axioms

MDPI

Article

Quasiconformal Homeomorphisms Explicitly Determining the Basic Curve Quasi-Invariants

Samuel L. Krushkal [1,2]

[1] Department of Mathematics, Bar-Ilan University, Ramat-Gan 5290002, Israel; krushkal@math.biu.ac.il
[2] Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA

Abstract: The classical Belinskii theorem implies that any sufficiently regular function $\mu(z)$ on the extended complex plane $\widehat{\mathbb{C}}$ with a small $C^{1+\alpha}$ norm generates via the two-dimensional Cauchy integral a quasiconformal automorphism w of $\widehat{\mathbb{C}}$ with the Beltrami coefficient $\widetilde{\mu} = \mu + O(\|\mu\|^2)$. We consider μ supported in arbitrary bounded quasiconformal disks and show that under appropriate assumptions of μ, this automorphism explicitly provides the basic curvelinear quasi-invariants associated with conformal and quasiconformal maps, advancing an old problem of quasiconformal analysis.

Keywords: univalent function; quasiconformal maps and reflections; quasicircle; universal Teichmüller space; the Belinskii theorem; abelian quadratic differential; Grunsky operator; the Teichmüller and Grunsky norms of univalent functions; Fredholm eigenvalue

MSC: 30C55; 30C62; 30F60; 32G15; 46G20

check for updates

Citation: Krushkal, S.L. Quasiconformal Homeomorphisms Explicitly Determining the Basic Curve Quasi-Invariants. *Axioms* **2023**, *12*, 944. https://doi.org/10.3390/axioms12100944

Academic Editor: Miodrag Mateljevic

Received: 4 September 2023
Revised: 21 September 2023
Accepted: 27 September 2023
Published: 30 September 2023

1. Preliminaries

1.1. Preamble

An important, open problem in geometric complex analysis establishing algorithms for explicit or approximate determination of the basic curvilinear and analytic quasi-invariant functionals intrinsically connected with conformal and quasiconformal maps, such as their Teichmüller and Grunsky norms, Fredholm eigenvalues and the quasireflection coefficients of associated quasicircles. It is important also for the potential theory. The investigations in this field of complex analysis were originated by the classical works of Ahlfors, Schiffer, Kühnau, Schober and continued by many other mathematicians. However, the problem has not been solved completely even for convex polygons.

This problem has intrinsic interest also in view of its connection with the geometry of Teichmüller spaces and with the approximation theory. It is crucial also for numerical aspects of quasiconformal analysis.

The present paper is connected with the author's investigations in this direction (see, for example, [1]) and considers the classes of univalent holomorphic functions not admitting the canonical Teichmüller extremal extensions. We give a complete solution of the indicated problem for some natural broad classes of Beltrami coefficients supported in the generic quasiconformal domains. All previous results were obtained only for the canonical unit disk $\mathbb{D} = \{|z| < 1\}$.

Our approach is different and involves the deep results from Teichmüller space theory and complex differential geometry.

1.2. Some Invariants and Norms of Univalent Functions with Quasiconformal Extension

Consider the collection Σ_Q of univalent functions on the disk $\mathbb{D}^* = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$ with expansions

$$f(z) = z + b_0 + b_1 z^{-1} + \dots$$

having quasiconformal extensions across the boundary unit circle $\mathbb{S}^1 = \partial\mathbb{D}^*$ to the whole Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

To have compactness of this class in the topology of locally uniform convergence on $\mathbb{C}$, we add the third normalization condition $f(0) = 0$.

The Beltrami coefficients of these extensions are supported in the unit disk $\mathbb{D}$ and run over the unit ball

$$\mathrm{Belt}(\mathbb{D})_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu(z)|\mathbb{D}^* = 0, \ \ \|\mu\|_\infty < 1\}.$$

Each $\mu \in \mathrm{Belt}(\mathbb{D})_1$ determines a unique homeomorphic solution to the Beltrami equation $\bar{\partial} w = \mu \partial w$ on $\mathbb{C}$ (quasiconformal automorphism of $\widehat{\mathbb{C}}$) normalized by the assumptions $w^\mu \in \Sigma_Q, w^\mu(0) = 0$.

One of the important invariants intrinsically connected with univalence is the **Schwarzian derivative**

$$S_w(z) = \Big(\frac{w''(z)}{w'(z)}\Big)' - \frac{1}{2}\Big(\frac{w''(z)}{w'(z)}\Big)^2, \quad z \in \mathbb{D}^*.$$

As it is well known (see [2]), these derivatives belong to the complex Banach space $\mathbf{B} = \mathbf{B}(\mathbb{D}^*)$ of hyperbolically bounded holomorphic functions in the disk $\mathbb{D}^*$ with norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\mathbb{D}} \, (|z|^2 - 1)^2 |\varphi(z)|.$$

In the case of functions $f^\mu \in \Sigma_Q$, their Schwarzians S_{f^μ} run over a bounded domain in $\mathbf{B}$ modeling the **universal Teichmüller space T**. The space $\mathbf{B}$ is dual to the Bergman space $A_1(\mathbb{D}^*)$, a subspace of $L_1(\mathbb{D}^*)$ formed by integrable holomorphic functions (quadratic differentials $\varphi(z)dz^2$) on $\mathbb{D}^*$. Note that $\varphi(z) = O(z^{-4})$ near $z = \infty$. For the needed results from Teichmüller space theory, see [3–5].

The importance of the differential invariant $S_w(z)$ in mathematics is essentially caused by its Moebius invariance. The chain rule for Schwarzians yields

$$S_{w_2 \circ w_1}(z) = S_{w_2}(w_1)(w_1'(z))^2 + S_{w_1}(z),$$

which, for a fractal linear map $\sigma(w)$, implies $S_{\sigma \circ w}(z) = S_w(z)$.

The Taylor coefficients b_j of $f \in \Sigma_Q$ reflect the fundamental intrinsic features of these functions following their conformality. There is also another important coefficient collection naturally prescribed to normalized univalent functions in the disk. Namely, one defines for any $f \in \Sigma_Q$ its **Grunsky coefficients** α_{mn} from the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\mathbb{D}^*)^2, \tag{1}$$

where the principal branch of the logarithmic function is chosen. These coefficients satisfy the inequality

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn} \, \alpha_{mn} x_m x_n \Big| \le 1 \tag{2}$$

for anysequence $\mathbf{x} = (x_n)$ from the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\| = \big(\sum\limits_1^\infty |x_n|^2\big)^{1/2}$; conversely, the inequality (2) also is sufficient for the univalence of a locally univalent function in $\mathbb{D}^*$ (cf. [6,7]).

The minimum $k(f)$ of dilatations $k(w^\mu) = \|\mu\|_\infty$ among all quasiconformal extensions $w^\mu(z)$ of f onto the whole plane $\widehat{\mathbb{C}}$ (forming the equivalence class of f) is called the **Teichmüller norm** of this function. Hence,

$$k(f) = \tanh d_{\mathbf{T}}(\mathbf{0}, S_f),$$

where $d_{\mathbf{T}}$ denotes the Teichmüller–Kobayashi distance on the space $\mathbf{T}$. This quantity dominates the **Grunsky norm**

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(f) x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\}$$

by $\varkappa(f) \le k(f)$. For most functions f, we have the strong inequality

$$\varkappa(f) < k(f)$$

(moreover, the functions satisfying this inequality form a dense subset of Σ, see [8]), while the functions with the equal norms play a crucial role in many applications of quasiconformal analysis. Thus, it is important to find some broad collections of univalent functions with $\varkappa(f) = k(f)$.

These norms coincide only when any extremal Beltrami coefficient μ_0 for f (i.e., with $\|\mu_0\|_\infty = k(f)$) satisfies

$$\|\mu_0\|_\infty = \sup \Big\{ \Big| \iint_{\mathbb{D}} \mu_0(z) \psi(z) dxdy \Big| : \psi \in A_1^2(\mathbb{D}),\ \|\psi\|_{A_1(\mathbb{D})} = 1 \Big\} = \varkappa(f) \quad (z = x + iy). \tag{3}$$

Here, $A_1(\mathbb{D})$ denotes the subspace in $L_1(\mathbb{D})$ formed by integrable holomorphic functions (quadratic differentials $\psi(z)dz^2$ on $\mathbb{D}$, and $A_1^2(\mathbb{D})$ is its subset consisting of ψ with zeros of even order on $\mathbb{D}$, i.e., of the squares of holomorphic functions (see [9–11]). Note that, due to [9], every $\psi \in A_1^2(\mathbb{D})$ has the form

$$\psi(z) = \frac{1}{\pi} \sum_{m+n=4}^{\infty} \sqrt{mn}\, x_m x_n z^{m+n-2} \tag{4}$$

and $\|\psi\|_{A_1(\mathbb{D})} = \|\mathbf{x}\|_{l^2} = 1$, $\mathbf{x} = (x_n)$.

Note that all notions introduced above are also valid for the univalent functions $F(z)$ in the unit disk $\mathbb{D}$ normalized by $F(0) = 0$, $F'(0) = 1$ (extending quasiconformally onto $\mathbb{D}^*$). Their inversions $f_F(z) = 1/F(1/z)$ belong to Σ_Q and are zero free on $\mathbb{D}^*$. The Grunsky coefficients of these functions (and other related notions) are defined similar to (1) and $\varkappa(F) = \varkappa(f_F)$, $k(F) = k(f_F)$. But it is technically more convenient to deal with the class Σ_Q.

1.3. Generalization of Grunsky Inequalities

The method of Grunsky inequalities has been generalized in several directions, even to bordered Riemann surfaces X with a finite number of boundary components (see [7,12,13]). We consider these inequalities in unbounded simply connected hyperbolic domains, for which a quasiconformal variant of this theory has been developed in [10].

Let $L \subset \mathbb{C}$ be an oriented bounded quasicircle separating the points 0 and ∞. Denote its interior and exterior domains by D and D^* (so $0 \in D$, $\infty \in D^*$). Then, if $\delta_D(z)$ denotes the Euclidean distance of z from the boundary of D and $\lambda_D(z)|dz|$ is its hyperbolic metric of Gaussian curvature -4, we have

$$\frac{1}{4} \le \lambda_D(z) \delta_D(z) \le 1$$

(the right-hand inequality follows from the Schwarz lemma and the left from the Koebe one-quarter theorem).

For such a domain $D^* \ni \infty$, one must use instead of (1) the expansion

$$-\log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{m,n=1}^{\infty} \frac{\beta_{mn}}{\chi(z)^m\, \chi(\zeta)^n},$$

where χ denotes a conformal map of D^* onto the disk $\mathbb{D}^*$ so that $\chi(\infty) = \infty$, $\chi'(\infty) > 0$. By Milin's univalence theorem [7], generalizing the Grunsky result for the disk, a holomorphic function

$$f(z) = z + \text{const} + O(z^{-1})$$

in a neighborhood of $z = \infty$ is extended to a univalent function on the domain D^* if and only if its coefficients β_{mn} satisfy

$$\sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \beta_{mn}\, x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\} \leq 1.$$

Accordingly, the **generalized Grunsky norm** is defined by

$$\varkappa_{D^*}(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \beta_{mn}\, x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\}.$$

The coefficients β_{mn} relate to holomorphic functions $\varphi_n(z)$ in D whose derivatives $\varphi_n'(z)$ form a complete orthonormal system in $A_2(D)$; in the case of $D = \mathbb{D}$, one can use the powers z^n, $n = 0, 1, \ldots$.

We now consider the class $\Sigma_Q(D^*)$ of univalent functions in domain D^* with expansions

$$f(z) = z + b_0 + b_1 z^{-1} + \ldots$$

near $z = \infty$, admitting quasiconformal extensions onto the complementary domain D. Similar to the above, we subject these extensions to $f(0) = 0$. Their Beltrami coefficients run over the ball

$$\text{Belt}(D)_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu(z)|D^* = 0, \ \|\mu\|_\infty < 1\}.$$

The corresponding Schwarzian derivatives S_f belong to the Banach space $\mathbf{B}(D^*)$ of holomorphic functions on D^* with finite norm

$$\|\varphi\|_{\mathbf{B}(\mathbb{D}^*)} = \sup_{D^*} \lambda_{D^*}(z)|\varphi(z)|$$

and fill its bounded subdomain modeling the universal Teichmüller space with the base point D^*.

For each $\mu \in \text{Belt}(D)_1$, we consider its Teichmüller equivalence class $[\mu]$ consisting of $\mu \in \text{Belt}(D)_1$ such that the maps f^ν coincide with f^μ on the boundary ∂D of domain D. These classes are in one-to-one correspondence with the points of the space $\mathbf{T}$, and the quotient space $\text{Belt}(D)_1/\sim$ with the defining projection $\mu \mapsto [\mu]$ is biholomorphically equivalent to $\mathbf{T}$.

The extremal Beltrami coefficients μ_0, minimizing the dilatation $\|\mu\|_\infty$ in the equivalence classes $[\mu]$, play a crucial norm in geometric function theory, Teichmüller space theory, numerical mathematics, etc.

There is a well-known criterion for extremality given by the Hamilton–Krushkal–Reich–Strebel theorem. We present this theorem and its relation to the Grunsky norm via the following theorem.

Proposition 1. *A coefficient $\mu_0 \in \text{Belt}(D)_1$ is extremal in its class (minimizes the dilatation $\|\mu|_\infty$) if and only if*

$$\|\mu_0\|_\infty = \sup \Big\{ \Big| \iint_D \mu_0(z)\psi(z)dxdy \Big| : \psi \in A_1(D), \ \|\psi\|_{A_1} = 1 \Big\} \quad (z = x + iy), \tag{5}$$

while the equality $\varkappa_{\mathbb{D}^*}(f^\mu) = k(f^\mu)$ is valid if and only if

$$\|\mu\|_\infty = \sup \Big\{ \Big| \iint_D \mu(z)\psi(z)dxdy \Big| : \psi \in A_1^2(D), \ \|\psi\|_{A_1(D)} = 1 \Big\}. \tag{6}$$

Here, $A_1(D)$ denotes the subspace in $L_1(D)$ formed by integrable holomorphic functions (quadratic differentials $\psi(z)dz^2$ on D, and $A_1^2(D)$ is its subset consisting of ψ with zeros of even order on D, i.e., of the squares of holomorphic functions.

In addition, if the equivalence class of f is a **Strebel point** of the space $\mathbf{T}$ with base point D^*, which means that this class contains the Teichmüller extremal extension $f^{k|\psi_0|/\psi_0}$ with $\psi_0 \in A_1(D)$, then necessarily, $\psi_0 = \omega^2 \in A_1^2$ (cf. [9,10,14,15]).

An important fact is that the Strebel points are dense in any Teichmüller space (see [4]), which yields, in particular, that the univalent function of any $f(z)$ on D is approximated in the strong topology of the space $\mathbf{T}$ by functions having qusiconformal extensions to $\widehat{\mathbb{C}}$ and constant regular dilatation in the complementary domain.

In the case of the generic quasidisk D, the representation of elements of the space $A_1^2(D)$ is much more complicated than (4). As was established in [10], every $\psi \in A_1^2(D)$ is of the form

$$\psi(z) = \frac{1}{\pi} \sum_{m,n=1}^{\infty} x_m x_n \, P_m'(z) P_n'(z) dxdy,$$

where $\|\mathbf{x}\|_{l^2} = \|\omega\|_{L_2}$ (here $\mathbf{x} = (x_n)$) and P_n are well-defined polynomials arising from the expansion

$$\frac{1}{w-z} = \sum_{1}^{\infty} P_n'(w) \varphi_n(z),$$

with $\varphi_n = \chi^n$ (given above); the degree of P_n equals n. It coincides with the representation (4) for $\psi \in A_1^2(\mathbb{D})$.

Even the case of ellipse is complicated. This case has its intrinsic interest, because for an ellipse, the orthonormal basis $\{\varphi_n\}$ indicated above can be given explicitly using the Chebyshev polynomials of the second kind. We shall describe this example in more detail in Section 3.

1.4. Substantial Boundary Points and Teichmüller Extremality

Assume that $\mu_0 \in \mathrm{Belt}(D)_1$ is extremal in its class but not of the Teichmüller type. A point $z_0 \in \partial D$ is called **substantial** (or essential) for μ_0 if for any $\varepsilon > 0$, there exists a neighborhood U_0 of z_0 such that

$$\sup_{D^* \setminus U_0} |\mu_0(z)| < \|\mu_0\|_\infty - \varepsilon;$$

so the maximal dilatation $k(w^{\mu_0}) = \|\mu\|_\infty$ is attained on D by approaching this point.

In addition, there exists a sequence $\{\psi_n\} \subset A_1(D)$ such that $\psi_n(z) \to 0$ locally uniformly on D but $\|\psi_n\| = 1$ for any n, and

$$\lim_{n\to\infty} \iint_D \mu_0(z) \psi_n(z) dxdy = \|\mu_0\|_\infty.$$

Such sequences are called **degenerated**.

The image of a substantial point is a common point of two quasiconformal arcs, which can be of the spiral type.

The Teichmüller extremal Beltrami coefficients do not admit degenerated maximizing sequences and substantial points (see [4,15]).

As was mentioned above, the equivalence classes $[\mu]$ containing the Teichmüller coefficients correspond to the Strebel points of the space $\mathbf{T}$ and are dense on this space.

1.5. Quasiconformal Reflections and Fredholm Eigenvalues

The Teichmüller and Grunsky norms are intrinsically connected with the quasiconformal reflections, Fredholm eigenvalues and other quasi-invariants of quasiconformal curves. We briefly outline the main notions and results; for the details, see [8,11,16–18].

The **quasiconformal reflections** (or quasireflections) are the orientation-reversing quasiconformal homeomorphisms of the sphere $\widehat{\mathbb{C}}$, which preserve point-wise some (oriented) quasicircle $L \subset \widehat{\mathbb{C}}$ and interchange its interior and exterior domains.

In other words, quasireflections are topological involutions of the sphere $\widehat{\mathbb{C}}$ whose fixed Jordan curves are the quasicircles.

One defines for L its **reflection coefficient**

$$q_L = \inf k(f) = \inf \|\partial_z f / \partial_{\bar{z}} f\|_\infty,$$

taking the infimum over all quasireflections across L. Due to [14,19], the dilatation

$$Q_L = (1 + q_L)/(1 - q_L) \geq 1$$

is equal to the quantity

$$Q_L = (1 + k_L)^2/(1 - k_L)^2,$$

where k_L is the minimal dilatation among all orientation-preserving quasiconformal automorphisms f_* of $\widehat{\mathbb{C}}$ carrying the unit circle onto L, and $k(f_*) = \|\partial_{\bar{z}} f_* / \partial_z f_*\|_\infty$.

The reflection with dilatation Q_L is extremal. A remarkable and very useful fact established by Ahlfors is that any quasicircle also admits a Lipschitz-continuous quasireflection with some coefficient $C(q_L)$ (see [19]).

The **Fredholm eigenvalues** ρ_n of an oriented smooth closed Jordan curve $L \subset \widehat{\mathbb{C}}$ are the eigenvalues of its double-layer potential, or equivalently, of the integral equation

$$u(z) + \frac{\rho}{\pi} \int\limits_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta = h(z),$$

where n_ζ denotes the outer normal and ds_ζ is the length element at $\zeta \in L$. These values are crucial in many applications in various fields of complex analysis, potential theory, continuum mechanics and physics (see [1,18,20–23]).

The least positive eigenvalue $\rho_L = \rho_1$ is naturally connected with conformal and quasiconformal maps and can be defined for any oriented closed Jordan curve L by

$$\frac{1}{\rho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)},$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions to be u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$.

The known basic tools for quantitative estimation of the Freholm eigenvalues ρ_L of quasicircles is given by Ahlfors' inequality

$$1/\rho_L \leq q_L,$$

where q_L denotes the minimal dilatation of quasireflections across L [16], and by the fundamental Kühnau–Schiffer theorem [11,24], which states that *the value ρ_L is reciprocal to the Grunsky norm $\varkappa(f)$ of the Riemann mapping function of the exterior domain of L.*

Unfortunately, the Ahlfors inequality gives only a rough upper bound for ρ_L, while the application of the Kühnau–Schiffer result requires knowledge of the exact value of the Grunsky norm. Thus, the explicit or even approximate determination of these quasi-invariants remains an important open problem.

For all functions $f \in S_Q$ (i.e., univalent in the disk $\mathbb{D}^*$) with $k(f) = \varkappa(f)$, we have the exact explicit values

$$q_{f(\mathbb{S}^1)} = \frac{1}{\rho_{f(\mathbb{S}^1)}} = \varkappa(f). \tag{7}$$

We do not touch here the topics concerning the quasireflections across quasi-intervals or across their finite collections, to which the notion of Fredholm eigenvalues can also be extended.

1.6. Metrics with Negative Generalized Gaussian Curvature

We shall apply the conformal metrics $ds = \lambda(t)|dt|$ on the disk $\mathbb{D}$ with $\lambda(t) \geq 0$ (called also semi-metrics), having the negative generalized Gaussian curvature. Such a curvature is defined for an upper semicontinuous Finsler metric $ds = \lambda|dt|$ in a domain $\Omega \subset \mathbb{C}$ by

$$\kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2}, \tag{8}$$

where Δ is the **generalized Laplacian**

$$\Delta\lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta}) d\theta - \lambda(t) \Big\}$$

(provided that $-\infty \leq \lambda(t) < \infty$).

Note that this is equivalent to regarding the differential operator $\Delta = 4\partial^2/\partial z \partial \bar{z}$ in the distributional sense.

Similarly to C^2 functions, for which Δ coincides with the usual Laplacian, one obtains that λ is subharmonic on Ω if and only if $\Delta\lambda(t) \geq 0$; hence, at the points t_0 of local maximum of λ with $\lambda(t_0) > -\infty$, we have $\Delta\lambda(t_0) \leq 0$. This gives rise to the sectional holomorphic curvature of a Finsler metric on a complex Banach manifold X, which is defined as the supremum of the curvatures (8) over appropriate collections of holomorphic maps from the disk into X for a given tangent direction in the image.

As is well known [25,26], the holomorphic curvature of the Kobayashi–Teichmüller metric $\mathcal{K}_{\mathbf{T}}(x, v)$ of the universal Teichmüller space $\mathbf{T}$ equals -4 at all points (x, v) of the tangent bundle $\mathcal{T}(\mathbf{T})$ over $\mathbf{T}$. Instead, the holomorphic curvature of metric $\lambda_\varkappa$ generated on $\mathbb{D}$ by the Grunsky Finsler structure satisfies the inequality $\Delta \log \lambda \geq 4\lambda^2$, where Δ is again the generalized Laplacian (see [8]).

We also shall apply the metrics whose generalized curvature satisfies a more general inequality

$$\Delta \log \lambda \geq K\lambda^2$$

with $K = \text{const} > 0$.

1.7. Basic Underlying Theorems

First of all, we essentially use the following remarkable result established by P.P. Belinskii [27], which gives rise to other investigations.

Theorem 1 ([27])**.** *Let a function $\mu(\zeta)$ be defined on the plane $\mathbb{C}$ and C^1-smooth, up to the jumps on a finite number of closed smooth curves. Let*

$$|\mu(\zeta)| < \varepsilon, \ |\partial_\zeta \mu| < \varepsilon, \ |\partial_{\bar{\zeta}} \mu| < \varepsilon,$$

and let either $\mu(1/\zeta)$ or $(\zeta/\bar{\zeta})^2 \mu(1/\bar{\zeta})$ satisfy, in a neighborhood of the point $\zeta = 0$, the same assumptions as the function $\mu(\zeta)$ in the finite points. Then, for sufficiently small $\varepsilon > 0$, the function

$$w(z) = z - \frac{z(z-1)}{\pi} \iint\limits_{|\zeta| < \infty} \frac{\mu(\zeta) d\xi d\eta}{\zeta(\zeta - 1)(\zeta - z)} \tag{9}$$

provides a quasiconformal homeomorphism of the whole plane $\widehat{\mathbb{C}}$, whose Beltrami coefficient is

$$\widetilde{\mu} = \mu + O(\|\mu\|_\infty^2), \tag{10}$$

and this map differs from the map with Beltrami coefficient $\mu(z)$ and the same normalization up to a quantity of order ε^2 uniformly in any bounded domain.

This theorem plays a crucial role in the variational calculus for quasiconformal maps. Its original proof in [27], especially of the estimate (10), is complicated and relates on the deep results from geometric function theory and from the potential theory. Now this proof can be essentially simplified and shortened by including the map (9) into a holomorphic motion of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ and applying the lambda lemma for such motions.

Theorem 1 involves only sufficiently smooth Beltrami coefficients μ with small norm. It was recently strengthened by the author and applied to the complex and potential geometry of the universal Teichmüller space.

Note that Theorem 1 relates to the problem of I.N. Vekua of 1961 on the homeomorphy of approximate solutions of the singular two-dimensional integral equation intrinsically connected with the Beltrami equation by constructing quasiconformal maps. Consider in the space $L_p(\mathbb{C})$ with $p > 2$ the well-known integral operators

$$T\rho(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{\zeta - z}, \quad \Pi\rho(\zeta) = -\frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\rho(\zeta) d\xi d\eta}{(\zeta - z)^2} = \partial_z T\rho(z)$$

assuming for simplicity that ρ has a compact support in $\mathbb{C}$. Then, the second integral exists as a Cauchy principal value, and the derivative $\partial_z T$ generically is understood to be distributional.

One of the fundamental results of quasiconformal theory is that every quasiconformal automorphism w^μ of $\widehat{\mathbb{C}}$ with $\|\mu\|_\infty = k < 1$ is represented in the form

$$w^\mu(z) = z + T\rho(z),$$

where ρ is the solution in L_p (for $2 < p < p_0(k)$) of the integral equation

$$\rho = \mu + \mu\Pi\rho,$$

given by the series

$$\rho = \mu + \mu\Pi\mu + \mu\Pi\mu(\Pi\mu) + \dots . \tag{11}$$

Denote by μ_n the n-th partial sum of the series (11), and set

$$f_n(z) = z - \frac{1}{\pi} \iint\limits_{\mathbb{C}} \frac{\mu_n(\zeta) d\xi d\eta}{\zeta - z}.$$

The question of Vekua was whether all f_n also are homeomorphisms.

Theorem 1 solves it positively for the first iteration $f_1(z)$, provided that Beltrami coefficient μ is sufficiently regular.

There is the counterexample of T. Iwaniec, which shows that the smoothness and smallness assumptions in the Belinskii theorem cannot be dropped completely. A simple modification of his construction allows us to define $\varepsilon \in (0, 1)$ and a Beltrami coefficient μ so that the second iteration

$$f_2(z) = z + T\mu(z) + T(\mu\Pi\mu)(z)$$

is not injective in $\mathbb{D}$. The details are exposed in survey [26].

We use here the special case of Theorem 1 for $\mu \in \mathrm{Belt}(\mathbb{D})_1$.

The next underlying result is the following theorem proven in [28].

Theorem 2. *Every Beltrami coefficient $\mu \in Belt(\mathbb{D})_1$, which belongs to the Sobolev space $W^{1,p}(\mathbb{D})$, $p > 2$, and has a substantial point $z_0 \in \mathbb{S}^1$, is extremal in its equivalence class, and the function $f^\mu|D^*$ has equal Teichmüller and Grunsky norms, and*

$$q_{L_1} = 1/\rho_{L_1} = \varkappa(f) = k(f) = \|\mu\|_\infty. \quad (12)$$

This theorem admits a weakened extension to arbitrary quasidisks as follows. Letting for $\mu \in \mathrm{Belt}(D)_1$, $\psi \in A_1(D)$,

$$\langle \mu, \psi \rangle_D = \iint_D \mu(z)\psi(z)dxdy,$$

we have the following.

Theorem 3 ([28])**.** *Let a Beltrami coefficient $\mu \in Belt(D)_1$ belong to $W^{1,p}(D)$ with $p \geq 2$ and have a substantial point z_0 on the boundary ∂D. Let $\mu(z) \to 0$ for z filling a subarc $\gamma \subset \partial D$ (which depends on μ) as z approaches γ from inside D. Then, μ is extremal in its equivalence class, and the Grunsky norm $\varkappa_{D^*}(f^\mu)$ of the function $f^\mu|D^*$ also equals $\|\mu\|_\infty$, i.e.,*

$$k(f^\mu) = \varkappa_{D^*}(f^\mu) = \|\mu\|_\infty = \sup_{\psi \in A_1^2(D), \|\psi\|_{A_1}=1} |\langle \mu, \psi \rangle_D|. \quad (13)$$

The difference between the hypotheses of Theorems 2 and 3 is caused by the fact that the first step in the proof of Theorem 2 is a special case of Theorem 3 (i.e., it concerns the Beltrami coefficients $\mu \in \mathrm{Belt}(\mathbb{D})_1$ vanishing on a subarc of $\mathbb{S}^1$), while the next steps of this proof essentially involve a result of Kühnau, which is established only for the canonical disk $\mathbb{D}^*$.

2. Main Theorem

The main result of this paper is the following theorem, which strengthens Theorem 1 and shows that quasiconformal map $w^{\widetilde{\mu}}$ defined by integral (9) inherits the main basic properties brought by the original coefficient μ.

Theorem 4. *Let a function $\mu \in \mathrm{Belt}(\mathbb{D})_1$ satisfy the smoothness conditions of Theorem 2. Then both homeomorphisms w^μ and $w^{\widetilde{\mu}}$ given by the integral*

$$z + T\mu(z) = z - \frac{1}{\pi} \iint_{\mathbb{D}} \frac{\mu(\zeta)d\xi d\eta}{\zeta - z}$$

via the Belinskii theorem have equal Teichmüller and Grunsky norms and satisfy the relations (12).

We also give some possible extensions of this important result. One of the interesting **open problems** here is to describe all bounded convex polygons which obey the relations (9).

We precede the proof of Theorem 4 by the following important remark. From (8), we have

$$w(z) = z + T\mu;$$

hence, $\partial_{\bar{z}} w = \mu$, $\partial_z = 1 + \Pi\mu$, and

$$\widetilde{\mu} = \frac{\partial_{\bar{z}} w}{\partial_z} = \frac{\mu}{1 + \Pi\mu}, \quad (14)$$

which implies, in view of well-known properties of the integral operators, that $\mathbf{T}$ and Φ simultaneously depend holomorphically on complex parameters.

Proof of Theorem 4. First observe that the Grunsky coefficients $\alpha_{mn}(f^\mu)$ of functions $f^\mu \in \Sigma_Q$ generate for each $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ the holomorphic maps

$$h_{\mathbf{x}}(f^\mu) = \sum_{m,n=1}^{\infty} \alpha_{mn}(f^\mu) x_m x_n : \ \mathrm{Belt}(\mathbb{D})_1 \to \mathbb{D}, \tag{15}$$

of the ball $\mathrm{Belt}(\mathbb{D})_1$ and of the universal Teichmüller space $\mathbf{T}$ into the unit disk $\mathbb{D}$, and

$$\sup_{\mathbf{x}} |h_{\mathbf{x}}(f^\mu)| = \varkappa_{D^*}(f^\mu). \tag{16}$$

This holomorphy follows from the holomorphy of coefficients α_{mn} with respect to $\mu \in \mathrm{Belt}(D)_1$ and $S_{f^\mu} \in \mathbf{T}$ and the well-known estimate

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \beta_{mn} x_m x_n \Big|^2 \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2 \tag{17}$$

which holds for any finite M, N and $1 \le j \le M$, $1 \le l \le N$ (see [29], p. 61). □

The same is valid for the generalized Grunsky coefficients $\beta_{mn}(S_{f^\mu})$ of the functions $f^\mu \in \Sigma(D^*)$ generating the holomorphic maps

$$h_{\mathbf{x}}(S_{f^\mu}) = \sum_{m,n=1}^{\infty} \beta_{mn}(f^\mu) x_m x_n : \ \mathrm{Belt}(D)_1 \to \mathbb{D} \tag{18}$$

with $\sup_{\mathbf{x}} |h_{\mathbf{x}}(f^\mu)| = \varkappa_{D^*}(f^\mu)$.

The holomorphy of these functions follows from the holomorphy of coefficients β_{mn} with respect to Beltrami coefficients $\mu \in \mathrm{Belt}(D)_1$ mentioned above using the corresponding estimate

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \beta_{mn} x_m x_n \Big|^2 \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2$$

which holds for any finite M, N and $1 \le j \le M$, $1 \le l \le N$; this estimate is a simple corollary of the Milin univalence theorem (cf. [7], p. 193).

Similar arguments imply that the maps (18) regarded as functions of points $\varphi^\mu = S_{f^\mu}$ in the universal Teichmüller space $\mathbf{T}$ (with the base point D^*) are holomorphic on $\mathbf{T}$.

Note also that both norms $\varkappa_{D^*}$ and $k(f)$ are continuous logarithmically plurisubharmonic functions on $\mathbf{T}$ [28].

Now, given a function $f \in \Sigma_Q$, take its extremal extension f^μ (i.e., such that $k(f) = \|\mu\|_\infty$) and set

$$\mu_* = \mu / \|\mu\|_\infty$$

and pass to maps $f^{t\mu_*}(z)$ with $|t| < 1$. It follows from Theorem 2 that the disk

$$\mathbb{D}(\mu) = \{t\mu/\|\mu\|_\infty : \ |t| < 1\} \subset \mathrm{Belt}(\mathbb{D})_1$$

and its image $\phi_{\mathbf{T}}(\mathbb{D}(\mu))$ are geodesic (extremal) either from the Teichmüller, Kobayashi and Carathéodory metrics on $\mathbf{T}$ (which are equal on this disk and have the holomorphic curvature -4). Denote these metrics by $\tau_{\mathbf{T}}$, $d_{\mathbf{T}}$, $c_{\mathbf{T}}$ and their infinitesimal (differential) forms by $F_{\mathbf{T}}(\psi, v)$, $\lambda_{\mathcal{K}}(\psi, v)$, $\lambda_{\mathcal{C}}(\psi, v)$, respectively.

Using the functions (15) determined by Beltrami coefficients from $\mathbb{D}(\mu)$, we pull back the hyperbolic metric

$$\lambda_{\mathbb{D}}(t)|dt| = |dt|/(1 - |t|^2)$$

of the disk $\mathbb{D}$, obtaining on this disk the conformal metrics $\lambda_{h_{\mathbf{x}}}(t)|dt|$ with

$$\lambda_{h_{\mathbf{x}}}(t) = |h'_{\mathbf{x}}(t)|/(1 - |h_{\mathbf{x}}(t)|^2).$$

All these metrics have at their noncritical points the Gaussian curvature -4. Now, take the upper envelope of these metrics

$$\lambda_{\varkappa}(t) = \sup\{\lambda_{h_{\mathbf{x}}}(t) : \ \mathbf{x} \in S(l^2)\} \tag{19}$$

followed by its upper semicontinuous regularization, which determines a logarithmically subharmonic metric $\lambda_{\varkappa}(t)$ on the unit disk. This metric is circularly symmetric, i.e., satisfies $\lambda_{\varkappa}(t) = \lambda_{\varkappa}(|t|)$. Its generalized Gaussian curvature

$$\kappa_{\lambda_{\varkappa}}(t) \leq -4$$

(cf. [8]). In fact, the metric (19) is the differential (infinitesimal) form of the norm $\varkappa_{D^*}(f)$ for $f = f^{\nu} \in \mathbb{D}(\mu)$.

We have the relation

$$\lambda_{\varkappa}(t) \leq \lambda_{\mathcal{K}}(t) \quad \text{for all } \ |t| \leq 1, \tag{20}$$

while the assumption of Theorem 4 gives, by Theorem 2 and equality (16), the relations

$$\lambda_{\varkappa}(0) = \lambda_{\mathcal{K}}(0) = 1. \tag{21}$$

A refined comparison of these metrics is obtained by applying **Minda's maximum principle** given by the following.

Lemma 1. *If a function $u : \ D \to [-\infty, +\infty)$ is upper semicontinuous in a domain $\Omega \subset \mathbb{C}$ and its generalized Laplacian satisfies the inequality*

$$\Delta u(z) \geq K u(z) \tag{22}$$

with some positive constant K at any point $z \in D$, where $u(z) > -\infty$, and if

$$\limsup_{z \to \zeta} u(z) \leq 0 \ \textit{ for all } \zeta \in \partial D,$$

then either $u(z) < 0$ for all $z \in D$ or $u(z) \equiv 0$ on D.

The proof of this lemma related to the Ahlfors–Schwarz lemma [16] is given in [30]; for its variations, see [31].

We choose a sufficiently small neighborhood U_0 of the origin $t = 0$ and put

$$M = \{\sup \lambda_{\mathcal{K}}(t) : t \in U_0\}.$$

Then, in this neighborhood, we have

$$\lambda_{\mathcal{K}}(t) + \lambda_{\varkappa}(t) \leq 2M.$$

Consider the function

$$u = \log \frac{\lambda_{\varkappa}}{\lambda_{\mathcal{K}}}.$$

Then (cf. [7,31]) for $t \in U_0$,

$$\Delta u(t) = \Delta \log \lambda_{\varkappa}(t) - \Delta \log \lambda_{\mathcal{K}}(t) \geq 4(\lambda_{\varkappa}^2 - \lambda_{\mathcal{K}}^2) \geq 8M(\lambda_{\varkappa} - \lambda_{\mathcal{K}}).$$

The elementary estimate

$$M \log(t/s) \geq t - s \quad \text{for } \ 0 < s \leq t < M$$

(with equality only for $t = s$) implies that

$$M \log \frac{\lambda_\alpha(t)}{\lambda_\varkappa(t)} \geq \lambda_\alpha(t) - \lambda_\varkappa(t),$$

and hence,

$$\Delta u(t) \geq 8M^2 u(t).$$

It follows that the function u satisfies on U_0 the inequality (21) with $K = 8M^2$. Applying Lemma 1 to u and noting that the equality (20) yields

$$u(0) = \lim_{t \to 0} \log \frac{\lambda_\varkappa(t)}{\lambda_\mathcal{K}(t)} = 0,$$

one derives by this lemma that both metrics $\lambda_\varkappa$ and $\lambda_\mathcal{K}$ must be equal on U_0, and in a similar way, their equality on the entire disk $\mathbb{D}$. This proves the infinitesimal version of Theorem 4.

Finally, to obtain the global version, we apply the following reconstruction lemma for the Grunsky norm proven in [8], which provides that this norm is the integrated form of $\lambda_\varkappa$ along the Teichmüller extremal disks (Functions (15) and (18) allow one to determine the metric $\lambda_\varkappa$ on the whole space $\mathbf{T}$; it corresponds to another canonical Finsler structure on the space $\mathbf{T}$ generated by the Grunsky coefficients).

Lemma 2. *On any Teichmüller extremal disk*

$$\mathbb{D}(\mu_0) = \{t\mu_0/\|\mu_0\|_\infty : |t| < 1\} \subset \mathrm{Belt}(\mathbb{D})_1,$$

we have the equality

$$\tanh^{-1}[\varkappa(f^{r\mu_0/\|\mu_0\|_\infty})] = \int_0^r \lambda_\varkappa(t)dt.$$

Integrating the metrics $\lambda_\varkappa$ and λ_α along the extremal disk $\mathbb{D}(\mu_*)$, one obtains the required right equality in (4). Other equalities in (4) follow from the classical relations for quasi-invariants q_L and ρ_L of the curves $L = w^\mu(\mathbb{S}^1)$ and $L = w^{\widetilde{\mu}}(\mathbb{S}^1)$, indicated in Section 1.3. This completes the proof of the theorem.

3. Extensions of Theorem 4

Theorem 4 can be extended in a weaker form to univalent functions on arbitrary unbounded quasidisks D^* taking the integrals of type (9) over the complementary quasidisks D with appropriate functions μ compatible with Theorem 3.

As an example, let us consider the integral

$$f^\mu(z) = z - \frac{1}{\pi} \iint_\mathcal{E} \frac{\mu(\zeta)d\xi d\eta}{\zeta - z},$$

over the interior $\mathcal{E}$ of ellipse with the foci at $-1, 1$ and semiaxes a, b $(a > b)$. Noting that an orthonormal basis in the Hilbert space $A_2(\mathcal{E})$ of the square integrable holomorphic functions on $\mathcal{E}$ is formed by the polynomials

$$P_n(z) = 2\sqrt{\frac{n+1}{\pi}}\,(r^{n+1} - r^{-n-1})\, U_n(z),$$

where $r = (a+b)^2$ and $U_n(z)$ are the Chebyshev polynomials of the second kind,

$$U_n(z) = \frac{1}{\sqrt{1-z^2}}\, \sin[(n+1)\arccos z], \quad n = 0, 1, \ldots$$

(see [32]), one concludes that for any Beltramu coefficient μ satisfying the assumptions of Theorem 3, the following equalities are valid:

$$\|\mu\|_\infty = \sup \Big\{ \Big| \iint_{\mathcal{E}} \mu(z) \Big(\sum_0^\infty c_n P_n(z)\Big)^2 dxdy \Big| : \Big\| \sum_0^\infty c_n P_n(z) \Big\|_{A_2(\mathcal{E})} = 1 \Big\} = k(f^\mu) = \varkappa_{\mathcal{E}^*}(f^\mu),$$

where $\mathcal{E}^*$ means the exterior of the indicated ellipse.

We also mention the following useful extension of Theorem 4. Let $L \subset \mathbb{C}$ be an oriented closed $C^{1+\sigma}$-smooth Jordan curve (hence, a quasicircle), separating the points 0 and ∞ ($\sigma > 0$). Denote its interior and exterior domains by D_L and D_L^*, respectively, and consider the corresponding spaces $A_1(D_L)$ and $A_1^2(D_L)$.

Theorem 5. *Let*

$$t \mapsto \mu(z,t) = t\mu_0(z) + t^2\mu_1(z) + O(t^2)$$

be a holomorphic map of the ball $\mathrm{Belt}(D_L)_1$ *of itself with* $\mu_0(z)$*, satisfying the assumptions of Theorem 3. Then, for sufficiently small* $|t| > 0$*, the function*

$$w_t(z) = z - \frac{1}{\pi} \iint_{D_L} \frac{\mu(\zeta,t)}{\zeta - z} d\xi d\eta = z - \frac{t}{\pi} \iint_{D_L} \frac{\mu_0(\zeta)}{\zeta - z} d\xi d\eta + O(t^2)$$

determines a quasiconformal automorphism $\widetilde{w}(z,t)$ *of the sphere* $\widehat{\mathbb{C}}$ *with the complex dilatation*

$$\mu(z,t)^* = t\mu_0(z) + O(t^2) \quad \text{for } z \in D_L$$

and conformal on D_L^**. Its restriction to* D_L^* *has equal Teichmüller and Grunsky norms satisfying*

$$k(\widetilde{w}(\cdot;t)) = \varkappa_{D_L^*}(\widetilde{w}(\cdot;t)) = |t| + O(|t|^2), \tag{23}$$

with a uniform estimate of the remainder for $|t| \le t_0' < t_0$*.*

In the case of the disk, this theorem simultaneously implies the approximate values of the reflection coefficients and Fredholm eigenvalues of quasicircles $L_t = \widetilde{w}(\mathbf{S}^1;t)$ and is valid for all μ_0 satisfying the assumptions of Theorem 2.

Some results of such type for Teichmüller coefficients $\mu_0 = |\psi_0|/\psi_0$ with $\psi_0 \in A_1^2$ were established in [1].

Proof of Theorem 5. We now have

$$w(z,t) = z + T\mu(\cdot,t),$$

and

$$\partial_{\bar{z}} w(z,t) = \mu, \quad \partial_z w = 1 + \Pi\mu(\cdot,),$$

which yields

$$\widetilde{\mu}(z,t) = \frac{\partial_{\bar{z}} w}{\partial_z w} = \frac{\mu(z,t)}{1 + \Pi\mu(z,t)}$$

and this function is holomorphic in t (in the L_∞-norm). This determines (at least for sufficiently small $|t|$) a holomorphic motion of $\widehat{\mathbb{C}}$ generating the holomorphic disks in the ball $\mathrm{Belt}(D_L)_1$ and in the space $\mathbf{T}$. □

Similar to the proof of Theorem 4, for either of these disks, one derives the equality $k(w^{\widetilde{\mu}(\cdot,t)}) = \varkappa_{D_L^*}(w^{\widetilde{\mu}(\cdot,t)})$ and the estimate (23), which proves Theorem 5.

4. Examples

The question of explicitly constructing the examples illustrating Theorems 4 and 5 (as well as the related results of some other papers in this direction devoted to the characterization of the associated curvelinear functionals) is complicated. The complications are caused by the requirement that the point $z_0 \in \mathbb{S}^1$, at which the modulus of a given Beltrami coefficient μ attains its maximum on the closed disk $\overline{\mathbb{D}}$, must be simultaneously a substantial point for the corresponding quasisymmetric function $f^\mu|\mathbb{S}^1$.

We provide here such examples combining Theorems 4 and 5, with the results presented in the survey [1].

Let $L \subset \widehat{\mathbb{C}}$ be a closed (oriented) unbounded quasicircle passing through the infinite point, with the convex interior, which is $C^{1+\delta}$ smooth at all finite points ($\delta > 0$) and has, at infinity, the asymptotes approaching the interior angle $\pi\alpha_\infty < 0$. It is proven in [1], by applying the holomorphic motions and the rather deep results from the Finsler geometry of universal Teichmüller space, that for any such curve, we have the equalities

$$\varkappa(f) = k(f) = q_L = 1/\rho_L = 1 - |\alpha_\infty|, \tag{24}$$

where f is the conformal mapping function of $\mathbb{D}^*$ onto the exterior domain of L.

These equalities imply that the modulus of the extremal Beltrami coefficient μ of the prescribed univalent function $f \in \Sigma_Q$ attains its maximal value, equal to $1 - |\alpha_\infty|$ at the point $z_0 \in \mathbb{S}^1$, which is the image of $w = \infty$ under the map f^{-1}, and this Beltrami coefficient is not of the Teichmüller type. Therefore, the point z_0 of its maximum is substantial.

Actually, Theorem 4 provides in this case another proof of the equalities (24), but it does not provide for the maximum the exact value $1 - |\alpha_\infty|$. This geometric quantity is intrinsically connected with the curve L.

A more complicate example is obtained taking the curves L with two angle points: $w_0 \in \mathbb{C}$ and $w = \infty$, with the angle openings $\pi\alpha_0 > 0$ and $\pi\alpha_\infty$ assuming again that the interior of L is convex. The indicated theorem from [1] provides for such curves the relations

$$\varkappa(f) = k(f) = q_L = 1/\rho_L = \max(1 - |\alpha_0|, 1 - |\alpha_\infty|).$$

These equalities yield simultaneously the exact values of the reflection coefficient and other functionals for the convex curvelinear lunes bounded by two smooth arcs with the common endpoints a, b because any such lune is a Moebius image of the interior domain for the above curve L.

Funding: This research received no external funding.

Data Availability Statement: No data are used in this article.

Acknowledgments: I am thankful to the referees for their comments and suggestions.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Krushkal, S.L. Fredholm eigenvalues and quasiconformal geometry of polygons. *J. Math. Sci.* **2021**, *252*, 472–501. [CrossRef]
2. Goluzin, G.M. *Geometric Theory of Functions of Complex Variables*; Mathematical Monographs, Translator; American Mathematical Society: Providence, RI, USA, 1969; Volume 26.
3. Earle, C.J.; Kra, I.; Krushkal, S.L. Holomorphic motions and Teichmüller spaces. *Trans. Am. Math. Soc.* **1994**, *944*, 927–948.
4. Gardiner, F.P.; Lakic, N. *Quasiconformal Teichmüller Theory*; American Mathematical Society: Providence, RI, USA, 2000.
5. Lehto, O. *Univalent Functions and Teichmüller Spaces*; Springer: New York, NY, USA, 1987.
6. Grunsky, H. Koeffizientenbedingungen für schlicht abbildende meromorphe Funktionen. *Math. Z.* **1939**, *45*, 29–61. [CrossRef]
7. Milin, I.M. *Univalent Functions and Orthonormal Systems*; Mathematical Monographs, Odnolistnye Funktcii i Normirovannie Systemy, Translators; American Mathematical Society: Providence, RI, USA, 1977; Volume 49.
8. Krushkal, S.L. Strengthened Moser's conjecture, geometry of Grunsky coefficients and Fredholm eigenval-ues. *Cent. Eur. J. Math.* **2007**, *5*, 551–580. [CrossRef]
9. Krushkal, S.L. Grunsky coefficient inequalities, Carathéodory metric and extremal quasiconformal mappings. *Comment. Math. Helv.* **1989**, *64*, 650–660. [CrossRef]

10. Krushkal, S.L. Strengthened Grunsky and Milin inequalities. *Contemp. Math.* **2016**, *667*, 159–179.
11. Kühnau, R. Quasikonforme Fortsetzbarkeit, Fredholmsche Eigenwerte und Grunskysche Koeffizien-tenbedingungen. *Ann. Acad. Sci. Fenn. Ser. AI Math.* **1982**, *7*, 383–391. [CrossRef]
12. Lebedev, N.A. *The Area Principle in the Theory of Univalent Functions*; Nauka: Moscow, Russia, 1975. (In Russian)
13. Schiffer, M.; Spencer, D.S. *Functionals of Finite Riemann Surfaces*; Princeton University Press: Princeton, NJ, USA, 1954.
14. Kühnau, R. Möglichst konforme Spiegelung an einer Jordankurve. *Jahresber. Deutsch. Math.* **1988**, *90*, 90–109.
15. Strebel, K. On the existence of extremal Teichmueller mappings. *J. Anal. Math.* **1976**, *30*, 464–480. [CrossRef]
16. Ahlfors, L. An extension of Schwarz's lemma. *Trans. Am. Math. Soc.* **1938**, *43*, 359–364.
17. Krushkal, S.L. Quasiconformal extensions and reflections. In *Handbook of Complex Analysis: Geometric Function Theory*; Kühnau, R., Ed.; Elsevier Science: Amsterdam, The Netherlands, 2005; Chapter 11, Volume II, pp. 507–553.
18. Schober, G. Estimates for Fredholm eigenvalues based on quasiconformal mapping. In *Numerische, Insbe-sondere Approximation-stheoretische Behandlung von Funktionalgleichungen*; Lecture Notes in Math. 333; Springer: Berlin/Heidelberg, Germany, 1973; pp. 211–217.
19. Ahlfors, L. *Lectures on Quasiconformal Mappings*; Van Nostrand: Princeton, NJ, USA, 1966.
20. Ahlfors, L. Remarks on the Neumann-Poincaré integral equation. *Pac. J. Math.* **1952**, *2*, 271–280. [CrossRef]
21. Gaier, D. *Konstruktive Methoden der Konformen Abbildung*; Springer: Berlin/Heidelberg, Germany, 1964.
22. Royden, H.L. The Ahlfors-Schwarz lemma: The case of equality. *J. Anal. Math.* **1986**, *46*, 261–270. [CrossRef]
23. Warschawski, S.E. On the effective determination of conformal maps. In *Contribution to the Theory of Rie-Mann Surfaces*; Ahlfors, L.; Calabi, E., Morse, M., Sario, l., Spencer, D.C., Eds.; Princeton Univ. Press: Princeton, NJ, USA, 1953; pp. 177–188.
24. Schiffer, M. Fredholm eigenvalues and Grunsky matrices. *Ann. Pol. Math.* **1981**, *39*, 149–164. [CrossRef]
25. Abate, M.; Patrizio, G. Isometries of the Teichmüller metric. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **1998**, *26*, 437–452.
26. Krushkal, S.L. Variational principles in the theory of quasiconformal maps. In *Handbook of Complex Analysis: Geometric Function Theory*; Kühnau, R., Ed.; Elsevier Science: Amsterdam, The Netherlands, 2005; Chapter 2, Volume 2, pp. 31–98.
27. Belinskii, P.P. *General Properties of Quasiconformal Mappings*; Nauka: Novosibirsk, Russia, 1974. (In Russian)
28. Krushkal, S.L. Extremal properties of Sobolev's Beltrami coefficients and distortion of curvelinear function-als. *J. Math. Sci.* 2023, *in press*.
29. Pommerenke, C. *Univalent Functions*; Vandenhoeck & Ruprecht: Göttingen, Germany, 1975.
30. Minda, D. The strong form of Ahlfors' lemma. *Rocky Mt. Math.* **1987**, *17*, 457–461. [CrossRef]
31. Dineen, S. *The Schwarz Lemma*; Clarendon Press: Oxford, UK, 1989.
32. Nehari, Z. *Conformal Mapping*; McGraw-Hill: New York, NY, USA, 1952.