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Solution of High-Order Nonlinear Integrable Systems Using Darboux Transformation

Xinhui Wu [1,2], Jiawei Hu [1,2] and Ning Zhang [1,2,*]

[1] College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, China
[2] Department of Fundamental Course, Shandong University of Science and Technology, Taian 271019, China
[*] Correspondence: skd991310@sdust.edu.cn

Abstract: The 4×4 trace-free complex matrix set is introduced in this study. By using it, we are able to create a novel soliton hierarchy that is reduced to demonstrate its bi-Hamiltonian structure. Additionally, we give the Darboux matrix T, whose elements are connected to the spectral parameter in accordance with the various positions and numbers of the spectral parameter λ. The Darboux transformation approach has also been successfully applicated to superintegrable systems.

Keywords: new soliton hierarchy; generalized Hamiltonian structures; Darboux transformation

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1. Introduction

Professor Tu [1], as is well known, presents a method for constructing the Lax equation systematically using Lie algebra as a tool. In this paper, we describe a novel 4×4 isospectral issue [2] with three potentials, as well as the related hierarchy of nonlinear evolution equations. A new coupled KdV equation [3] is generated in particular. The trace identity is also used to explore their generalized bi-Hamiltonian structures [4–9]. Furthermore, the nonlinearization of the associated Lax pair yields a new finite-dimensional Hamiltonian system.

Soliton equations in nonlinear science have important applications in many fields, such as nonlinear optics, deep water wave theory, plasma physics, etc. For many soliton equations, we have many methods to obtain their exact solutions, such as the backscattering method, bilinear method, Darboux transform method, algebraic geometry method and so on. Many interesting exact solutions have been found, among which the famous ones are the pure soliton solution, finite band potential solution and pole expansion solution. Among these methods, Darboux transformation has been paid more and more attention and has developed rapidly in the theoretical study of soliton and polarizable systems. In Refs. [10–13], we know that Darboux transformation is a completely algebraic and powerful approach for obtaining a new solution to a nonlinear problem from an existing one.

Over the last several decades, many linear and nonlinear equations have been constructed and extended via the Darboux transformation, including the Korteweg–de-Vries equation, nonlinear Schrodinger equations and many others. Darboux transformations of linear Schrodinger equations all have one thing in common: the solution of an auxiliary equation must be obtained in order to complete the transformation. Normally, this auxiliary equation must have the same form as the underlying Schrodinger equation; however, in a few recent studies, this restriction has been removed. Discrete integrable systems have also been successfully discretized using the Darboux transformation [5,14–22] technique.

We begin by introducing a 4×4 complex matrix set, whose trace is zero. As a result of its application, we gain a soliton hierarchy in Section 2 that is reduced to its Hamiltonian structures. In addition, in Section 3, we offer the Darboux matrix T for one of the equations in the hierarchy, in which each element has a connection to the spectral parameter λ depending on its position and number. Every element of T is assumed to be polynomial in terms of λ. This paper aims to examine the use of the Darboux transformation method in superintegrable systems.

2. A New Soliton Hierarchy

2.1. A Hierarchy of New cKdV Equations

$$\phi_x = M\phi, \qquad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ p-\lambda & 0 & r & 0 \\ 0 & 0 & 0 & 1 \\ q & 0 & p-\lambda & 0 \end{pmatrix}; \tag{1}$$

then, the stationary zero curvature equation

$$N_x = [M, N], \quad N = (N_{ij})_{4\times 4}. \tag{2}$$

Let

$$\begin{array}{lll} N_{12} + N_{34} = 2X, & N_{14} = Y, & N_{32} = Z, \\ X = \sum_{j\geq 0} X_{j-1}\lambda^{-j}, & Y = \sum_{j\geq 0} Y_{j-1}\lambda^{-j}, & Z = \sum_{j\geq 0} Z_{j-1}\lambda^{-j}. \end{array} \tag{3}$$

Substituting Equation (3) into Equation (2), we have

$$\begin{array}{l} 2N_{11} = \partial^{-1}(rZ - qY) - X_x, N_{12} = X, \\ 2N_{13} = -Y_x, N_{14} = Y, \\ 2N_{21} = rZ + qY - X_{xx} + 2(p-\lambda)X, \\ 2N_{22} = \partial^{-1}(rZ - qY) + X_x, \\ 2N_{23} = 2rX + 2(p-\lambda)Y - Y_{xx}, 2N_{24} = Y_x, \\ 2N_{31} = -Z_x, N_{32} = Z, \\ 2N_{33} = \partial^{-1}(qY - rZ) - X_x, N_{34} = X, \\ 2N_{41} = 2qX - Z_{xx} + 2(p-\lambda)Z, 2N_{42} = Z_x, \\ 2N_{43} = qY + rZ - X_{xx} + 2(p-\lambda)X, 2N_{44} = \partial^{-1}(qY - rZ) + X_x. \end{array}$$

Then, substituting the above equations into the following ones,

$$\begin{array}{l} 2N_{21x} = -(p-\lambda)2N_{12x} + r2N_{31} - q2N_{24}, \\ 2N_{23x} = -(p-\lambda)2N_{14x} + r(2N_{33} - q2N_{22}), \\ 2N_{41x} = -(p-\lambda)2N_{32x} + q(2N_{11} - 2N_{44}), \\ 2N_{43x} = -(p-\lambda)2N_{34x} + q2N_{13} - r2N_{42} \end{array}$$

Equation (2) becomes

$$(-\partial^3 + 4p\partial + 2\partial p)X + (q\partial + \partial q)Y + (r\partial + \partial r)Z = 4\lambda X_x. \tag{4}$$

$$2(q\partial + \partial q)X + 2q\partial^{-1}qY + (2p_x + 4p\partial - 2q\partial^{-1}r - \partial^3)Z = 4\lambda Z_x, \tag{5}$$

$$2(r\partial + \partial r)X + (2p_x + 4p\partial - 2r\partial^{-1}q - \partial^3)Y + 2r\partial^{-1}rZ = 4\lambda Y_x, \tag{6}$$

Thus, from Equations (4) and (5), we obtain the Lenard gradient sequence

$$A(2X, Y, Z)^T = \lambda B(2X, Y, Z)^T, \tag{7}$$

where

$$A = \begin{pmatrix} p\partial + \partial p - \frac{1}{2}\partial^3 & q\partial + \partial q & r\partial + \partial r \\ q\partial + \partial q & 2q\partial^{-1}q & 2p\partial + 2\partial p - 2q\partial^{-1}r - \partial^3 \\ r\partial + \partial r & 2p\partial + 2\partial p - 2q\partial^{-1}r - \partial^3 & 2r\partial^{-1}r \end{pmatrix},$$

$$B = \begin{pmatrix} 2\partial & 0 & 0 \\ 0 & 4\partial & 0 \\ 0 & 0 & 6\partial \end{pmatrix},$$

$$H_j = (2X_j, Y_j, Z_j), j \geq 0.$$

Substituting Equation (2) into Equation (7) yields the recursion relation

$$BH_0 = 0, AH_j = BH_{j+1}, j \geq 0. \tag{8}$$

Using the initial value,

$$H_0 = (2, 0, 0)^T.$$

H_j is defined only by the recursion relation in Equation (8). Particularly, we obtain the following equations:

$$H_1 = \begin{pmatrix} p \\ \frac{1}{2}r \\ \frac{1}{2}q \end{pmatrix},$$

$$H_2 = \begin{pmatrix} -\frac{1}{4}p_{xx} + \frac{3}{4}p^2 + \frac{3}{4}qr \\ -\frac{1}{8}r_{xx} + \frac{3}{4}pr \\ -\frac{1}{8}q_{xx} + \frac{3}{4}pq \end{pmatrix}.$$

Make an assumption:

$$\phi_{t_n} = N^{(n)}\phi, N^{(n)} = (\lambda^{(n+1)}N)_+, n \geq 0, \tag{9}$$

where the symbol $+$ represents the choice of a non-negative power of λ. The zero-curvature representation is then produced by the compatibility condition of Equations (1) and (9).

$$M_{t_n} - N_x^{(n)} + [M, N^{(n)}] = 0,$$

which is equivalent to

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}_{t_n} = BH_n = AH_{n-1}, \tag{10}$$

where A and B are given by Equation (7), when

$$n = 1, N^{(1)} = \begin{pmatrix} \frac{-p_x}{4} & \lambda + \frac{p}{2} & \frac{-r_x}{2} & \frac{r}{2} \\ \alpha & \frac{p_x}{4} & \frac{r\lambda}{2} - \frac{r_x x}{4} + pr & \frac{r_x}{4} \\ \frac{-q_x}{4} & \frac{q}{2} & \frac{-p_x}{4} & \lambda + \frac{p}{2} \\ \frac{q\lambda}{2} - \frac{q_x x}{4} + pq & \frac{q_x}{4} & \alpha & \frac{p_x}{4} \end{pmatrix}, \tag{11}$$

where

$$\alpha = -\lambda^2 + \frac{p\lambda}{2} - \frac{p_{xx}}{4} + \frac{p^2}{2} + \frac{qr}{2}.$$

As a result, a novel coupled KdV equation associated with Equations (1) and (11) is presented:

$$\begin{aligned} p_t &= -\tfrac{1}{4}p_{xxx} + \tfrac{3}{2}pp_x + \tfrac{3}{4}(qr)_x, \\ v_t &= -\tfrac{1}{4}q_{xxx} + \tfrac{3}{2}(pq)_x, \\ w_t &= -\tfrac{1}{4}r_{xxx} + \tfrac{3}{2}(pr)_x. \end{aligned} \tag{12}$$

When $p = q = r$, Equation (12) is reduced to the KdV equation

$$p_t = -\frac{1}{4}p_{xxx} + 3pp_x. \tag{13}$$

2.2. Generalized Hamiltonian Structures

We use the King–Cartan form $\langle A, B\rangle$ as $tr(A, B)$to examine the hierarchy's generalized Hamiltonian structures (10). So, we can conclude from direct calculations that

$$\left\langle N, \frac{\partial M}{\partial \lambda}\right\rangle = -N_{12} - N_{34}, \left\langle N, \frac{\partial M}{\partial p}\right\rangle = N_{12} + N_{34}, \left\langle N, \frac{\partial M}{\partial q}\right\rangle = N_{14}, \left\langle N, \frac{\partial M}{\partial r}\right\rangle = N_{32}. \tag{14}$$

According to the trace identity

$$(\frac{\delta}{\delta p}, \frac{\delta}{\delta q}, \frac{\delta}{\delta r})\left\langle N, \frac{\partial M}{\partial \lambda}\right\rangle = (\lambda^{-\gamma}(\frac{\partial}{\partial \lambda})\lambda^{\gamma})(\left\langle N, \frac{\partial M}{\partial p}\right\rangle, \left\langle N, \frac{\partial M}{\partial q}\right\rangle, \left\langle N, \frac{\partial M}{\partial r}\right\rangle). \tag{15}$$

Equations (3) and (14) are substituted into it to obtain

$$(\frac{\delta}{\delta p}, \frac{\delta}{\delta q}, \frac{\delta}{\delta r})(-2X_{j+1}) = (\gamma - j - 1)(2X_j, Y_j, Z_j), j \geq 0. \tag{16}$$

By comparing the coefficient of $j = 0$ in the previous equation, we can find the constant, which is $\gamma = \frac{1}{2}$.

By combining it with Equation (16), we obtain

$$(\frac{\delta}{\delta p}, \frac{\delta}{\delta q}, \frac{\delta}{\delta r})K_j = H_j^T, \quad K_j = \frac{4X_{j+1}}{2j+1}. \tag{17}$$

As a result, we obtain the desired generalized Hamiltonian hierarchy structures of Equation (10)

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}_{t_n} = A(\frac{\delta K_(n-1)}{\delta p}, \frac{\delta K_(n-1)}{\delta q}, \frac{\delta K_(n-1)}{\delta r})^T = B(\frac{\delta K_(n)}{\delta p}, \frac{\delta K_(n)}{\delta q}, \frac{\delta K_(n)}{\delta r})^T. \tag{18}$$

where A and B are given by Equation (7).

3. Darboux Transformation

3.1. Spatial Scales of the Darboux Transformation

We introduce the temporal part

$$\phi_t = N\phi, \tag{19}$$

where

$$N = \begin{pmatrix} \frac{-p_x}{4} & \lambda + \frac{p}{2} & \frac{-r_x}{2} & \frac{r}{2} \\ -\lambda^2 + \frac{p\lambda}{2} - \frac{p_{xx}}{4} + \frac{p^2}{2} + \frac{qr}{2} & \frac{p_x}{4} & \frac{r\lambda}{2} - \frac{r_x x}{4} + pr & \frac{r_x}{4} \\ \frac{-q_x}{4} & \frac{q}{2} & \frac{-p_x}{4} & \lambda + \frac{p}{2} \\ \frac{q\lambda}{2} - \frac{q_x x}{4} + pq & \frac{q_x}{4} & -\lambda^2 + \frac{p\lambda}{2} - \frac{p_{xx}}{4} + \frac{p^2}{2} + \frac{qr}{2} & \frac{p_x}{4} \end{pmatrix}.$$

The compatibility condition

$$\phi_{xt} = \phi_{tx},$$

give rise to a zero curvature equation

$$M_t - N_x + [M, N] = 0. \tag{20}$$

Consider the Darboux transformation

$$\bar{\phi} = T\phi, \tag{21}$$

where T is defined by

$$T_x + TM = \bar{M}T; \tag{22}$$

at the same time,

$$T_t + TN = \bar{N}T, \tag{23}$$

A novel spectral issue is as follows:

$$\bar{\phi}_x = \bar{M}\bar{\phi}, \qquad \bar{\phi}_t = \bar{N}\bar{\phi}, \tag{24}$$

where $\bar{M}$ and $\bar{N}$ have the same form as M and N, except replacing p, q and r with $\bar{p}$, $\bar{q}$ and $\bar{r}$.

Now, we consider the basic form of T. First of all, we assume that

$$T = \lambda(a_{ij})_{4\times 4} + (b_{ij})_{4\times 4}, \tag{25}$$

in which a_{ij} and b_{ij} (i,j = 1, 2, 3, 4) are functions of x and t. When Equation (25) is inserted into Equation (22), the coefficients matrix for λ^2 is shown to be

$$\begin{pmatrix} -a_{12} & 0 & -a_{14} & 0 \\ -a_{22} + a_{11} & a_{12} & -a_{24} + a_{13} & a_{14} \\ -a_{31} & 0 & -a_{34} & 0 \\ -a_{42} + a_{31} & a_{32} & -a_{44} + a_{33} & a_{34} \end{pmatrix}. \tag{26}$$

The following are the simplest non-trivial versions of T:

$$a_{21} \neq 0, a_{43} \neq 0 \; and \; a_{i,j} = 0 \; in \; otherwise. \tag{27}$$

We reinsert Equation (25) into Equation (22) under condition Equation (27), and compare the coefficients of λ^j(j = 2, 1, 0). We can easily obtain

$$\begin{aligned} &a_{21x} = 0, \quad a_{43x} = 0, \quad b_{12} = -a_{21}, \quad b_{13} = b_{24}, \\ &b_{14} = b_{32} = 0, \quad b_{11} = b_{22}, \\ &b_{34} = -a_{43}, \quad b_{31} = b_{42}, \quad b_{33} = b_{44}, \end{aligned} \tag{28}$$

When j = 1, the coefficients matrix of λ is the following:

$$\begin{pmatrix} -b_{12} - a_{21} & 0 & -b_{14} & 0 \\ a_{21x} - b_{22} + b_{11} & a_{21} + b_{12} & -b_{24} + b_{13} & b_{14} \\ -b_{32} & 0 & -b_{34} - a_{43} & 0 \\ -b_{42} + b_{31} & b_{32} & a_{43x} - b_{44} + b_{33} & a_{43} + b_{34} \end{pmatrix}. \tag{29}$$

The subsequent equations result from j = 0:

$$b_{11x} + b_{12}p + b_{14}q - b_{21} = 0, \tag{30}$$

$$b_{12x} + b_{11} - b_{22} = 0, \tag{31}$$

$$b_{13x} + b_{12}r + b_{14}p - b_{23} = 0, \tag{32}$$

$$b_{14x} + b_{13} - b_{24} = 0, \tag{33}$$

$$b_{21x} + b_{22}p + b_{24}q - \bar{p}b_{11} - \bar{r}b_{31} = 0, \tag{34}$$

$$b_{22x} + b_{21} - \bar{p}b_{12} - \bar{r}b_{32} = 0, \tag{35}$$

$$b_{23x} + b_{22}r + b_{24}p - \bar{p}b_{13} - \bar{r}b_{33} = 0, \tag{36}$$

$$b_{24x} + b_{23} - \bar{p}b_{14} - \bar{r}b_{34} = 0, \tag{37}$$

$$b_{31x} + pb_{32} + qb_{34} - b_{41} = 0, \tag{38}$$

$$b_{32x} + b_{31} - b_{42} = 0, \tag{39}$$

$$b_{33x} + b_{32}r + b_{34}p - b_{43} = 0, \tag{40}$$

$$b_{34x} + b_{33} - b_{44} = 0, \tag{41}$$

$$b_{41x} + b_{42}p + b_{44}q - \bar{q}b_{11} - \bar{p}b_{31} = 0, \tag{42}$$

$$b_{42x} + b_{41} - \bar{q}b_{12} - \bar{p}b_{32} = 0, \tag{43}$$

$$b_{43x} + b_{42}r + b_{44}p - \bar{q}b_{13} - \bar{p}b_{33} = 0, \tag{44}$$

$$b_{44x} + b_{43} - \bar{q}b_{14} - \bar{p}b_{34} = 0. \tag{45}$$

Both $det(\phi)$ and $det(\bar{\phi})$ are constants since the solutions to Equations (1) and (24) are two 4×4 matrices. $Tr(M) = Tr(\bar{M}) = 0$ indicates that Equation (22) has a constant $\lambda = \lambda_1$ and a solution $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ that satisfies the condition

$$b_{11}\phi_1 + b_{12}\phi_2 + b_{13}\phi_3 + b_{14}\phi_4 = 0, \tag{46}$$

$$(a_{21}\lambda + b_{21})\phi_1 + b_{22}\phi_2 + b_{23}\phi_3 + b_{24}\phi_4 = 0, \tag{47}$$

$$b_{31}\phi_1 + b_{32}\phi_2 + b_{33}\phi_3 + b_{34}\phi_4 = 0, \tag{48}$$

$$b_{41}\phi_1 + b_{42}\phi_2 + (a_{43}\lambda + b_{43})\phi_3 + b_{44}\phi_4 = 0. \tag{49}$$

Substituting Equations (30), (31), (35), (41) and (34) for (28), we obtain

$$b_{11x} = b_{12x} = b_{22x} = b_{34x} = 0, \tag{50}$$

Combining Equation (28) with Equations (46)–(50), we obtain

$$\begin{aligned} &b_{11} = 1, \quad b_{12} = -a_{21}, \quad b_{13} = 1, \quad b_{14} = 0, \\ &b_{21} = -a_{21}p, \quad b_{22} = 1, \quad b_{23} = -a_{21}r, \quad b_{24} = 1, \\ &b_{31} = 1, \quad b_{32} = 0, \quad b_{33} = 1, \quad b_{34} = -a_{43}, \\ &b_{41} = -a_{43}q, \quad b_{42} = 1, \quad b_{43} = -a_{43}p, \quad b_{44} = 1. \end{aligned} \tag{51}$$

The undefined functions $\bar{p}$, $\bar{q}$, $\bar{r}$ can be determined using Equations (45), (42) and (36):

$$\bar{p} = p, \tag{52}$$

$$\bar{q} = -a_{43}q_x + q, \tag{53}$$

$$\bar{r} = -a_{21}r_x + r, \tag{54}$$

The others of Equations (30)–(45) can be verified to be automatically satisfied.

3.2. Temporal Scales of the Darboux Transformation

The compatibility condition $\bar{\phi}_{xt} = \bar{\phi}_{tx}$ holds if the transformation Equation (25) maps Equation (19) into $\bar{\phi}_t$=$\bar{N}\bar{\phi}$, in which $\bar{N}$ has the same form as N in (19) except that p, q and r have been replaced with $\bar{p}$, $\bar{q}$ and $\bar{r}$:

$$\bar{M}_t - \bar{N}_x + [\bar{M}, \bar{N}] = 0. \tag{55}$$

Note that Equation (19) has a new solution given by $(\bar{p}, \bar{q}, \bar{r})$.

The following is the fundamental point of the proof. We need to demonstrate that the equation $\bar{\phi}_t$=$\bar{N}\bar{\phi}$ holds.

Comparing the coefficient of λ^j (j = 3, 2, 1, 0), when j = 2, the coefficients matrix is

$$\begin{pmatrix} -b_{12} - a_{21} & 0 & -b_{14} & 0 \\ -b_{22} + b_{11} & a_{21} + b_{12} & -b_{24} + b_{13} & b_{14} \\ -b_{32} & 0 & -b_{34} - a_{43} & 0 \\ -b_{42} + b_{31} & b_{32} & -b_{44} + b_{33} & a_{43} + b_{34} \end{pmatrix}. \tag{56}$$

From Equation (51), we can easily determine that Equation (56) is correct.

We have the same form as Equation (29) when $a_{21x} = 0$ and $a_{43x} = 0$.

The subsequent equations result from j = 1:

$$b_{12}\frac{p}{2} + b_{14}\frac{q}{2} - a_{21}\frac{\bar{p}}{2} - b_{21} = 0, \tag{57}$$

$$b_{11} - b_{22} = 0, \tag{58}$$

$$b_{12}\frac{r}{2} + b_{14}\frac{p}{2} - a_{43}\frac{\bar{r}}{2} - b_{23} = 0, \tag{59}$$

$$b_{13-}b_{24} = 0, \tag{60}$$

$$a_{21t} - a_{21}\frac{p_x}{4} - b_{22}\frac{p}{2} + b_{24}\frac{q}{2} - a_{21}\frac{\bar{p}_x}{4} - b_{11}\frac{\bar{p}}{2} - b_{31}\frac{\bar{r}}{2} = 0, \tag{61}$$

$$\frac{p}{2}a_{21} + b_{21} - b_{12}\frac{\bar{p}}{2} - b_{32}\frac{\bar{r}}{2} = 0, \tag{62}$$

$$-a_{21}\frac{r_x}{4} + b_{22}\frac{r}{2} + b_{24}\frac{p}{2} - a_{43}\frac{\bar{r}_x}{4} - b_{13}\frac{\bar{p}}{2} - b_{33}\frac{\bar{r}}{2} = 0, \tag{63}$$

$$a_{21}\frac{r}{2} + b_{23} - b_{14} - b_{34}\frac{\bar{r}}{2} = 0, \tag{64}$$

$$b_{32}\frac{p}{2} + b_{34}\frac{q}{2} - a_{21}\frac{\bar{q}}{2} - b_{41} = 0, \tag{65}$$

$$b_{31} - b_{42} = 0, \tag{66}$$

$$b_{32}\frac{r}{2} + b_{34}\frac{p}{2} - b_{43} - a_{43}\frac{\bar{p}}{2} = 0, \tag{67}$$

$$b_{33} - b_{44} = 0, \tag{68}$$

$$-a_{43}\frac{q_x}{4} + b_{42}\frac{p}{2} + b_{44}\frac{q}{2} - a_{21}\frac{\bar{q}_x}{4} - b_{11}\frac{\bar{q}}{2} - b_{31}\frac{\bar{p}}{2} = 0, \tag{69}$$

$$a_{43}\frac{r}{2} + b_{41} - b_{12}\frac{\bar{q}}{2} - b_{32}\frac{\bar{p}}{2} = 0, \tag{70}$$

$$a_{43t} - a_{43}\frac{p_x}{4} + b_{42}\frac{r}{2} + b_{44}\frac{p}{2} - b_{43}\frac{\bar{p}_x}{4} - b_{13}\frac{\bar{q}}{2} - b_{33}\frac{\bar{p}}{2} = 0, \tag{71}$$

$$a_{43}\frac{p}{2} + b_{43} - b_{14}\frac{\bar{q}}{2} - b_{34}\frac{\bar{p}}{2} = 0. \tag{72}$$

From Equation (37), we know that

$$\bar{r}a_{43} = a_{21}r, \tag{73}$$

Substituting Equation (51) for Equation (59), and replacing Equation (59) with Equation (73), Equation (59) is valid. Substituting Equation (51) for Equation (72), we find that Equation (72) is valid. Equations (57)–(72) can be verified to be correct in a similar way.

The subsequent equations result from $j = 0$:

$$\begin{gathered} b_{11t} - \tfrac{p_x}{4}b_{11} - (\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2})b_{12} - \tfrac{q_x}{4}b_{13} \\ -(\tfrac{q_{xx}}{4} - pq)b_{14} + \tfrac{\bar{p}_x}{4}b_{11} - \tfrac{\bar{p}}{2}b_{21} + \tfrac{\bar{r}_x}{4}b_{31} - \tfrac{\bar{r}}{2}b_{14} = 0, \end{gathered} \tag{74}$$

$$b_{12t} + \frac{p}{2}b_{11} + \frac{p_x}{4}b_{12} + \frac{q}{2}b_{13} + \frac{q_x}{4}b_{14} + \frac{\bar{p}_x}{4}b_{12} - \frac{\bar{p}}{2}b_{22} + \frac{\bar{r}_x}{4}b_{32} - \frac{\bar{r}}{2}b_{42} = 0, \tag{75}$$

$$\begin{gathered} b_{13t} - \tfrac{r_x}{4}b_{11} - (\tfrac{r_{xx}}{4} - pr)b_{12} - \tfrac{p_x}{4}b_{13} \\ -(\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2})b_{14} + \tfrac{\bar{p}_x}{4}b_{13} - \tfrac{\bar{p}}{2}b_{23} + \tfrac{r_x}{4}b_{33} - \tfrac{\bar{r}}{2}b_{43} = 0, \end{gathered} \tag{76}$$

$$b_{14t} + \frac{r}{2}b_{11} + b_{12}\frac{r_x}{4} + b_{13}\frac{p}{2} + b_{14}\frac{p_x}{4} + \frac{\bar{p}_x}{4}b_{11} - \frac{\bar{p}}{2}b_{24} + \frac{\bar{r}_x}{4}b_{34} - \frac{\bar{r}}{2}b_{44} = 0, \tag{77}$$

$$\begin{gathered} b_{21t} - \tfrac{p_x}{4}b_{21} - b_{22}(\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2}) - \tfrac{q_x}{4}b_{23} - (\tfrac{q_{xx}}{4} - pq)b_{24} \\ +(\tfrac{\bar{p}_{xx}}{4} - \tfrac{\bar{p}^2}{2} - \tfrac{\bar{q}\bar{r}}{2})b_{11} - \tfrac{\bar{p}_x}{4}b_{21} + (\tfrac{\bar{r}_{xx}}{4} - \bar{p}\bar{r})b_{31} - \tfrac{\bar{r}_x}{4}b_{41} = 0, \end{gathered} \tag{78}$$

$$\begin{gathered} b_{22t} + b_{21}\tfrac{p}{2} + b_{22}\tfrac{p_x}{4} + b_{23}\tfrac{q}{2} + b_{24}\tfrac{q_x}{4} + (\tfrac{\bar{p}_{xx}}{4} - \tfrac{\bar{p}^2}{2} - \tfrac{\bar{q}\bar{r}}{2})b_{12} \\ -\tfrac{\bar{p}_x}{4}b_{22} + (\tfrac{\bar{r}_{xx}}{4} - \bar{p}\bar{r})b_{32} - \tfrac{\bar{r}_x}{4}b_{42} = 0, \end{gathered} \tag{79}$$

$$\begin{gathered} b_{23t} - \tfrac{r_x}{4}b_{21} - (\tfrac{r_{xx}}{4} - pr)b_{22} - \tfrac{p_x}{4}b_{23} - (\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2})b_{24} \\ +(\tfrac{\bar{p}_{xx}}{4} - \tfrac{\bar{p}^2}{2} - \tfrac{\bar{q}\bar{r}}{2})b_{13} - \tfrac{\bar{p}_x}{4}b_{23} + (\tfrac{\bar{r}_{xx}}{4} - \bar{p}\bar{r})b_{33} - \tfrac{\bar{r}_x}{4}b_{43} = 0, \end{gathered} \tag{80}$$

$$\begin{gathered} b_{24t} + b_{21}\tfrac{r}{2} + b_{22}\tfrac{r_x}{4} + b_{23}\tfrac{p}{2} + b_{24}\tfrac{p_x}{4} \\ +(\tfrac{\bar{p}_{xx}}{4} - \tfrac{\bar{p}^2}{2} - \tfrac{\overline{qr}}{2})b_{14} - \tfrac{\bar{p}_x}{4}b_{24} + (\tfrac{\bar{r}_{xx}}{4} - \overline{pr})b_{34} + \tfrac{\bar{r}_x}{4}b_{44} = 0, \end{gathered} \tag{81}$$

$$\begin{gathered} b_{31t} - \tfrac{p_x}{4}b_{31} - (\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2})b_{32} - \tfrac{q_x}{4}b_{33} \\ -(\tfrac{q_{xx}}{4} - pq)b_{34} + \tfrac{\bar{q}_x}{4}b_{11} - \tfrac{\bar{q}}{2}b_{21} + \tfrac{\bar{p}_x}{4}b_{31} - \tfrac{\bar{p}}{2}b_{41} = 0, \end{gathered} \tag{82}$$

$$b_{32t} + \frac{p}{2}b_{31} + \frac{p_x}{4}b_{32} + \frac{q}{2}b_{33} + \frac{q_x}{4}b_{34} + \frac{\bar{q}_x}{4}b_{12} - \frac{\bar{q}}{2}b_{22} + \frac{\bar{p}_x}{4}b_{32} - \frac{\bar{p}}{2}b_{42} = 0, \tag{83}$$

$$\begin{gathered} b_{33t} - \tfrac{r_x}{4}b_{31} - (\tfrac{r_{xx}}{4} - pr)b_{32} - \tfrac{p_x}{4}b_{33} \\ -(\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2})b_{34} + \tfrac{\bar{q}_x}{4}b_{13} - \tfrac{\bar{q}}{2}b_{23} + \tfrac{\bar{u}_x}{4}b_{33} - \tfrac{\bar{u}}{2}b_{43} = 0, \end{gathered} \tag{84}$$

$$b_{34t} + b_{31}\frac{r}{2} + b_{32}\frac{r_x}{4} + b_{33}\frac{p}{2} + b_{34}\frac{p_x}{4} + \frac{\bar{q}_x}{4}b_{14} - \frac{\bar{q}}{2}b_{24} + \frac{\bar{p}_x}{4}b_{34} - \frac{\bar{p}}{2}b_{44} = 0, \tag{85}$$

$$\begin{gathered} b_{41t} - \tfrac{p_x}{4}b_{41} - (\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2})b_{42} - \tfrac{q_x}{4}b_{43} - (\tfrac{q_{xx}}{4} - pq)b_{44} \\ +(\tfrac{\bar{q}_{xx}}{4} - \bar{p}\bar{q})b_{11} + \tfrac{\bar{q}_x}{4}b_{21} + (\tfrac{\bar{p}_{xx}}{4} - \tfrac{\bar{p}^2}{2} - \tfrac{\bar{q}\bar{r}}{2})b_{31} - \tfrac{\bar{p}_x}{4}b_{41} = 0, \end{gathered} \tag{86}$$

$$\begin{gathered} b_{42t} + \tfrac{p}{2}b_{41} + \tfrac{p_x}{4}b_{42} + \tfrac{q}{2}b_{43} + \tfrac{q_x}{4}b_{44} + (\tfrac{\bar{q}_{xx}}{4} - \bar{p}\bar{q})b_{12} \\ -\tfrac{\bar{q}_x}{4}b_{22} + (\tfrac{\bar{p}_{xx}}{4} - \tfrac{\bar{p}^2}{2} - \tfrac{\bar{q}\bar{r}}{2})b_{32} - \tfrac{\bar{p}_x}{4}b_{42} = 0, \end{gathered} \tag{87}$$

$$\begin{gathered} b_{43t} - \tfrac{r_x}{4}b_{41} - (\tfrac{r_{xx}}{4} - pr)b_{42} - \tfrac{p_x}{4}b_{43} - (\tfrac{p_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{qr}{2})b_{44} \\ +(\tfrac{\bar{q}_{xx}}{4} - \bar{p}\bar{q})b_{13} - \tfrac{\bar{q}_x}{4}b_{23} + (\tfrac{\bar{p}_{xx}}{4} - \tfrac{\bar{p}^2}{2} - \tfrac{\bar{q}}{\bar{r}})b_{33} - \tfrac{\bar{p}_x}{4}b_{43} = 0, \end{gathered} \tag{88}$$

$$\begin{gathered} b_{44t} + \tfrac{r}{2}b_{41} + \tfrac{r_x}{4}b_{42} + \tfrac{p}{2}b_{43} + \tfrac{p_x}{4}b_{44} + (\tfrac{\bar{q}_{xx}}{4} - \bar{p}\bar{q})b_{14} \\ -\tfrac{\bar{q}_x}{4}b_{24} + (\tfrac{\bar{p}_{xx}}{4} - \tfrac{p^2}{2} - \tfrac{\bar{q}\bar{r}}{2})b_{34} - \tfrac{\bar{p}_x}{4}b_{44} = 0. \end{gathered} \tag{89}$$

From Equation (43), we know that

$$a_{21}\bar{q} = a_{43}q. \tag{90}$$

Then, we can obtain

$$a_{21}\bar{q}_x = a_{43}q_x. \tag{91}$$

Substituting Equation (51) for Equation (83), and replacing Equation (83) with Equations (91) and (53), we find that Equation (83) is valid.

Equations (74)–(89) can be verified to be correct in a similar way.

4. Conclusions and Remarks

The related hierarchy of nonlinear evolution equations is presented for a novel isospectral problem with three potentials in a 4×4 matrix context. Notably, this approach yields a new coupled KdV equation. The trace identity is used to explore their generalized bi-Hamiltonian structures. Furthermore, the related Lax pair is subjected to a nonlinearization process in order to create a new finite-dimensional Hamiltonian system. The Lax operator generates sufficient conserved integrals that are involutionary and functionally independent to ensure the Hamiltonian system's Liouville integrability. In addition, we offer the Darboux matrix T for one of the equations in the hierarchy, in which each element has a connection to the spectral parameter λ depending on its position and number. And, every element of T is assumed to be polynomial in terms of λ. We have also verified that T is correct.

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