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Existence and Uniqueness Results of Fractional Differential Inclusions and Equations in Sobolev Fractional Spaces

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Abstract: In this work, by using the iterative method, we discuss the existence and uniqueness of solutions for multiterm fractional boundary value problems. Next, we examine some existence and uniqueness returns for semilinear fractional differential inclusions and equations for multiterm problems by using some notions and properties on set-valued maps and give some examples to explain our main results. We explore and use in this paper the fundamental properties of set-valued maps, which are needed for the study of differential inclusions. It began only in the mid-1900s, when mathematicians realized that their uses go far beyond a mere generalization of single-valued maps.

Keywords: semilinear fractional differential inclusion; boundary value problem; fixed-point theorem; set-valued maps; iterative methods; Riemann–Liouville operator; differential equations

MSC: 26A33; 34A08; 34A60; 34B10; 34B15

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1. Introduction

Modern mathematical theories are devoted to the search for new tools for studying various real processes. On the one hand, this is caused by the sufficient completeness and completeness of the study of known mathematical models and, on the other hand, by new tasks and new capabilities of information technologies. The theory of fractional calculus, which has been actively developing in recent decades, has made it possible to discover new properties of systems that describe complex physical processes: processes with memory, processes in fractal environments, and many more. Many works are devoted to the use of fractional calculus for various applications (see [1–3] and the references therein). Theoretical aspects of fractional integrodifferential calculus were studied in the articles [4–6]. Fractional calculus is considered one of the most important areas of mathematics, which plays an important role in applications in many fields of science such as physics, biology, engineering, and others. Using different mathematical analysis techniques, many research papers were published on integral differential equations, as well as fractional differential equations (see [7–14]). For more explanations and notions related to the definitions and various issues of fractional integrals and derivatives, please see [15–17].

The study of functional differential inclusions dates back to the works of [18], in which conditions for the existence of solutions were found for various classes of initial and boundary value problems for inclusions of retarded types of integer and fractional orders of derivatives.

Inclusions and fractional differential equations generalize inclusions and ordinary differential equations to non-integer random orders. It always appears in various fields such as physics, chemistry, biophysics, biology, engineering, control theory, and others. Recently, many works have been published on inclusions and fractional differential equations by applying the fixed-point theorem to prove some existence and singularity properties. Many articles have been published in this direction (see, for instance, [19–26]). In [27], the authors proposed a nonlinear fractional differential equation of the type

$$\begin{cases} {}^cD^{\varpi}w(t) = f(t, w(t), {}^cD^{\eta}w(t)), & \text{for a.e. } 0 \leq t \leq T, \\ \\ \alpha w(0) - \beta w'(0) = \int_0^T g(r, w)dr, \\ \gamma w(T) - \delta w'(T) = \int_0^T h(r, w)dr, \end{cases} \tag{1}$$

The existence and uniqueness results were discussed with Caputo fractional derivatives by using appropriate standard fixed-point theorems. For fractional differential inclusions, we mention the work by [28], where a boundary value problem of fractional differential inclusions with fractional separated boundary conditions is given

$$\begin{cases} {}^cD^{\varpi}w(t) \in \mathcal{F}(t, w(t)), & \text{for a.e. } 0 \leq t \leq 1,\ 1 < \varpi < 2, \\ \\ \alpha_1 w(0) + \beta_1^c D^{\kappa}w(0) = \gamma_1, \\ \alpha_2 w(1) + \beta_2^c D^{\kappa}w(1) = \gamma_2. \end{cases} \tag{2}$$

Owing to the standard contraction mapping theory, the question of existence and uniqueness are obtained. Next, it is improved by Cernea [29], where a multipoint boundary value problem for a fractional-order differential inclusion with the standard Riemann–Liouville fractional derivative

$$\begin{cases} D^{\varpi}w(t) \in \mathcal{F}(t, w(t), w'(t)), & \text{for a.e. } 0 \leq t \leq 1,\ 1 < \varpi < 3, \\ \\ w(0) = w'(0) = 0, \\ w(1) - \sum\limits_{i=1}^{m} a_i w(y_i) = \gamma. \end{cases} \tag{3}$$

was studied, and the existence of a unique solution was obtained. Motivated by the papers cited above and other related papers, in this paper, we extend all previous results and consider a multiterm fractional boundary value problem with the generalized Riemann–Liouville fractional derivative. To begin with, we consider the existence and uniqueness of a solution for the following problem:

$$\begin{cases} D^{\varpi}_{0^+}w(t) \in \mathcal{F}(t, w(t)), & \text{for a.e. } 0 \leq t \leq 1,\ 1 < \varpi < 2, \\ \\ w(0) = 0, \\ w(1) = pI^{\mu_1}_{0^+}h_1(\xi, w(\xi)) + qI^{\mu_2}_{0^+}h_2(\eta, w(\eta)). \end{cases} \tag{4}$$

Moreover, we will dispute the resolution of some results of existence to the following semilinear fractional differential equations for the boundary value problem:

$$\begin{cases} D^{\varpi}_{0^+}w(t) = f(t, w(t)),\ \ 1 < \varpi < 2,\ \ 0 \leq t \leq 1, \\ w(0) = 0, \\ \\ w(1) = pI^{\mu_1}_{0^+}h_1(\xi, w(\xi)) + qI^{\mu_2}_{0^+}h_2(\eta, w(\eta)), \end{cases} \tag{5}$$

where $1 < \varpi < 2, p, q \geq 0,\ \mu_1, \mu_2 \geq 1,\ \ 0 < \xi, \eta \leq 1,\ f, h_j : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $1 \leq j \leq 2$, and $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued function.

Conditions for the existence of solutions to Problem (4) are indicated in Theorem 1 in Section 3.1. In Section 3.2, the question of the existence of solutions for (5) is studied and stated in Theorem 3. We also introduce and give relevance ti our subject in Section 1 and then state preliminary results and definition in Section 2. This paper is finished with a discussion and conclusion where textual explanations are clear enough.

2. Notions and Preliminaries

We recall here useful tools and materials that will be used later. Let $\varpi > 0$ and $l \in L^1([0,1];\mathbb{R})$. The integral

$$I_{0^+}^{\varpi} l(t) = \frac{1}{\Gamma(\varpi)} \int_0^t (t-\zeta)^{\varpi-1} l(\zeta) d\zeta,$$

is the Riemann–Liouville integral of order ϖ.

If $n-1 \leq \varpi < n$, then the derivative of Riemann–Liouville to a function $l : [0,1] \to \mathbb{R}$ is

$$D_{0^+}^{\varpi} l(t) = \frac{1}{\Gamma(n-\varpi)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\zeta)^{n-\varpi-1} l(\zeta) d\zeta,$$

$$= \left(\frac{d}{dt}\right)^n I_{0^+}^{n-\varpi} l(t).$$

If $n = [\varpi] + 1$, $[\varpi]$ denotes the greatest integer number less than ϖ.

Lemma 1 ([15])**.** *Let $u \in L^1(0,1)$, $\delta > \varpi > 0$. We have,*

(i) $I_{0^+}^{\delta} I_{0^+}^{\varpi} w(t) = I_{0^+}^{\delta+\varpi} w(t)$,

(ii) $D_{0^+}^{\varpi} I_{0^+}^{\delta} w(t) = I_{0^+}^{\delta-\varpi} w(t)$,

(iii) $D_{0^+}^{\delta} I_{0^+}^{\delta} w(t) = w(t)$.

Lemma 2 ([15])**.** *Let $\varpi > 0$ and $\nu > 0$, then*

(i) $D_{0^+}^{\varpi} t^{\nu-1} = \begin{cases} \frac{\Gamma(\nu)}{\Gamma(\nu-\varpi)} t^{\nu-\varpi-1}, \\ 0, \;\; if \;\; \nu-\varpi \in \mathbb{Z}^- \end{cases}$

(ii) $I_{0^+}^{\varpi} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\varpi+1)} t^{\nu+\varpi}$.

Lemma 3. *Let $\lambda > 0$, $n = [\lambda] + 1$. We have*

$$D_{0^+}^{\lambda} \varphi(t) = 0,$$

equivalent to

$$\varphi(t) = \sum_{j=1}^{n} a_j t^{\lambda - j}.$$

Remark 1. *For each $l \in L^1([0,1],\mathbb{R})$ the solution of*

$$D_{0^+}^{\varpi} \phi(t) = l(t),$$

is given by

$$\phi(t) = I_{0^+}^{\varpi} l(t) + \sum_{j=1}^{n} c_j t^{\varpi - j},$$

where $n = [\varpi] + 1$.

We define some initial symbols and concepts that will be used in this research. Let $(\mathbb{X}, \|.\|)$ be a normed space. We note by

$$\mathcal{P}(\mathbb{X}) = \{\mathcal{Y} \subseteq \mathbb{X} : \mathcal{Y} \neq \emptyset\}$$
$$\mathcal{P}_b(\mathbb{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathbb{X}) : \mathcal{Y}, \text{ bounded }\}$$
$$\mathcal{P}_{cl}(\mathbb{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathbb{X}) : \mathcal{Y} \text{ closed }\}$$
$$\mathcal{P}_{cp,c}(\mathbb{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathbb{X}) : \mathcal{Y} \text{ compact, convex}\}$$
$$\mathcal{P}_{cp}(\mathbb{X}) = \{\mathcal{Y} \in \mathcal{P}(\mathbb{X}) : \mathcal{Y} \text{ compact }\}.$$

Let $A, B \in \mathcal{P}_{cl}(\mathbb{X})$. The Pompieu–Hausdorff distance of A, B is defined as

$$\mathcal{H}_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.$$

A multivalued

$$\mathcal{F} : \mathbb{X} \to \mathcal{P}(\mathbb{X}),$$

is convex (closed) valued if $\mathcal{F}(x)$ is convex (closed) for all $x \in \mathbb{X}$. $\mathcal{F}$ is called upper semicontinuous on $\mathbb{X}$ if, for every open set $\mathcal{O}$ of $\mathbb{X}$ containing $\mathcal{F}(x_0)$, there exists an open neighborhood $\mathcal{U}_0$ of x_0 such that $\mathcal{F}(\mathcal{U}_0) \subseteq \mathcal{O}$. Equivalently, $\mathcal{F}$ is upper semicontinuous if the set

$$\{x \in \mathbb{X} : F(x) \subseteq \mathcal{O}\},$$

is open for any open set $\mathcal{O}$ of $\mathbb{X}$.

A set-valued map

$$f : [0, 1] \to \mathcal{P}(\mathbb{X}),$$

is measurable if, for every $x \in \mathbb{X}$, the function

$$t \mapsto d(x, f(t)) = \inf\{d(x, y) : y \in f(t)\},$$

is a measurable function.

Let $\mathbb{X}, \mathbb{Y}$ be two normed spaces and $l : \mathbb{X} \to \mathbb{Y}$ a set-valued upper semicontinuous. Then, for all $y_0 \in \mathbb{X}, \varepsilon > 0$, there are $\delta > 0$ with

$$l(y) \subseteq l(y_0) + B(0, \varepsilon) \text{ for each } y \in B(y_0, \delta).$$

Definition 1 ([30])**.** *A set-valued map*

$$\mathcal{F} : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}),$$

is Carathéodory if

(1) $t \mapsto \mathcal{F}(t, x)$ *is measurable for each* $x \in \mathbb{R}$.
(2) $x \to \mathcal{F}(t, x)$ *is upper semicontinuous for a.e.* $0 \leq t \leq 1$.

Let X, Y be two normed spaces and $L : X \to Y$ a set-valued map. The set-valued L is Lipschitzean if there are $r > 0$ with

$$L(w) \subseteq L(v) + B(0, r\|w - v\|) \text{ for each } w, v \in X.$$

If the constant $r < 1$, we say that the set-valued L is contraction.

Proposition 1 ([30])**.** *Let* $W \subset L^1([0, 1], \mathbb{R})$ *such that*

(i) $W(t)$ *are relatively compact for a.e* $0 \leq t \leq 1$.
(ii) *There exists* $l \in L^1([0, 1], \mathbb{R})$ *with* $v(t) \leq l(t) \quad \forall v \in W.$ *and* $0 \leq t \leq 1$.

Then, W is weakly compact in $L^1([0,1];\mathbb{R})$.

Proposition 2 ([30]). *Let*

$$L:[0,1]\times X\to \mathcal{P}_{cp}(Y),$$

be a Carathéodory multifunction and $l:[0,1]\to X$ a measurable function. So, the multifunction

$$\zeta\in[0,1]\mapsto L(\zeta,l(\zeta)),$$

is measurable.

Let

$$\mathcal{F}:[0,1]\times\mathbb{R}^n\to\mathcal{P}_{cp}(\mathbb{R}),$$

be integrable bounded if there exists $\varphi\in L^1(\mathbb{R})$; $\forall v\in S_{\mathcal{F},\tilde{w}}$, and we have

$$|v(t)|\leq\varphi(t)\ \text{ for almost } 0\leq t\leq 1,$$

where

$$S_{G,\tilde{w}}=\{w\in L^1([0,1],\mathbb{R}); w(t)\in\mathcal{F}(t,\tilde{w}(t))\ \text{ for a.e } \ 0\leq t\leq 1\}.$$

Let

$$\mathbb{X}=\{w\in L^2([0,1],\mathbb{R}); D^{\beta}_{0^+}w\in L^2([0,1],\mathbb{R})\}\ \text{ with } 0<\beta<\varpi-1.$$

The space $(\mathbb{X},\|.\|_{\mathbb{X}})$ is Banach reflexive space [31], where

$$\|w\|_{\mathbb{X}}=\|w\|_2+\|D^{\beta}_{0^+}w\|_2.$$

3. Contents and Main Results

3.1. Results of Existence and Uniqueness in Sobolev Fractional Space

Definition 2. *A function w is a solution of (4) if there exists a function $v\in L^1([0,1],\mathbb{R})$; $v(t)\in\mathcal{F}(t,w(t))$ a.e., $0\leq t\leq 1$, where*

$$\begin{cases} D^{\varpi}_{0^+}w(t)=v(t),\ \ 1<\varpi<2,\\ w(0)=0,\\ \\ w(1)=pI^{\mu_1}_{0^+}h_1(\xi,w(\xi))+qI^{\mu_2}_{0^+}h_2(\eta,w(\eta)).\end{cases}$$

Lemma 4. *For a given $y\in \mathrm{L}^1([0,1],\mathbb{R})$, a function w is a solution to*

$$\begin{cases} D^{\varpi}_{0^+}w(t)=y(t),\ \ 1<\varpi<2,\ \ \textit{for a.e.}\ \ 0\leq t\leq 1,\\ \\ w(0)=0,\\ w(1)=pI^{\mu_1}_{0^+}h_1(\xi,w(\xi))+qI^{\mu_2}_{0^+}h_2(\eta,w(\eta)),\end{cases}\tag{6}$$

if and only if

$$\begin{aligned} w(t) \;=\; & \frac{1}{\Gamma(\varpi)}\int_0^t(t-\zeta)^{\varpi-1}y(\zeta)d\zeta-\left[\frac{1}{\Gamma(\varpi)}\int_0^1(1-\zeta)^{\varpi-1}y(\zeta)d\zeta\right.\\ & -\frac{p}{\Gamma(\mu_1)}\int_0^{\xi}(\xi-\zeta)^{\mu_1-1}h_1(\zeta,w(\zeta))d\zeta-\\ & \left.\frac{q}{\Gamma(\mu_2)}\int_0^{\eta}(\eta-\zeta)^{\mu_2-1}h_2(\zeta,w(\zeta))d\zeta\right]t^{\varpi-1}. \end{aligned}\tag{7}$$

Proof. Assuming that w satisfies (6), from Remark 1, we obtain

$$w(t) = I_{0^+}^{\varpi} y(t) - c_1 t^{\varpi-1} - c_2 t^{\varpi-2}, \tag{8}$$

where $c_1, c_2 \in \mathbb{R}$.

We obtain from the first condition $c_2 = 0$. Also, we obtain from the second condition

$$w(1) \quad = \quad I_{0^+}^{\varpi} y(1) - c_1 = p I_{0^+}^{\mu_1} h_1(\xi, w(\xi)) + q I_{0^+}^{\mu_2} h_2(\eta, w(\eta)).$$

This means that

$$\begin{aligned} c_1 \quad = \quad & \frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi-1} y(\zeta) d\zeta \\ & - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi} (\xi-\zeta)^{\mu_1-1} h_1(\zeta, w(\zeta)) d\zeta \\ & - \frac{q}{\Gamma(\mu_2)} \int_0^{\eta} (\eta-\zeta)^{\mu_2-1} h_2(\zeta, w(\zeta)) d\zeta. \end{aligned} \tag{9}$$

We replace the value of c_1 with the value obtained in (9). We obtain the integral equation (7).

Conversely, if w satisfies (7) by Lemmas 1 and 2, we obtain $D_{0^+}^{\varpi} w(t) = y(t)$. By simple calculation, we obtain

$$w(1) = p I_{0^+}^{\mu_1} h_1(\xi, w(\xi)) + q I_{0^+}^{\mu_2} h_2(\eta, w(\eta)),$$

and by (8), we have $w(0) = 0$. □

Now, we study the existence of the solution for (4)

Theorem 1. *We assume that*

(D1) $\mathcal{F} : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ *is a Carathéodory set-valued map and integrable bounded with* $\phi \in \mathcal{C}([0,1], \mathbb{R})$*.*

(D2) There exists $C_1, C_2 > 0$ *such that*

$$|h_j(t,x) - h_j(t,y)| \le C_j |x-y|,$$

and

$$h_j(t,0) = 0,$$

for $0 \le t \le 1$ *and* $j \in \{1,2\}$*.*

(D3) $\dfrac{C_1 p \xi^{\mu_1-1}}{\Gamma(\mu_1)} + \dfrac{C_2 q \eta^{\mu_2-1}}{\Gamma(\mu_2)} < 1.$

So, Problem (4) *has a solution in* $\mathbb{X}$*.*

Proof. For each measurable function u, the set $S_{\mathcal{F},u}$ is nonempty.

We use the iterative method. Let (w_n) be a sequence of measurable function with $w_0 \in \mathbb{X}$ such that

$$\begin{aligned} w_{n+1}(t) \quad = \quad & \frac{1}{\Gamma(\varpi)} \int_0^t (t-\zeta)^{\varpi-1} v_n(\zeta) d\zeta - \Bigg[\frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi-1} v_n(\zeta) d\zeta \\ & - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi} (\xi-\zeta)^{\mu_1-1} h_1(\zeta, z_n(\zeta)) d\zeta \\ & - \frac{q}{\Gamma(\mu_2)} \int_0^{\eta} (\eta-\zeta)^{\mu_2-1} h_2(\zeta, w_n(\zeta)) d\zeta \Bigg] t^{\varpi-1}, \end{aligned} \tag{10}$$

with $v_n \in S_{F,w_n}$ for each $n \in \mathbb{N}$.

Step 1 ($w_n \in \mathbb{X}$).

We prove by recurrence, since w integrable bounded, then there exists a function $\phi \in L^1([0,1],\mathbb{R})$ with

$$v_n(t) \leq \phi(t) \ \text{ a.e } 0 \leq t \leq 1.$$

Then, $w_0 \in \mathbb{X}$, and if $w_n \in \mathbb{X}$, we obtain

$$|w_{n+1}(t)| \leq \frac{2\|\phi\|_1}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} \int_0^{\xi} |w_n(\zeta)| d\zeta + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \int_0^{\eta} |w_n(\zeta)| d\zeta,$$

hence $w_{n+1} \in L^2([0,1],\mathbb{R})$ and

$$\begin{aligned} D_{0^+}^{\beta} w_{n+1}(t) &= \frac{1}{\Gamma(\varpi - \beta)} \int_0^t (t-\zeta)^{\varpi - \beta - 1} v_n(\zeta) d\zeta \\ & -\frac{\Gamma(\varpi)}{\Gamma(\varpi - \beta)} \left[\frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi - 1} v_n(\zeta) d\zeta \right. \\ & -\frac{p}{\Gamma(\mu_1)} \int_0^{\xi} (\xi - \tau)^{\mu_1 - 1} h_1(\zeta, w_n(\zeta)) d\zeta - \\ & \left. \frac{q}{\Gamma(\mu_2)} \int_0^{\eta} (\eta - \zeta)^{\mu_2 - 1} h_2(\zeta, w_n(\zeta)) d\zeta \right] t^{\varpi - \beta - 1}, \end{aligned}$$

that means

$$|D_{0^+}^{\beta} w_{n+1}(t)| \leq \frac{2\|v_n\|_1}{\Gamma(\varpi - \beta)} + \frac{\Gamma(\varpi)}{\Gamma(\varpi - \beta)} \big(\frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \big) \|w_n\|_2,$$

hence $D_{0^+}^{\beta} w_{n+1} \in L^2([0,1],\mathbb{R})$.

Then, for all $n \in \mathbb{N}$ the sequence (w_n) belongs in the space $\mathbb{X}$.

Step 2 ((w_n) bounded in $\mathbb{X}$).

Let $n \geq 1$ and $0 \leq t \leq 1$, then

$$|w_{n+1}(t)| \leq \frac{2}{\Gamma(\varpi + 1)} \int_0^1 |v_n(\zeta)| d\zeta + \left(\frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \right) \|w_n\|_2,$$

and by simple calculation, we have

$$\begin{aligned} |w_{n+1}(t)|^2 &\leq \left(\frac{2\|\phi\|_1}{\Gamma(\varpi + 1)} \sum_{j=0}^{n} \big(\frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \big)^j \right. \\ & \left. + \big(\frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \big)^{n+1} \|w_0\|_2 \right)^2. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \|w_{n+1}\|_2 &\leq \frac{2\|\phi\|_1}{\Gamma(\varpi + 1)} \sum_{j=0}^{n} \Big(\frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \Big)^j \\ &+ \Big(\frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \Big)^{n+1} \|w_0\|_2, \end{aligned} \tag{11}$$

i.e., the sequence (w_n) is bounded in $L^2([0,1];\mathbb{R})$, and

$$\begin{aligned} D^{\beta}_{0^+} w_{n+1}(t) &= \frac{1}{\Gamma(\varpi-\beta)}\int_0^t (t-\zeta)^{\varpi-\beta-1} v_n(\zeta)d\zeta \\ &-\frac{\Gamma(\varpi)}{\Gamma(\varpi-\beta)}\left[\frac{1}{\Gamma(\varpi)}\int_0^1 (1-\zeta)^{\varpi-1} v_n(\zeta)d\zeta\right. \\ &-\frac{p}{\Gamma(\mu_1)}\int_0^{\xi} (\xi-\zeta)^{\mu_1-1} h_1(\zeta, w_n(\zeta))d\zeta \\ &\left.-\frac{q}{\Gamma(\mu_2)}\int_0^{\eta} (\eta-\zeta)^{\mu_2-1} h_2(\zeta, w_n(\zeta))d\zeta\right] t^{\varpi-\beta-1}. \end{aligned}$$

So,

$$|D^{\beta}_{0^+} w_{n+1}(t)|^2 \leq \left(\frac{2\|v_n\|_1}{\Gamma(\varpi-\beta)} + \frac{\Gamma(\varpi)}{\Gamma(\varpi-\beta)}\left(\frac{C_1 p \xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2-1}}{\Gamma(\mu_2)}\right)\|w_n\|_2\right)^2.$$

Since (w_n) is bounded in $L^2([0,1];\mathbb{R})$ and $\mathcal{F}$ integrable bounded, it is clear that there exists $\gamma > 0$ with

$$\|D^{\beta}_{0^+} w_n\|_2 \leq \gamma \ \text{ for each } n \geq 1. \tag{12}$$

From (11) and (12) we conclude that the sequence (w_n) is bounded in $\mathbb{X}$.

Step 3 (Passage to the limit).

Since (w_n) is bounded in $\mathbb{X}$ and $\mathbb{X}$ is reflexive Banach space, then the subsequence (w_{n_k}) converges weakly to an element in $\mathbb{X}$ noted by $\bar{w}$. Now, we show that $\bar{w}$ is a solution to Problem (4).

Let (v_{n_k}) be a sequence in $L^1([0,1],\mathbb{R})$ with

$$v_{n_k}(t) \in \mathcal{F}(t, w_{n_k}(t)) \ \text{ a.e. } 0 \leq t \leq 1.$$

By Proposition 1 the sequence (v_{n_k}) has a subsequence converge weakly to $\bar{v}$ in $L^1([0,1],\mathbb{R})$.

The sequence $w_{n_k}(t)$ is bounded in $\mathbb{R}$, and it has a subsequence noted by $w_{n_k}(t)$ that converges to $w(t)$ and $w(t) = \bar{w}(t)$ for each $0 \leq t \leq 1$.

The sequence $(v_{n_k}(t))$ is bounded in $\mathbb{R}$ (because ϕ is bounded), and it has a subsequence noted by $v_{n_k}(t)$ converge to $w(t)$ and $w(t) = \bar{v}(t)$.

The upper semicontinuous of $\mathcal{F}$ dictates that

$$\bar{v}(t) \in \mathcal{F}(t, \bar{w}(t)) \ \text{ a.e. } 0 \leq t \leq 1.$$

Passing to the limit of Equation (10), we obtain

$$\begin{aligned} \bar{w}(t) &= \frac{1}{\Gamma(\varpi)}\int_0^t (t-\zeta)^{\varpi-1}\bar{v}(\zeta)d\zeta - \left[\frac{1}{\Gamma(\varpi)}\int_0^1 (1-\zeta)^{\varpi-1}\bar{v}(\zeta)d\zeta\right. \\ &-\frac{p}{\Gamma(\mu_1)}\int_0^{\xi} (\xi-\zeta)^{\mu_1-1} h_1(\zeta, \bar{w}(\zeta))d\zeta \\ &\left.-\frac{q}{\Gamma(\mu_2)}\int_0^{\eta} (\eta-\zeta)^{\mu_2-1} h_2(\zeta, \bar{w}(\zeta))d\zeta\right] t^{\varpi-1}, \end{aligned}$$

which means $\bar{w}$ is a solution to Problem (4), and $\bar{w} \in \mathbb{X}$. □

Example 1. *We consider the fractional problem*

$$\begin{cases} D_{0^+}^{1.9}w(t) \in \mathcal{F}(t, w(t)), \\ \\ w(0) = 0, \\ w(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 s(1-\zeta)^3 w(\zeta) d\zeta, \end{cases} \tag{13}$$

where

$$\mathcal{F} : (t, x) \in [0,1] \times \mathbb{R} \mapsto \left[\frac{|x \sin t|}{2}; \frac{t|x|}{2} + \arctan t \right] \in \mathcal{P}(\mathbb{R}).$$

In this problem, we have

(i) *The set-valued $\mathcal{F}$ is Carathéodory set-valued, and $\mathcal{F}(t, x)$ is a nonempty and compact set in $\mathbb{R}$.*
(ii) *$h_1(t, x) = tx$ and $h_2(t, x) = 0$.*
(iii) *$\varpi = 1.9, p = \mu_1 = 4, q = 0$ and $C_1 = 1$.*

Since

$$\frac{C_1 p \xi^{\mu_1 - 1}}{\varpi \Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\varpi \Gamma(\mu_2)} < 1.$$

Consequently, by Theorem 1, the considered (13) admits a solution.

Remark 2. *If the function ϕ is not continuous functions, but ϕ is measurable and bounded on $[0,1]$, then the result of Theorem 1 is still valid.*

Theorem 2. *Let the conditions below hold*

(D1)* *$\mathcal{F} : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is integrable bounded with $\phi \in \mathcal{C}([0,1], \mathbb{R})$ with*
- *The multivalued map $\mathcal{F}(t, .)$ is L-Lipschitzean.*
- *The multivalued map $\mathcal{F}(., x)$ is measurable for each $x \in \mathbb{R}$.*

(D2)* *There exists $C_1, C_2 > 0$ with*

$$|h_j(t, x) - h_j(t, y)| \leq C_j |x - y|;$$

$$h_j(t, 0) = 0,$$

for $0 \leq t \leq 1$ and $j \in \{1, 2\}$.

(D3)* $\frac{2L}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} < 1.$

So, (4) has one solution in $\mathbb{X}$.

Proof. From Theorem 1, (4) has a solution. We prove now it is unique.

Let w_1 and w_2 be two solutions for Problem (4), so $\exists v_1 \in S_{\mathcal{F}, w_1}$, $v_2 \in S_{\mathcal{F}, w_2}$ with

$$\begin{aligned} w_j(t) &= \frac{1}{\Gamma(\varpi)} \int_0^t (t-\zeta)^{\varpi - 1} v_j(\zeta) d\zeta - \left[\frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi - 1} v_j(\zeta) d\zeta \right. \\ &\quad - \frac{p}{\Gamma(\mu_1)} \int_0^{\xi} (\xi - \zeta)^{\mu_1 - 1} h_1(\zeta, w_j(\zeta)) d\zeta \\ &\quad \left. - \frac{q}{\Gamma(\mu_2)} \int_0^{\eta} (\eta - \zeta)^{\mu_2 - 1} h_2(\zeta, w_j(\zeta)) d\zeta \right] t^{\varpi - 1}, \end{aligned}$$

for $1 \leq j \leq 2$, then

$$\begin{aligned} |w_2(t) - w_1(t)| &\leq \frac{2L}{\Gamma(\varpi)} \int_0^1 |w_2(\zeta) - w_1(\zeta)| d\zeta \\ &+ \left(\frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \right) \int_0^1 |w_2(\zeta) - w_1(\zeta)| d\zeta. \end{aligned}$$

After that,

$$\|w_2 - w_1\|_2 \leq \left(\frac{2L}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \right) \|w_2 - w_1\|_2,$$

while

$$\frac{2L}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} < 1,$$

then $w_1 = w_2$. □

Example 2. *Let us consider the problem*

$$\begin{cases} D_{0^+}^{1.9} w(t) = t + \frac{1.9 \arctan(w(t))}{2}, \\ \\ w(0) = 0, \\ w(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 \tau(1-\tau)^3 w(\tau) d\zeta, \end{cases} \tag{14}$$

where $\mathcal{F}(t, x) = \{t + \frac{\arctan(x)}{2}\}$.

The set-valued $\mathcal{F}$ *is* $\frac{1.9}{2}$*-Lipschitz.*

In this problem, we have

(i) $h_1(t, x) = 1.9tx$ *and* $h_2(t, x) = 0$.
(ii) $\varpi = 1.9, p = \mu_1 = 4, q = 0$.
(iii) $C_1 = 1.9$.

Since

$$\frac{2L}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1 - 1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2 - 1}}{\Gamma(\mu_2)} \approx 0.8981 < 1.$$

Consequently, from Theorem, 2 the considered (14) has a unique solution.

3.2. Results of Existence for Fractional Differential Equation

Lemma 5. *w is a solution to*

$$\begin{cases} D_{0^+}^{\varpi} w(t) = f(t, w(t)), \ \ 1 < \varpi < 2, \ \ 0 \leq t \leq 1, \\ w(0) = 0, \\ w(1) = p I_{0^+}^{\mu_1} h_1(\xi, w(\xi)) + q I_{0^+}^{\mu_2} h_2(\eta, w(\eta)), \end{cases} \tag{15}$$

if and only if

$$\begin{aligned} w(t) &= \frac{1}{\Gamma(\varpi)} \int_0^t (t-\zeta)^{\varpi-1} f(\zeta, w(\zeta)) d\zeta \\ &- \left[\frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi-1} f(\zeta, w(\zeta)) d\zeta \right. \\ &- \frac{p}{\Gamma(\mu_1)} \int_0^{\xi} (\xi-\zeta)^{\mu_1-1} h_1(\zeta, w(\zeta)) d\zeta \\ &- \left. \frac{q}{\Gamma(\mu_2)} \int_0^{\eta} (\eta-\zeta)^{\mu_2-1} h_2(\zeta, w(\zeta)) d\zeta \right] t^{\varpi-1}. \end{aligned} \tag{16}$$

Proof. Assuming that w satisfies (15), from Remark 1, we have

$$w(t) = I_{0^+}^{\varpi} f(t, w(t)) - c_1 t^{\varpi-1} - c_2 t^{\varpi-2},$$

where $c_1, c_2 \in \mathbb{R}$.

In the first condition, we have $c_2 = 0$, and from the second condition, we obtain

$$w(1) = I_{0^+}^{\varpi} f(1, w(1)) - c_1 = p I_{0^+}^{\mu_1} h_1(\xi, w(\xi)) + q I_{0^+}^{\mu_2} h_2(\eta, w(\eta)).$$

This means that

$$\begin{aligned} c_1 &= \frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi-1} f(\zeta, w(\zeta)) d\zeta \\ &- \frac{p}{\Gamma(\mu_1)} \int_0^{\xi} (\xi-\zeta)^{\mu_1-1} h_1(\zeta, w(\zeta)) d\zeta \\ &- \frac{q}{\Gamma(\mu_2)} \int_0^{\eta} (\eta-\tau)^{\mu_2-1} h_2(\zeta, w(\zeta)) d\zeta. \end{aligned}$$

We replace the value of c_1 with the value obtained in (17). We obtain the integral Equation (16)

Conversely, if w satisfies (16) by Lemmas 1 and 2, we obtain

$$D_{0^+}^{\varpi} w(t) = f(t, w(t)).$$

By simple calculation, we find

$$w(1) = p I_{0^+}^{\mu_1} h_1(\xi, w(\xi)) + q I_{0^+}^{\mu_2} h_2(\eta, w(\eta)),$$

and then $w(0) = 0$. □

Theorem 3. *Let the conditions below hold*

(S1) $f \in L^2([0,1] \times \mathbb{R}, \mathbb{R})$ *with*

$$|f(t,x) - f(t,y)| \leq s|x-y|, s > 0.$$

(S2) There exists $C_1, C_2 > 0$ *such that*

$$|h_j(t,x) - h_j(t,y)| \leq C_j|x-y|,$$

and

$$h_j(t,0) = 0,$$

(S3)

$$\begin{aligned}\gamma &= \max\left\{\frac{2\zeta}{\Gamma(\varpi-\beta)}+\frac{\Gamma(\varpi)}{\Gamma(\varpi-\beta)}\left(\frac{C_1p\xi^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\eta^{\mu_2-1}}{\Gamma(\mu_2)}\right);\right.\\ &\left.\frac{2\zeta}{\Gamma(\varpi)}+\frac{C_1p\xi^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\eta^{\mu_2-1}}{\Gamma(\mu_2)}\right\}<1.\end{aligned}$$

Then, (5) *has a solution in* $\mathbb{X}$*.*

Proof. We use the iterative method. Let (w_n) be a sequence of function with $w_0 \in \mathbb{X}$ such that

$$\begin{aligned}w_{n+1}(t) &= \frac{1}{\Gamma(\varpi)}\int_0^t (t-\zeta)^{\varpi-1}f(\zeta,w_n(\zeta))d\zeta\\ &-\left[\frac{1}{\Gamma(\varpi)}\int_0^1(1-\zeta)^{\varpi-1}f(\zeta,w_n(\zeta))d\zeta\right.\\ &-\frac{p}{\Gamma(\mu_1)}\int_0^\xi(\xi-\zeta)^{\mu_1-1}h_1(\zeta,w_n(\zeta))d\zeta\\ &\left.-\frac{q}{\Gamma(\mu_2)}\int_0^\eta(\eta-\zeta)^{\mu_2-1}h_2(\zeta,w_n(\zeta))d\zeta\right]t^{\varpi-1},\end{aligned} \tag{17}$$

Step 1 ($w_n \in \mathbb{X}$). We use the proof by recurrence of the element $w_0 \in \mathbb{X}$, and supposing that $w_n \in \mathbb{X}$, we will prove that $w_{n+1} \in \mathbb{X}$, indeed

$$\begin{aligned}|w_{n+1}(t)| &\le \frac{1}{\Gamma(\varpi)}\int_0^t (t-\zeta)^{\varpi-1}|f(\zeta,w_n(\zeta))|d\zeta\\ &+\frac{1}{\Gamma(\varpi)}\int_0^1(1-\zeta)^{\varpi-1}|f(\zeta,w_n(\zeta))|d\zeta\\ &+\frac{p}{\Gamma(\mu_1)}\int_0^\xi(\xi-\zeta)^{\mu_1-1}|h_1(\zeta,w_n(\zeta))|d\zeta\\ &+\frac{q}{\Gamma(\mu_2)}\int_0^\eta(\eta-\tau)^{\mu_2-1}|h_2(\zeta,w_n(\zeta))|d\zeta\\ &\le \frac{2\|f\|_2}{\Gamma(\varpi)}+\left(\frac{C_1p\xi^{\mu_1-1}}{\Gamma(\mu_1)}+\frac{C_2q\eta^{\mu_2-1}}{\Gamma(\mu_2)}\right)\|w_n\|_2,\end{aligned}$$

then $w_{n+1} \in L^2([0,1],\mathbb{R})$, and

$$\begin{aligned}|D^\beta_{0^+}w_{n+1}(t)| &\le \frac{1}{\Gamma(\varpi-\beta)}\int_0^t (t-\zeta)^{\varpi-1}|f(\zeta,w_n(\zeta))|d\zeta\\ &+\frac{1}{\Gamma(\varpi-\beta)}\int_0^1(1-\zeta)^{\varpi-1}|f(\zeta,w_n(\zeta))|d\zeta\\ &+\frac{p\Gamma(\varpi)}{\Gamma(\mu_1)\Gamma(\varpi-\beta)}\int_0^\xi(\xi-\zeta)^{\mu_1-1}|h_1(\zeta,w_n(\zeta))|d\zeta\\ &+\frac{q\Gamma(\varpi)}{\Gamma(\mu_2)\Gamma(\varpi-\beta)}\int_0^\eta(\eta-\zeta)^{\mu_2-1}|h_2(\zeta,w_n(\zeta))|d\zeta\\ &\le \frac{2\|f\|_2}{\Gamma(\varpi-\beta)}+\left(\frac{C_1p\xi^{\mu_1-1}\Gamma(\varpi)}{\Gamma(\mu_1)\Gamma(\varpi-\beta)}+\frac{C_2q\eta^{\mu_2-1}\Gamma(\varpi)}{\Gamma(\mu_2)\Gamma(\varpi-\beta)}\right)\|w_n\|_2,\end{aligned}$$

then $D^\beta_{0^+}w_{n+1} \in L^2([0,1],\mathbb{R})$, i.e., $w_n \in \mathbb{X}$ for each $n \in \mathbb{N}$.

Step 2 (w_n is a Cauchy sequence). For all $n \in \mathbb{N}$, we find

$$
\begin{aligned}
& |w_{n+1}(t) - w_n(t)| \\
\leq\ & \frac{1}{\Gamma(\varpi)} \int_0^t (t-\zeta)^{\varpi-1} |f(\zeta, w_n(\zeta)) - f(\zeta, w_{n-1}(\zeta))| d\zeta \\
& + \frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi-1} |f(\zeta, w_n(\zeta)) - f(\zeta, w_{n-1}(\zeta))| d\zeta \\
& + \frac{p}{\Gamma(\mu_1)} \int_0^\xi (\xi-\zeta)^{\mu_1-1} |h_1(\zeta, w_n(\zeta)) - h_1(\zeta, w_{n-1}(\zeta))| d\zeta \\
& + \frac{q}{\Gamma(\mu_2)} \int_0^\eta (\eta-\zeta)^{\mu_2-1} |h_2(\zeta, w_n(\zeta)) - h_2(\zeta, w_{n-1}(\zeta))| d\zeta \\
& \leq \left(\frac{2\zeta}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2-1}}{\Gamma(\mu_2)} \right) |w_n(t) - w_{n-1}(t)|.
\end{aligned}
$$

By recurrence, we write

$$
|w_{n+1}(t) - w_n(t)| \leq \left(\frac{2\zeta}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2-1}}{\Gamma(\mu_2)} \right)^n |w_1(t) - w_0(t)|.
$$

Let $p \in \mathbb{N}$ and $n \geq 1$, then

$$
\begin{aligned}
& |w_{n+p}(t) - w_n(t)| \\
\leq\ & \sum_{k=1}^{p} |w_{n+k}(t) - w_{n+k-1}(t)| \\
& \leq \sum_{k=1}^{p} \left(\frac{2\zeta}{\Gamma(\varpi)} + \frac{C_1 p \xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q \eta^{\mu_2-1}}{\Gamma(\mu_2)} \right)^{k-1} |w_n(t) - w_{n-1}(t)|. \\
\leq\ & \frac{|w_n(t) - w_{n-1}(t)|}{1 - \frac{2\zeta}{\Gamma(\varpi)} - \frac{C_1 p \xi^{\mu_1-1}}{\Gamma(\mu_1)} - \frac{C_2 q \eta^{\mu_2-1}}{\Gamma(\mu_2)}},
\end{aligned}
$$

as $n \to +\infty$, we have

$$
|w_{n+p}(t) - w_n(t)| \to 0. \tag{18}
$$

This means $(w_n(t))$ is a Cauchy sequence in $\mathbb{R}$ for each $t \in [0,1]$, and then by (18), we have

$$
\|w_{n+p} - w_n\|_2 \to 0. \tag{19}
$$

$$
\begin{aligned}
D_{0^+}^{\beta} w_{n+1}(t) = & \frac{1}{\Gamma(\varpi-\beta)} \int_0^t (t-\zeta)^{\varpi-\beta-1} f(\zeta, w_n(\zeta)) d\zeta \\
& - \frac{\Gamma(\varpi)}{\Gamma(\varpi-\beta)} \left[\frac{1}{\Gamma(\varpi)} \int_0^1 (1-\zeta)^{\varpi-1} f(\zeta, w_n(\zeta)) d\zeta \right. \\
& - \frac{p}{\Gamma(\mu_1)} \int_0^\xi (\xi-\zeta)^{\mu_1-1} h_1(\zeta, w_n(\zeta)) d\zeta \\
& \left. - \frac{q}{\Gamma(\mu_2)} \int_0^\eta (\eta-\zeta)^{\mu_2-1} h_2(\zeta, w_n(\zeta)) d\zeta \right] t^{\varpi-\beta-1},
\end{aligned}
$$

then

$$
\begin{aligned}
& |D^{\beta}_{0^+} w_{n+1}(t) - D^{\beta}_{0^+} w_n(t)| \\
\leq \; & \frac{1}{\Gamma(\varpi-\beta)} \int_0^t (t-\zeta)^{\varpi-\beta-1} |f(\zeta, w_n(\zeta)) - f(\zeta, w_{n-1}(\zeta))| d\zeta \\
& + \frac{1}{\Gamma(\varpi-\beta)} \int_0^1 (1-\zeta)^{\varpi-\beta-1} |f(\zeta, w_n(\zeta)) - f(\zeta, w_{n-1}(\zeta))| d\zeta \\
& + \frac{p\Gamma(\varpi)}{\Gamma(\mu_1)\Gamma(\varpi-\beta)} \int_0^{\xi} (\xi-\zeta)^{\mu_1-1} |h_1(\zeta, w_n(\zeta)) - h_1(\zeta, w_{n-1}(\zeta))| d\zeta \\
& + \frac{q\Gamma(\varpi)}{\Gamma(\mu_2)\Gamma(\varpi-\beta)} \int_0^{\eta} (\eta-\zeta)^{\mu_2-1} |h_2(\zeta, w_n(\zeta)) - h_2(\zeta, w_{n-1}(\zeta))| d\zeta \\
& \leq \left(\frac{2\zeta}{\Gamma(\varpi-\beta)} + \frac{C_1 p \xi^{\mu_1-1}\Gamma(\varpi)}{\Gamma(\mu_1)\Gamma(\varpi-\beta)} + \frac{C_2 q \eta^{\mu_2-1}\Gamma(\varpi)}{\Gamma(\mu_2)\Gamma(\varpi-\beta)} \right) |w_n(t) - w_{n-1}(t)|.
\end{aligned}
$$

By recurrence, we write

$$
\begin{aligned}
& |D^{\beta}_{0^+} w_{n+1}(t) - D^{\beta}_{0^+} w_n(t)| \\
& \leq \left(\frac{2\zeta}{\Gamma(\varpi-\beta)} + \frac{C_1 p \xi^{\mu_1-1}\Gamma(\varpi)}{\Gamma(\mu_1)\Gamma(\varpi-\beta)} + \frac{C_2 q \eta^{\mu_2-1}\Gamma(\varpi)}{\Gamma(\mu_2)\Gamma(\varpi-\beta)} \right)^n |w_1(t) - w_0(t)|.
\end{aligned}
$$

Let $p \in \mathbb{N}$ and $n \geq 1$, then

$$
\begin{aligned}
& |D^{\beta}_{0^+} w_{n+p}(t) - D^{\beta}_{0^+} w_n(t)| \\
\leq \; & \sum_{k=1}^{p} |D^{\beta}_{0^+} w_{n+k}(t) - D^{\beta}_{0^+} w_{n+k-1}(t)| \\
& \leq \sum_{k=1}^{p} \left(\frac{2\zeta}{\Gamma(\varpi-\beta)} + \frac{C_1 p \xi^{\mu_1-1}\Gamma(\varpi)}{\Gamma(\mu_1)\Gamma(\varpi-\beta)} + \frac{C_2 q \eta^{\mu_2-1}\Gamma(\varpi)}{\Gamma(\mu_2)\Gamma(\varpi-\beta)} \right)^{k-1} |w_n(t) - w_{n-1}(t)| \\
& \leq \frac{|w_n(t) - w_{n-1}(t)|}{1 - \frac{2\zeta}{\Gamma(\varpi-\beta)} - \frac{C_1 p \xi^{\mu_1-1}\Gamma(\varpi)}{\Gamma(\mu_1)\Gamma(\varpi-\beta)} - \frac{C_2 q \eta^{\mu_2-1}\Gamma(\varpi)}{\Gamma(\mu_2)\Gamma(\varpi-\beta)}},
\end{aligned}
$$

as $n \rightarrow +\infty$, we have

$$
|D^{\beta}_{0^+} w_{n+p}(t) - D^{\beta}_{0^+} w_n(t)| \rightarrow 0,
$$

that means $(D^{\beta}_{0^+} w_n(t))$ is a Cauchy sequence in $\mathbb{R}$. This is easy to see

$$
\| D^{\beta}_{0^+} w_{n+p} - D^{\beta}_{0^+} w_n \|_2 \rightarrow 0. \tag{20}
$$

From (19) and (20), the sequence (w_n) is a Cauchy sequence in $\mathbb{X}$, so there exists an element noted by $\tilde{w}$, which represents a limit of this sequence. For all $0 \leq t \leq 1$, we obtain $w_n(t) \rightarrow \tilde{w}(t)$.

Then, $\tilde{w}$ represents a solution for (16). □

Example 3. *Let us consider the following problem*

$$
\begin{cases}
D^{1.8}_{0^+} w(t) = f(t, w(t)), \\
\\
w(0) = 0, \\
w(1) = \dfrac{4}{\Gamma(4)} \displaystyle\int_0^1 \tau(1-\tau)^3 w(\tau) d\tau,
\end{cases} \tag{21}
$$

where

$$f : (t, x) \in [0,1] \times \mathbb{R} \mapsto (t+1)\frac{\cos(x)}{20}.$$

In this problem, we have

$$\max\left\{\frac{2\zeta}{\Gamma(\varpi-\beta)} + \frac{\Gamma(\varpi)}{\Gamma(\varpi-\beta)}(\frac{C_1 p\xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q\eta^{\mu_2-1}}{\Gamma(\mu_2)}); \frac{2\zeta}{\Gamma(\varpi)} + \frac{C_1 p\xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q\eta^{\mu_2-1}}{\Gamma(\mu_2)}\right\}$$
$$\approx 0.9149$$
$$< 1$$

Then, from precedent Theorem 3 Problem (21), taking one solution in the space

$$\mathbb{X} = \{w \in L^2([0,1], \mathbb{R}); D_{0^+}^{0.8} w \in L^2([0,1], \mathbb{R})\}.$$

Example 4. *Let us consider the following problem:*

$$\begin{cases} D_{0^+}^{1.7} w(t) = \frac{t \sin x}{3}, \\ \\ w(0) = 0, \\ w(1) = \frac{1}{6}\int_0^1 \tau(1-\tau)^4 w(\tau) d\tau, \end{cases} \tag{22}$$

In this problem, we have

$$\max\left\{\frac{2\zeta}{\Gamma(\varpi-\beta)} + \frac{\Gamma(\varpi)}{\Gamma(\varpi-\beta)}(\frac{C_1 p\xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q\eta^{\mu_2-1}}{\Gamma(\mu_2)}); \frac{2\zeta}{\Gamma(\varpi)} + \frac{C_1 p\xi^{\mu_1-1}}{\Gamma(\mu_1)} + \frac{C_2 q\eta^{\mu_2-1}}{\Gamma(\mu_2)}\right\}$$
$$\approx 0.9004$$
$$< 1$$

One can see that the function

$$(t; x) \mapsto \frac{t \sin x}{3} \in L^2([0,1] \times \mathbb{R}; \mathbb{R}),$$

and

$$|f(t,x)| \leq \frac{1}{3}.$$

If we take $h_1(t,x) = tx$ *and* $p = 4, q = 0, \mu_1 = 5$ *and* $s = \frac{1}{3}$*. Then, from Theorem 3, Problem (22) has a solution in*

$$\mathbb{X} = \{w \in L^2([0,1], \mathbb{R}); D_{0^+}^{0.7} w \in L^2([0,1], \mathbb{R})\}.$$

Remark 3. *Here, we are interested only in the mathematical point of view, making mathematical contributions to support a rapidly developing literature. Since the differential inclusions are usually applied to deal with differential equations with a discontinuous right-hand side or an inaccurately known right-hand side, this can be seen as a generalization of the notion of ordinary differential equations. Knowing that many phenomena from physics, chemistry, mechanics, and electricity can be modeled by ordinary and partial differential equations involving fractional derivatives gives a clear, precise, and accurate idea about the scope of application of this model in real-life problems. Some mathematical examples satisfying our problem with its assumptions are given to illustrate the obtained results and help the reader and the field of applied sciences benefit from our results.*

4. Conclusions

Over the past decades, the theory of functional–differential inclusions has received significant development, primarily the functional–differential inclusion of a multiterm type. Scientists from different countries are conducting research in the field of the theory of initial boundary value problems for various classes of differential, integrodifferential and functional–differential inclusions in partial derivatives with integer and fractional orders of derivatives. Our paper includes several new contributions:

1. This work is devoted to the multiterm fractional boundary value problem and semi-linear fractional differential inclusions and equations, which occupy models in many applied sciences areas.
2. Our systems inherit many properties of the classical earlier results; they are a natural generalization.
3. Sufficient conditions for the existence and uniqueness of solutions are established where newly developed methods of fractional integrodifferential calculus and the theory of fixed points of multivalued mappings form the basis of this study.

It is known that the dynamics of economic, social, and environmental macrosystems is a multivalued dynamic process and that fractional-order differential and integrodifferential inclusions are natural models of macrosystem dynamics. Such inclusions are also used to describe certain physical and mechanical systems.

The existence and stability (Ulam–Hyers–Rassias stability and asymptotic stability) of solutions for such classes of systems involving the Hadamard or Hilfer fractional derivative will be very interesting. The same equation/inclusion with the presence of delay, which may be finite, infinite, or state-dependent, will also be a very interesting subject. Other subjects to impulsive effect, which may be fixed or non-instantaneous are open problems in this direction.

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