axioms

Article

New Majorized Fractional Simpson Estimates

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Abstract: Fractional calculus has been a concept used to acquire new variants of some well-known integral inequalities. In this study, our primary goal is to develop majorized fractional Simpson's type estimates by employing a differentiable function. Practicing majorization theory, we formulate a new auxiliary identity by utilizing fractional integral operators. In order to obtain new bounds, we employ the idea of convex functions on the Niezgoda–Jensen–Mercer inequality for majorized tuples, along with some fundamental inequalities including the Hölder, power mean, and Young inequalities. Some applications to the quadrature rule and examples for special functions are provided as well. Interestingly, the main findings are the generalizations of many known results in the existing literature.

Keywords: convex functions; fractional calculus; Simpson inequality; Hölder's inequality; Young's inequality; majorization theory; special functions

MSC: 26D15; 26D20; 26D99

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1. Introduction

Mathematical inequalities are today acknowledged and taught as some of the most practical areas of mathematics as they have successfully influenced numerous scientific and engineering domains. Their practical applications can be found in information theory, economics, engineering, and finance [1–3]. They have particularly contributed to the study of numerical and partial differential equations. Fractional analysis is a striking idea used to explore hidden information that cannot be explored while dealing with classical analysis. Fractional integral inequalities are crucial in establishing the existence and uniqueness of solutions for certain fractional differential equations. The subject of fractional analysis has been studied by many mathematicians. These mathematicians have studied several fractional derivatives and integrals using a variety of methods and techniques.

Convex functions have witnessed a surge in popularity during the past few years. They are utilized frequently in a variety of contemporary analytic fields across the spectrum of mathematical specializations. They are particularly useful in terms of optimization theory due to the fact that they exhibit a large number of useful properties in this field.

Theory of inequalities pertaining to convex functions exhibit strong influence on one another. Convex functions have allowed for the discovery of a large number of noteworthy and beneficial inequalities that are quite useful in applied sciences. The most noteworthy classical definition of a convex function on convex sets in terms of line segments can be considered as:

$$\mathfrak{f}(\eta\mathfrak{a} + (1-\eta)\mathfrak{b}) \leq \eta\mathfrak{f}(\mathfrak{a}) + (1-\eta)\mathfrak{f}(\mathfrak{b}) \tag{1}$$

where $\mathfrak{f} : [\varsigma_1, \varsigma_2] \subseteq \Re \to \Re$ is a mapping valid for all $\mathfrak{a}, \mathfrak{b} \in [\varsigma_1, \varsigma_2]$ and $\eta \in [0, 1]$.

A fundamental and well-known inequality for convex functions is the Jensen inequality and its related inequalities. This is because the Jensen inequality and its associated inequalities can be used in a wide variety of fields, including optimization, probability theory, information theory, and computer challenges [4–7]. Numerous well-known inequalities for convex functions are frequently used in current research.

Fractional calculus is an effective vision that may be used to explain physical events and issues that occur in daily life. This branch of mathematics is included in the broad category of applied mathematics. The fractional integral and derivative operators helps with improving the relationships between mathematics and other specializations by providing solutions that are more closely related to real-world problems. Fractional integral operators and fractional derivative operators have contributed new concepts to fractional analysis in terms of their application areas and spaces [8–10]. They do not share the same singularity, locality, or kernel qualities with one another. One way to define fractional calculus is an extension of the notion of the derivative operator, where the notion of the derivative operator can be extended from the integer order to any rational order. With the help of fractional integrals, one may solve a wide range of issues that come up in the fields of science and engineering. The theory is developed through the introduction of the most useful fractional integral operators, named Riemann–Liouville fractional integral operators, which lay the foundation of fractional calculus in the broader sense.

Definition 1 ([8]). *The formula for the left- and right-sided Riemann–Liouville fractional integral operators of order $\alpha > 0$ is as follows, where $\mathfrak{f} \in L[\mathfrak{a}, \mathfrak{b}]$:*

$$J^{\alpha}_{\mathfrak{a}^+}\mathfrak{f}(\eta) = \frac{1}{\Gamma(\alpha)} \int_{\mathfrak{a}}^{\eta} \frac{\mathfrak{f}(\tau)\, d\tau}{(\eta - \tau)^{1-\alpha}}, \quad \eta > \mathfrak{a}$$

and

$$J^{\alpha}_{\mathfrak{b}^-}\mathfrak{f}(\eta) = \frac{1}{\Gamma(\alpha)} \int_{\eta}^{\mathfrak{b}} \frac{\mathfrak{f}(\tau)\, d\tau}{(\tau - \eta)^{1-\alpha}}, \quad \eta < \mathfrak{b}.$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1} dt$. Here, $J^0_{\mathfrak{a}^+}\mathfrak{f}(\eta) = J^0_{\mathfrak{b}^-}\mathfrak{f}(\eta) = \mathfrak{f}(\mathfrak{a})$. The fractional integral simplifies to the classical integral for $\alpha = 1$.

The definition of majorization that we use to explain our results is as follows:

Definition 2 ([11]). *Let $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_{\beta})$ and $\mathfrak{b} = (\mathfrak{b}_1, \dots, \mathfrak{b}_{\beta})$ be two β-tuple of real numbers such that $\mathfrak{a}_{[\beta]} \leq \mathfrak{a}_{[\beta-1]} \leq \dots \leq \mathfrak{a}_{[1]}$, $\mathfrak{b}_{[\beta]} \leq \mathfrak{b}_{[\beta-1]} \dots \leq \mathfrak{b}_{[1]}$, then $\mathfrak{a}$ is said to be majorized by $\mathfrak{b}$ (or $\mathfrak{b}$ is a said to be majorized by $\mathfrak{a}$, symbolically $\mathfrak{b} \prec \mathfrak{a}$), if*

$$\sum_{\varrho=1}^{\mathfrak{s}} \mathfrak{b}_{[\varrho]} \leq \sum_{\varrho=1}^{\mathfrak{s}} \mathfrak{a}_{[\varrho]} \qquad for \;\; \mathfrak{s} = 1, 2, \dots, \beta - 1 \tag{2}$$

and

$$\sum_{\varrho=1}^{\beta} \mathfrak{b}_{[\varrho]} = \sum_{\varrho=1}^{\beta} \mathfrak{a}_{[\varrho]}.$$

Definition 3 ([12]). *Let $\mathfrak{f}$ be a convex function on the given interval $[\mathfrak{a}, \mathfrak{b}]$ and $0 < \mathfrak{a}_1 \leq \mathfrak{a}_2 \leq \dots \leq \mathfrak{a}_t$, and let$\omega = (\omega_1, \omega_2, \dots, \omega_t)$ be the non-negative weight such that $\sum_{\iota=1}^{t} \omega_\iota = 1$, Then, Jensen's inequality can be given as:*

$$\mathfrak{f}\left(\sum_{\imath=1}^{t}\omega_\imath \mathfrak{a}_\imath\right) \leq \left(\sum_{\imath=1}^{t}\omega_\imath \mathfrak{f}(\mathfrak{a}_\imath)\right) \tag{3}$$

for all $\mathfrak{a}_\imath \in [\zeta_1, \zeta_2]$, $\omega_\imath \in [0,1]$ *for* $(\imath = 1, 2, \ldots, t)$.

Jensen's inequality (3) recaptures the concept of a convex function when t = 2. Jensen's inequality has numerous significant applications in the fields of optimization, statistics, finance, and economics, but it is particularly useful in foretelling the estimates of the limits of distance functions in information theory [13–15].

The Jensen–Mercer inequality, an important mathematical inequality deducted from Jensen's inequality, is widely used in the fields of optimization and convex analysis. By giving the convex combination an upper limit, it provides a limitation on the convex combination of a function over a set of variables where the weights of the variables form a probability distribution. A wide range of disciplines, including economics, statistics, and machine learning, have used the Jensen–Mercer inequality. It is frequently used in various fields to create crucial boundaries and validate important conclusions.

A fundamental idea in convex analysis, the Jensen–Mercer inequality has several applications in various branches of mathematics. It provides a result that is unavoidably necessary in order to understand how convex functions behave when they are used with convex combinations. McD Mercer in 2003 [16] first proposed an intriguing variation of Jensen's inequality, namely, the Jensen–Mercer inequality, which is described as follow:

Definition 4 ([16])**.** *Under the assumption of Definition 3, if* $\mathfrak{f}$ *is a convex function on* $[\zeta_1, \zeta_2]$*, then*

$$\mathfrak{f}\left(\zeta_1 + \zeta_2 - \sum_{\imath=1}^{t}\omega_\imath\ \mathfrak{a}_\imath\right) \leq \mathfrak{f}(\zeta_1) + \mathfrak{f}(\zeta_2) - \sum_{\imath=1}^{t}\omega_\imath\ \mathfrak{f}(\mathfrak{a}_\imath), \tag{4}$$

holds for all finite positive increasing sequences $\mathfrak{a}_\imath \in [\zeta_1, \zeta_2]$.

Numerous academics have conducted in-depth research on the Jensen–Mercer inequality. Several methods have been used, including increasing its dimension, obtaining it for convex operators with all of its multiple refinements, obtaining operator variations for super-quadratic functions, upgrading, and performing several generalizations with implications for information theory [17–19].

In [20], Niezgoda introduced the extended version of the Jensen–Mercer inequality by deploying a majorization scheme as follows:

Theorem 1. *Consider* $\mathfrak{f}$ *as the continuous convex function defined on I and a real* $t \times ß$ *matrix* $\mathfrak{a}_{\imath s}$ *with* $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_ß)$ *as a ß-tuple such that* $\zeta_\varrho, \mathfrak{a}_{\imath\varrho} \in I$ *for all* $\imath = 1, 2, \ldots, t$ *and* $\varrho = \{1, 2, 3, \ldots, ß\}$*. Moreover, let weight functions* $\omega_\imath \geq 0$ *such that* $\sum_{\imath=1}^{t}\omega_\imath = 1$*. If* ζ *majorizes every row of* $\mathfrak{a}_{\imath s}$*, then*

$$\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\sum_{\imath=1}^{t}\omega_\imath \mathfrak{a}_{\imath s}\right) \leq \sum_{\varrho=1}^{ß}\mathfrak{f}(\zeta_\varrho) - \sum_{\varrho=1}^{ß-1}\sum_{\imath=1}^{t}\omega_\imath \mathfrak{f}(\mathfrak{a}_{\imath s}). \tag{5}$$

Taking into account the above notable Niezgoda inequality, Faisal et al. [21] recently introduced the below-mentioned Hermite–Hadamard-type inequality for majorized tuples:

Theorem 2. *Let* $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_ß)$*,* $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_ß)$ *and* $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_ß)$ *be three ß-tuples such that* $\zeta_\varrho, \mathfrak{a}_\varrho, \mathfrak{b}_\varrho \in I$*, for all* $\varrho = \{1, \ldots, ß\}$*,* $\mathfrak{a} > \mathfrak{b}$*,* $\alpha > 0$*, and let* $\mathfrak{f}$ *be a continuous convex function defined on I. If* $\mathfrak{a} \prec \zeta$ *and* $\mathfrak{b} \prec \zeta$*, then:*

$$\mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho-\sum_{\varrho=1}^{\beta-1}\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2\left(\sum\limits_{\varrho=1}^{\beta-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)\right)^{\alpha}}\left\{J^{\alpha}_{\left(\sum\limits_{\varrho=1}^{\beta}\zeta_\varrho-\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho\right)^{+}}\mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho-\sum_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho\right)\right.$$
$$\left.+J^{\alpha}_{\left(\sum\limits_{\varrho=1}^{\beta}\zeta_\varrho-\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho\right)^{-}}\mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho-\sum_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho\right)\right\}$$
$$\leq\frac{\mathfrak{f}\left(\sum\limits_{\varrho=1}^{\beta}\zeta_\varrho-\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho\right)+\mathfrak{f}\left(\sum\limits_{\varrho=1}^{\beta}\zeta_\varrho-\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho\right)}{2}$$
$$\leq\sum_{\varrho=1}^{\beta}\mathfrak{f}(\zeta_\varrho)-\frac{\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{f}(\mathfrak{a}_\varrho)+\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{f}(\mathfrak{b}_\varrho)}{2} \tag{6}$$

Researchers that work with various integrals or convex functions have a unique opportunity to apply the notion of majorization. Experts from a range of fields are fascinated by this occurrence. Numerous majorization ideas have been adopted and used in a variety of academic fields, including graph theory, optimization, and economics. Within the discipline of mathematics, the idea of majorization makes up a sizable domain.

The famous Simpson's inequality error estimates are described as follows:

Definition 5 ([22])**.** *Consider $\mathfrak{f}:[\zeta_1,\zeta_2]\to\Re$ as a continuously differentiable mapping of order four on (ζ_1,ζ_2) and $\left|\left|\mathfrak{f}^{(4)}\right|\right|_\infty=\sup_{x\in(\zeta_1,\zeta_2)}\left|\mathfrak{f}^{(4)}(\mathfrak{a})\right|<\infty$. Then, the following estimation holds:*

$$\left|\frac{1}{3}\left\{\frac{\mathfrak{f}(\zeta_1)+\mathfrak{f}(\zeta_2)}{2}+2\mathfrak{f}\left(\frac{\zeta_1+\zeta_2}{2}\right)-\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\mathfrak{f}(\mathfrak{a})dx\right\}\right|$$
$$\leq\frac{1}{2880}\left|\left|\mathfrak{f}^{(4)}\right|\right|_\infty(\zeta_2-\zeta_1)^4.$$

Numerous mathematicians have explored different possibilities to give estimations of Simpson-type inequalities. One of the possibilities is to use convex mappings because convexity theory provides a consistent and effective method for addressing a wide variety of problems. A wide variety of phenomena can be attributed to the several subfields that make up pure and applied sciences. For closely approximating the results of convex functions in theoretical and applied mathematics, the Simpson and Newton approaches have become popular due to their usefulness and variety. Specifically, a range of Simpson-type inequalities, applicable to first-, second-, and third-order differentiable functions have been formulated by utilizing fractional integral operators. Dragomir et al. presented a noteworthy finding [22] when they utilized the quadrature formula for numerical integration in conjunction with the Simpson-type inequality. Alomari et al. [23] and Sarikaya et al. [24] have developed several additional Simpson-type inequalities for s-convex functions, which offer enhanced bounds. Later, researchers have utilized Riemann–Liouville fractional integral operators to establish Simpson-type inequalities for diverse families of differentiable and convex functions. For the first time, Chen and Huang presented some new inequalities of the Simpson type by employing s-convexity via fractional integrals [25]. Later on, Kermausuor [26] employed Katugampola fractional integral operators to investigate Simpson-type integral inequalities for s-convex functions in the second sense. In recent years, research on Simpson-type inequalities for twice-differentiable functions has seen significant growth. Sarikaya et al. [27] first presented some Simpson-type inequalities for functions whose second derivatives absolute values are convex. Hezanci et al. [28] established an identity involving twice-differentiable functions using Riemann–Liouville fractional integral operators, from which a series of Simpson-type inequalities were derived. Butt et al. [29] explored a new fractional Mercer–Simpson-type inequality for twice-differentiable convex functions pertaining to nonsingular or nonlocal kernels with various applications. Also, there are a number of results reported in the literature on integral inequalities involving three-times-differentiable convex functions to give estimations of Simpson-type integral inequalities. Liu and Chun et al. investigated Simpson-type inequalities with respect to third derivatives being h-convex and extended s-convex [30], considering Riemann–Liouville integrals. Similarly, using Riemann–Liouville fractional integrals, Hezenci et al. [31] established Simpson-type inequalities containing three-times-differentiable convex functions. For the first time, Niezgoda [20] used the concept of majorization and extended the Jensen–Mercer inequality. Based on this extended concept of majorization, Butt et al. gave new bounds for Newton–Simpson-type inequalities in [32]. There are scarce results presented in the literature concerning Simpson's inequality for majorized tuples.

The primary objective of this study is to employ majorization-type outcomes for Simpson-type inequalities through the use of Riemann–Lioville fractional integrals, wherein a convex function is implicated. We present novel estimates of Simpson's fractional inequalities utilizing convexity and the Niezgoda–Jensen–Mercer inequality, which are related to majorization. The present paper presents Simpson's estimations expressed in terms of special q-digamma and Bessel functions, which establish elegant connections.

2. Main Results

We start by establishing novel auxiliary identity for the Riemann–Liouville integral operator using majorized tuples, which can be used to derive future advancements.

Lemma 1. *Let* $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_\beta)$, $\mathfrak{a} = (\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_\beta)$ *and* $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_\beta)$ *be three tuples such that* $\zeta_\varrho, \mathfrak{a}_\varrho, \mathfrak{b}_\varrho \in I$ *for all* $\varrho \in \{1,2,3,\ldots,\beta\}$ *and* $\mathfrak{a}_\beta > \mathfrak{b}_\beta$ *with* $\alpha > 0$ *and* $\mathfrak{f}$ *being the continuous differentiable functions on interval* $I \subseteq \Re$. *If* $\mathfrak{f}' \in L(I)$ *and* ζ *majorize* $\mathfrak{a}$, $\mathfrak{b}$ ($\zeta \succ \mathfrak{a}$, *and* $\zeta \succ \mathfrak{b}$), *then the following identity:*

$$S_\alpha(\mathfrak{a}_\varrho, \mathfrak{b}_\varrho, \zeta_\varrho, \beta; \mathfrak{f}) = \frac{\sum_{\varrho=1}^{\beta-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)}{2}\left[\int_0^1 \left(\frac{\eta^\alpha}{2} - \frac{1}{3}\right)\mathfrak{f}'\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \left[\frac{1+\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho + \frac{1-\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho\right]\right)d\eta\right.$$
$$\left. + \int_0^1 \left(\frac{1}{3} - \frac{\eta^\alpha}{2}\right)\mathfrak{f}'\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \left[\frac{1+\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho + \frac{1-\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho\right]\right)d\eta\right] \tag{7}$$

is satisfied for $\eta \in [0,1]$, *where*

$$S_\alpha(\mathfrak{a}_\varrho, \mathfrak{b}_\varrho, \zeta_\varrho, \beta; \mathfrak{f}) = \frac{1}{6}\left\{\mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \sum_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho\right) + 4\mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \sum_{\varrho=1}^{\beta-1}\frac{\mathfrak{a}_\varrho + \mathfrak{b}_\varrho}{2}\right) + \mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \sum_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho\right)\right\}$$
$$- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\sum_{\varrho=1}^{\beta-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)\right)^\alpha} \times \left\{J^\alpha_{\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \sum_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho\right)^-}\mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \sum_{\varrho=1}^{\beta-1}\frac{\mathfrak{a}_\varrho + \mathfrak{b}_\varrho}{2}\right)\right.$$
$$\left. + J^\alpha_{\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \sum_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho\right)^+}\mathfrak{f}\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho - \sum_{\varrho=1}^{\beta-1}\frac{\mathfrak{a}_\varrho + \mathfrak{b}_\varrho}{2}\right)\right\}.$$

Proof. Let us start with

$$\frac{\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\Bigg[\int_0^1\left(\frac{\eta^\alpha}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right]\right)d\eta+$$
$$\int_0^1\left(\frac{1}{3}-\frac{\eta^\alpha}{2}\right)\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho\right]\right)d\eta\Bigg]$$

$$=\sum_{\varrho=1}^{ß-1}\frac{(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\{I_1+I_2\}. \tag{8}$$

where I_1 and I_2 are given as

$$I_1=\int_0^1\left(\frac{\eta^\alpha}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right]\right)d\eta. \tag{9}$$

$$I_2=\int_0^1\left(\frac{1}{3}-\frac{\eta^\alpha}{2}\right)\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho\right]\right)d\eta. \tag{10}$$

Now, by applying integration in parts on I_1, we obtain

$$I_1=\left(\frac{\eta^\alpha}{2}-\frac{1}{3}\right)\frac{\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right]\right)}{\sum_{\varrho=1}^{ß-1}\frac{\mathfrak{b}_\varrho-\mathfrak{a}_\varrho}{2}}\Bigg|_0^1-$$
$$\int_0^1\frac{\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left[\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right]\right)}{\sum_{\varrho=1}^{ß-1}\frac{\mathfrak{b}_\varrho-\mathfrak{a}_\varrho}{2}}\frac{\alpha\eta^{\alpha-1}}{2}d\eta$$

$$=\frac{1}{3\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}\left[\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho\right)\right]+\frac{2}{3\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}\left[\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\frac{\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2}\right)\right]$$
$$-\frac{\alpha}{\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}\int_0^1\mathfrak{f}\left[\sum_{\varrho=1}^{ß}\zeta_\varrho-\left(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right)\right]\eta^{\alpha-1}d\eta.$$

By substituting the variables, we obtain

$$I_1=\frac{1}{3\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}\left[\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho\right)\right]+\frac{2}{3\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}\left[\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\frac{\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2}\right)\right]$$
$$-\frac{\alpha.2^\alpha}{\left(\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)\right)^{\alpha+1}}\int_{\sum_{\varrho=1}^{ß}\zeta_\varrho-\sum_{\varrho=1}^{ß-1}\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2}}^{\sum_{\varrho=1}^{ß}\zeta_\varrho-\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho}\left[P-[\sum_{\varrho=1}^{ß}\zeta_\varrho-\frac{\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2}]\right]^{\alpha-1}\mathfrak{f}(P)dP.$$

Similarly, by applying integration on I_2, we have

$$I_2 = \left[\int_0^1 \left(\frac{1}{3} - \frac{\eta^\alpha}{2}\right)\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho + \frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho\right]\right)d\eta\right.$$

$$= \frac{1}{3\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)}\left[\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right)\right] + \frac{2}{3\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)}\left[\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \frac{\sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho + \mathfrak{b}_\varrho}{2}\right)\right]$$

$$- \frac{\alpha.2^\alpha}{\left(\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)\right)^{\alpha+1}}\int_{\sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho}^{\sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2}}\left[[\sum_{\varrho=1}^{ß}\zeta_\varrho - \frac{\sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho + \mathfrak{b}_\varrho}{2}] - P\right]^{\alpha-1}\mathfrak{f}(P)dP.$$

Before using the concept of the fractional integral definition, it is necessary to show that $\sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2} > \sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho$ and $\sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2} < \sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho$.
Given that $\mathfrak{a}_ß > \mathfrak{b}_ß \Longrightarrow \mathfrak{a}_ß - \mathfrak{b}_ß > 0$. Furthermore, $\mathfrak{a} \prec \zeta$ and $\mathfrak{b} \prec \zeta$. Then $\sum\limits_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho + \mathfrak{b}_ß = \sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho + \mathfrak{a}_ß \Longrightarrow \sum\limits_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho - \sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho = \mathfrak{a}_ß - \mathfrak{b}_ß$. Moreover, $-\sum\limits_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho < -\sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho \Longrightarrow -\sum\limits_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho < \sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho - 2\sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho \Longrightarrow -\sum\limits_{\varrho=1}^{ß-1}\frac{(\mathfrak{a}_\varrho+\mathfrak{b}_\varrho)}{2} < -\sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho$.
Adding $\sum\limits_{\varrho=1}^{ß}\zeta_\varrho$ on both sides yields

$$\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\frac{(\mathfrak{a}_\varrho + \mathfrak{b}_\varrho)}{2} < \sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho.$$

Similarly, one can construct

$$\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho < \sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\frac{(\mathfrak{a}_\varrho + \mathfrak{b}_\varrho)}{2}.$$

Adding I_1 and I_2, we have

$$\frac{1}{3\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)}\left[\mathfrak{f}[\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho] + 4\mathfrak{f}[\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\frac{(\mathfrak{a}_\varrho + \mathfrak{b}_\varrho)}{2}] + \mathfrak{f}[\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho]\right]$$
$$- \frac{2^\alpha\Gamma(\alpha+1)}{\left(\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)\right)^{\alpha+1}}\left[J^\alpha_{(\sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho)^+}\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\frac{\mathfrak{a}_\varrho + \mathfrak{b}_\varrho}{2}\right)\right.$$
$$\left. + J^\alpha_{(\sum\limits_{\varrho=1}^{ß}\zeta_\varrho - \sum\limits_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho)^-}\mathfrak{f}\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \sum_{\varrho=1}^{ß-1}\frac{\mathfrak{a}_\varrho + \mathfrak{b}_\varrho}{2}\right)\right] \tag{11}$$

Multiply by $\sum\limits_{\varrho=1}^{ß-1}\frac{\mathfrak{b}_\varrho - \mathfrak{a}_\varrho}{2}$ on both sides of (11) and then simplifying the identity, we obtain (7) which is desired result. □

Remark 1. *If we set $\beta = 2$ in (7) , we get the subsequent equality:*

$$S_{\alpha}(\mathfrak{a}_{\varrho}, \mathfrak{b}_{\varrho}, \zeta_{\varrho}, 2; \mathfrak{f}) = \frac{1}{6}\left[\mathfrak{f}(\zeta_1+\zeta_2-\mathfrak{a}) + 4\mathfrak{f}\left(\zeta_1+\zeta_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) + \mathfrak{f}(\zeta_1+\zeta_2-\mathfrak{b})\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{b}-\mathfrak{a})^{\alpha}}$$
$$\times\left[J^{\alpha}_{(\zeta_1+\zeta_2-\mathfrak{a})^-}\mathfrak{f}\left(\zeta_1+\zeta_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) + J^{\alpha}_{(\zeta_1+\zeta_2-\mathfrak{b})^+}\mathfrak{f}\left(\zeta_1+\zeta_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)\right]$$
$$= \frac{\mathfrak{b}-\mathfrak{a}}{2}\left[\int_0^1\left(\frac{\eta^{\alpha}}{2}-\frac{1}{3}\right)\mathfrak{f}'\left(\zeta_1+\zeta_2-(\frac{1+\eta}{2}\mathfrak{a}+\frac{1-\eta}{2}\mathfrak{b})\right)d\eta + \right.$$
$$\left.\int_0^1\left(\frac{1}{3}-\frac{\eta^{\alpha}}{2}\right)\mathfrak{f}'\left(\zeta_1+\zeta_2-(\frac{1+\eta}{2}\mathfrak{b}+\frac{1-\eta}{2}\mathfrak{a})\right)d\eta\right],$$

which is a Mercer equality via theRiemann–Liouville fractional integral, which is new in the literature.

Remark 2. *If we set $\alpha = 1$ in Remark 1, we obtain the following classical Simpson–Mercer result:*

$$S_1(\mathfrak{a}_{\varrho}, \mathfrak{b}_{\varrho}, \zeta_{\varrho}, 2; \mathfrak{f}) = \frac{1}{6}\left[\mathfrak{f}(\zeta_1+\zeta_2-\mathfrak{a}) + 4\mathfrak{f}\left(\zeta_1+\zeta_2-\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) + \mathfrak{f}(\zeta_1+\zeta_2-\mathfrak{b})\right] - \frac{1}{\mathfrak{b}-\mathfrak{a}}\int_{\zeta_1+\zeta_2-\mathfrak{b}}^{\zeta_1+\zeta_2-\mathfrak{a}}\mathfrak{f}(P)dP$$
$$= \frac{\mathfrak{b}-\mathfrak{a}}{2}\left[\int_0^1(\frac{\eta}{2}-\frac{1}{3})\mathfrak{f}'(\zeta_1+\zeta_2-(\frac{1+\eta}{2}\mathfrak{a}+\frac{1-\eta}{2}\mathfrak{b}))d\eta + \right.$$
$$\left.\int_0^1(\frac{1}{3}-\frac{\eta}{2})\mathfrak{f}'(\zeta_1+\zeta_2-(\frac{1+\eta}{2}\mathfrak{b}+\frac{1-\eta}{2}\mathfrak{a}))d\eta\right],$$

which is a new identity in the literature.

Remark 3. *For $\zeta_1 = \mathfrak{a}$ and $\zeta_2 = \mathfrak{b}$ in Remark 1, we obtain a lemma via a Riemann–Liouville fractional integral operator, which was for the first time proved by Chen and Huang in [25].*

If we set $\alpha = 1$, $\zeta_1 = \mathfrak{a}$ and $\zeta_2 = \mathfrak{b}$ in Remark 1, one can recapture the classical Simpson lemma that was proved by Sarikaya et al. in [24].

Some new fractional Simpson-type inequalities via majorization based on Lemma 1 for convex function are presented hereafter.

Theorem 3. *Under the consideration of Lemma 1, if $|\mathfrak{f}'|$ is continuous convex function on I, then for all $\alpha > 0$, one obtains the following fractional integral inequality:*

$$\left|S_{\alpha}(\mathfrak{a}_{\varrho}, \mathfrak{b}_{\varrho}, \zeta_{\varrho}, \beta; \mathfrak{f})\right| \leq \frac{\sum_{\varrho=1}^{\beta-1}(\mathfrak{b}_{\varrho}-\mathfrak{a}_{\varrho})}{2}\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} - \frac{2\alpha-1}{6(\alpha+1)}\right)\left[2\sum_{\varrho=1}^{\beta}|\mathfrak{f}'(\zeta_{\varrho})|\right.$$
$$\left.-\left\{\sum_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{a}_{\varrho})| + \sum_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{b}_{\varrho})|\right\}\right]. \tag{12}$$

Proof. By utilizing Lemma 1, along with the modulus property, we have

$$\left|S_{\alpha}(\mathfrak{a}_{\varrho}, \mathfrak{b}_{\varrho}, \zeta_{\varrho}, \beta; \mathfrak{f})\right| \leq \frac{\sum_{\varrho=1}^{\beta-1}(\mathfrak{b}_{\varrho}-\mathfrak{a}_{\varrho})}{2}\left[\int_0^1\left|\frac{\eta^{\alpha}}{2}-\frac{1}{3}\right|\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{\beta}\zeta_{\varrho}-(\frac{1+\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{a}_{\varrho}+\frac{1-\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{b}_{\varrho})\right)\right|d\eta + \right.$$
$$\left.\int_0^1\left|\frac{1}{3}-\frac{\eta^{\alpha}}{2}\right|\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{\beta}\zeta_{\varrho}-(\frac{1+\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{b}_{\varrho}+\frac{1-\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{a}_{\varrho})\right)\right|d\eta\right]. \tag{13}$$

By utilizing Theorem 1, for $t = 2$, $\omega_1 = \frac{1-\eta}{2}$, and $\omega_2 = \frac{1+\eta}{2}$, we have

$$
\begin{aligned}
&\leq \frac{\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|\left\{\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|-\left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|\right]\right\}d\eta \\
&+\int_0^1\left|\frac{1}{3}-\frac{\eta^\alpha}{2}\right|\left\{\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|-\left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|\right]\right\}d\eta \\
&\leq \frac{\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left[\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right]\left\{\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|\right. \qquad (14)\\
&\left.-\left(\frac{1+\eta+1-\eta}{2}\right)\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|-\left(\frac{1+\eta+1-\eta}{2}\right)\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|\right\} \\
&\leq \frac{\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left[\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right]\left[2\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|-\left\{\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|+\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|\right\}\right].
\end{aligned}
$$

□

Remark 4. *If we put ß = 2 in Theorem 3, we obtain the following inequality:*

$$
\left|S_\alpha(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f})\right|\leq\frac{\mathfrak{b}-\mathfrak{a}}{2}\left[\frac{\alpha}{\alpha+1}(\frac{2}{3})^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right]\left\{2\mid\mathfrak{f}'(\zeta_1)\mid+2\mid\mathfrak{f}'(\zeta_2)\mid-\mid\mathfrak{f}'(\mathfrak{a})\mid-\mid\mathfrak{f}'(\mathfrak{b})\mid\right\}
$$

which is the majorized estimation of the fractional Mercer inequality in the literature.

Remark 5. *If we put $\alpha=1$ and ß $=2$ into Theorem 3, we acquire the classical Simpson–Mercer estimates proved by Butt et al. in [32].*

If we put $\zeta_1=\mathfrak{a}$ and $\zeta_2=\mathfrak{b}$ in Remark 4, we obtain fractional estimates of the Simpson-type inequality given in [25] for $s=1$.

Putting $\alpha=1$, $\zeta_1=\mathfrak{a}$, and $\zeta_2=\mathfrak{b}$ in Remark 4, we acquire classical Simpson estimates for the differentiable convex function given by Sarikaya et al. in [24] for $s=1$.

Now, we give some new fractional estimates of the Simpson-type inequality by using some well-known inequalities, i.e, Hölder's inequality, the power mean inequality, and Young's inequality.

Theorem 4. *Under the considerations of Lemma 1, if $|\mathfrak{f}'|^{\mathfrak{q}}$ is continuous convex on I and $\mathfrak{q}>1$, then for all $\alpha>0$, we obtain the below-mentioned majorized fractional integral inequality:*

$$
\begin{aligned}
\left|S_\alpha(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,ß;\mathfrak{f})\right|&\leq\frac{\sum_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left[\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{1/\mathfrak{p}}\times\left\{\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left[\frac{3}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}+\frac{1}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right]\right\}^{1/q}\right.\\
&\left.+\left\{\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left[\frac{1}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}+\frac{3}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right]\right\}^{1/q}\right] \qquad (15)
\end{aligned}
$$

where $\eta\in[0,1]$ and $\frac{1}{\mathfrak{p}}+\frac{1}{\mathfrak{q}}=1$.

Proof. By applying the modulus and Hölder's inequality to Lemma 1, we have:

$$\left|S_\alpha\left(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,ß;\mathfrak{f}\right)\right|\leq\frac{\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left[\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{\frac{1}{\mathfrak{p}}}\right.$$
$$\left(\int_0^1\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho)\right)\right|^{\mathfrak{q}}d\eta\right)^{\frac{1}{\mathfrak{q}}}+\left(\int_0^1\left|\frac{1}{3}-\frac{\eta^\alpha}{2}\right|^{\mathfrak{p}}d\eta\right)^{\frac{1}{\mathfrak{p}}}\tag{16}$$
$$\left.\left(\int_0^1\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho)\right)\right|^{\mathfrak{q}}d\eta\right)^{\frac{1}{\mathfrak{q}}}\right]$$

By utilizing the convexity of $|\mathfrak{f}'|^{\mathfrak{q}}$ and applying Theorem 1, for $t=2$, $\omega_1=\frac{1-\eta}{2}$, and $\omega_2=\frac{1+\eta}{2}$, we have

$$\leq\frac{\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left[\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{\frac{1}{\mathfrak{p}}}\left[\int_0^1\left(\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}})\right)\right]^{\frac{1}{\mathfrak{q}}}\right.$$
$$\left.+\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{\frac{1}{\mathfrak{p}}}\left[\int_0^1\left(\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}})\right)\right]^{\frac{1}{\mathfrak{q}}}\right]$$
$$=\frac{\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left[\left(\left(\frac{1}{3}\right)^{\mathfrak{p}}\left[2\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}-1\right]+\frac{1}{2^{\mathfrak{p}}(\alpha p+1)}\left[1-2\left(\frac{2}{3}\right)^{\frac{\alpha p+1}{\alpha}}\right]\right)^{\frac{1}{\mathfrak{p}}}\times\right.$$
$$\left\{\left[\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left(\frac{3}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}+\frac{1}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right)\right]^{\frac{1}{\mathfrak{q}}}+\right.$$
$$\left.\left.\left[\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left(\frac{3}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}+\frac{1}{4}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}\right)\right]^{\frac{1}{\mathfrak{q}}}\right\}\right].$$

Here, we use

$$(A-B)^{\mathfrak{p}}\leq A^{\mathfrak{p}}-B^{\mathfrak{p}}$$

for $A>B>0$ and $\mathfrak{p}\geq 1$. □

Remark 6. *Here, by substituting ß $=2$ in Theorem 4, we have the following inequality:*

$$\left|S_\alpha\left(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f}\right)\right|\leq\frac{\mathfrak{b}-\mathfrak{a}}{2}\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{\frac{1}{\mathfrak{p}}}\times\left[\left[|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-[\frac{3}{4}|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}+\frac{1}{4}|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}]\right]^{\frac{1}{\mathfrak{q}}}\right.$$
$$\left.+\left[|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-[\frac{3}{4}|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}+\frac{1}{4}|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}]\right]^{\frac{1}{\mathfrak{q}}}\right]$$

which gives new fractional Mercer estimates.

Remark 7. *Selecting $\alpha=1$ in Remark 6, we obtain the following bound of classical Mercer inequality in terms of conjugate exponent:*

$$\begin{aligned}&\left|S_1\big(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f}\big)\right|\\&\leq\frac{\mathfrak{b}-\mathfrak{a}}{2}\left(\int_0^1\left|\frac{\eta}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{\frac{1}{\mathfrak{p}}}\times\left[\left[|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-[\frac{3}{4}|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}+\frac{1}{4}|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}]\right]^{\frac{1}{\mathfrak{q}}}\right.\\&\left.+\left[|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-[\frac{3}{4}|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}+\frac{1}{4}|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}]\right]^{\frac{1}{\mathfrak{q}}}\right].\end{aligned}$$

Remark 8. *By substituting $\zeta_1 = \mathfrak{a}$ and $\zeta_2 = \mathfrak{b}$ in the previous Remark 6, we recapture the fractional Simpson estimates proved in [25] for $s = 1$.*

If we choose $\alpha = 1$, $\zeta_1 = \mathfrak{a}$, and $\zeta_2 = \mathfrak{b}$ in Remark 6, we obtain the classical results that were proved by Sarikaya et al. [24] for $s = 1$.

For convex mappings, we give a new approach to construct the upper bound for the right-hand side of the majorized Simpson's inequality in the following Theorem.

Theorem 5. *Under consideration of Lemma 1, if $|\ \mathfrak{f}'\ |^{\mathfrak{q}}$ is continuous convex on I and $\mathfrak{q} > 1$, then for all $\alpha > 0$, one obtains the undermentioned fractional integral inequality:*

$$\begin{aligned}\left|S_\alpha\big(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,\beta;\mathfrak{f}\big)\right|&\leq\frac{\sum\limits_{\varrho=1}^{\beta-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{1/\mathfrak{p}}\times\left[\left\{|\mathfrak{f}'(\sum_{\varrho=1}^{\beta}\zeta_\varrho)|^{\mathfrak{q}}-\left[\frac{|\mathfrak{f}'(\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho)|^{\mathfrak{q}}+|\mathfrak{f}'(\sum\limits_{\varrho=1}^{\beta-1}\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2})|^{\mathfrak{q}}}{2}\right]\right\}^{1/\mathfrak{q}}\right.\\&\left.+\left\{|\mathfrak{f}'(\sum_{\varrho=1}^{\beta}\zeta_\varrho)|^{\mathfrak{q}}-\left[\frac{|\mathfrak{f}'(\sum\limits_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho)|^{\mathfrak{q}}+|\mathfrak{f}'(\sum\limits_{\varrho=1}^{\beta-1}\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2})|^{\mathfrak{q}}}{2}\right]\right\}^{1/\mathfrak{q}}\right]\end{aligned}\tag{17}$$

that is satisfied for $\eta \in [0,1]$ and $\frac{1}{\mathfrak{p}}+\frac{1}{\mathfrak{q}}=1$.

Proof. It is easy to obtain the inequality (16) as mentioned in the proof of Theorem 4. Since $|\mathfrak{f}'|^{\mathfrak{q}}$ is convex on I, by employing Theorem 2 for $\alpha = 1$, it is easy to obtain

$$\begin{aligned}&\int_0^1\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho-(\frac{1+\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho)\right)\right|^{\mathfrak{q}}d\eta\\&\qquad\leq\sum_{\varrho=1}^{\beta}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left[\frac{\sum\limits_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}+\sum\limits_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2})|^{\mathfrak{q}}}{2}\right],\end{aligned}\tag{18}$$

$$\begin{aligned}&\int_0^1\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{\beta}\zeta_\varrho-(\frac{1-\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{a}_\varrho+\frac{1+\eta}{2}\sum_{\varrho=1}^{\beta-1}\mathfrak{b}_\varrho)\right)\right|^{\mathfrak{q}}d\eta\\&\qquad\leq\sum_{\varrho=1}^{\beta}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left[\frac{\sum\limits_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}+\sum\limits_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\frac{\mathfrak{a}_\varrho+\mathfrak{b}_\varrho}{2})|^{\mathfrak{q}}}{2}\right].\end{aligned}\tag{19}$$

By substituting (18) and (19) into (16), we obtain (17), which is the required result. □

Remark 9. *Here, by using $\beta = 2$ in Theorem 5, we obtain the following fractional Mercer-type estimates,*

$$\left|S_\alpha(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f})\right| \leq \frac{\mathfrak{b}-\mathfrak{a}}{2}\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta\right)^{\frac{1}{\mathfrak{p}}} \times \left\{\left(|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-\left[\frac{|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}+\left|\mathfrak{f}'\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)\right|^{\mathfrak{q}}}{2}\right]\right)^{\frac{1}{\mathfrak{q}}}\right.$$
$$\left.+\left(|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-\left[\frac{|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}+\left|\mathfrak{f}'\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)\right|^{\mathfrak{q}}}{2}\right]\right)^{\frac{1}{\mathfrak{q}}}\right\}.$$

Remark 10. *By substituting $\zeta_1=\mathfrak{a}$ and $\zeta_2=\mathfrak{b}$ into Remark 9, we obtain the estimates of the fractional Simpson inequality pertaining to Riemann–Liouville integral operators proved by Chen and Huang in [25] for $s=1$.*

If we substitute $\zeta_1=\mathfrak{a}$, $\zeta_2=\mathfrak{b}$ and $\alpha=1$ in Remark 9, we deduce the classical form of the inequality given by Sarikaya et al. in [24] for $s=1$.

Theorem 6. *In consideration of Lemma 1, if $|\mathfrak{f}'|^{\mathfrak{q}}$ is a continuous convex function on I and $\mathfrak{q}>1$, then, for all $\alpha>0$, one obtains the following fractional integral inequality:*

$$\begin{aligned}
&\left|S_\alpha(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,\beta;\mathfrak{f})\right| \leq \frac{\sum_{\varrho=1}^{\beta-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right)^{1-\frac{1}{\mathfrak{q}}}\times\\
&\left[\left(\sum_{\varrho=1}^{\beta}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}\right)\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right)-\right.\\
&\left\{\sum_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}}-\frac{\alpha^2+\alpha-1}{4(\alpha+1)(\alpha+2)}\right)+\sum_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right.\\
&\left.\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}}-\frac{\alpha^2+3\alpha-1}{12(\alpha+1)(\alpha+2)}\right)\right\}^{\frac{1}{\mathfrak{q}}}\\
&+\left\{\left(\sum_{\varrho=1}^{\beta}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}\right)\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right)\right\}\\
&-\left\{\sum_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}}-\frac{\alpha^2+3\alpha-1}{12(\alpha+1)(\alpha+2)}\right)+\sum_{\varrho=1}^{\beta-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right.\\
&\left.\left.\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}+\frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}}-\frac{\alpha^2+\alpha-1}{4(\alpha+1)(\alpha+2)}\right)\right\}^{\frac{1}{\mathfrak{q}}}\right]
\end{aligned} \tag{20}$$

that is satisfied for $\eta\in[0,1]$ and $\frac{1}{\mathfrak{p}}+\frac{1}{\mathfrak{q}}=1$.

Proof. By utilizing the power mean inequality on Lemma 1, we have

$$\begin{aligned}
&\left|S_\alpha(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,\text{ß};\mathfrak{f})\right| \leq \frac{\sum_{\varrho=1}^{\text{ß}-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{2}\left[\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|^{\mathfrak{q}}d\eta\right)^{1-\frac{1}{\mathfrak{q}}}\right.\\
&\times\left(\int_0^1\left|\frac{\eta^\alpha}{2}-\frac{1}{3}\right|\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{\text{ß}}\zeta_\varrho-\left(\frac{1+\eta}{2}\sum_{\varrho=1}^{\text{ß}-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{\text{ß}-1}\mathfrak{b}_\varrho\right)\right)\right|^{\mathfrak{q}}d\eta\right)^{\frac{1}{\mathfrak{q}}}+\\
&\left.\int_0^1\left|\frac{1}{3}-\frac{\eta^\alpha}{2}\right|^{\mathfrak{q}}\right)^{1-\frac{1}{\mathfrak{q}}}\left(\int_0^1\left|\frac{1}{3}-\frac{\eta^\alpha}{2}\right|\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{\text{ß}}\zeta_\varrho-\left(\frac{1+\eta}{2}\sum_{\varrho=1}^{\text{ß}-1}\mathfrak{b}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{\text{ß}-1}\mathfrak{a}_\varrho\right)\right)\right|^{\mathfrak{q}}d\eta\right)^{\frac{1}{\mathfrak{q}}}\right].
\end{aligned} \tag{21}$$

Since $|\,\mathfrak{f}'\,|^{\mathfrak{q}}$ is convex on I, by applying Theorem 1, for $t = 2$, $\omega_1 = \frac{1-\eta}{2}$ and $\omega_2 = \frac{1+\eta}{2}$, we obtain

$$\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \left(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho + \frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho\right)\right)\right|^{\mathfrak{q}} \leq \sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}} - \left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}} + \frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}\right] \tag{22}$$

$$\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho - \left(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho + \frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right)\right)\right| \leq \sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}} - \left[\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}} + \frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right]. \tag{23}$$

Now, by substituting (22) and (23) into (21) and then by simplifying the integrals, we obtain (20), which finishes the proof. □

Remark 11. *The above inequality (20) leads to develop the following new variant of fractional Simpson–Mercer inequality by substituting $ß = 2$ in Theorem 6.*

$$\begin{aligned}
&\left|S_\alpha(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f})\right| \leq \frac{(\mathfrak{b}-\mathfrak{a})}{2}\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} - \frac{2\alpha-1}{6(\alpha+1)}\right)^{1-\frac{1}{\mathfrak{q}}}\times\\
&\left[\left\{\left(|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}} + |\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}\right)\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} - \frac{2\alpha-1}{6(\alpha+1)}\right) - \left\{|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}\right.\right.\right.\\
&\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\alpha+1} + \frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}} - \frac{\alpha^2+\alpha-1}{4(\alpha+1)(\alpha+2)}\right)\\
&\left.\left.+|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} - \frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}} - \frac{\alpha^2+3\alpha-1}{12(\alpha+1)(\alpha+2)}\right)\right\}\right\}^{\frac{1}{\mathfrak{q}}} +\\
&\left\{\left(|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}} + |\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}\right)\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} - \frac{2\alpha-1}{6(\alpha+1)}\right) - \left\{|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}\right.\right.\\
&\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} + \frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}} - \frac{\alpha^2+\alpha-1}{4(\alpha+1)(\alpha+2)}\right)\\
&\left.\left.\left.+|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}\left(\frac{\alpha}{2(\alpha+1)}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} - \frac{\alpha}{4(\alpha+2)}\left(\frac{2}{3}\right)^{\frac{\alpha+2}{\alpha}} - \frac{\alpha^2+3\alpha-1}{12(\alpha+1)(\alpha+2)}\right)\right\}\right\}^{\frac{1}{\mathfrak{q}}}\right],
\end{aligned}$$

which is new in the literature.

Remark 12. *By substituting $\alpha = 1$ in Remark 11, we obtain the following classical form of the inequality:*

$$\begin{aligned}
&\left|S_1(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f})\right| \leq \frac{\mathfrak{b}-\mathfrak{a}}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{\mathfrak{q}}} \times \left[\left[\left(\frac{5}{36}\right)|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}} + \left(\frac{5}{36}\right)|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}} - \left[\frac{29}{648}|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}} + \frac{61}{648}|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}\right]\right]^{\frac{1}{\mathfrak{q}}}\right.\\
&\left.+\left[\left(\frac{5}{36}\right)|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}} + \left(\frac{5}{36}\right)|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}} - \left[\frac{29}{648}|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}} + \frac{61}{648}|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}\right]\right]^{\frac{1}{\mathfrak{q}}}\right],
\end{aligned}$$

which is a new result in the literature.

Remark 13. *From the previous result for $\alpha = 1$, $\zeta_1 = \mathfrak{a}$, and $\zeta_2 = \mathfrak{b}$, we obtain the bounds for the classical Simpson inequality for the differentiable convex functions results proved by Sarikaya et al. in [25] for $s = 1$.*

Theorem 7. *In consideration of Lemma 1, if $|\mathfrak{f}'|^{\mathfrak{q}}$ is a continuous convex function on I and $\mathfrak{q} > 1$, for all $\alpha > 0$, we obtain the fractional integral inequality given as:*

$$\left|S_\alpha\left(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,ß;\mathfrak{f}\right)\right| \leq \frac{\sum\limits_{\varrho=1}^{ß-1}\left(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho\right)}{2}\left[\frac{2}{p}\int_0^1\left|\frac{\eta^{\mathfrak{p}}}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta+\frac{1}{\mathfrak{q}}\left[2\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left\{\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}+\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right\}\right]\right]. \quad (24)$$

Proof. From Lemma 1, by applying Young's inequality on (13), we have

$$\left|S_\alpha\left(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,ß;\mathfrak{f}\right)\right| \leq \frac{\sum\limits_{\varrho=1}^{ß-1}\left(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho\right)}{2}\left[\frac{1}{\mathfrak{p}}\int_0^1\left|\frac{\eta^{\alpha}}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta+\frac{1}{\mathfrak{q}}\int_0^1\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho\right)\right)\right|^{\mathfrak{q}}d\eta\right.$$
$$\left.+\frac{1}{\mathfrak{p}}\int_0^1\left|\frac{1}{3}-\frac{\eta^{\alpha}}{2}\right|^{\mathfrak{p}}d\eta+\frac{1}{\mathfrak{q}}\int_0^1\left|\mathfrak{f}'\left(\sum_{\varrho=1}^{ß}\zeta_\varrho-\left(\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{b}_\varrho+\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}\mathfrak{a}_\varrho\right)\right)\right|^{\mathfrak{q}}d\eta\right].$$

By applying the convexity of $|\mathfrak{f}'|^{\mathfrak{q}}$ and using Theorem 1, for $t=2$, $\omega_1=\frac{1+\eta}{2}$ and $\omega_2=\frac{1-\eta}{2}$, we have

$$\leq \frac{\sum\limits_{\varrho=1}^{ß-1}\left(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho\right)}{2}\left[\frac{2}{p}\int_0^1\left|\frac{\eta^{\alpha}}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta+\frac{1}{\mathfrak{q}}\left[\int_0^1\left[\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left\{\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}\right.\right.\right.\right.$$
$$\left.\left.-\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right\}\right]d\eta$$
$$\left.+\frac{1}{\mathfrak{q}}\int_0^1\left[\sum_{\varrho=1}^{ß}|\mathfrak{f}'(\zeta_\varrho)|^{\mathfrak{q}}-\left\{\frac{1-\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{a}_\varrho)|^{\mathfrak{q}}-\frac{1+\eta}{2}\sum_{\varrho=1}^{ß-1}|\mathfrak{f}'(\mathfrak{b}_\varrho)|^{\mathfrak{q}}\right\}\right]d\eta\right]$$

By simplifying the integrals in (25), we deduce (24), which completes the proof. □

Remark 14. *If we substitute ß = 2 in Theorem 6, then we obtain the following new Mercer estimates using Young's inequality:*

$$\left|S_\alpha\left(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f}\right)\right| \leq \frac{(\mathfrak{b}-\mathfrak{a})}{2}\left[\frac{2}{p}\int_0^1\left|\frac{\eta^{\alpha}}{2}-\frac{1}{3}\right|^{\mathfrak{p}}d\eta+\frac{1}{\mathfrak{q}}\left\{2|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+2|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-\left\{|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}+|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}\right\}\right\}\right].$$

Remark 15. *If we set $\alpha=1$ and $ß=2$ in Theorem 6, then we obtain the following bounds by using Young's inequality:*

$$\left|S_1\left(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,2;\mathfrak{f}\right)\right| \leq \frac{\mathfrak{b}-\mathfrak{a}}{2}\left[\frac{2}{p}\left(\frac{(1+2^{p+1})}{6^{\mathfrak{p}}(3p+3)}\right)+\frac{1}{\mathfrak{q}}\left[2|\mathfrak{f}'(\zeta_1)|^{\mathfrak{q}}+2|\mathfrak{f}'(\zeta_2)|^{\mathfrak{q}}-\left[|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}+|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}\right]\right]\right].$$

Remark 16. *If we set $\alpha=1$, $\zeta_1=\mathfrak{a}$, and $\zeta_2=\mathfrak{b}$ in Remark 15, we obtain the following bounds of Simpson's inequality in the classical sense:*

$$\left|\frac{1}{6}\left[\mathfrak{f}(\mathfrak{a})+4\mathfrak{f}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\mathfrak{f}(\mathfrak{b})\right]-\frac{1}{\mathfrak{b}-\mathfrak{a}}\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{f}(P)dP\right|$$
$$\leq \frac{\mathfrak{b}-\mathfrak{a}}{2}\left[\frac{2}{\mathfrak{p}}\left(\frac{(1+2^{\mathfrak{p}+1})}{6^{\mathfrak{p}}(3^{\mathfrak{p}}+3)}\right)+\frac{1}{\mathfrak{q}}\left\{|\mathfrak{f}'(\mathfrak{a})|^{\mathfrak{q}}+|\mathfrak{f}'(\mathfrak{b})|^{\mathfrak{q}}\right\}\right].$$

3. Simpson's Estimates for Numerical Quadrature and Special Functions

3.1. Numerical Quadrature Rule

In this section, we examine the application of the integral inequalities involving fractional integral operators, as discussed in the preceding section, for the purpose of approximating composite quadrature rules. This approach yields a substantially reduced error compared to conventional methods.

Proposition 1. *Let $\mathfrak{f} : [\zeta_1, \zeta_2] \to \Re$ be a bounded function. If $I_\epsilon \in \zeta_1 = \vartheta_0, \vartheta_1, \ldots, \vartheta_{\epsilon-1}, \vartheta_\epsilon = \zeta_2$ is the interval and $\vartheta_{\beta,1}, \vartheta_{\beta,2} \in [\vartheta_\beta, \vartheta_{\beta+1}]$ with $\hbar_\beta = \vartheta_{\beta+1} - \vartheta_\beta$ for all $\beta = 0, 1, \ldots, \epsilon - 1$, then we have,*

$$\int_{\vartheta_0+\vartheta_\epsilon-\vartheta_2}^{\vartheta_0+\vartheta_\epsilon-\vartheta_1} \mathfrak{f}(P)dP = B(I_\epsilon, \mathfrak{f}) + R(I_\epsilon, \mathfrak{f})$$

where

$$B(I_\epsilon, \mathfrak{f}) = \frac{1}{6}\Bigg[\sum_{\beta=0}^{\epsilon-1} \mathfrak{f}[\vartheta_\beta + \vartheta_{\beta+1} - \vartheta_{\beta,1}]\hbar_\beta + 4\sum_{\beta=0}^{\epsilon-1} \mathfrak{f}[\vartheta_\beta + \vartheta_{\beta+1} - \frac{\vartheta_{\beta,1} + \vartheta_{\beta,2}}{2}]\hbar_\beta + \sum_{\beta=0}^{\epsilon-1} \mathfrak{f}[\vartheta_\beta + \vartheta_{\beta+1} - \vartheta_{\beta,2}]\hbar_\beta\Bigg]$$

and remainder term satisfies

$$|R(I_\epsilon, \mathfrak{f})| \leq \frac{5}{72}\left[\hbar_\beta^2 2\sum_{\beta=0}^{\epsilon-1}\Big[|\mathfrak{f}'(\vartheta_\beta)| + |\mathfrak{f}'(\vartheta_{\beta+1})|\Big] - \Big[\hbar_\gamma^2 \sum_{\beta=0}^{\epsilon-1}|\mathfrak{f}'(\vartheta_{\beta,1})| + \hbar_\beta^2 \sum_{\beta=0}^{\epsilon-1}|\mathfrak{f}'(\vartheta_{\beta,2})|\Big]\right].$$

Proof. Applying Theorem 3 with ß = 2 and $\alpha = 1$ on interval $[\vartheta_\beta, \vartheta_{\beta+1}]$, $\beta = 0, 1, \ldots, \epsilon - 1$, we obtain

$$\left|\frac{1}{6}\left[\mathfrak{f}[\vartheta_\beta + \vartheta_{\beta+1} - \vartheta_{\beta,1}]h_\beta + 4\mathfrak{f}[\vartheta_\beta + \vartheta_{\vartheta+1} - \frac{\vartheta_{\beta,1} + \vartheta_{\beta,2}}{2}]\hbar_\beta + \mathfrak{f}[\vartheta_\beta + \vartheta_{\beta+1} - \vartheta_{\beta,2}]\hbar_\beta\right] - \int_{\vartheta_\beta+\vartheta_{\beta+1}-\vartheta_{\beta,2}}^{\vartheta_\beta+\vartheta_{\beta+1}-\vartheta_{\beta,1}} \beta(P)dP\right| \leq$$

$$\frac{5}{72}\left[\hbar_\beta^2 2\sum_{\beta=0}^{\epsilon-1}\Big[|\mathfrak{f}'(\vartheta_\beta)| + |\mathfrak{f}'(\vartheta_{\beta+1})|\Big] - \Big[\hbar_\beta^2 \sum_{\beta=0}^{\epsilon-1}|\mathfrak{f}'(\vartheta_{\beta,1})| + \hbar_\beta^2 \sum_{\beta=0}^{\epsilon-1}|\mathfrak{f}'(\vartheta_{\beta,2})|\Big]\right]$$

For all $\beta = 0, 1, \ldots, \epsilon - 1$, summing over 0 to $\epsilon - 1$ and utilizing the triangular inequality, we obtain the required result. □

3.2. *q*-Digamma Function

The function known as $\Psi_{\mathbf{q}}$-*digamma* is defined as the logarithmic derivative of the $\mathbf{q}$-*gamma* function. It plays a crucial role in relation to the $\mathbf{q}$-*gamma* function. Several researchers have explored the monotonicity and complete monotonicity properties of the $\mathbf{q}$-*gamma* and $\mathbf{q}$-*digamma* functions in various applications, leading to notable inequalities [33].

Let us consider $0 < \mathbf{q} < 1$. The $\mathbf{q}$-digamma (psi) function $\Psi_{\mathbf{q}}$ is the $\mathbf{q}$ analogue of the psi or digamma function Ψ (see [34]), given as:

$$\begin{aligned}\Psi_{\mathbf{q}} &= -\ln(1-\mathbf{q}) + \ln \mathbf{q} \sum_{k=0}^{\infty} \frac{\mathbf{q}^{k+\xi}}{1-\mathbf{q}^{k+\xi}} \\ &= -\ln(1-\mathbf{q}) + \ln \mathbf{q} \sum_{k=0}^{\infty} \frac{\mathbf{q}^{k\xi}}{1-\mathbf{q}^{k\xi}}.\end{aligned}$$

For $\mathbf{q} > 1$ and $\xi > 0$, $\mathbf{q}-$digamma function $\Psi_{\mathbf{q}}$ can be written as:

$$\begin{aligned}\Psi_{\mathbf{q}} &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[\xi - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+\xi)}}{1-\mathbf{q}^{-(k+\xi)}}\right] \\ &= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[\xi - \frac{1}{2} - \sum_{k=0}^{\infty}\frac{\mathbf{q}^{-k\xi}}{1-\mathbf{q}^{-k\xi}}\right].\end{aligned}$$

Proposition 2. *Suppose $\zeta_\varrho, \mathfrak{a}_s, \mathfrak{b}_s$ are real numbers such that $\zeta_\varrho \succ \mathfrak{a}_\varrho$ $\zeta_\varrho \succ \mathfrak{b}_\varrho$, and $0 < q < 1$. Then, the following inequality holds*

$$\begin{aligned}\left|S_\alpha\left(\mathfrak{a}_\varrho, \mathfrak{b}_\varrho, \zeta_\varrho, ß; \Psi_q\right)\right| \leq \frac{\sum\limits_{\varrho=1}^{ß-1}(\mathfrak{b}_\varrho - \mathfrak{a}_\varrho)}{2}\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}} - \frac{2\alpha-1}{6(\alpha+1)}\right)\left[2\sum_{\varrho=1}^{ß}|\Psi_q'(\zeta_\varrho)|\right. \\ \left. - \left\{\sum_{\varrho=1}^{ß-1}|\Psi_q'(\mathfrak{a}_\varrho)| + \sum_{\varrho=1}^{ß-1}|\Psi_q'(\mathfrak{b}_\varrho)|\right\}\right].\end{aligned} \tag{25}$$

Proof. From the definition of $\mathbf{q}$-digamma function, it is simple to see that if the function $\mathfrak{f} \to \Psi$, then $\mathfrak{f}'(\xi) = \Psi_{\mathbf{q}}'(\xi)$ is completely monotonic on $(0,\infty)$ for each $\mathbf{q} \in (0,1)$ and is convex and non-negative. By using this substitution for Theorem 3, we obtain the desired outcome. □

Remark 17. *If we set $ß = 2$ and $\alpha = 1$ in Proposition 2, we have following estimation of the classical Mercer inequality:*

$$\left|S_1\left(\mathfrak{a}_\varrho, \mathfrak{b}_\varrho, \zeta_\varrho, 2; \Psi_q\right)\right| \leq \frac{5}{72}(\mathfrak{b}-\mathfrak{a})\left[2\mid\Psi_q'(\zeta_1)\mid + 2\mid\Psi_q'(\zeta_2)\mid - \mid\Psi_q'(\mathfrak{a})\mid - \mid\Psi_q'(\mathfrak{b})\mid\right].$$

which is a new result in the literature.

Remark 18. *If we set $\zeta_1 = \mathfrak{a}$ and $\zeta_2 = \mathfrak{b}$ in Remark 17, we obtain the below-mentioned new result in the literature.*

$$\left|\frac{1}{6}\left[\Psi_q(\mathfrak{a}) + 4\Psi_q\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) + \Psi_q(\mathfrak{b})\right] - \frac{1}{\mathfrak{b}-\mathfrak{a}}\int_{\mathfrak{a}}^{\mathfrak{b}}\Psi_q(P)dP\right| \leq \frac{5}{72}(\mathfrak{b}-\mathfrak{a})\left[\mid\Psi_q'(\mathfrak{a})\mid + \mid\Psi_q'(\mathfrak{b})\mid\right]$$

3.3. Modified Bessel Function

Friedrich Wilhelm Bessel gave the Bessel functions their name, while Daniel Bernoulli is usually recognized as having first proposed it in 1732. By using its generating function, several conclusions concerning Bessel functions have been obtained. We recall the original first kind of modified Bessel function $\Im_\varpi$ (see [35]), which has a series representation as:

$$\Im_\varpi(\xi) = \sum_{n\geq 0}\frac{\left(\frac{\xi}{2}\right)^{\varpi+2n}}{n!\Gamma(\varpi+n+1)},$$

where $\xi \in \Re$ and $\varpi > -1$. The second kind of modified Bessel function $\hbar_\varpi$ (see [35], p. 77) is usually defined as

$$\hbar_\varpi(\xi) = \frac{\pi}{2}\,\frac{\Im_{-\varpi}(\xi) - \Im_\varpi(\xi)}{\sin\varpi\pi}.$$

Consider the function $\Theta_\varpi(\xi) : \Re \to [1,\infty)$, defined by

$$\Theta_\varpi(\xi) = 2^\varpi\Gamma(\varpi+1)\xi^{-\varpi}F_\varpi(\xi),$$

where Γ is the gamma function.

The first-order derivative formula of $\Theta_{\varpi}(\xi)$ is given by [35]:

$$\Theta'_{\varpi}(\xi) = \frac{\xi}{2(\varpi+1)}\Theta_{\varpi+1}(\xi) \tag{26}$$

and the second-order derivative can be easily calculated from (26) as

$$\Theta''_{\varpi}(\xi) = \frac{\xi^2\Theta_{\varpi+2}(\xi)}{4(\varpi+1)(\varpi+2)} + \frac{\Theta_{\varpi+1}(\xi)}{2(\varpi+1)}. \tag{27}$$

Proposition 3. *Suppose $\varpi > -1$, $\zeta_\varrho \succ \mathfrak{a}_\varrho$, and $\zeta_\varrho \succ \mathfrak{b}_\varrho$, then*

$$\begin{aligned}\left|S_\alpha(\mathfrak{a}_\varrho,\mathfrak{b}_\varrho,\zeta_\varrho,\text{ß};\Theta_\varpi)\right| &\leq \frac{\sum_{\varrho=1}^{\text{ß}-1}(\mathfrak{b}_\varrho-\mathfrak{a}_\varrho)}{4(\varpi+1)}\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right)\left[2\sum_{\varrho=1}^{\text{ß}}|\zeta_\varrho\Theta_{\varpi+1}(\zeta_\varrho)|\right.\\ &\left.-\left\{\sum_{\varrho=1}^{\text{ß}-1}|\mathfrak{a}_\varrho\Theta_{\varpi+1}(\mathfrak{a}_\varrho)|+\sum_{\varrho=1}^{\text{ß}-1}|\mathfrak{b}_\varrho\Theta_{\varpi+1}(\mathfrak{b}_\varrho)|\right\}\right].\end{aligned} \tag{28}$$

Proof. The required result can be obtained by using the Bessel function $\mathfrak{f} \to \Theta_\varpi$. It is simple to see that by using $\Theta'_\varpi(\xi)$, $\xi > 0$ on $[0,\infty]$ is a convex function. By considering this substitution and (26) in Theorem 3, we obtain the desired outcome. □

Remark 19. *If we put ß $= 2$, $\zeta_1 = \mathfrak{a}$ and $\zeta_2 = \mathfrak{b}$ in Proposition 3, we obtain:*

$$\begin{aligned}&\left|\frac{1}{6}\left\{\Theta_\varpi(\mathfrak{a})+4\Theta_\varpi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+\Theta_\varpi(\mathfrak{b})\right\}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mathfrak{b}-\mathfrak{a})^\alpha}\times\right.\\ &\left.\left\{J^\alpha_{\mathfrak{b}^-}\Theta_\varpi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)+J^\alpha_{\mathfrak{a}^+}\Theta_\varpi\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right)\right\}\right|\\ &\leq \frac{\mathfrak{b}-\mathfrak{a}}{4(\varpi+1)}\left(\frac{\alpha}{\alpha+1}\left(\frac{2}{3}\right)^{\frac{\alpha+1}{\alpha}}-\frac{2\alpha-1}{6(\alpha+1)}\right)\left[|\mathfrak{a}\Theta_{\varpi+1}(\mathfrak{a})|+|\mathfrak{b}\Theta_{\varpi+1}(\mathfrak{b})|\right].\end{aligned}$$

which is a new result in the literature.

4. Conclusions

To the best of our knowledge, the current investigation is the first one with respect to the Rieman–Liouville fractional Simpson-type integral inequality involving differentiable functions along with a majorization scheme. We introduced a new fractional integral identity for differentiable functions by utilizing the concept of majorization theory. Taking advantage of the established identity, and in combination with convexity, we obtained a series of fractional Simpson-type integral inequalities. It is also worth noting that the primary results obtained here transform into the results of the fractional integral type when the parameter ß $= 2$ is taken and subjected to appropriate transformations. Meanwhile, these turn into the findings for the Simposn–Mercer inequality for differentiable convex functions by using particular substitutions. Finally, some applications to the Simpson quadrature rule were presented and our findings provide Simpson-type estimations for special functions such as q-digamma functions and Bessel functions. Our approach may have further implementations in the theory of majorization. It would be interesting to extend such findings for other convexities in the literature. By using generalized convexities, it is possible to consider Simpson's inequalities with majorization for generalized integral operators with nonlocal and nonsingular kernels. With the ideas developed in this paper, we hope to motivate interested researchers to further explore other types of fractional integral operators, local fractional integrals, and fractal-fractional integrals to similarly construct new identities and to derive its related integral inequalities. Also, it would be interesting to extend this idea for twice- and thrice-differentiable functions to locate the respective Simpson estimations in terms of majorized tuples.

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