R}Ge_\psi)$ is a subring of the center $Z(kGe_\psi)$, the algebra $\mathcal{R}Ge_\psi$ is a primitive G-interior $\mathcal{R}$-algebra (see [4]).

The anchor group of an irreducible character appeared for the first time in [45], defined as the defect group of the primitive G-interior $\mathcal{R}$-algebra $\mathcal{R}Ge_\psi$ for any irreducible character ψ of G. As the anchor group of an irreducible character is a defect group, it is a p-subgroup of G (see [46]).

Let us present the most important characteristics of the anchor group of irreducible characters that we use in this paper. The following theorem appears in ([45], Theorems 1.2 and 1.3).

Theorem 1. *Consider B to be a p-block of a finite group G with a defect group D. Let $\psi \in Irr(B)$ with anchor group A_ψ. Suppose L is an $\mathcal{R}G$-lattice affording ψ. The following holds:*

1. *The anchor group of ψ is a subgroup of the defect group D (up to G-conjugacy) of B.*
2. *The anchor group of ψ contains a vertex of L.*
3. *If the defect group D is abelian, then D is an anchor group of ψ.*
4. *If ψ has a full defect (height zero), then A_ψ is the defect group of B.*
5. *If $\psi^0 \in IBr(G)$, then L is unique up to isomorphism and A_ψ is a vertex of L.*

Theorem 2 ([47])**.** *Let G be a finite group and B be a p-block of G with a defect group D_B. Suppose $\psi \in Irr(B)$ such that $\psi^0 \in IBr(B)$. Then, the anchor group A_ψ of ψ is cyclic if and only if the defect group D_B is cyclic. In particular, if A_ψ is cyclic, then it is the defect group of B.*

Lemma 1 ([46])**.** *Let G be a finite group. If $\psi \in Irr(G)$ with a degree prime to p, then the anchor group of ψ is a Sylow p-subgroup of G.*

1.5. Methodology

Our main methods are based on classical and standard theories on the direct product of finite groups [22,27], block theory [5,32,48], and character theory [6,49]. In addition, the Green correspondence theorem is key for studying block theory and calculating the anchor groups of irreducible characters. In fact, given a p-subgroup P of a finite group G, let $N_G(P)$ be the normalizer of P in G, $Ind(\mathcal{R}G|P)$ be the set of all isomorphism classes of the indecomposable $\mathcal{R}G$-lattices with vertex P, and $Ind(\mathcal{R}N_G(P)|P)$ be the set of all isomorphism classes of the indecomposable $\mathcal{R}N_G(P)$-lattices with vertex P. The following is the Green correspondence theorem, which appears in [1–5,37,50].

Theorem 3. *Consider the hypotheses in the above paragraph. There is a bijection between $Ind(\mathcal{R}G|P)$ and $Ind(\mathcal{R}N_G(P)|P)$. We say that the lattice $L \in Ind(\mathcal{R}G|P)$ corresponds to the lattice $\acute{L} \in Ind(\mathcal{R}N_G(P)|P)$ if and only if $\acute{L}$ is the unique (up to isomorphism) direct summand of the restriction $Res^G_{N_G(P)}(L)$ with vertex P or L is the unique (up to isomorphism) direct summand of the induction $Ind^G_{N_G(P)}(\acute{L})$ with vertex P.*

We recall that the vertex of an indecomposable $\mathcal{R}G$-lattice L is a unique (up to G-conjugacy) minimal p-subgroup P of G, such that L is P-projective of G. Consequently, L is a direct summand of the induced $Ind^G_P(N)$ for some $\mathcal{R}P$-lattice N.

2. Preliminaries

In this section, we present the classical and standard theories regarding the direct product of finite groups. We detail some characteristics of the ordinary irreducible characters used throughout the paper.

The following propositions are crucial for the representation of direct products of finite groups.

Proposition 1. *Let G be a direct product of the finite groups H_1 and H_2. Let B be a p-block of G with defect group D_B. If b_i is a p-block of H_i with defect group D_{b_i}, $i = 1,2$, then the following holds:*

(a) *$b_1 \otimes b_2$ is a p-block of G and $BL(G)$ is of the form $\{b_i \otimes b_j | b_i \in BL(H_1), b_j \in BL(H_2)\}$.*

(b) *$K(B) = K(b_1)K(b_2)$ and $L(B) = L(b_1)L(b_2)$.*

(c) *$D_B =_G D_{b_1} \times D_{b_2}$.*

Proof. See ([48], Propositions 2.3, 2.4, and 2.6). □

We offer the classical and standard theories of the direct product of finite groups in the following result.

Proposition 2. *Let G be a direct product of the finite groups H_1 and H_2. Then, the following holds:*

(a) *G is abelian if and only if each of H_1 and H_2 are abelian.*

(b) *The center $Z(G) = Z(H_1) \times Z(H_2)$.*

(c) *The commutator $\acute{G} = \acute{H}_1 \times \acute{H}_2$.*

Proof. For (a), see ([27], Chapter 9, Exercise 7). For (b), see ([22], Section 5.1, Exercise 1). For (c), see ([28], Chapter 3, Exercise 165). □

Theorem 4. *Let $G = H_1 \times H_2$ be a direct product of the finite groups H_1 and H_2. Then,*

$$Irr(G) = \{\psi \otimes \phi | \psi \in Irr(H_1), \phi \in Irr(H_2)\}.$$

Proof. We write $\psi \otimes \phi := \psi.\phi$. See ([6], Chapter 4, Theorem 4.21). □

Now, we mention some properties of the ordinary irreducible characters (see ([6], Chapter 2)). The ordinary irreducible character is a homomorphism if it is only linear (i.e., of degree one). Furthermore, the ordinary irreducible character has a kernel. It also has a center, although it is not a group.

Definition 1. *Consider G to be a finite group and $\psi \in Irr(G)$.*

- *The kernel of ψ is defined as $ker(\psi) := \{x \in G : \psi(x) = \psi(1)\}$. It can easily be proven that $ker(\psi)$ is a normal subgroup of G. If $ker(\psi) = \{1_G\}$, then we say that ψ is a faithful character.*
- *The center of ψ is a subgroup of G, defined as $Z(\psi) := \{x \in G : |\psi(x)| = \psi(1)\}$.*

Lemma 2. *The group G is abelian if and only if every irreducible character of G is of degree one.*

Lemma 3. *Consider G to be a finite group and ψ be a character of G with $\psi = \sum n_j\psi_j$ for $\psi_j \in Irr(G)$. Then, $ker(\psi) = \bigcap\{ker(\psi_j)|n_j > 0\}$.*

Lemma 4. *Let G be a finite group with a commutator subgroup $\acute{G}$. Then,*

$$\acute{G} = \bigcap\{ker(\gamma)|\gamma \in Irr(G), \gamma(1) = 1\}.$$

Lemma 5. *Let G be a finite group. Then, $Z(G) = \bigcap\{Z(\psi)|\psi \in Irr(G)\}$.*

Theorem 5. *Let G be a finite group with an abelian Sylow p-subgroup. Suppose G has a faithful irreducible character ψ of degree $\psi(1) = p^a$. Then, $\psi(1)$ is the exact power of p which divides $[G : Z(G)]$.*

Proof. See ([6], Theorem 3.13). □

3. Some Conjectures on Direct Products

In this section, we deal with BHCZ, RC, and OC. We prove that the direct product $H_1 \times H_2$ of the finite groups H_1 and H_2 satisfies these conjectures if and only if H_1 and H_2 satisfy these conjectures.

Proposition 3. *Let G be a direct product of the finite groups H_1 and H_2. Then, G satisfies RC if and only if H_1 and H_2 satisfy RC.*

Proof. Suppose H_i, $i = 1, 2$, are finite groups that satisfy RC. If $\chi_i \in Irr(H_i)$, which belongs to a p-block b_i of H_i with a defect group D_i for $i = 1, 2$, then $p^{def(\chi_i)} \geq | Z(D_i) |$. Moreover, the equality holds if and only if D_i is abelian for $i = 1, 2$. We need to show that $p^{def(\chi_1 \otimes \chi_2)} \geq | Z(D_1 \times D_2) |$, where equality holds if and only if $D_1 \times D_2$ is abelian. From Proposition 1(a), (c), $b_1 \otimes b_2$ is the p-block of the direct product $H_1 \times H_2$ and has a defect group that is equal up to G-conjugacy to $D_1 \times D_2$. Per Proposition 2(b), the center of a direct product of groups is the direct product of their centers. Now, from the definition of the defect number of irreducible characters and Theorem 4, we have

$$\begin{aligned} def(\chi_1 \otimes \chi_2) &= v_p\left(\frac{|H_1 \times H_2|}{\chi_1 \otimes \chi_2(1)}\right), \\ &= v_p\left(\frac{|H_1| \cdot |H_2|}{\chi_1(1) \cdot \chi_2(1)}\right), \\ &= v_p(|H_1| \cdot |H_2|) - v_p(\chi_1(1) \cdot \chi_2(1)), \\ &= v_p(|H_1|) + v_p(|H_2|) - v_p(\chi_1(1)) - v_p(\chi_2(1)), \\ &= v_p\left(\frac{|H_1|}{\chi_1(1)}\right) + v_p\left(\frac{|H_2|}{\chi_2(1)}\right). \end{aligned}$$

Hence,

$$def(\chi_1 \otimes \chi_2) = def(\chi_1) + def(\chi_2). \tag{4}$$

Therefore,

$$p^{def(\chi_1 \otimes \chi_2)} = p^{def(\chi_1)}.p^{def(\chi_2)} \geq |Z(D_1)| \cdot |Z(D_2)| = |Z(D_1 \times D_2)|.$$

As Proposition 2(a) states, the direct product of finite groups is abelian if and only if each of them is abelian; thus, the equality holds. The other direction is easily achieved through the same steps and citations. □

Remark 1. *Let B be a p-block of the finite group $H_1 \times H_2$ with a defect group D. Then, from Proposition 1(a), (c), there exists a p-block b_i of H_i with a defect group D_i for $i = 1, 2$ such that $B = b_1 \otimes b_2$ is the p-block of $H_1 \times H_2$ with defect group $D =_G D_1 \times D_2$. We have*

$$p^{def(B)} = p^{def(b_1 \otimes b_2)} = |D_1 \times D_2| = |D_1| \cdot |D_2| = p^{def(b_1)} \cdot p^{def(b_2)}.$$

Hence,

$$def(b_1 \otimes b_2) = def(b_1) + def(b_2). \tag{5}$$

Now, from Equations (4) and (5), the height of the irreducible character $\chi_1 \otimes \chi_2$ can be calculated as follows:

$$h(\chi_1 \otimes \chi_2) = h(\chi_1) + h(\chi_2). \tag{6}$$

Proposition 4. *Let G be a direct product of the finite groups H_1 and H_2. Then, G satisfies BHZC if and only if H_1 and H_2 satisfy BHZC.*

Proof. Suppose H_i, $i = 1, 2$, are finite groups that satisfy BHZC. Let D_i be a defect group of a p-block b_i of H_i for $i = 1, 2$. Suppose the defect group D of a p-block $b_1 \otimes b_2$ of a finite group $H_1 \times H_2$ is abelian. Then, $D =_G D_1 \times D_2$ and, per Proposition 2(a), the direct product of groups is abelian if and only if each of them is abelian. Thus, the defect group D_i of a p-block b_i of H_i is abelian for $i = 1, 2$. As H_1 and H_2 satisfy BHZC, for all $\chi_i \in Irr(b_i)$, we have $h(\chi_i) = 0$, $i = 1, 2$. Then, from Equation (6), we obtain the height of all irreducible characters in the p-block $b_1 \otimes b_2$ as zero. For the converse implication, suppose all irreducible characters $\chi_1 \otimes \chi_2$ in the p-block $b_1 \otimes b_2$ of $H_1 \times H_2$ have height zero. From Equation (6) and the fact that the height of an irreducible character is a non-negative integer by definition, we find that all irreducible characters in a p-block b_i of H_i for $i = 1, 2$ have height zero. Hence, per BHZC, the defect group D_i of a p-block b_i of H_i is abelian for $i = 1, 2$. Now, also per Proposition 2(a), the defect group $D_1 \times D_2$ of the p-block $b_1 \otimes b_2$ of $H_1 \times H_2$ is abelian. The same steps and citations can also be used to obtain the result in the other direction. □

Proposition 5. *Let G be a direct product of the finite groups H_1 and H_2. Then, G satisfies OC if and only if H_1 and H_2 satisfy OC.*

Proof. Suppose $B \in BL(G)$ with defect group D_B. From Proposition 1(a), (c), there exists a p-block b_i of H_i with a defect group D_{b_i}, $i = 1, 2$, such that $B = b_1 \otimes b_2$ is the p-block of $H_1 \times H_2$ with defect group $D_B =_G D_{b_1} \times D_{b_2}$. First, we need to show that $K_0(B) = K_0(b_1)K_0(b_2)$. Let $\chi \in Irr(G)$, which belongs to the p-block B of G. From Theorem 4, $\chi = \psi \otimes \phi$, where $\psi \in Irr(H_1)$ and $\phi \in Irr(H_2)$. As $B = b_1 \otimes b_2$, $\psi \in Irr(b_1)$ and $\phi \in Irr(b_2)$. Suppose χ has height zero. From Equation (6) and the fact that the height of an irreducible character is a non-negative integer by definition, the irreducible characters ψ and ϕ have height zero. From Proposition 1(b), we can infer that

$$K_0(B) = K_0(b_1)K_0(b_2).$$

Now, suppose G satisfies OC. Then, $K_0(B) \leq [D_B : \acute{D}_B]$. Hence, per Propositions 1(c) and 2(c), this is equivalent to

$$\begin{aligned} \Leftrightarrow K_0(B) &\leq [D_B : \acute{D}_B], \\ \Leftrightarrow K_0(b_1)K_0(b_2) &\leq [D_{b_1} \times D_{b_2} : \acute{D}_{b_1} \times \acute{D}_{b_2}], \\ \Leftrightarrow K_0(b_1)K_0(b_2) &\leq [D_{b_1} : \acute{D}_{b_1}][D_{b_2} : \acute{D}_{b_2}]. \end{aligned}$$

Hence, $K_0(b_i) \leq [D_{b_i} : \acute{D}_{b_i}]$ for $i = 1, 2$. Thus, H_1 and H_2 satisfy OC. The other direction is proven similarly. □

4. Relative Versions of Conjectures and the Green Correspondence Theorem

In this section, we give the conjectures MARC, MHZC, and MAOC, which are related to the algebraic concept of "the anchor group of an irreducible character," which are the relative versions of RC, BHZC, and OC, respectively. By restricting these conjectures to the anchor group instead of the defect group, we prove MARC in some cases. We introduce the relative version of the Green correspondence theorem for a finite simple group G that contains the irreducible character of G with degree p, where p is an odd prime. We give suitable examples of this type of theory.

First, we give the relative version of RC.

MARC: Suppose G is a finite group. Let $\chi \in Irr(G)$ with anchor group A_χ. Then, $p^{def(\chi)} \geq |Z(A_\chi)|$, and equality holds if and only if A_χ is abelian.

In the following results, we verify MARC in special cases.

Proposition 6. *Consider G to be a finite group. Let $\chi \in Irr(G)$ with anchor group A_χ such that the order $|Z(A_\chi)| = p$. Then, MARC holds for χ.*

Proof. Suppose $\chi \in Irr(G)$, which belongs to the p-block B of G with defect group D. If the defect group D is abelian or the irreducible character χ is of height zero, then the anchor group of χ is D, per Theorem 1(4), (5). Thus, the result holds by ([24], Lemma 3.1). If χ has defect zero, then it is lying in a p-block $B = \{\chi\}$ with abelian defect group $D = \{1_G\}$ per ([5], Theorem 6.29) (see also ([3], Theorem 2.3.2)). Thus,

$$p^{def(\chi)} = p^0 = |Z(A_\chi)| = 1.$$

If χ has defect n, $n \geq 1$. Thus,

$$p^{def(\chi)} = p^n \geq |Z(A_\chi)| = p.$$

□

Assume $\overline{G} = G/Q$ and Q is a normal subgroup of G. Let $\overline{\psi} \in Irr(\overline{G})$; we say that the character ψ is the lift of $\overline{\psi}$ to G if it satisfies $\psi(g) = \overline{\psi}(gQ)$, where $g \in G$. From ([49], Theorem 17.3), $\overline{\psi} \in Irr(\overline{G})$ if and only if $\psi \in Irr(G)$ and $\ker(\psi)$ contains Q. So, we have $Irr(\overline{G}) \subseteq Irr(G)$. From ([33], p. 137), there exists a unique p-block B of G that contains the p-block $\overline{B}$ of $\overline{G}$, and we write $Irr(B) \supseteq Irr(\overline{B})$.

Proposition 7. *Using the same hypotheses as above, let Q be a normal $\acute{p}$-subgroup of G and $\overline{\psi} \in Irr(\overline{G})$. Suppose $\psi \in Irr(G)$ is the lift of $\overline{\psi}$ to G. Let $\psi^0 \in IBr(G)$ with a cyclic anchor group. If ψ satisfies MARC, then so does $\overline{\psi}$.*

Proof. Suppose B is a p-block of G that contains ψ and $\overline{B}$ is a p-block of $\overline{G}$ that contains $\overline{\psi}$. From the details above, $Irr(\overline{B}) \subseteq Irr(B)$. From ([33], Theorem 9.9(c)), the defect groups of $\overline{B}$ and B are isomorphic. Since the anchor group of ψ is cyclic, it is the defect group of B per Theorem 2. Hence, the anchor groups of ψ and $\overline{\psi}$ are isomorphic. □

If we restrict BHZC to the anchor group instead of the defect group, then the statement is not true. In particular, the "if" implication is not true.

Example 2. *Let $p = 2$, $G = S_4$ be the symmetric group of degree four. From Example 1, there is only one 2-block B_0 of S_4. From ([45], Example 5.8. (2)), there exists $\chi \in Irr(S_4)$ of degree two with anchor group V_4, which is an abelian group, but the height of χ is not zero.*

The relative version of BHZC is as follows:

MHZC: If every irreducible character in a p-block has height zero, then their anchor group is abelian.

Furthermore, we can reduce OC to the anchor group of the irreducible character (MAOC) as follows:

MAOC: Let $\chi \in Irr(G)$ with an anchor group A_χ. Suppose χ belongs to the p-block B of G. Then,

$$K_0(B) \leq [A_\chi : \acute{A}_\chi],$$

where $\acute{A}_\chi$ is the commutator subgroup of A_χ.

Remark 2. *Let D be an abelian defect group of the p-block B. We know that OC leads to Brauer's $K(B)$ conjecture, which states that $K(B) \leq |D_B|$. However, this statement is not true in the case of the anchor group of irreducible characters; that is, for any $\chi \in Irr(B)$, $K(B) \leq |A_\chi|$ is not true in general. From Examples 1 and 2, there is only one 2-block B_0 of S_4 that contains the irreducible character χ of degree two with anchor group V_4. We have $K(B_0) = 5 > |A_\chi|$.*

We focus on a simple finite group that contains the irreducible character with degree p, where p is an odd prime.

Theorem 6. *Let G be a simple finite group. Let $\psi \in Irr(G)$ with degree $\psi(1) = p$, where p is an odd prime number. Then, the anchor group of ψ is the trivial group.*

Proof. We have the degree $\psi(1) = p$, which divides the order of G, per ([5], Theorem 2.4) and ([6], Theorem 3.11). Thus, G has a non-trivial Sylow p-subgroup P of G. As G is a simple group, either $\ker(\psi) = G$ or $\ker(\psi) = 1_G$. If $\ker(\psi) = G$, then ψ is the trivial character of G, which is not the case. Thus, ψ is a faithful irreducible character of G. Furthermore, from Lemma 2, the group G is non-abelian. If P is non-abelian, then the commutator $\acute{P} \neq \{1_P\}$ and the center $Z(P) \neq \{1_G\}$. Consider $Res^G_P(\psi) = \sum_{\chi_i \in Irr(P)} d_i\chi_i$ for a positive integer d_i. Since $\psi(1) = p = Res^G_P(\psi)(1)$, then $1 \leq \chi_i(1) \leq \psi(1)$. As $\chi_i(1)$ divides the order of P, the degree of χ_i; $\chi_i(1)$ is a power of p. We conclude that either $Res^G_P(\psi)$ is the sum of the linear characters of P or $Res^G_P(\psi)$ is the irreducible character of P. Let $Res^G_P(\psi) = d_{i_1}\chi_{i_1} + d_{i_2}\chi_{i_2} + \ldots + d_{i_t}\chi_{i_t}$, where $d_{i_K} > 0$ and $\chi_{i_K}(1) = 1$. As is well-known, $\ker(Res^G_P(\psi)) \subseteq \ker(\psi)$. Hence, per Lemma 3, $\ker(Res^G_P(\psi)) = \bigcap_{1 \leq j \leq t} \ker\chi_{i_j}$. Therefore, via Lemma 4,

$$\{1_P\} \neq \acute{P} \subseteq \bigcap_{1 \leq j \leq t} \ker(\chi_{i_j}) \subseteq \ker\psi.$$

This contradicts the fact that ψ is faithful. Thus, $Res^G_P(\psi)$ is an irreducible character of P. From Lemma 5, we have

$$\{1_G\} \neq Z(P) = \bigcap_{\chi \in Irr(P)} Z(\chi) \subseteq Z(Res^G_P(\psi)) \subseteq Z(\psi).$$

Hence, $Z(\psi) \neq \{1_G\}$. Since G is simple, $Z(\psi) = G$ and G is abelian. This leads us to another contradiction. Thus, P is abelian, G is a non-abelian simple group, and $Z(G) = \{1_G\}$. Hence, from Theorem 5, p is the exact power of p which divides $[G : Z(G)] = |G|$. We can infer that a Sylow p-subgroup of G is cyclic of order p. Now, the defect of ψ is defined as $p^{def(\psi)} = \frac{|G|_p}{\psi(1)_p} = 1$ and $def(\psi) = 0$. Hence, per ([3], Theorem 2.3.2), ψ belongs to the singleton p-block, and the defect group of the singleton p-block is the trivial group $\{1_G\}$. Then, the result is obtained from Theorem 1 (1). □

Remark 3. *In Theorem 6, we exclude $p = 2$, as no simple group exists with an irreducible character of degree 2, as in ([49], Corollary 22.13).*

The following corollary immediately follows from Theorem 6.

Corollary 1. *Let G be a simple finite group that has an irreducible character of degree p, where p is an odd prime. If $\chi \in Irr(G)$ with $\chi(1)_p = p$, then the anchor group of χ is the trivial group.*

We introduce the relative version of the Green correspondence theorem (Theorem 3) in a simple finite group G, which contains the irreducible character ψ of degree p, where p is an odd prime. Let B be a p-block of G. We define $Ind(B|A)$ to be the set of all isomorphism classes of the indecomposable $\mathcal{R}G$-lattices with vertex A, which belong to B. We write

$$Irr(B|A) := \{\chi \in Irr(B) | \chi^0 \in IBr(B) \text{ and } A \text{ is the anchor group of } \chi\}.$$

Lemma 6. *Per the same hypotheses as above, let $\chi \in Irr(G)$ with the non-trivial anchor group A and $\chi^0 \in IBr(G)$. We write $N = N_G(A)$ to be the normalizer of A in G. Let $\theta \in Irr(N)$ with $\theta^0 \in IBr(N)$ such that θ lies under ψ; that is, $\langle Res_N^G(\psi), \theta\rangle \neq 0$. Then, the irreducible characters χ and θ have the same anchor group. However, if χ belongs to the p-block B of G and θ belongs to the p-block b of N, then $|Irr(B|A)| = |Irr(b|A)|$.*

Proof. Assume that L is the indecomposable $\mathcal{R}G$-lattice affording χ and $\acute{L}$ is the indecomposable $\mathcal{R}N$-lattice affording θ. Then, from Theorem 1 (5), L is unique up to isomorphism and A is a vertex of L. Per Theorem 6, G possesses a cyclic Sylow p-subgroup that contains all p-subgroups of G. Hence, the vertex of L is equal to the anchor group of an irreducible character χ, which is equal to the defect group of the p-block B (see ([47], proof of Theorem 5)). Hence, a one-to-one correspondence exists between $Irr(B|A)$ and $Ind(B|A)$. Likewise, there is a one-to-one correspondence between $Irr(b|A)$ and $Ind(b|A)$. The condition $\langle Res_N^G(\psi), \theta\rangle \neq 0$, is equivalent to $\acute{L}$ being a direct summand of the restriction $Res_N^G(L)$ with vertex A. Per the Green correspondence theorem [1], $\acute{L}$ has a vertex A. Thus, $\acute{L} \in Ind(b|A)$. Therefore, the irreducible character θ has anchor group A, and $|Irr(B|A)| = |Irr(b|A)|$. □

We extracted the Brauer character tables for the following examples from ([2], Appendix B). These tables can also be obtained for some examples (but not all) from GAP [51]. One can also extract the degree of the irreducible characters, the structure of the defect group of a p-block of G, and its normalizer in the group G from GAP [51].

Example 3. *Consider G to be a simple group $GL(3,2)$, the general linear group of order $168 = 2^3 \cdot 3 \cdot 7$. The number of irreducible characters is $|Irr(GL(3,2))| = 6$.*

<u>In the case of $p = 3$</u>

<u>We have four 3-blocks of G.</u> The principal 3-block B_0 of $GL(3,2)$ has defect 1 and contains three irreducible characters, all of degree prime to 3. Hence, the anchor group A_χ of each irreducible character χ in B_0 is a Sylow 3-subgroup of $GL(3,2)$ per Lemma 1. The Sylow 3-subgroup of $GL(3,2)$ is isomorphic to C_3, a cyclic group of order 3. The two irreducible characters of $GL(3,2)$ are of degree three, and their anchor groups are the trivial group $\{1_{GL(3,2)}\}$ per Theorem 6. The irreducible character ψ of $GL(3,2)$ with $\psi_p(1) = 3$ has the trivial anchor group per Corollary 1. The normalizer of A_χ in $GL(3,2)$ is $N_{GL(3,2)}(A_\chi) = S_3$, the symmetric group of degree three. We have that C_3 is a normal 3-subgroup of S_3 and the centralizer $C_{S_3}(C_3) = C_3$. From ([32], Chapter V, Corollary 3.11), there is only one 3-block b_0 of S_3 with $def(b_0) = 1$ that contains the irreducible character θ lying under χ. Note that $|Irr(B_0|A_\chi)| = 2 = |Irr(b_0|A_\chi)|$. The application of the relative versions of the conjectures is detailed in the following: the center of A_χ

is isomorphic to C_3, a cyclic group of order 3. Thus, for each $\chi \in Irr(GL(3,2))$, MARC holds because of Proposition 6. As all irreducible characters in the principal 3-block B_0 have height zero, the defect group of B_0 is abelian because of BHZC. Hence, their anchor groups are abelian based on Theorem 1(3). Thus, MHZC holds. As $A_\chi \cong C_3$ is an abelian group, the commutator $\acute{A}_\chi = \{1_{C_3}\}$. We have $K_0(B_0) = 3 = [C_3 : \{1_{C_3}\}]$, so MAOC holds.

***In the case of** $p = 7$*

We have two 7-blocks of G. The principal 7-block B_0 of $GL(3,2)$ has defect 1 and contains five irreducible characters, all of degree prime to 7. Hence, the anchor group A_χ of each irreducible character χ in B_0 is a Sylow 7-subgroup of $GL(3,2)$, which is isomorphic to C_7, a cyclic group of order 7. The singleton 7-block with the trivial defect group $\{1_{GL(3,2)}\}$. The normalizer of A_χ in $GL(3,2)$ is $N_{GL(3,2)}(A_\chi) \cong (C_7 : C_3)$, the non-abelian group of order 21. Let b_0 be the principal 7-block of $(C_7 : C_3)$ which contains θ lying under χ. Note that $|Irr(B_0|A_\chi)| = 3 = |Irr(b_0|A_\chi)|$. The application of the relative versions of the conjectures is detailed in the following: for each $\chi \in Irr(B_0)$, the center of A_χ is isomorphic to C_7. Then, per Proposition 6, MARC holds. We have that all irreducible characters in the principal 7-block B_0 have height zero. Hence, their anchor groups are abelian, and MHZC holds. As $A_\chi \cong C_7$ is an abelian group, the commutator $\acute{A}_\chi = \{1_{C_7}\}$. We have $K_0(B_0) = 5 < [C_7 : \{1_{C_7}\}] = 7$, so MAOC holds.

Example 4. *Consider G to be a simple group A_5, the alternating group of degree five of order $60 = 2^2 \cdot 3 \cdot 5$. The number of irreducible characters is $|Irr(A_5)| = 5$.*

***In the case of** $p = 3$*

We have three 3-blocks of A_5. The principal 3-block B_0 of A_5 has defect 1 and contains three irreducible characters, all of the degree prime to 3. Hence, the anchor group A_χ of each irreducible character χ in B_0 is a Sylow 3-subgroup of A_5, which is isomorphic to C_3, a cyclic group of order 3. As the two irreducible characters of A_5 are of degree three, their anchor groups are the trivial group $\{1_{A_5}\}$ per Theorem 6. The normalizer of A_χ in A_5 is $N_{A_5}(A_\chi) = S_3$, the symmetric group of degree three. As in the previous example, there is only one 3-block b_0 of S_3, which contains the irreducible character θ lying under χ. We have $|Irr(B_0|A_\chi)| = 2 = |Irr(b_0|A_\chi)|$. The application of the relative versions of the conjectures is as follows: the center of A_χ is isomorphic to C_3, a cyclic group of order 3. Thus, for each $\chi \in Irr(A_5)$, MARC holds. Note that all irreducible characters in the principal 3-block B_0 have height zero. Hence, their anchor groups are abelian, and MHZC holds. As $A_\chi \cong C_3$ is an abelian group, the commutator $\acute{A}_\chi = \{1_{C_3}\}$. We have $K_0(B_0) = 3 = [C_3 : \{1_{C_3}\}]$, and MAOC holds.

***In the case of** $p = 5$*

We have two 5-blocks of A_5. The principal 5-block B_0 of A_5 has defect 1 and contains four irreducible characters, all of degree prime to 5. Hence, the anchor group A_χ of each irreducible character χ in B_0 is a Sylow 5-subgroup of A_5 per Lemma 1. The Sylow 5-subgroup of A_5 is isomorphic to C_5, a cyclic group of order 5. The normalizer of A_χ in A_5 is $N_{A_5}(A_\chi) \cong D_{10}$, the dihedral group of order 10. We have that C_5 is a normal 5-subgroup of D_{10} and the centralizer $C_{D_{10}}(C_5) = C_5$. From ([32], Chapter V, Corollary 3.11), there is only one 5-block b_0 of D_{10} with $def(b_0) = 1$. Let $\theta \in Irr(b_0)$ lies under χ. Then, we have $|Irr(B_0|C_5)| = 2 = |Irr(b_0|C_5)|$. The application of the relative versions of the conjectures is as follows: the center of A_χ is isomorphic to C_5. Thus, for each $\chi \in Irr(A_5)$, MARC holds. Note that all irreducible characters in the principal 5-block B_0 have height zero. Hence, their anchor groups are abelian, and MHZC holds. As $A_\chi \cong C_5$ is an abelian group, the commutator $\acute{A}_\chi = \{1_{C_5}\}$. We have that $K_0(B_0) = 4 < [C_5 : \{1_{C_5}\}] = 5$, and MAOC holds.

Remark 4. *If the simple group G does not satisfy the condition stated in Theorem 6, then there is no cyclic Sylow p-subgroup of G, and it does not satisfy Lemma 6, as shown in the following example.*

For the following example, we used the Magma computational algebra system [52] to find the Brauer irreducible characters for the group $SL(3,3)$.

Example 5. *Let $p = 3$, $G = SL(3,3)$ be the special linear group of order 5616. The degrees of the irreducible characters of $SL(3,3)$ are*

ψ_i	ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}	ψ_{12}
$\psi_i(1)$	1	12	13	16	16	16	16	26	26	26	27	39

Note that $|Irr(SL(3,3))| = 12$, which belong in two 3-blocks. The principal 3-block B_0 has defect 3 and contains 11 irreducible characters, 9 of which are of degree prime to 3 and two of which are of degree 12 and 39, namely, ψ_2 and ψ_{12}, respectively. The defect group D of B_0 is the extraspecial 3-group ($C_3 \times C_3 : C_3$) of order 27, which is a Sylow 3-subgroup P of G. Thus, from Lemma 1, the anchor group of each irreducible character with degree prime to 3 is a Sylow 3-subgroup. It remains to calculate the anchor groups of ψ_2 and ψ_{12}. We have that $N := N_G(P) = (C_3 \times C_3 : C_3) : (C_2 \times C_2)$ is the normalizer of P in G, which is the group of order 108. We can see that $Res^G_N(\psi_2) = 2\phi_1 + \phi_6 + \phi_8 + \phi_{11}$, where $\phi_1, \phi_6, \phi_8, \phi_{11} \in Irr(N)$, as follows:

	1*a*	**3*a***	**2*a***	**6*a***	**3*b***	**3*c***	**3*d***	**2*b***	**2*c***	**6*b***	**6*c***
$\boldsymbol{Res^G_N(\psi_2)}$	**12**	**3**	**4**	**1**	**3**	**3**	**0**	**4**	**4**	**1**	**1**
ϕ_1	1	1	1	1	1	1	1	1	1	1	1
ϕ_6	2	−1	2	−1	2	2	−1	0	0	0	0
ϕ_8	2	2	0	0	2	−1	−1	0	2	0	−1
ϕ_{11}	6	0	0	0	−3	0	0	2	0	−1	0

The notation in the first row above is as provided in the Atlas of Finite Groups [53]. Let L be the indecomposable $\mathcal{R}G$-lattice affording ψ_2. Let M_1, M_6, M_8, and M_{11} be the $\mathcal{R}N$-lattices that afford ϕ_1, ϕ_6, ϕ_8, and ϕ_{11}, respectively. Hence, $Res^G_N(L) = M_1 \oplus M_6 \oplus M_8 \oplus M_{11}$. We can see that M_1 is the direct summand of $Res^G_N(L)$. Then, per the Green correspondence Theorem 3, the two lattices M_1 and L have the same vertex. We have that the reduction $\overline{M_1}$ is the trivial FG-module. Then, per ([54], Corollary 1), $\overline{M_1}$ has a Sylow 3-subgroup of N as a vertex. Thus, the Sylow 3-subgroup of N is a vertex of the indecomposable $\mathcal{R}N$-lattice M_1 per ([2], Chapter 11, Exercise 21). It follows that the Sylow 3-subgroup of N is a vertex of L. We know that the Sylow 3-subgroup of N is equal to the Sylow 3-subgroup P of G in this example. Per Theorem 1(2), the vertex of L is contained in an anchor group of ψ_2. Therefore, the anchor group of ψ_2 is a Sylow 3-subgroup P of G. To calculate the anchor group of ψ_{12}, we use the fact that $\psi_3 \in Irr(G)$ is of degree 13. Suppose $\acute{L}, \acute{\acute{L}}$ are the indecomposable $\mathcal{R}G$-lattices that afford ψ_{12}, ψ_3, respectively. Consider $\theta \in Irr(N)$ to be of degree 1, such that $Ind^G_N(\theta) = \psi_{12} + \psi_3$, as follows:

	1*a*	**3*a***	**2*a***	**6*a***	**3*b***	**3*c***	**3*d***	**2*b***	**2*c***	**6*b***	**6*c***
θ	1	1	1	1	1	1	1	−1	−1	−1	−1

	1*a*	**3*a***	**3*b***	**13*a***	**13*b***	**13*c***	**13*d***	**2*a***	**6*a***	**8*a***	**8*b***	**4*b***
$\boldsymbol{Ind^G_N(\theta)}$	**52**	**7**	**1**	**0**	**0**	**0**	**0**	**−4**	**−1**	**0**	**0**	**0**
ψ_3	13	4	1	0	0	0	0	−3	0	−1	−1	1
ψ_{12}	39	3	0	0	0	0	0	−1	−1	1	1	−1

Suppose M is the indecomposable $\mathcal{R}N$-lattice that affords θ. Hence, $Ind^G_N(M) = \acute{L} \oplus, \acute{\acute{L}}$ and the two lattices M and $\acute{L}$ correspond to each other. Per the Green correspondence theorem, they have the same vertex. As the reduction $\overline{M}$ of M has dimension prime to 3, the vertex of $\overline{M}$ is a Sylow p-subgroup of N. As shown in the case of ψ_2, we conclude that the anchor group of ψ_{12} is a Sylow p-subgroup P of G. There is only one 3-block b_0 of N. Note that $1 = |Irr(B_0|P)| \neq |Irr(b_0|P)| = 4$, which does not satisfy Lemma 6. The application of the relative versions of the conjectures is as follows: the center of the extraspecial 3-group is isomorphic to C_3, a cyclic group of order 3. Thus, for any $\chi \in Irr(SL(3,3))$, MARC holds. Note that the defect group of B_0 is non-abelian group and there exist $\psi_2, \psi_{12} \in Irr(B_0)$, which are not of height zero. Thus, MHZC *holds. The commutator subgroup of the extraspecial* 3*-group* P *is isomorphic to* C_3*. We have that* $K_0(B_0) = 9 = [(C_3 \times C_3 : C_3) : C_3]$*, and MAOC holds.*

5. Discussion

In this paper, we consider the application of RC, BHZC, and OC to a direct product of finite groups. We use classical and standard theories for the direct product of finite groups, block theory, and character theory to accomplish these results. In fact, Propositions 2.3 and 2.6 in [48] are crucial in block theory for a direct product of finite groups. We also discuss the restriction of these conjectures to anchor groups of irreducible characters instead of defect groups. As the anchor group of an irreducible character ψ of a finite group G is a defect group of the primitive G-interior $\mathcal{R}$-algebra $\mathcal{R}Ge_{\psi}$, the previous conclusion is logical. We give suitable examples of this reduction.

The review of these conjectures in Sections 1.1–1.3 can be compared to our results in a simple finite group. Our discussion revolves around the anchor groups of the irreducible character with degree p, where p is an odd prime. In [50], J. A. Green proved the Green correspondence theorem. In this work, we introduce the relative version of the Green correspondence theorem in a simple finite group G that contains the irreducible character ψ of degree p, where p is an odd prime. To achieve this result, we use Theorem 1 (5): if $\psi \in Irr(G)$ such that $\psi^0 \in IBr(G)$ with anchor group A_{ψ}, then there is a unique (up to isomorphism) $\mathcal{R}G$-lattice L affording ψ and A_{ψ} is a vertex of L. The outcomes of this paper are important for the modular representation theory of a direct product of finite groups, including an attempt to develop reductions of the RC, BHZC, and OC to the algebraic concept "anchor group of irreducible characters", as well as a relative version of the Green correspondence theorem. We plan to study more conjectures regarding the modular representation of a direct product of finite groups, including an assessment of how reductions can be formed for these conjectures in an attempt to solve them.

6. Conclusions

This work focuses on BHZC [10], RC [21], and OC [26] (see Sections 1.1–1.3). We prove that the direct product $H_1 \times H_2$ of two finite groups H_1 and H_2 satisfies these conjectures if and only if H_1 and H_2 both satisfy these conjectures. We provide relative versions of RC (MARC), BHZC (MHZC), and OC (MAOC) with respect to the algebraic concept of "the anchor group of an irreducible character." We prove the relative version of RC (MARC) in the case of the center of the anchor group of χ, A_{χ} with order $|Z(A_{\chi})| = p$ and for $\overline{\psi} \in Irr(\overline{G})$ with some conditions. Consider G to be a simple finite group. We prove that the anchor group of the irreducible character with degree p is the trivial group, where p is an odd prime. Finally, we present suitable examples of these conjectures and theories in simple finite groups. Many questions and conjectures remain in modular representation theory. We will study more conjectures related to the modular representation of a direct product of finite groups and attempt to develop reductions for these conjectures to solve them.

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