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Existence and Uniqueness Results for a Pantograph Boundary Value Problem Involving a Variable-Order Hadamard Fractional Derivative

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Abstract: This paper discusses the problem of the existence and uniqueness of solutions to the boundary value problem for the nonlinear fractional-order pantograph equation, using the fractional derivative of variable order of Hadamard type. The main results are proved through the application of fractional calculus and Krasnoselskii's fixed-point theorem. Moreover, the Ulam–Hyers–Rassias stability of the nonlinear fractional pantograph equation is analyzed. To conclude this paper, we provide an example illustrating our findings and approach.

Keywords: fractional operators of variable order; Krasnoselskii's fixed-point result; piecewise continuous functions.

MSC: 34B05; 26A33; 34D99

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1. Introduction

In recent decades, the evolution of fractional calculus has been remarkable (for more details, see [1–9]) due to its valuable tools for constructing models for many different phenomena occurring in real life. This has formed a solid foundation for the notion of fractional operators with variable order (i.e., derivative and integral operators), each with several different definitions. Indeed, different concepts have been proposed by Grunwald–Letnikov, Erdlyi–Kober, and Riesz, as well as the widely used Riemann–Liouville and Caputo notions, and those by Hadamard and Hilfer. We consider general notions representing fractional-type integral and derivative operators, where the order is no longer constant but is a function of specific variables (for details, see [10]). These, along with some of their significant applications, have prompted a thorough analysis focused on the study of existence and uniqueness problems in equations based on this type of operator.

One of the notations frequently used for pantograph equations involves differential equations with proportional delays, making them a relevant exemplification of differential equations with delay. These types of problems have attracted a lot of attention in both pure and applied fields, such as quantum mechanics, dynamical systems, number theory, etc. In recent years, numerous researchers have focused on this type of problem [11–14]. An interesting application was presented by Ockendon and Taylor [15], who considered the pantograph of an electric locomotive to study the electric current, constituting an important work in the study of the equations bearing this name.

In [16], S. Harikrishnan considered the following pantograph equation with fractional operators of the $\psi-$Hilfer type, subject to nonlocal initial conditions:

$$\mathcal{D}_{a^+}^{\alpha,\beta;\psi}\varphi(t) = g(t,\varphi(t),\varphi(\lambda t)), \quad 0<\lambda<1, \quad t \in J := (a,T], \tag{1}$$

$$I_{a^+}^{1-\gamma;\psi}\varphi(t)\Big|_{t=a} = \sum_{k=1}^{n} C_k\varphi(t_k), \quad t_k \in (a,T], \tag{2}$$

where $\mathcal{D}_{a^+}^{\alpha,\beta;\psi}$ represents the $\psi-$Hilfer fractional-order derivative with order α, where $\alpha \in (0,1)$, and type β with $\beta \in [0,1]$, such that $\alpha+\beta-\alpha\beta =: \gamma$. We use $\mathbb{R}$ to represent the real line and let the function $f : J\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be continuous and t_k be some predefined points ordered as follows: $a < t_1 \le \cdots \le t_k < T$. The real numbers C_k are fixed.

In [17], the authors used two fixed-point results: one from Krasnoselskii and another involving extended contractive mappings. Their aim was to investigate the stability properties of the following discrete-type pantograph fractional equation:

$$\begin{cases} \Delta_*^{\beta}[K](t) = f(t+\beta, K(t+\beta), K(\lambda(t+\beta))), \\ K(0) = p[K], \end{cases}$$

for $t \in \mathbb{N}_{1-\beta}$, where $\beta \in (0,1]$, $\lambda \in (0,1)$, Δ_*^{β} represents a Caputo-type difference operator, K represents the pantograph motion, $f : E \longrightarrow \mathbb{R}$ is continuous, and $p : C([0,+\infty),\mathbb{C}) \longrightarrow \mathbb{R}$ is Lipschitzian in K. Here, $E := [0,+\infty)\times C([0,+\infty),\mathbb{C})\times C([0,+\infty),\mathbb{C})$, and $\mathbb{N}_t := \{t, t+1, t+2, \ldots\}$.

Using the Hilfer operator, stability properties were investigated in [18] for a certain generalized nonlinear pantograph equation with fractional order and discrete time. In addition, stability conditions were established using Ulam and Hyers' results. Recently, in [19], the authors investigated some properties of positive solutions, such as existence and uniqueness, for nonlinear pantograph differential equations with fractional operators of the Caputo–Hadamard type.

The above-mentioned paper was the inspiration for the present work, where we consider the following boundary value problem for the nonlinear pantograph fractional differential equation of variable order of the Hadamard type, and study the existence and uniqueness properties:

$$\begin{cases} {}^{H}\mathcal{D}_{1^+}^{\psi(t)}\varphi(t) = g(t,\varphi(t),\varphi(\lambda t)), & t \in [1,T], \\ \varphi(1) = \varphi(T) = 0, \end{cases} \tag{3}$$

where $T < +\infty$, $\psi(t) \in (1,2)$, $\lambda \in (0,1)$, the function $g : [1,T]\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous, and ${}^{H}\mathcal{D}_{1^+}^{\psi(t)}$ represents the left-hand Hadamard-type fractional derivative with variable order.

This paper is divided into several sections. Section 2 outlines the notions and previous auxiliary lemmas that demonstrate interesting properties related to the problem that is the subject of this study. Section 3 presents the main results concerning the existence and uniqueness properties of the solutions to the problem (3), using the fixed-point result of Krasnoselskii as the fundamental tool. Section 4 focuses on the analysis of stability in terms of Ulam–Hyers–Rassias for the considered problem. Lastly, Section 5 illustrate an example of the results.

2. Preliminary Results

Here, we introduce some basic concepts and auxiliary preliminary results that are crucial for the development of the rest of this paper.

Definition 1 ([20]). *Let the constants $a,\ b$ be such that $1 \leq a < b < +\infty$, and consider the function $\psi : [a,b] \longrightarrow (0,+\infty)$. The left-hand Hadamard fractional integral of order $\psi(t)$ (variable order) for a given function φ is defined as*

$$ {}^{H}I_{1^+}^{\psi(t)}\varphi(t) = \frac{1}{\Gamma(\psi(t))}\int_1^t \left(\ln\frac{t}{s}\right)^{\psi(t)-1}\frac{\varphi(s)}{s}ds, \quad t \in (a,+\infty). $$

Definition 2 ([20]). *We fix $n \in \mathbb{N}$ and consider $\psi : [a,b] \longrightarrow (n-1,n)$. The left-hand Hadamard derivative of order $\psi(t)$ (variable order) for a given function φ is defined as*

$$ {}^{H}\mathcal{D}_{1^+}^{\psi(t)}\varphi(t) = \frac{t^n}{\Gamma(n-\psi(t))}\frac{d^n}{dt^n}\left[\int_1^t \left(\ln\frac{t}{s}\right)^{n-1-\psi(t)}\frac{\varphi(s)}{s}ds\right], \quad t \in (a,+\infty). $$

It is a well-known fact that when we consider a fractional order given by a constant function $\psi(t) \equiv \psi$, the Hadamard fractional operators (i.e., integral and derivative) of variable order are identical to their constant-order counterparts; thus, the semi-group property yields the following properties

$$ {}^{H}I_{1^+}^{\psi_1}\,{}^{H}I_{1^+}^{\psi_2} = {}^{H}I_{1^+}^{\psi_2}\,{}^{H}I_{1^+}^{\psi_1} = {}^{H}I_{1^+}^{\psi_1+\psi_2}. $$

Using these properties as a basis, we can transform fractional-order differential equations into integral equations, and the corresponding transformation is an equivalence. On the other hand, to prove the existence and uniqueness of solutions to integral equations, we can apply some fixed-point results. However, recent studies have proved that this type of semi-group property is not generally true in the case of variable-order fractional operators. This makes it difficult to transform differential equations into integral equations in an equivalent way. In addition,

$$ \begin{aligned} {}^{H}I_{1^+}^{\psi_1(t)}\,{}^{H}I_{1^+}^{\psi_2(t)} &\neq {}^{H}I_{1^+}^{\psi_2(t)}\,{}^{H}I_{1^+}^{\psi_1(t)} \\ &\neq {}^{H}I_{1^+}^{\psi_1(t)+\psi_2(t)}, \end{aligned} $$

where $\psi_1(t)$ and $\psi_2(t)$ are general non-negative functions. We provide some examples to prove these claims.

Example 1. *In this example, we prove that*

$$ {}^{H}I_{1^+}^{\psi_1(t)}\,{}^{H}I_{1^+}^{\psi_2(t)}\varphi(t) \neq {}^{H}I_{1^+}^{\psi_1(t)+\psi_2(t)}\varphi(t). $$

Let $\psi_1(t) = \begin{cases} t+1, & t \in [1,3], \\ 1-t, & t \in (3,4], \end{cases} \quad \psi_2(t) = \begin{cases} 1, & t \in [1,3], \\ 2, & t \in (3,4], \end{cases} \quad \varphi(t) = 1, \quad 1 \leq t \leq 4,$

$$ \begin{aligned} {}^{H}I_{1^+}^{\psi_1(t)}\,{}^{H}I_{1^+}^{\psi_2(t)}\varphi(t) &= \frac{1}{\Gamma(\psi_1(t))}\int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{\psi_1(t)-1} \\ &\quad\times\left[\frac{1}{\Gamma(\psi_2(s))}\int_1^s \frac{1}{h}\left(\ln\frac{s}{h}\right)^{\psi_2(s)-1}\varphi(h)\,dh\right]ds \\ &= \frac{1}{\Gamma(\psi_1(t))}\int_1^3 \frac{1}{s}\left(\ln\frac{t}{s}\right)^{\psi_1(t)-1}\left[\frac{1}{\Gamma(1)}\int_1^s \frac{1}{h}\left(\ln\frac{s}{h}\right)^{1-1}dh\right]ds \\ &\quad+\frac{1}{\Gamma(\psi_1(t))}\int_3^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{\psi_1(t)-1}\left[\frac{1}{\Gamma(2)}\int_1^s \frac{1}{h}\left(\ln\frac{s}{h}\right)^{2-1}dh\right]ds. \end{aligned} $$

Therefore

$$ {}^{H}I_{1^+}^{\psi_1(t)}\, {}^{H}I_{1^+}^{\psi_2(t)}\varphi(t)\Big|_{t=3} = \frac{1}{\Gamma(4)}\int_1^3 \left(\ln\frac{3}{s}\right)^3 \ln s ds\Big|_{t=3} \approx 0,06069. $$

However,

$$ \begin{aligned} {}^{H}I_{1^+}^{\psi_1(t)+\psi_2(t)}\varphi(t)\Big|_{t=3} &= \frac{1}{\Gamma(\psi(t)+\psi_2(t))}\int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{\psi_1(t)+\psi_2(t)-1} ds\Big|_{t=3} \\ &= \frac{1}{\Gamma(4+1)}\int_1^t \frac{1}{s}\left(\ln\frac{3}{s}\right)^{4+1-1} ds \\ &= \frac{1}{\Gamma(6)}(\ln 3)^5 \approx 0.01333. \end{aligned} $$

Example 2. *In this example, we prove that*

$$ \begin{aligned} {}^{H}\mathcal{D}_{1^+}^{\psi_1(t)}\, {}^{H}\mathcal{D}_{1^+}^{\psi_2(t)}\varphi(t) &\neq {}^{H}\mathcal{D}_{1^+}^{\psi_2(t)}\, {}^{H}\mathcal{D}_{1^+}^{\psi_1(t)}\varphi(t) \\ &\neq {}^{H}\mathcal{D}_{1^+}^{\psi_1(t)+\psi_2(t)}\varphi(t). \end{aligned} $$

Let $0 < \psi_1(t) < 1$, $0 < \psi_2(t) < 1$, *where* $\psi_1(t) \neq \psi_2(t)$, *and let* $\varphi(t) = 1$, $t \in [1, T]$. *Then,*

$$ \begin{aligned} {}^{H}\mathcal{D}_{1^+}^{\psi_1(t)}\, {}^{H}\mathcal{D}_{1^+}^{\psi_2(t)}\varphi(t) &= \frac{t}{\Gamma(1-\psi_1(t))}\frac{d}{dt}\int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{-\psi_1(t)} \\ &\quad \times \left[\frac{s}{\Gamma(1-\psi_2(s))}\frac{d}{ds}\int_1^s \frac{1}{h}\left(\ln\frac{s}{h}\right)^{-\psi_2(s)} dh\right] ds \\ &= \frac{t}{\Gamma(1-\psi_1(t))}\frac{d}{dt}\int_1^t \left(\ln\frac{t}{s}\right)^{-\psi_1(t)} \\ &\quad \times \left[\frac{1}{\Gamma(1-\psi_2(s))}\frac{d}{ds}\left(\frac{(-\ln s)^{1-\psi_2(s)}}{1-\psi_2(s)}\right)\right] ds \\ &= \frac{-t}{\Gamma(1-\psi_1(t))}\frac{d}{dt}\int_1^t \left(\ln\frac{t}{s}\right)^{-\psi_1(t)} \frac{(\ln s)^{-\psi_2(s)}}{s\Gamma(1-\psi_2(s))} ds \\ &= -\, {}^{H}\mathcal{D}_{1^+}^{\psi_1(t)}\left(\frac{(\ln t)^{-\psi_2(t)}}{\Gamma(1-\psi_2(t))}\right), \\ {}^{H}\mathcal{D}_{1^+}^{\psi_2(t)}\, {}^{H}\mathcal{D}_{1^+}^{\psi_1(t)}\varphi(t) &= \frac{t}{\Gamma(1-\psi_2(t))}\frac{d}{dt}\int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{-\psi_2(t)} \\ &\quad \times \left[\frac{s}{\Gamma(1-\psi_1(s))}\frac{d}{ds}\int_1^s \frac{1}{h}\left(\ln\frac{s}{h}\right)^{-\psi_1(s)} dh\right] ds \\ &= \frac{t}{\Gamma(1-\psi_2(t))}\frac{d}{dt}\int_1^t \left(\ln\frac{t}{s}\right)^{-\psi_2(t)} \\ &\quad \times \left[\frac{1}{\Gamma(1-\psi_1(s))}\frac{d}{ds}\left(\frac{(-\ln s)^{1-\psi_1(s)}}{1-\psi_1(s)}\right)\right] ds \\ &= \frac{-t}{\Gamma(1-\psi_2(t))}\frac{d}{dt}\int_1^t \left(\ln\frac{t}{s}\right)^{-\psi_2(t)} \frac{(\ln s)^{-\psi_1(s)}}{s\Gamma(1-\psi_1(s))} ds \\ &= -\, {}^{H}\mathcal{D}_{1^+}^{\psi_2(t)}\left(\frac{(\ln t)^{-\psi_1(t)}}{\Gamma(1-\psi_1(t))}\right), \end{aligned} $$

$$
\begin{aligned}
{}^{H}\mathcal{D}_{1^+}^{\psi_1(t)+\psi_2(t)}\varphi(t) &= \frac{t^2}{\Gamma(2-\psi_1(t)-\psi_2(t))}\frac{d^2}{dt^2}\int_1^t \frac{1}{s}\left(\ln\frac{t}{s}\right)^{1-\psi_1(t)-\psi_2(t)}\varphi(s)ds \\
&= \frac{t^2}{\Gamma(2-\psi_1(t)-\psi_2(t))}\frac{d^2}{dt^2}\left[\frac{-(\ln t)^{2-\psi_1(t)-\psi_2(t)}}{2-\psi_1(t)-\psi_2(t)}\right] \\
&= \frac{(\psi_1(t)+\psi_2(t))(\ln t)^{-\psi_1(t)-\psi_2(t)}}{\Gamma(2-\psi_1(t)-\psi_2(t))}.
\end{aligned}
$$

In this example, we illustrate that variable-order fractional calculus does not inherit the semigroup property exhibited by constant-order fractional calculus.

Definition 3. *Consider E a Banach space, and let $\Omega \subset E$ be a subset. We denote the set of continuous functions as $\varphi : \Omega \longrightarrow E$ by $C(\Omega, E)$, endowed with the usual supremum norm*

$$
||\varphi||_\infty := \sup\{|\varphi(t)|, \quad t \in \Omega\},
$$

which ensures a Banach space structure. We also consider $L^1(\Omega, E)$ as the Banach space of measurable functions $\varphi : \Omega \longrightarrow E$ that are Bochner-integrable, endowed with the norm

$$
||\varphi||_{L^1} := \int_\Omega |\varphi(t)|dt, \qquad t \in \Omega.
$$

We can also deduce that $PC(\Omega, E) := \{\psi : \Omega \longrightarrow E\ ,\ \psi : (t_{i-1}, t_i] \longrightarrow E$ is piecewise continuous for all $i = 1, \dots, n\}$,

$$
\begin{aligned}
\psi_i : (t_{i-1}, t_i] &\longrightarrow E \\
t &\longrightarrow \psi_i(t) = \psi(t).
\end{aligned}
$$

$PC(\Omega, E)$ also has the structure of a Banach space with the norm

$$
||\psi||_{PC} := \sup_{i=\overline{1,n}} ||\psi_i||.
$$

Definition 4 ([21])**.** *Consider a subset S of the space $\mathbb{R}$:*

(i) *It is said that S is a generalized interval if it is either a standard interval, a point, or $\emptyset$.*
(ii) *If S is a generalized interval, the finite set $\mathcal{P}$ consisting of generalized intervals contained in S is called a partition of S, provided that every $x \in S$ belongs to exactly one of the generalized intervals in the finite set $\mathcal{P}$.*
(iii) *Obviously, the function $\psi : t \mapsto \mathbb{R}$ is piecewise constant with respect to the partition $\mathcal{P}$ of S. In other words, for every $W \in S$, ψ is constant on W.*

We state the following propositions in order to prove the definition and the continuity of the left-hand Hadamard variable-order integral.

Proposition 1. *Consider $\psi \in C([1, T], (1, 2])$ and the space*

$$
C_\alpha([1, T], \mathbb{R}) = \{\varphi(t) \in C([1, T], \mathbb{R}),\ (\ln t)^\alpha \varphi(t) \in C([1, T], \mathbb{R}), \quad \alpha \in (0, 1)\}.
$$

If $\varphi \in C_\alpha([1, T], \mathbb{R})$, then the left-hand Hadamard variable-order integral ${}^{H}I_{1^+}^{\psi(t)}\varphi(t)$ exists for every $t \in [1, T]$.

Proof. We take the function $\Gamma(\psi(t))$, which is continuous and non-zero on $[1,T]$. Let $M_\psi := \max_{t\in[1,T]}\left|\frac{1}{\Gamma(\psi(t))}\right| > 0$. Thus, for $1 \le s \le t \le T$, we have

$$\begin{cases} \left(\ln \frac{t}{s}\right)^{\psi(t)-1} \le 1, & \text{if} \quad 1 \le \frac{t}{s} \le e, \\ \left(\ln \frac{t}{s}\right)^{\psi(t)-1} \le \left(\ln \frac{t}{s}\right)^{\psi_{\max}-1}, & \text{if} \quad \frac{t}{s} > e. \end{cases}$$

Obviously, for $1 \le \frac{t}{s} < +\infty$, we obtain

$$\left(\ln \frac{t}{s}\right)^{\psi(t)-1} \le \max\left\{1, \left(\ln \frac{t}{s}\right)^{\psi_{\max}-1}\right\} = M^*.$$

Then, by applying the function $(\ln(\cdot))^\alpha$ for any $\alpha \in (0,1)$, it is possible to conclude that

$$\begin{aligned} \left|{}^H I_{1^+}^{\psi(t)}\varphi(t)\right| &= \frac{1}{\Gamma(\psi(t))}\int_1^t \left(\ln \frac{t}{s}\right)^{\psi(t)-1} \frac{|\varphi(s)|}{s} ds \\ &\le M_\psi \int_1^t \left(\ln \frac{t}{s}\right)^{\psi(t)-1} (\ln s)^{-\alpha} (\ln s)^{\alpha} \frac{|\varphi(s)|}{s} \\ &\le M_\psi M^* \sup_{t\in[1,T]} |(\ln t)^\alpha \varphi(t)| \int_1^t \frac{(\ln s)^{-\alpha}}{s} ds \\ &\le M_\psi M^* \sup_{t\in[1,T]} |(\ln t)^\alpha \varphi(t)| \frac{\ln T}{1-\alpha} < +\infty. \end{aligned}$$

This implies that the fractional variable-order integral ${}^H I_{1^+}^{\psi(t)}$ is well defined for each $t \in [1,T]$. $\square$

Proposition 2. *Let $\psi \in C([1,T],(1,2])$. Then, ${}^H I_{1^+}^{\psi(t)}\varphi(t) \in C([1,T],\mathbb{R})$ for every $\varphi \in C([1,T],\mathbb{R})$.*

Proof. For $t_1, t_2 \in [1,T]$, $t_1 \le t_2$, and $\varphi \in C([1,T],\mathbb{R})$, we obtain

$$\begin{aligned} \left|{}^H I_{1^+}^{\psi(t_1)}\varphi(t_1) - {}^H I_{1^+}^{\psi(t_2)}\varphi(t_2)\right| = &\left|\int_1^{t_1} \frac{1}{\Gamma(\psi(t_1))}\left(\ln \frac{t_1}{s}\right)^{\psi(t_1)-1} \frac{\varphi(s)}{s} ds \right. \\ &\left. - \int_1^{t_2} \frac{1}{\Gamma(\psi(t_2))}\left(\ln \frac{t_2}{s}\right)^{\psi(t_2)-1} \frac{\varphi(s)}{s} ds\right|. \end{aligned}$$

Consider the following change of variables: $s = \tau(t_i - 1) + 1$, where $i = 1,2$:

$$\begin{aligned} = &\left|\int_0^1 \frac{1}{\Gamma(\psi(t_1))} \frac{(t_1-1)}{\tau(t_1-1)+1}\left(\ln \frac{t_1}{\tau(t_1-1)+1}\right)^{\psi(t_1)-1} \varphi(\tau(t_1-1)+1) d\tau \right. \\ &\left. - \int_0^1 \frac{1}{\Gamma(\psi(t_2))} \frac{(t_2-1)}{\tau(t_2-1)+1}\left(\ln \frac{t_2}{\tau(t_2-1)+1}\right)^{\psi(t_2)-1} \varphi(\tau(t_2-1)+1) d\tau\right| \end{aligned}$$

$$
\begin{aligned}
&= \Bigg| \int_0^1 \Bigg[\frac{1}{\Gamma(\psi(t_1))} \frac{(t_1-1)}{\tau(t_1-1)+1} \left(\ln \frac{t_1}{\tau(t_1-1)+1} \right)^{\psi(t_1)-1} \varphi(\tau(t_1-1)+1) \\
&\quad - \frac{1}{\Gamma(\psi(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_1}{\tau(t_1-1)+1} \right)^{\psi(t_1)-1} \varphi(\tau(t_1-1)+1) \Bigg] d\tau \\
&\quad + \int_0^1 \Bigg[\frac{1}{\Gamma(\psi(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_1}{\tau(t_1-1)+1} \right)^{\psi(t_1)-1} \varphi(\tau(t_1-1)+1) \\
&\quad - \frac{1}{\Gamma(\psi(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} \varphi(\tau(t_1-1)+1) \Bigg] d\tau \\
&\quad + \int_0^1 \Bigg[\frac{1}{\Gamma(\psi(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} \varphi(\tau(t_1-1)+1) \\
&\quad - \frac{1}{\Gamma(\psi(t_2))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} \varphi(\tau(t_1-1)+1) \Bigg] d\tau \\
&\quad + \int_0^1 \Bigg[\frac{1}{\Gamma(\psi(t_2))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} \varphi(\tau(t_1-1)+1) \\
&\quad - \frac{1}{\Gamma(\psi(t_2))} \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} \varphi(\tau(t_2-1)+1) \Bigg] d\tau \Bigg| \\
&\leq \sup_{t\in[1,T]} |\varphi(t)| \int_0^1 \frac{1}{\Gamma(\psi(t_1))} \left(\ln \frac{t_1}{\tau(t_1-1)+1} \right)^{\psi(t_1)-1} \left| \frac{(t_1-1)}{\tau(t_1-1)+1} - \frac{(t_2-1)}{\tau(t_2-1)+1} \right| d\tau \\
&\quad + \sup_{t\in[1,T]} |\varphi(t)| \int_0^1 \frac{1}{\Gamma(\psi(t_1))} \frac{(t_2-1)}{\tau(t_2-1)+1} \\
&\quad \times \left| \left(\ln \frac{t_1}{\tau(t_1-1)+1} \right)^{\psi(t_1)-1} - \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} \right| d\tau \\
&\quad + \sup_{t\in[1,T]} |\varphi(t)| \int_0^1 \frac{(t_2-1)}{\tau(t_2-1)+1} \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} \left| \frac{1}{\Gamma(\psi(t_1))} - \frac{1}{\Gamma(\psi(t_2))} \right| d\tau \\
&\quad + \int_0^1 \frac{1}{\Gamma(\psi(t_2))} \frac{(t_2-1)}{\tau(t_2-1)+1} \\
&\quad \times \left(\ln \frac{t_2}{\tau(t_2-1)+1} \right)^{\psi(t_2)-1} |\varphi(\tau(t_1-1)+1) - \varphi(\tau(t_2-1)+1)| d\tau.
\end{aligned}
$$

Since the following functions are continuous $\frac{(t-1)}{\tau(t-1)+1}$, $\left(\ln\left(\frac{t}{\tau(t-1)+1}\right)\right)^{\psi(t)-1}$ $\frac{1}{\Gamma(\psi(t))}$, and $\varphi(t)$, we deduce that the left-hand Hadamard fractional integral is continuous for all $t \in [1,T]$. Therefore, ${}^{H}I_{1^+}^{\psi(t)}\varphi(t)$ belongs to the space $C([1,T],\mathbb{R})$ for each $\varphi \in C([1,T],\mathbb{R})$. □

The following Lemma is crucial in this paper for establishing the connection between a differential equation and its integral counterpart.

Lemma 1 ([10]). *Let $\alpha > 0$, and let a and b be constants such that $1 < a < b$. Suppose $\varphi \in L^1(a,b)$ is such that ${}^{H}\mathcal{D}_{a^+}^{\alpha}\varphi \in L^1(a,b)$. Then, we have the following properties:*

The fractional differential equation

$$ {}^{H}\mathcal{D}_{a^+}^{\alpha}\varphi = 0 $$

has, as a solution,

$$ \varphi(t) = C_1\left(\ln\frac{t}{a}\right)^{\alpha-1} + C_2\left(\ln\frac{t}{a}\right)^{\alpha-2} + \cdots + C_n\left(\ln\frac{t}{a}\right)^{\alpha-n} = \sum_{i=1}^{n} C_i\left(\ln\frac{t}{a}\right)^{\alpha-i}, $$

where $C_i \in \mathbb{R}$. Additionally,

$$ {}^{H}I^{\alpha}_{a^+}({}^{H}\mathcal{D}^{\alpha}_{a^+})\varphi(t) = \varphi(t) + \sum_{i=1}^{n} C_i \left(\ln \frac{t}{a}\right)^{\alpha-i}, $$

where we have taken $n = [\alpha] + 1$.

Furthermore,

$$ {}^{H}\mathcal{D}^{\alpha}_{a^+}\,({}^{H}I^{\alpha}_{a^+})\varphi(t) = \varphi(t). $$

Theorem 1 ([22])**.** *Suppose that X is a non-empty closed and convex subset of a Banach space E. Let $f : X \longrightarrow E$ be a continuous and condensing mapping satisfying either [22] (Equation (1), p 461) or [22] (Equation (2), p 462). If the range of f is bounded, then it has a fixed point.*

Theorem 2 ([10])**.** *Suppose that Ω is a non-empty bounded closed and convex subset of a real Banach space E, and consider F_1, F_2 two operators that are defined on Ω and satisfy the following hypotheses:*

(a) $F_1(\Omega) + F_2(\Omega) \subset \Omega$.
(b) F_1 is continuous on Ω, and $F_1(\Omega)$ is a relatively compact subset of E.
(c) F_2 is a (strict) contraction on Ω, that is, there exists a constant $k \in [0, 1)$ such that

$$ ||F_2(x) - F_2(y)|| \leq k||x - y||, \quad \text{for all } x, y \in \Omega. $$

Then, the equation $F_1(x) + F_2(x) = x$ admits a solution on Ω.

3. Existence of the Solution

In this section, we include the main results concerning the existence and uniqueness of the solution.

We consider $P = [1, t_1], (t_1, t_2], (t_2, t_3], \ldots, (t_n, T]$ (where n is a fixed natural number) as a partition of the compact interval $[1, T]$. We then select a function $\psi : [1, T] \longrightarrow (1, 2)$ that is piecewise constant with respect to the partition P, which means that

$$ \psi(t) = \sum_{i=1}^{n} \psi_i \mathbb{I}_i(t), \quad t \in [1, T], $$

where the constants $\psi_i \in (1, 2)$ for all $i \in \{1, 2, \ldots, n\}$, and $\mathbb{I}_i$ represents the characteristic function of $[t_{i-1}, t_i]$, for all $i \in \{1, 2, \ldots, n\}$, that is,

$$ \mathbb{I}_i(t) = \begin{cases} 1 & t \in [t_{i-1}, t_i], \\ 0 & \text{otherwise}. \end{cases} $$

Hence, we obtain

$$ \begin{aligned} {}^{H}\mathcal{D}^{\psi(t)}_{1^+}\varphi(t) &= \frac{t^2}{\Gamma\left(2 - \sum_{i=1}^{n} \psi_i \mathbb{I}_i(t)\right)} \frac{d^2}{dt^2} \left[\int_1^t \left(\ln \frac{t}{s}\right)^{1-\sum_{i=1}^{n} \psi_i \mathbb{I}_i(t)} \frac{\varphi(s)}{s} ds \right] \\ &= \frac{t^2}{\Gamma(2 - \psi(t))} \frac{d^2}{dt^2} \left[\int_1^{t_1} \left(\ln \frac{t}{s}\right)^{1-\psi_1} \frac{\varphi(s)}{s} ds \right] + \frac{t^2}{\Gamma(2 - \psi(t))} \frac{d^2}{dt^2} \left[\int_{t_1}^{t_2} \left(\ln \frac{t}{s}\right)^{1-\psi_2} \frac{\varphi(s)}{s} ds \right] \\ &\quad + \cdots + \frac{t^2}{\Gamma(2 - \psi(t))} \frac{d^2}{dt^2} \left[\int_{t_n}^{t} \left(\ln \frac{t}{s}\right)^{1-\psi_n} \frac{\varphi(s)}{s} ds \right] \\ &= \frac{t^2}{\Gamma(2 - \psi(t))} \frac{d^2}{dt^2} \left[\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t}{s}\right)^{1-\psi_i} \frac{\varphi(s)}{s} ds + \int_{t_n}^{t} \left(\ln \frac{t}{s}\right)^{1-\psi_n} \frac{\varphi(s)}{s} ds \right] \end{aligned} $$

$$= \frac{t^2}{\Gamma(2-\psi(t))}\left[\sum_{i=1}^{n}\frac{d^2}{dt^2}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t}{s}\right)^{1-\psi_i}\frac{\varphi(s)}{s}ds + \frac{d^2}{dt^2}\int_{t_n}^{t}\left(\ln\frac{t}{s}\right)^{1-\psi_n}\frac{\varphi(s)}{s}ds\right].$$

Thus, the equation of the problem (3) is rewritten in the following way:

$$\begin{aligned}{}^{H}\mathcal{D}_{1^+}^{\psi(t)}\varphi(t) =& \frac{t^2}{\Gamma(2-\psi(t))}\left[\sum_{i=1}^{n}\frac{d^2}{dt^2}\int_{t_{i-1}}^{t_i}\left(\ln\frac{t}{s}\right)^{1-\psi_i}\frac{\varphi(s)}{s}ds + \frac{d^2}{dt^2}\int_{t_n}^{t}\left(\ln\frac{t}{s}\right)^{1-\psi_n}\frac{\varphi(s)}{s}ds\right]\\ =& g(t,\varphi(t),\varphi(\lambda t)).\end{aligned}$$

In particular, if we consider the interval $[1,t_1]$, the expression is adapted as

$$ {}^{H}\mathcal{D}_{1^+}^{\psi_1}\widehat{\varphi}(t) = \frac{t^2}{\Gamma(2-\psi_1)}\frac{d^2}{dt^2}\left[\int_1^t\left(\ln\frac{t}{s}\right)^{1-\psi_1}\frac{\widehat{\varphi}(s)}{s}ds\right]. \tag{4}$$

Again, in the interval $(t_1,t_2]$, the expression can be written as

$$ {}^{H}\mathcal{D}_{1^+}^{\psi_2}\widehat{\varphi}(t) = \frac{t^2}{\Gamma(2-\psi_1)}\left[\frac{d^2}{dt^2}\int_1^{t_1}\left(\ln\frac{t}{s}\right)^{1-\psi_1}\frac{\widehat{\varphi}(t)}{s}ds + \frac{d^2}{dt^2}\int_{t_1}^{t}\left(\ln\frac{t}{s}\right)^{1-\psi_2}\frac{\widehat{\varphi}(t)}{s}ds\right]. \tag{5}$$

In the same way, if we consider the particular interval $(t_{i-1},t_i]$, the expression reduces to

$$ {}^{H}\mathcal{D}_{1^+}^{\psi_i}\widehat{\varphi}(t) = \frac{t^2}{\Gamma(2-\psi_i)}\left[\sum_{k=1}^{i-1}\frac{d^2}{dt^2}\int_1^{t_k}\left(\ln\frac{t}{s}\right)^{1-\psi_k}\frac{\widehat{\varphi}(t)}{s}ds + \frac{d^2}{dt^2}\int_{t_{i-1}}^{t}\left(\ln\frac{t}{s}\right)^{1-\psi_i}\frac{\widehat{\varphi}(s)}{s}ds\right]. \tag{6}$$

Consider the set of functions $E_i := C([t_{i-1},t_i],\mathbb{R})$, and define the norm

$$||\varphi||_{E_i} := \sup_{t\in[t_{i-1},t_i]}|\varphi(t)|, \quad i\in\{1,2,\dots,n\},$$

which gives E_i the structure of a Banach space.

We consider the function $\widehat{\varphi}\in E_i$, which satisfies the property that $\widehat{\varphi}(t)=0$ for all $t\in[1,t_{i-1}]$ for each $i\in\{2,\dots,n\}$. This function serves as a solution to the above equations for any $i=1,\dots,n$. Thus, we consider the following auxiliary boundary value problems for Hadamard-type constant-order fractional equations

$$\begin{cases} {}^{H}\mathcal{D}_{t_{i-1}^+}^{\psi_i}\widehat{\varphi}(t) = \dfrac{t^2}{\Gamma(2-\psi_i)}\dfrac{d^2}{dt^2}\left[\displaystyle\int_{t_{i-1}}^{t}\left(\ln\frac{t}{s}\right)^{1-\psi_i}\frac{\widehat{\varphi}(s)}{s}ds\right] = g(t,\widehat{\varphi}(t),\widehat{\varphi}(\lambda t)), \ \ t_{i-1}<t\le t_i,\\ \widehat{\varphi}(t_{i-1}) = \widehat{\varphi}(t_i) = 0.\end{cases} \tag{7}$$

Definition 5. *It is said that the problem (3) has a solution φ if there exist functions φ_i, with $\varphi_1\in E_1$ satisfying Equation (4); $\varphi_1(1)=\varphi_1(t_1)=0$; $\varphi_2\in E_2$ satisfying Equation (5); $\varphi_2(t_1)=\varphi_2(t_2)=0$; $\varphi_i\in E_i$ satisfying Equation (6); and $\varphi_i(t_{i-1})=\varphi_i(t_i)=0$ $(i=3,\dots,n)$.*

Remark 1. *The problem (3) is said to have a unique solution if the above-mentioned functions φ_i are unique.*

By combining all the previous information, we can prove the following results.

Lemma 2. *Suppose that $i\in\{1,\dots,n\}$ is a natural number, and suppose that there exists $\alpha\in(0,1)$ such that $(\ln t)^{\alpha}g\in C((t_{i-1},t_i]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$. Then, the function $\widehat{\varphi}$ is a solution to (7) if and only if $\widehat{\varphi}$ is the solution to the following integral equation*

$$\widehat{\varphi}(t) = -\left(\ln\frac{t_i}{t_{i-1}}\right)^{1-\psi_i}\left(\ln\frac{t}{t_{i-1}}\right)^{\psi_i-1}\left({}^{H}I_{t_{i-1}^+}^{\psi_i}g(t_i,\widehat{\varphi}(t_i),\widehat{\varphi}(\lambda t_i))\right) + {}^{H}I_{t_{i-1}^+}^{\psi_i}g(t,\widehat{\varphi}(t),\widehat{\varphi}(\lambda t)).$$

Proof. Supposing that $\widehat{\varphi}$ satisfies (7). Then, we can transform (7) into an equivalent integral equation. For $t \in (t_{i-1}, t_i]$, according to Lemma 1, we have

$$\begin{aligned} {}^{H}I^{\psi_i}_{t^+_{i-1}} \, {}^{H}\mathcal{D}^{\psi_i}_{t^+_{i-1}} \widehat{\varphi}(t) &= {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t, \widehat{\varphi}(t), \widehat{\varphi}(\lambda t)), \\ \widehat{\varphi}(t) + \sum_{k=1}^{2} C_k \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i - k} &= {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t, \widehat{\varphi}(t), \widehat{\varphi}(\lambda t)), \\ \widehat{\varphi}(t) &= \sum_{k=1}^{2} C_k \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i - k} + {}^{H}I^{\psi}_{t^+_{i-1}} g(t, \widehat{\varphi}(t), \widehat{\varphi}(\lambda t)). \end{aligned}$$

From the boundary conditions $\widehat{\varphi}(t_i) = \widehat{\varphi}(t_{i-1}) = 0$, we deduce that

$$\begin{cases} C_1 &= -\left(\ln \dfrac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left[C_2 \left(\ln \dfrac{t}{t_{i-1}}\right)^{\psi_i - 2} + {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t_i, \widehat{\varphi}(t_i), \widehat{\varphi}(\lambda t_i)) \right], \\ 0 &= C_2 \left[-\left(\ln \dfrac{t_i}{t_{i-1}}\right)^{\psi_i - 2} {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t_i, \widehat{\varphi}(t_i), \widehat{\varphi}(\lambda t_i)) \right]. \end{cases}$$

Therefore, we obtain

$$\begin{cases} C_2 = 0, \\ C_1 = -\left(\ln \dfrac{t_i}{t_{i-1}}\right)^{1-\psi_i} {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t_i, \widehat{\varphi}(t_i), \widehat{\varphi}(\lambda t_i)). \end{cases}$$

In conclusion, the expression of the solution for the auxiliary boundary value problem (7) is

$$\begin{aligned} \widehat{\varphi}(t) = &-\left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i - 1} {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t_i, \widehat{\varphi}(t_i), \widehat{\varphi}(\lambda t_i)) \\ &+ {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t, \widehat{\varphi}(t), \widehat{\varphi}(\lambda t)), \quad t_{i-1} < t \le t_i. \end{aligned}$$

□

In order to prove the most relevant results in this section, we establish some hypotheses.

Hypothesis 1. *For $0 \le \alpha \le 1$, let $(\ln(\cdot))^{\alpha} g : [1, T] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, such that there exists $L > 0$, which satisfies*

$$(\ln t)^{\alpha} |g(t, x_1, y_1) - g(t, x_2, y_2)| \le L(|x_1 - x_2| + |y_1 - y_2|),$$

for every $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Hypothesis 2. *Suppose that*

$$\frac{4L}{(1-\alpha)\Gamma(\psi_i)} \left[(\ln t_i)^{1-\alpha} - (\ln t_{i-1})^{1-\alpha} \right] \left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\alpha} \le 1.$$

Theorem 3. *Suppose that Hypothesis 1 and Hypothesis 2 hold. Under these circumstances, the boundary value problem (3) has exactly one solution in the space $C([1, T], \mathbb{R})$.*

Proof. Define the following mapping $F : E_i \longrightarrow E_i$, which is well defined and given by

$$(F\widehat{\varphi})(t) = -\left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i - 1} {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t_i, \widehat{\varphi}(t_i), \widehat{\varphi}(\lambda t_i)) + {}^{H}I^{\psi_i}_{t^+_{i-1}} g(t, \widehat{\varphi}(t), \widehat{\varphi}(\lambda t)). \quad (8)$$

Let the ball $B_{R_i} = \{\varphi \in E_i : ||\varphi||_{E_i} \leq R_i\}$, where

$$R_i \geq \frac{\frac{2}{\Gamma(\psi_i+1)} \sup_{t\in[1,T]} |g(t,0,0)| \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i}}{1 - \frac{4L}{(1-\alpha)\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i - 1} [(\ln t_i)^{\alpha-1} - (\ln t_{i-1})^{\alpha-1}]}.$$

Note that B_{R_i} is a non-empty, closed, bounded, and convex subset of the space E_i. We split the mapping F into the mappings F_1 and F_2 defined on B_{R_i} in the following way

$$\begin{cases} (F_1\varphi)(t) &= -\left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i-1} {}^H I^{\psi_i}_{t^+_{i-1}} g(t_i, \varphi(t_i), \varphi(\lambda t_i)), \\ (F_2\varphi)(t) &= {}^H I^{\psi_i}_{t^+_{i-1}} g(t, \varphi(t), \varphi(\lambda t)). \end{cases}$$

Step 1: $F_1(B_{R_i}) + F_2(B_{R_i}) \subset B_{R_i}$.

$$\begin{aligned} |F_1(\varphi)(t) + F_2(\varphi)(t)| &= \left| -\left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i-1} {}^H I^{\psi_i}_{t^+_{i-1}} g(t_i, \varphi(t_i), \varphi(\lambda t_i)) \right. \\ &\quad \left. + {}^H I^{\psi_i}_{t^+_{i-1}} g(t, \varphi(t), \varphi(\lambda t)) \right| \\ &\leq \left| {}^H I^{\psi_i}_{t^+_{i-1}} g(t_i, \varphi(t_i), \varphi(\lambda t_i)) + {}^H I^{\psi_i}_{t^+_{i-1}} g(t, \varphi(t), \varphi(\lambda t)) \right| \\ &\leq \left| {}^H I^{\psi_i}_{t^+_{i-1}} g(t_i, \varphi(t_i), \varphi(\lambda t_i)) \right| + \left| {}^H I^{\psi_i}_{t^+_{i-1}} g(t, \varphi(t), \varphi(\lambda t)) \right| \\ &\leq \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\psi_i-1} \frac{|g(s, \varphi(s), \varphi(\lambda s))|}{s} ds \\ &\quad + \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t} \left(\ln \frac{t}{s}\right)^{\psi_i-1} \frac{|g(s, \varphi(s), \varphi(\lambda s))|}{s} ds \\ &\leq \frac{2}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\psi_i-1} (\ln s)^{-\alpha} (\ln s)^{\alpha} \\ &\quad \times \frac{|g(s, \varphi(s), \varphi(\lambda s)) - g(s,0,0)|}{s} ds \\ &\quad + \frac{2}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\psi_i-1} \frac{|g(s,0,0)|}{s} ds \\ &\leq \frac{2}{\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i-1} \int_{t_{i-1}}^{t_i} \frac{(\ln s)^{-\alpha}}{s} L(|\varphi(s)| + |\varphi(\lambda s)|)\, ds \\ &\quad + \frac{2}{\Gamma(\psi_i)} \sup_{t\in[t_i, t_{i-1}]} |g(t,0,0)| \int_{t_{i-1}}^{t_i} \left(\ln \frac{t_i}{s}\right)^{\psi_i-1} ds \\ &\leq \frac{4L||\varphi||_{E_i}}{\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i-1} \left.\frac{(\ln s)^{1-\alpha}}{1-\alpha}\right|_{t_{i-1}}^{t_i} \\ &\quad + \frac{2}{\Gamma(\psi_i+1)} \sup_{t\in[t_{i-1}, t_i]} |g(t,0,0)| \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i} \\ &\leq \frac{4L||\varphi||_{E_i}}{(1-\alpha)\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i-1} [(\ln t_i)^{1-\alpha} - (\ln t_{i-1})^{1-\alpha}] \end{aligned}$$

$$+\frac{2}{\Gamma(\psi_i+1)}\sup_{t\in[t_{i-1},t_i]}|g(t,0,0)|\left(\ln\frac{t_i}{t_{i-1}}\right)^{\psi_i}$$
$$\le R_i.$$

Step 2: F_1 is a contraction:

$$\begin{aligned}|F_1(\varphi_1)(t)-F_1(\varphi_2)(t)| &\le \frac{\left(\ln\frac{t_i}{t_{i-1}}\right)^{1-\psi_i}\left(\ln\frac{t}{t_{i-1}}\right)^{\psi_i-1}}{\Gamma(\psi_i)}\\ &\quad\times\left[\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\psi_i-1}\frac{|g(s,\varphi_1(s),\varphi_1(\lambda s))|}{s}ds\right.\\ &\quad\left.-\int_{t_{i-1}}^{t_i}\left(\ln\frac{t_i}{s}\right)^{\psi_i-1}\frac{|g(s,\varphi_2(s),\varphi_2(\lambda s))|}{s}ds\right]\\ &\le\frac{1}{\Gamma(\psi_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{\psi_i-1}\\ &\quad\times\int_{t_{i-1}}^{t_i}\frac{1}{s}(\ln s)^{-\alpha}[L(|\varphi_1(s)-\varphi_2(s)|+|\varphi_1(\lambda s)-\varphi_2(\lambda s)|)]ds\\ &\le\frac{2L||\varphi_1(t)-\varphi_2(t)||_{E_i}}{(1-\alpha)\Gamma(\psi_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{\psi_i-1}[(\ln t_i)^{1-\alpha}-(\ln t_{i-1})^{1-\alpha}].\end{aligned}$$

Therefore, the mapping F_1 is a contraction mapping thanks to Hypothesis 2.
Step 3: F_2 is continuous and $F_2(B_R)$ is relatively compact:

1. F_2 is a continuous mapping.
   We consider a sequence $\{\varphi_n\}$ with $\varphi_n\longrightarrow\varphi$ in B_R. Then, for every $t\in[t_{i-1},t_i]$, $i\in\{1,2,\dots,n\}$, we have

$$\begin{aligned}|F_2(\varphi_n)(t)-F_2(\varphi)(t)| &\le\frac{1}{\Gamma(\psi_i)}\int_{t_{i-1}}^{t_i}\frac{1}{s}\left(\ln\frac{t}{t_{i-1}}\right)^{\psi_i-1}\\ &\quad\times|g(s,\varphi_n(s),\varphi_n(\lambda s))-g(s,\varphi(s),\varphi(\lambda s))|ds\\ &\le\frac{L}{\Gamma(\psi_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{\psi_i-1}\int_{t_{i-1}}^{t_i}\frac{(\ln s)^{-\alpha}}{s}\\ &\quad\times(|\varphi_n(s)-\varphi(s)|+|\varphi_n(\lambda s)-\varphi(\lambda s)|)ds\\ &\le\frac{2L}{\Gamma(\psi_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{\psi_i-1}\frac{(\ln s)^{1-\alpha}}{(1-\alpha)}\bigg|_{t_{i-1}}^{t_i}||\varphi_n-\varphi||_{E_i}\\ &\le\frac{2L}{\Gamma(\psi_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{\psi_i-1}[(\ln t_i)^{1-\alpha}-(\ln t_{i-1})^{1-\alpha}]||\varphi_n-\varphi||_{E_i}.\end{aligned}$$

   This implies that

$$||F_2(\varphi_n)(t)-F_2(\varphi)(t)||_{E_i}\longrightarrow 0\quad\text{as}\quad n\to+\infty.$$

2. F_2 is equicontinuous:

$$\begin{aligned}|F_2(\varphi)(t_2)-F_2(\varphi)(t_1)| &=\left|\frac{1}{\Gamma(\psi_i)}\int_{t_{i-1}}^{t_2}\frac{1}{s}\left(\ln\frac{t_2}{s}\right)^{\psi_i-1}g(s,\varphi(s),\varphi(\lambda s))ds\right.\\ &\quad\left.-\frac{1}{\Gamma(\psi_i)}\int_{t_{i-1}}^{t_1}\frac{1}{s}\left(\ln\frac{t_1}{s}\right)^{\psi_i-1}g(s,\varphi(s),\varphi(\lambda s))\right|ds\end{aligned}$$

$$= \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_1} \frac{g(s,\varphi(s),\varphi(\lambda s))}{s} \left[\left(\ln \frac{t_2}{s} \right)^{\psi_i - 1} - \left(\ln \frac{t_1}{s} \right)^{\psi_i - 1} \right] ds$$
$$+ \frac{1}{\Gamma(\psi_i)} \int_{t_1}^{t_2} \frac{1}{s} \left(\ln \frac{t_2}{s} \right)^{\psi_i - 1} g(s,\varphi(s,\varphi(\lambda s)))ds.$$

Provided that $t_1 \longrightarrow t_2$, the right-hand term in the previous inequality tends to zero. As a consequence, F_2 is an equicontinuous mapping, and it is also uniformly bounded by Step 1.

Therefore, $F_2(B_{R_i})$ is relatively compact according to the Ascoli–Arzelà theorem.

From Theorem 2, we know that the auxiliary boundary value problem (7) has at least one solution in the set B_{R_i} for every $i \in \{1, 2, \ldots, n\}$.

To conclude, the boundary value problem (3) has at least one solution in the space $C([1, T], \mathbb{R})$, which is defined as

$$\varphi(t) = \begin{cases} \varphi_1(t) = \widehat{\varphi_1}(t), & \text{for } t \in [1, t_1], \\ \varphi_2(t) = \begin{cases} 0, & \text{for } t \in [1, t_1], \\ \widehat{\varphi_2}(t), & \text{for } t \in (t_1, t_2], \end{cases} \\ \vdots \\ \varphi_n(t) = \begin{cases} 0, & \text{for } t \in [1, t_{n-1}], \\ \widehat{\varphi_n}(t), & \text{for } t \in (t_{n-1}, T]. \end{cases} \end{cases}$$

With the help of the Grönwall Lemma, we can deduce the uniqueness of this solution. Indeed, let φ_i, φ_i^* be two solutions to problem (7). Therefore, for every $i \in \{1, 2, \ldots, n\}$, we obtain

$$\begin{aligned} |\varphi_i(t) - \varphi_i^*(t)| &= \Big| {}^H I_{1^+}^{\psi(t)} g(t_i, \varphi_i(t_i), \varphi_i(\lambda t_i)) + {}^H I_{1^+}^{\psi(t)} g(t, \varphi_i(t), \varphi_i(\lambda t)) \\ &\quad - {}^H I_{1^+}^{\psi(t)} g(t_i, \varphi_i^*(t_i), \varphi_i^*(\lambda t_i)) - {}^H I_{1^+}^{\psi(t)} g(t_i, \varphi_i^*(t), \varphi_i^*(\lambda t)) \Big| \\ &\le \frac{2}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s} \right)^{\psi_i - 1} |g(s, \varphi_i(s), \varphi_i(\lambda s)) - g(s, \varphi_i^*(s), \varphi_i^*(\lambda s))| ds \\ &\le \frac{2L}{\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}} \right)^{\psi_i - 1} \int_{t_{i-1}}^{t_i} \frac{1}{s} (\ln s)^{-\alpha} (|\varphi_i(s) - \varphi_i^*(s)| + |\varphi_i(\lambda s) - \varphi_i^*(\lambda s)|) ds \\ &\le \frac{4L}{(1-\alpha)\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}} \right)^{\psi_i - 1} \int_{t_{i-1}}^{t_i} \frac{1}{s} (\ln s)^{-\alpha} |\varphi_i(s) - \varphi_i^*(s)| ds. \end{aligned}$$

Therefore,

$$0 \le |\varphi_i(t) - \varphi_i^*(t)| \le 0, \quad \text{which guarantees that} \quad \varphi_i(t) = \varphi_i^*(t) \quad \text{for every } t \in [t_{i-1}, t_i].$$

This provides the uniqueness of φ_i. Now, according to Remark 1, the uniqueness of the solution to (3) is derived. □

4. Stability of the Solutions in Terms of Ulam–Hyers–Rassias

When analyzing the solutions to boundary value problems for differential equations, one of the most relevant qualitative results is the study of the stability of the solutions. Therefore, we consider here the boundary value problem of interest and investigate the stability of its solutions in terms of Ulam–Hyers–Rassias.

Definition 6 ([23])**.** *The boundary value problem* (3) *is considered stable from the perspective of Ulam–Hyers–Rassias with respect to the function* $\Phi \in C([1,T],\mathbb{R}_+)$ *if there exists* $\zeta_g \in \mathbb{R}$ *such that* $\forall \xi > 0$ *and* $\forall \Psi \in C([1,T],\mathbb{R})$ *that satisfies*

$$| \, {}^{H}\mathcal{D}_{1^+}^{\psi(t)} \Psi(t) - g(t, \Psi(t), \Psi(\lambda t))| \leq \xi \Phi(t), \quad t \in [1,T],$$

there exists $\varphi \in C([1,T],\mathbb{R})$*, which is a solution to the boundary value problem* (3) *such that*

$$|\Psi(t) - \varphi(t)| \leq \zeta_g \xi \Phi(t), \quad t \in [1,T].$$

Theorem 4. *Let Hypothesis 1 and Hypothesis 2 be satisfied, and we assume that:*
***Hypothesis* 3.** *There exists* $\Phi \in C([t_{i-1},t_i],\mathbb{R}_+)$ *an increasing mapping, and there exists* $\mu_\Phi > 0$*, such that for all* $t \in [t_{i-1},t_i]$*,*

$${}^{H}I_{t_{i-1}^+}^{\psi_i} \Phi(t) \leq \mu_{\Phi(t)} \Phi(t).$$

Then, the boundary value problem (3) *is stable in terms of Ulam–Hyers–Rassias with respect to* Φ*.*

Proof. Consider $\epsilon > 0$ and $\Psi \in C([t_{i-1},t_i],\mathbb{R})$, such that

$$|{}^{H}\mathcal{D}_{t_{i-1}^+}^{\psi_i} \Psi(t) - g(t, \Psi(t), \Psi(\lambda t))| \leq \xi \Phi(t), \quad t \in [t_{i-1},t_i]. \tag{9}$$

For every $i \in \{1,2,\ldots,n\}$, let us propose the following definitions

$$\Psi_i(t) = \begin{cases} 0, & t \in [1,t_{i-1}], \\ \Psi(t), & t \in (t_{i-1},t_i]. \end{cases}$$

By integrating both sides of the Equation (9), for $t \in (t_{i-1},t_i]$, we obtain

$$\begin{aligned}
&\left| {}^{H}I_{t_{i-1}^+}^{\psi_i} \left[{}^{H}\mathcal{D}_{t_{i-1}^+}^{\psi_i} \Psi_i(t) - g(t, \Psi_i(t), \Psi_i(\lambda t)) \right] \right| \\
&= \Bigg| \Psi_i(t) - \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s} \right)^{\psi_i - 1} g(s, \Psi_i(s), \Psi_i(\lambda s)) ds \\
&\quad + \left(\ln \frac{t_i}{t_{i-1}} \right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}} \right)^{\psi_i - 1} \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s} \right)^{\psi_i - 1} g(s, \Psi_i(s), \Psi_i(\lambda s)) ds \Bigg| \\
&\leq \xi \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s} \right)^{\psi_i - 1} \Phi_i(s) ds.
\end{aligned}$$

Similarly to previous arguments, the boundary value problem (3) admits a solution φ defined as

$$\varphi(t) = \begin{cases} \varphi_1(t) = \widehat{\varphi_1}(t), & \text{for } t \in [1,t_1], \\ \varphi_2(t) = \begin{cases} 0, & \text{for } t \in [1,t_1], \\ \widehat{\varphi_2}(t), & \text{for } t \in (t_1,t_2], \end{cases} \\ \vdots \\ \varphi_n(t) = \begin{cases} 0, & \text{for } t \in [1,t_{n-1}], \\ \widehat{\varphi_n}(t), & \text{for } t \in (t_{n-1},T]. \end{cases} \end{cases}$$

Then, for every $t \in (t_{i-1}, t_i]$, $i \in \{1, \ldots, n\}$, we have

$$
\begin{aligned}
|\Psi(t) - \varphi(t)| &= |\Psi_i(t) - \widehat{\varphi}_i(t)| \\
&= \left| \Psi_i(t) + \left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i - 1} \right. \\
&\quad \times \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s}\right)^{\psi_i - 1} g(s, \widehat{\varphi}_i(s), \widehat{\varphi}_i(\lambda s)) ds \\
&\quad \left. - \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s}\right)^{\psi_i - 1} g(s, \widehat{\varphi}_i(s), \widehat{\varphi}_i(\lambda s)) ds \right| \\
&\leq \left| \Psi_i(t) + \left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i - 1} \right. \\
&\quad \times \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_l} \frac{1}{s} \left(\ln \frac{t_i}{s}\right)^{\psi_i - 1} g(s, \Psi_i(s), \Psi_i(\lambda s)) ds \\
&\quad \left. - \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s}\right)^{\psi_i - 1} g(s, \Psi_i(s), \Psi_i(\lambda s)) ds \right| \\
&\quad + \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t} \frac{1}{s} \left(\ln \frac{t}{s}\right)^{\psi_i - 1} |g(s, \Psi_i(s), \Psi_i(\lambda s)) - g(s, \widehat{\varphi}_i(s), \widehat{\varphi}_i(\lambda s))| ds \\
&\quad + \frac{1}{\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{1-\psi_i} \left(\ln \frac{t}{t_{i-1}}\right)^{\psi_i - 1} \\
&\quad \times \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s}\right)^{\psi_i - 1} |g(s, \Psi_i(s), \Psi_i(\lambda s)) - g(s, \widehat{\varphi}_i(s), \widehat{\varphi}_i(\lambda s))| ds \\
&\leq \mu_{\Phi(t)} \xi \Phi(t) \\
&\quad + \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t} \frac{L}{s} \left(\ln \frac{t}{s}\right)^{\psi_i - 1} (\ln s)^{-\alpha} (|\Psi_i(s) - \widehat{\varphi}_i(s)| + |\Psi_i(\lambda s) - \widehat{\varphi}_i(\lambda s)|) ds \\
&\quad + \frac{1}{\Gamma(\psi_i)} \int_{t_{i-1}}^{t_i} \frac{1}{s} \left(\ln \frac{t_i}{s}\right)^{\psi_i - 1} (\ln s)^{-\alpha} L (|\Psi_i(s) - \widehat{\varphi}_i(s)| + |\Psi_i(\lambda s) - \widehat{\varphi}_i(\lambda s)|) ds \\
&\leq \mu_{\Phi(t)} \xi \Phi(t) + \frac{1}{\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i - 1} \\
&\quad \times \int_{t_{i-1}}^{t_i} \frac{L}{s} (\ln s)^{-\alpha} (|\Psi_i(s) - \widehat{\varphi}_i(s)| + |\Psi_i(\lambda s) - \widehat{\varphi}_i(\lambda s)|) ds \\
&\quad + \frac{1}{\Gamma(\psi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{\psi_i - 1} \\
&\quad \times \int_{t_{i-1}}^{t_l} \frac{1}{s} (\ln s)^{-\alpha} L (|\Psi_i(s) - \widehat{\varphi}_i(s)| + |\Psi_i(\lambda s) - \widehat{\varphi}_i(\lambda s)|) ds \\
&\leq \mu_{\Phi(t)} \xi \Phi(t) + \frac{4L||\Psi_i - \widehat{\varphi}_i||_{E_i}}{(1-\alpha)\Gamma(\varpi_i)} \left(\ln \frac{t_i}{t_{i-1}}\right)^{\varpi_t - 1} \left[(\ln t_i)^{1-\alpha} - (\ln t_{i-1})^{1-\alpha} \right] \\
&\leq \mu_{\Phi(t)} \xi \, \Phi(t) + \rho ||\Psi - \varphi||,
\end{aligned}
$$

where

$$\rho = \max_{i\in\{1,2,\ldots,n\}} \frac{4L}{(1-\alpha)\Gamma(\psi_i)}\left(\ln\frac{t_i}{t_{i-1}}\right)^{\omega_i-1}\left[(\ln t_i)^{1-\alpha}-(\ln t_{i-1})^{1-\alpha}\right].$$

Thus,

$$(1-\rho)||\Psi-\varphi|| \leq \mu_{\Phi(t)}\xi\Phi(t).$$

Therefore, for every $t\in[1,T]$, we obtain

$$|\Psi(t)-\varphi(t)| \leq \frac{\mu_{\Phi(t)}\xi\Phi(t)}{1-\rho} = \zeta_g\xi\,\Phi(t).$$

Then, the boundary value problem (3) is stable in terms of Ulam–Hyers–Rassias with respect to Φ. □

5. Example

In the final section, we provide an example to illustrate the theoretical results included in this paper. We consider the following boundary value problem

$$\begin{cases} {}^H\mathcal{D}_{1^+}^{\psi(t)}\varphi(t) = \dfrac{\tan t}{7\sqrt[3]{\pi}}\cosh(t)^{\psi(t)} + \dfrac{(\ln t)^{-\frac{1}{3}}}{t^3+7}[\varphi(t)+\varphi(\lambda t)], \quad 1\leq t\leq e, \\ \varphi(1)=\varphi(e)=0, \end{cases} \tag{10}$$

where

$$\psi(t) = \begin{cases} \dfrac{13}{10}, & \text{for } t\in[1,2], \\ \\ \dfrac{17}{10}, & \text{for } t\in(2,e], \end{cases}$$

$$\begin{aligned} &(\ln t)^{\frac{1}{3}}|g(t,\varphi_1(t),\varphi_1(\lambda t)) - g(t,\varphi_2(t),\varphi_2(\lambda t))| \\ &= \left|\frac{1}{t^3+7}[\varphi_1(t)+\varphi_1(\lambda t)] - \frac{1}{t^3+7}[\varphi_2(t)+\varphi_2(\lambda t)]\right| \\ &\leq \frac{1}{t^3+7}(|\varphi_1(t)-\varphi_2(t)| + |\varphi_1(\lambda t)-\varphi_2(\lambda t)|) \\ &\leq \frac{1}{8}(|\varphi_1(t)-\varphi_2(t)| + |\varphi_1(\lambda t)-\varphi_2(\lambda t)|). \end{aligned}$$

We consider the following auxiliary boundary value problems

$$\begin{cases} {}^H\mathcal{D}_{1^+}^{\frac{13}{10}}\varphi_1(t) = \dfrac{\tan t}{7\sqrt[3]{\pi}}\cosh(t)^{\frac{13}{10}} + \dfrac{(\ln t)^{-\frac{1}{3}}}{t^3+7}[\varphi_1(t)+\varphi_1(\lambda t)], \quad 1\leq t\leq 2, \\ \varphi_1(1)=0,\ \varphi_1(2)=0, \end{cases}$$

$$\begin{cases} {}^H\mathcal{D}_{1^+}^{\frac{17}{10}}\varphi_2(t) = \dfrac{\tan t}{7\sqrt[3]{\pi}}\cosh(t)^{\frac{17}{10}} + \dfrac{(\ln t)^{-\frac{1}{3}}}{t^3+7}[\varphi_2(t)+\varphi_2(\lambda t)], \quad 2< t\leq e, \\ \varphi_2(2)=0,\ \varphi_2(e)=0. \end{cases}$$

Simply, one can check that Hypothesis 2 is valid for $L = \dfrac{1}{8}$, $\alpha = \dfrac{1}{3}$, $i \in \{1,2\}$:

$$\begin{cases} \dfrac{4L}{(1-\alpha)\Gamma(\psi_i)}\left(\ln \dfrac{t_i}{t_{i-1}}\right)^{\psi_i - 1} & [(\ln t_i)^{1-\alpha} - (\ln t_{i-1})^{1-\alpha}] \\ & = \dfrac{1}{2\left(\dfrac{2}{3}\right)\Gamma\left(\dfrac{13}{10}\right)}\left(\ln \dfrac{2}{1}\right)^{\frac{13}{10}-1}[(\ln 2)^{\frac{2}{3}} - (\ln 1)^{\frac{2}{3}}] \\ & \approx 0.77662 < 1, \\ \dfrac{4L}{(1-\alpha)\Gamma(\psi_i)}\left(\ln \dfrac{t_i}{t_{i-1}}\right)^{\psi_i - 1} & [(\ln t_i)^{1-\alpha} - (\ln t_{i-1})^{1-\alpha}] \\ & = \dfrac{1}{2\left(\dfrac{2}{3}\right)\Gamma\left(\dfrac{17}{10}\right)}\left(\ln \dfrac{e}{2}\right)^{\frac{17}{10}-1}[1 - (\ln 2)^{\frac{2}{3}}] \\ & \approx 0.36102 < 1. \end{cases}$$

As a result, according to Theorem 3, the above-mentioned boundary value problem for Hadamard-type fractional equations has exactly one solution given by

$$\varphi(t) = \begin{cases} \varphi_1(t) = \widehat{\varphi_1}(t), & \text{for } t \in [1,2], \\ \varphi_2(t) = \begin{cases} 0, & \text{for } t \in [1,2], \\ \widehat{\varphi_2}(t), & \text{for } t \in (2,e]. \end{cases} \end{cases}$$

Let $\Phi(t) = \sqrt{\ln t}$,

$$\begin{aligned} {}^{H}I_{1^+}^{\omega_1}\Phi(t) &= \frac{1}{\Gamma\left(\frac{13}{10}\right)}\int_1^t \frac{1}{s}\left(\ln \frac{t}{s}\right)^{\frac{13}{10}-1}\sqrt{\ln s}ds \\ &\leq \frac{1}{\Gamma\left(\frac{13}{10}\right)}\int_1^t \frac{1}{s}\left(\ln \frac{t}{s}\right)^{\frac{3}{10}} ds \\ &\leq \frac{1}{\Gamma\left(\frac{23}{10}\right)}\sqrt{\ln t} := \mu_\Phi \Phi(t). \end{aligned}$$

Thus, Hypothesis 3 is satisfied for $\Phi(t) = \sqrt{\ln t}$ and $\mu_{\Phi(t)} = \dfrac{1}{\Gamma\left(\dfrac{23}{10}\right)}$,

$$\begin{aligned} {}^{H}I_{2^+}^{\omega_2}\Phi(t) &= \frac{1}{\Gamma\left(\frac{17}{10}\right)}\int_2^t \frac{1}{s}\left(\ln \frac{t}{s}\right)^{\frac{17}{10}-1}\sqrt{\ln s}ds \\ &\leq \frac{1}{\Gamma\left(\frac{17}{10}\right)}\int_2^t \frac{1}{s}\left(\ln \frac{t}{s}\right)^{\frac{7}{10}} ds \\ &\leq \frac{1}{\Gamma\left(\frac{27}{10}\right)}\sqrt{\ln t} := \mu_\Phi \Phi(t). \end{aligned}$$

Thus, Hypothesis 3 holds for $\Phi(t) = \sqrt{\ln t}$ and $\mu_{\Phi(t)} = \dfrac{1}{\Gamma\left(\dfrac{27}{10}\right)}$.

Therefore, the variable-order Hadamard pantograph boundary value problem (10) satisfies the stability property in terms of Ulam–Hyers–Rassias with respect to Φ.

6. Results and Discussion

The variable-order fractional pantograph equation represents a generalized version of the classical pantograph equation, where the differentiation operator is changed to a variable-order fractional derivative operator. This equation has gained significant attention in recent years since it can be used in modeling various real phenomena with complex dynamics and memory effects.

When comparing the results obtained for the variable-order pantograph fractional equation with previous ones, several aspects can be considered. Firstly, the inclusion of non-constant-order fractional derivatives allows more flexibility in modeling systems with non-local characteristics. Additionally, it provides a richer mathematical framework for analyzing the behavior of dynamical systems.

Overall, the utilization of variable-order fractional calculus yields a more comprehensive and accurate modeling framework compared to previous approaches. It provides enhanced capabilities for capturing complex dynamics, accounting for memory effects, and advancing our understanding of systems with intricate behaviors.

7. Conclusions

The results in this paper are connected to the existence, uniqueness, and stability of the solutions to a class of boundary value problems associated with nonlinear pantograph equations with fractional derivatives of the Hadamard type and variable order. In particular, we have derived the main existence results from the properties of the fractional operators and Krasnoselskii's fixed-point theorem. Following this, we have developed some stability results in terms of Ulam–Hyers–Rassias for the considered boundary value problem. Due to the importance of variable-order fractional calculus, both from a theoretical and an applied perspective, we think that the results obtained can be of interest to the research pursuits of many readers.

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