axioms

MDPI

Article

Existence and Uniqueness of Variable-Order φ-Caputo Fractional Two-Point Nonlinear Boundary Value Problem in Banach Algebra

Yahia Awad [1,*,†], Hussein Fakih [1,2,3,†] and Yousuf Alkhezi [4,†]

[1] Department of Mathematics and Physics, Bekaa Campus, Lebanese International University (LIU), Bekaa P.O. Box 5, Lebanon; hussein.fakih@liu.edu.lb
[2] Department of Mathematics and Physics, Beirut Campus, The International University of Beirut (BIU), Beirut P.O. Box 1001, Lebanon
[3] Khawarizmi Laboratory for Mathematics and Applications, Department of Mathematics, Lebanese University, Beirut P.O. Box 1001, Lebanon
[4] Mathematics Department, College of Basic Education, Public Authority for Applied Education and Training (PAAET), P.O. Box 34053, Kuwait City 70654, Kuwait; ya.alkhezi@paaet.edu.kw
* Correspondence: yehya.awad@liu.edu.lb
† These authors contributed equally to this work.

Abstract: Using variable-order fractional derivatives in differential equations is essential. It enables more precise modeling of complex phenomena with varying memory and long-range dependencies, improving our ability to describe real-world processes reliably. This study investigates the properties of solutions for a two-point boundary value problem associated with φ-Caputo fractional derivatives of variable order. The primary objectives are to establish the existence and uniqueness of solutions, as well as explore their stability through the Ulam-Hyers concept. To achieve these goals, Banach's and Krasnoselskii's fixed point theorems are employed as powerful mathematical tools. Additionally, we provide numerical examples to illustrate results and enhance comprehension of theoretical findings. This comprehensive analysis significantly advances our understanding of variable-order fractional differential equations, providing a strong foundation for future research. Future directions include exploring more complex boundary value problems, studying the effects of varying fractional differentiation orders, extending the analysis to systems of equations, and applying these findings to real-world scenarios, all of which promise to deepen our understanding of Caputo fractional differential equations with variable order, driving progress in both theoretical and applied mathematics.

Keywords: boundary value problem; φ-Caputo derivatives with variable order; piecewise constant functions; fixed point theorems; Green's function; Banach algebra

MSC: 26A33; 34A08; 34B15; 47H10

Citation: Awad, Y.; Fakih, H.; Alkhezi, Y. Existence and Uniqueness of Variable-Order φ-Caputo Fractional Two-Point Nonlinear Boundary Value Problem in Banach Algebra. *Axioms* **2023**, *12*, 935. https://doi.org/10.3390/axioms12100935

Academic Editors: Daniela Marian, Ali Shokri, Daniela Inoan, Kamsing Nonlaopon and Zacharias A. Anastassi

Received: 7 August 2023
Revised: 23 September 2023
Accepted: 27 September 2023
Published: 29 September 2023

1. Introduction

Fractional calculus has gained significant attention due to its ability to generalize classical calculus to non-integer orders, providing a powerful mathematical framework for modeling complex phenomena with non-local and memory-dependent behaviors (see [1–3]). The significance of fractional calculus transcends its theoretical elegance; it holds practical importance in phenomena marked by non-locality, memory effects, and long-range interactions. To emphasize the relevance of fractional calculus in contemporary scientific pursuits, we draw attention to several influential studies that have harnessed fractional derivatives to tackle intricate challenges. Noteworthy examples encompass the numerical solution of chemical kinetics' traveling waves (refer to [4]), the exploration of disease transmission dynamics involving asymptomatic carriers via fractal–fractional differential operators (as detailed in [5]), the modeling of COVID-19 pandemic dynamics using fractional order models (as illustrated in [6]), the examination of chaotic systems employing Mittag-Leffler kernel-based fractional methodologies (as explored in [7]), the analysis of nonlinear reaction–diffusion equations, and even the mathematical modeling of intricate biological systems like the human liver (as demonstrated in [8,9]). In addition, it is worth noting that there are many other related works that could be found in the literature. In recent years, the exploration of variable order fractional calculus has emerged, allowing the differentiation order to vary as a function of the independent variable or other relevant variables. This extension offers increased flexibility in modeling dynamic systems with evolving characteristics (see [10–19] and others). Within the filed of variable-order fractional calculus, fractional differential equations play a crucial role. These equations involve derivatives of non-integer orders and are instrumental in capturing memory effects, anomalous diffusion, and other intricate dynamics encountered in physics, engineering, and scientific fields (see [16,20–23] and the references therein).

The φ-Caputo derivatives have been introduced to extend the standard Caputo derivative by incorporating a variable function, enabling a more flexible definition of fractional derivatives. This generalization allows for a more accurate representation of complex systems with memory effects, enhancing modeling capabilities (see [24–32] and the references therein).

The application of φ-Caputo derivatives with variable order to two-point boundary value problems is gaining significant attention. By incorporating these derivatives in the formulation of boundary value problems, researchers can analyze the dynamics and properties of systems with evolving fractional behaviors and memory effects. This approach provides valuable insights into the behavior of solutions at distinct points and contributes to a comprehensive understanding of the system's overall behavior.

This research aims to enhance our understanding of fractional calculus with variable order and its application to two-point boundary value problems. Specifically, we investigate the behavior and properties of solutions for φ-Caputo fractional differential equations with variable order in this context. By exploring solution existence and uniqueness, we deepen our comprehension of system dynamics and lay the foundation for advancements in modeling and analysis across scientific and engineering domains. Our study extends previous research by considering a broader class of nonlinear fractional boundary value problems with variable order, providing fresh insights and paving the way for future investigations.

In the subsequent discussions, we present a summary of recent related works: In Ref. [33], the authors explored the basic theory of an initial value problem involving a fixed-order fractional differential equation using the classical approach. They considered the equation

$$\begin{cases} \mathfrak{D}^{\alpha}_{0^+}x(\zeta) = \mathcal{F}(\zeta, x(\zeta)), \\ x(0) = x_0, \end{cases} , \tag{1}$$

where $0 < \alpha < 1, 0 < \zeta < T < +\infty$, $\mathcal{F} \in C([0,T],\mathcal{R})$, $+D^{\alpha}_{0^+}$ is with the Riemann-Liouville fractional derivative.

In Ref. [31], the authors presented findings on the existence and uniqueness of solutions concerning the subsequent fractional differential equation:

$$\begin{cases} \mathfrak{D}^{\alpha,\psi}_{0^+}u(\zeta) = f(\zeta, u(\zeta)),\ \zeta \in \Delta = [0, T], \\ u(0) = u'(0) = 0,\ u(T) = u_T, \end{cases} \tag{2}$$

where $\mathfrak{D}^{\alpha,\psi}_{0^+}$ is the ψ-Caputo fractional derivative of u at order $\alpha \in (2, 3)$, $T > 0$, $f \in C(\Delta \times \mathcal{R}, \mathcal{R})$, and $u_T \in \mathcal{R}$.

On the other hand, [22] established the existence and uniqueness of solutions for the following variable-order fractional differential equation using the monotone iterative method:

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha(\zeta)} x(\zeta) = \mathcal{F}(\zeta, x(\zeta)), \\ x(0) = 0, \end{cases} \tag{3}$$

where $0 < \alpha(\zeta) < 1$, $\mathcal{F} \in C([0,T], \mathcal{R})$, $\mathfrak{D}_{0^+}^{\alpha(\zeta)}$ is with the Riemann-Liouville fractional derivative of variable order.

In Ref. [12], the authors considered the existence and uniqueness of the solution of another variable-order fractional differential equation using the Banach contraction mapping principle.

$$\begin{cases} \mathfrak{D}_{0^+}^{\alpha(\zeta)} x(\zeta) = \mathcal{F}(\zeta, x(\zeta)), 0 < \zeta \le T, \\ \zeta^{2-\alpha(\zeta)} x(\zeta)|_{\zeta=0} = 0, \end{cases} \tag{4}$$

where $1 < \alpha(\zeta) < 2 \; \forall \zeta \in [0,T]$ with $0 < T < +\infty$, $\mathfrak{D}_{0^+}^{\alpha(\zeta)}$ denotes the i^{th} Riemann-Liouville fractional derivative of variable-order $\alpha(\zeta)$.

In Ref. [16], the author obtained the existence, uniqueness, and the Ulam-Hyers stability of a solution to the following Caputo type variable-order fractional differential equation.

$$\begin{cases} {}^c\mathfrak{D}_{0^+}^{\alpha(\zeta)} x(\zeta) = \mathcal{F}(\zeta, x(\zeta)), 0 < \zeta \le T, \\ u(\zeta)|_{\zeta=0} = u_0, \end{cases} \tag{5}$$

where $0 < \alpha(\zeta) < 1 \; \forall \zeta \in [0,T]$ with $0 < T < +\infty$, ${}^c\mathfrak{D}_{0^+}^{\alpha(\zeta)}$ denotes the Caputo derivative of variable-order $\alpha(\zeta)$.

In Ref. [23], the concept of an approximate solution was introduced for the following differential equation of variable order with a derivative argument on the half-axis:

$$\begin{cases} {}^c\mathfrak{D}_{0^+}^{\alpha(\zeta)} x(\zeta) = \mathcal{F}(\zeta, x(\zeta), \mathfrak{D}_{0^+}^{\beta(\zeta)} x(\zeta)), \\ x(0) = 0, \end{cases} \tag{6}$$

where $0 < \beta(\zeta) < \alpha(\zeta) < 1 \; \forall \zeta \in [0,T]$ with $0 < T < +\infty$, $\mathcal{F}(\zeta, x(\zeta), \mathfrak{D}_{0^+}^{\beta(\zeta)} x(\zeta))$ are given real functions, and $\mathfrak{D}_{0^+}^{\alpha(\zeta)}$, $\mathfrak{D}_{0^+}^{\beta(\zeta)}$ denote the Riemann-Liouville fractional derivative of respective variable-order $\alpha(\zeta)$ and $\beta(\zeta)$.

Lastly, in [15], authors studied the existence and stability criteria of solutions for the following Hadamard fractional differential equations of variable order, where the results were based on the Kuratowski measure of noncompactness.

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\alpha(\zeta)} x(\zeta) = \mathcal{F}(\zeta, x(\zeta), {}^H\mathfrak{I}_{0^+}^{\alpha(\zeta)} x(\zeta)), 1 < \zeta < T < \infty, \\ x(1) = x(T) = 0, \end{cases} \tag{7}$$

where $1 < \alpha(\zeta) < 2 \; \forall \zeta \in [0,T]$ with $0 < T < +\infty$, $\mathcal{F}(\zeta, x(\zeta), {}^H\mathfrak{I}_{0^+}^{\alpha(\zeta)} x(\zeta))$ are given real functions, and ${}^H\mathfrak{D}_{0^+}^{\alpha(\zeta)}$, ${}^H\mathfrak{I}_{0^+}^{\alpha(\zeta)}$ denote the Hadamard fractional derivative of variable-order $\alpha(\zeta)$.

Inspired by previous research, this paper investigates the existence and uniqueness of solutions for the subsequent φ-Caputo fractional nonlinear two-point boundary value problems that contain variable order (VOFBVPs for short).

$$\begin{cases} {}^c\mathfrak{D}_{0^+}^{\alpha(\zeta),\varphi} x(\zeta) = \mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta(\zeta),\varphi} x(\zeta)), \\ x(\zeta)|_{\zeta=0} = u_0, \\ x(\zeta)|_{\zeta=T} = u_T. \end{cases} \tag{8}$$

where $1 < \beta(\zeta) < \alpha(\zeta) < 2\ \forall\zeta \in \mathcal{I} = [0, T]$ with $0 < T < +\infty$, $\mathfrak{D}_{0^+}^{\alpha(\zeta),\varphi}$ and $\mathfrak{D}_{0^+}^{\beta(\zeta),\varphi}$ denote the φ-Caputo fractional derivative of variable-order $\alpha(\zeta)$ and $\beta(\zeta)$, which is defined in (10), and $\mathcal{F}$ is a continuous function such that $\mathcal{F} : \mathcal{I} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}$.

In conclusion, the primary motivation driving this study for such a φ-Caputo fractional nonlinear boundary value problem is the growing need for a deeper understanding of variable-order fractional differential equations. These equations have gained prominence in various fields, including physics, engineering, and biology, due to their ability to model complex systems with memory and non-local behaviors. As such, elucidating their properties, establishing solution uniqueness, and exploring stability aspects are critical endeavors. By addressing these challenges, our research aims to contribute valuable insights that can aid in the accurate modeling and analysis of real-world phenomena.

To achieve this, our paper is structured as follows: We begin with an introduction in Section 1. Next, in Section 2, we present various notations, definitions, lemmas, and theorems that will be utilized throughout our study. Moving on to Section 3, we establish the existence and uniqueness of a mild solution for the VOFBVP (8) by employing the fixed point theorems of Banach and Krasnoselskii. Finally, Section 4 provides a numerical example, illustrating the practical application of our main findings.

2. Preliminaries

In this section, we introduce some notations, definitions, lemmas and theorems that are considered as prerequisites for our work. In particular, they are derived from the approaches in [22,24,26,31,34] and the references therein.

Definition 1. *Let $\alpha(\zeta) : \mathcal{I} \to (1, 2)$ with $\mathcal{I} = [a, b]$ and $-\infty < a < \zeta < b < +\infty$, $x \in C(\mathcal{I}, \mathcal{R})$, and let $\varphi \in C^1(\mathcal{I}, \mathcal{R})$ be an increasing function such that $\varphi'(\zeta) \neq 0$, $\forall\zeta \in \mathcal{I}$. Then, the φ-Riemann-Liouville fractional integral of $x(\zeta)$ with variable order $\alpha(\zeta)$ is defined as:*

$$\mathfrak{I}_{a^+}^{\alpha(\zeta),\varphi} x(\zeta) := \frac{1}{\Gamma(\alpha(\zeta))} \int_a^\zeta \varphi'(s)(\varphi(\zeta) - \varphi(s))^{\alpha(\zeta)-1} x(s) ds. \tag{9}$$

Definition 2. *Let $\alpha(\zeta) : \mathcal{I} \to (n-1, n)$ for all $\zeta \in \mathcal{I} = [a, b]$ and let φ, $x \in C^n(\mathcal{I}, \mathcal{R})$ be two functions such that φ is an increasing function with $\varphi'(\zeta) \neq 0$, $\forall\zeta \in \mathcal{I}$. Then, the left-sided φ-Caputo fractional derivative of $x(\zeta)$ with variable order $\alpha(\zeta)$ is defined as:*

$$\begin{aligned} {}^c\mathfrak{D}_{a^+}^{\alpha(\zeta),\varphi} x(\zeta) &= \mathfrak{I}_{a^+}^{n-\alpha(\zeta),\varphi} \left(\frac{1}{\varphi'(\zeta)} \frac{d}{d\zeta} \right)^n x(\zeta), \qquad (10) \\ &= \frac{1}{\Gamma(n - \alpha(\zeta))} \int_a^\zeta \varphi'(s)(\varphi(\zeta) - \varphi(s))^{n-\alpha(\zeta)-1} x_\varphi^{[n]}(s) ds, \end{aligned}$$

with

$$x_\varphi^{[n]}(s) = \left(\frac{1}{\varphi'(s)} \frac{d}{ds} \right)^n x(s), \tag{11}$$

where $n = [\alpha(\zeta)] + 1$ for all $\zeta \in [a, b]$.

It is clear that (10) yields the Caputo fractional derivative operator with variable order when $\varphi(\zeta) = \zeta$.

Remark 1. *The variable-order fractional derivatives and integrals are considered as extensions of the constant order fractional derivatives and integers. That is, if $\alpha(\zeta) = \alpha$, where α is finite positive constant real number, then $\mathfrak{I}_{a^+}^{\alpha(\zeta),\varphi}$ and ${}^c\mathfrak{D}_{a^+}^{\alpha(\zeta),\varphi}$ are the usual Riemann-Liouville fractional integrals and Caputo derivatives. In addition, as usual, in order to study the existence of solutions of a fractional differential equation, we have to transform it into an equivalent integral equation using some fundamental properties of $\mathfrak{I}_{a^+}^{\alpha,\varphi}$ and ${}^c\mathfrak{D}_{a^+}^{\alpha,\varphi}$.*

Lemma 1 ([26])**.** *The fractional integral $\mathfrak{I}_{0^+}^{\alpha,\varphi}x(\zeta)$, with $\zeta \in \mathcal{I}$ exists almost everywhere.*

Lemma 2 ([26])**.** *If $1 < \beta < \alpha < 2$, and $x \in L(0,b)$ with $0 < b < +\infty$, then the semigroup property for the Riemann-Liouville fractional integrals hold, i.e., $\mathfrak{I}_{0^+}^{\alpha,\varphi}\mathfrak{I}_{0^+}^{\beta,\varphi}x(\zeta) = \mathfrak{I}_{0^+}^{\beta,\varphi}\mathfrak{I}_{0^+}^{\alpha,\varphi}x(\zeta) = \mathfrak{I}_{0^+}^{\alpha+\beta,\varphi}x(\zeta)$ for all $\zeta \in \mathcal{I}$.*

Lemma 3 ([26])**.** *If $1 < \alpha < 2$, and $x \in L^1(0,b)$ with $0 < b < +\infty$, then ${}^c\mathfrak{D}_{a^+}^{\alpha,\varphi}\mathfrak{I}_{a^+}^{\alpha,\varphi}x(\zeta) = x(\zeta)$ for all $\zeta \in \mathcal{I}$.*

Lemma 4 ([26])**.** *Let $\alpha > 0$, then the differential equation ${}^c\mathfrak{D}_{0+}^{\alpha,\varphi}x(\zeta) = u(\zeta)$ has solution*

$$\begin{aligned} x(\zeta) \quad = \quad & c_0 + c_1(\varphi(\zeta) - \varphi(0)) + c_2(\varphi(\zeta) - \varphi(0))^2 + \ldots + c_{n-1}(\varphi(\zeta) - \varphi(0))^{n-1} \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{\zeta} \varphi'(s)(\varphi(\zeta) - \varphi(s))^{\alpha-1} u(s) ds, \end{aligned} \tag{12}$$

where $c_i \in \mathcal{R}$, $i = 0,1,2,\ldots,n-1$, and $n = [\alpha] + 1$.

For more information, one can see [24–26,31] and the references therein.

Remark 2. *The fractional integrals of variable order do not satisfy the low of exponents (semigroup property), in general, we have*

$$\mathfrak{I}_{a^+}^{\alpha(\zeta),\varphi}\mathfrak{I}_{a^+}^{\beta(\zeta),\varphi}x(\zeta) \neq \mathfrak{I}_{a^+}^{\alpha(\zeta)+\beta(\zeta),\varphi}x(\zeta) \text{ for all } \alpha(\zeta) > 0,\ \beta(\zeta) > 0, x \in C(\mathcal{I},\mathcal{R}), \tag{13}$$

where $\mathfrak{I}_{a^+}^{\alpha(\zeta),\varphi}$ and $\mathfrak{I}_{a^+}^{\beta(\zeta),\varphi}$ denote one of the fractional integrals of variable order defined in (9). For example, if $\varphi(\zeta) = \zeta$ where $\mathfrak{I}_{a^+}^{\alpha(\zeta),\varphi} = I_{a^+}^{\alpha(\zeta)}$ (Riemann-Liouville variable-order fractional integral defined in [22]), then

$$\mathfrak{I}_{a^+}^{\alpha(\zeta),\varphi}\mathfrak{I}_{a^+}^{\beta(\zeta),\varphi}x(\zeta) = I_{a^+}^{\alpha(\zeta)}I_{a^+}^{\beta(\zeta)}x(\zeta) \neq I_{a^+}^{\alpha(\zeta)+\beta(\zeta),\varphi}x(\zeta), \tag{14}$$

for all $\alpha(\zeta) > 0$, $\beta(\zeta) > 0, \zeta \in \mathcal{I}, x \in C(\mathcal{I},\mathcal{R})$ (see [12,15,22] and the references therein). Thus, it is obtained that there are severe challenges to consider the existence of solutions for differential equations with fractional derivatives of variable orders as in those of fixed order fractional derivatives by means of nonlinear fractional analysis. Moreover, any differential equation with variable-order fractional derivatives can not be directly transformed into an equivalent integral equation without using the above lemmas related to fractional derivatives with fixed orders and the following definitions.

Thus, we need to introduce some definitions which are used throughout this paper.

Definition 3 ([12])**.** *A generalized interval is a subset $\mathcal{I}$ of $\mathcal{R}$ which is either an interval , a point, or the empty set.*

Definition 4 ([12])**.** *If $\mathcal{I}$ is a generalized interval, then set P is a partition of $\mathcal{I}$ if P is a finite set of generalized intervals contained in $\mathcal{I}$, such that every x in $\mathcal{I}$ lies in exactly one of the generalized intervals $\mathcal{I}$ in P.*

Definition 5 ([12])**.** *Let $\mathcal{I}$ be a generalized interval, let $\mathcal{F} : \mathcal{I} \to \mathcal{R}$ be a function, and let P be a partition of $\mathcal{I}$. Then, $\mathcal{F}$ is said to be piecewise constant with respect to P if for every $\mathcal{I} \in P$, $\mathcal{F}$ is constant on $\mathcal{I}$.*

Definition 6 ([12])**.** *Let $\mathcal{I}$ be a generalized interval. The function $\mathcal{F} : \mathcal{I} \to \mathcal{R}$ is called piecewise constant on $\mathcal{I}$, if there exists a partition P of $\mathcal{I}$ such that $\mathcal{F}$ is piecewise constant with respect to P.*

Definition 7 ([34]). *Let $(X, \|.\|)$ be a Banach space. A mapping $\wp : X \to X$ is called a contraction on X if there exists a positive real number $c < 1$ such that $\|\wp(x) - \wp(y)\| \leq c\|x - y\|$, for all $x, y \in X$.*

Theorem 1 ([34]). *Let $(X, \|.\|)$ be a Banach space and let $\wp : X \to X$ be a contraction on X. Then, $\wp$ has a unique fixed point $x \in X$ (i.e., $\wp(x) = x$).*

Definition 8 ([34]). *Denote by $C(\mathcal{I}, \mathcal{R})$ the Banach space of continuous functions $\wp : \mathcal{I} \to \mathcal{R}$ with the norm $\|\wp\| = \sup\{|\wp(\zeta)|;\ \zeta \in \mathcal{I}\}$.*

3. Main Results

Throughout this paper, we consider the following assumptions:

(A_1) $\mathcal{F} : \mathcal{I} \times \mathcal{R}^2 \to \mathcal{R}$ is continuous and there exists $\mu \in C(\mathcal{I}, \mathcal{R}^+)$, with L_1 norm $\|\mu\|$, such that:

$$|\mathcal{F}(\zeta, u_1, u_2) - \mathcal{F}(\zeta, v_1, v_2)| \leq \mu(\zeta)(|u_1 - v_1| + |u_2 - v_2|), \tag{15}$$

$\forall\, \zeta \in \mathcal{I},\ u_i, v_i \in \mathcal{R},\ (i = 1, 2).$

(A_2) If $\alpha(\zeta) : [0, T] \to (1, 2)$ and $\beta(\zeta) : [0, T] \to (1, 2)$ are piecewise constant functions with partition $P = \{[0, T_1], [T_1, T_2], [T_2, T_3], \ldots, [T_{\mathbb{N}^*-1}, T_{\mathbb{N}^*}]\}$ ($\mathbb{N}^*$ is a given natural number) of the finite interval $[0, T]$, then

$$\alpha(\zeta) = \sum_{k=1}^{N^*} \alpha_k \mathcal{I}_k(\zeta), \zeta \in [0, T], \tag{16}$$

$$\beta(\zeta) = \sum_{k=1}^{N^*} \beta_k \mathcal{I}_k(\zeta), \zeta \in [0, T], \tag{17}$$

where $1 < \alpha_k; \beta_k < 2, k = 1, 2, 3, \ldots, \mathbb{N}^*$, $\mathcal{I}_k(\zeta)$ is the indicator of the interval $[T_{k-1}, T_k]$ for $k = 1, 2, 3, \ldots, \mathbb{N}^*$, where $T_0 = 0$ and $T_{\mathbb{N}^*} = T$, i.e.,

$$\mathcal{I}_k(\zeta) = \begin{cases} 1 \text{ for } \zeta \in [T_{k-1}, T_k], \\ 0 \qquad \text{elsewhere.} \end{cases} \tag{18}$$

(A_3) If $0 \leq r \leq \alpha_i - 1$, $i = 1, 2, \ldots, \mathbb{N}^*$, and $\varphi(\zeta)^r \mathcal{F} : [0, T] \times \mathcal{R}^2 \to \mathcal{R}$ is a continuous function, then there exists a positive constant L such that

$$L = \frac{\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_i) - \varphi(0))^{\alpha_i - \beta_i}}{\Gamma(\alpha_i - \beta_i + 1)}\right)}, \tag{19}$$

with

$$L\left(\frac{(\varphi(T_i) - \varphi(0))^{\alpha_i - 1}}{T_i \Gamma(\alpha_i + 1)} + \mathcal{G}_0(\varphi(T_i) - \varphi(0))\right) < 1, \tag{20}$$

and

$$\varphi(\zeta)^r |\mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta(\zeta),\varphi} x(\zeta)) - \mathcal{F}(\zeta, y(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta(\zeta),\varphi} y(\zeta))| \leq (1 + L)|x(\zeta) - y(\zeta)|, \tag{21}$$

for all $0 \leq \zeta \leq T$ and $x, y \in \mathcal{R}$.

Remark 3. *From assumption (A_1), we have*

$$|\mathcal{F}(\zeta, u_1, u_2)| - |\mathcal{F}(\zeta, 0, 0)| \leq |\mathcal{F}(\zeta, u_1, u_2) - \mathcal{F}(\zeta, 0, 0)| \leq \mu(\zeta)(|u_1| + |u_2|), \tag{22}$$

and

$$|\mathcal{F}(\zeta, u_1, u_2)| \leq \|\mu\|(|u_1| + |u_2|) + \mathfrak{F}, \text{ where } \mathfrak{F} = \sup_{\zeta \in \mathcal{I}} |\mathcal{F}(\zeta, 0, 0)|. \quad (23)$$

Now, in order to study the existence, uniqueness, and stability of solutions for the VOF-BVP (8), we have to carry out the following essential analysis.

From assumption (A_2), we have

$$\alpha(\zeta) = \sum_{k=1}^{N^*} \alpha_k \mathcal{I}_k(\zeta), \zeta \in [0, T], \quad (24)$$

$$\beta(\zeta) = \sum_{k=1}^{N^*} \beta_k \mathcal{I}_k(\zeta), \zeta \in [0, T], \quad (25)$$

Hence, we have for every $\zeta \in [0, T]$

$$\int_0^\zeta \frac{\varphi'(s)(\varphi(\zeta) - \varphi(s))^{1-\alpha(\zeta)}}{\Gamma(2 - \alpha(\zeta))} x_\varphi^{[2]}(s) ds = \sum_{k=1}^{N^*} \mathcal{I}_k(\zeta) \int_0^\zeta \frac{\varphi'(s)(\varphi(\zeta) - \varphi(s))^{1-\alpha_k}}{\Gamma(2 - \alpha_k)} x_\varphi^{[2]}(s) ds. \quad (26)$$

So, the VOFBVP (8) can be written as

$$\sum_{k=1}^{N^*} \mathcal{I}_k(\zeta) \int_0^\zeta \frac{\varphi'(s)(\varphi(\zeta) - \varphi(s))^{1-\alpha_k}}{\Gamma(2 - \alpha_k)} x_\varphi^{[2]}(s) ds = \mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta, \varphi} x(\zeta)), \ \zeta \in [0, T], \quad (27)$$

where

$${}^c\mathfrak{D}_{0^+}^{\beta, \varphi} x(\zeta) = \sum_{k=1}^{N^*} \mathcal{I}_k(\zeta) \int_0^\zeta \frac{\varphi'(s)(\varphi(\zeta) - \varphi(s))^{1-\beta_k}}{\Gamma(2 - \beta_k)} x_\varphi^{[2]}(s) ds), \ \zeta \in [0, T]. \quad (28)$$

Now, Equation (27) in the interval $[0, T_1]$ is written as:

$$\int_0^\zeta \frac{\varphi'(s)(\varphi(\zeta) - \varphi(s))^{1-\alpha_1}}{\Gamma(2 - \alpha_1)} x_\varphi^{[2]} ds = \mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_1, \varphi} x(\zeta)), \ \zeta \in [0, T_1]. \quad (29)$$

Again, Equation (27) in the interval $(T_1, T_2]$ is written as:

$$\int_0^\zeta \frac{\varphi'(s)(\varphi(\zeta) - \varphi(s))^{1-\alpha_2}}{\Gamma(2 - \alpha_2)} x_\varphi^{[2]} ds = \mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_2, \varphi} x(\zeta)), \ \zeta \in (T_1, T_2]. \quad (30)$$

If we complete in the following manner, we obtain that Equation (27) in the interval $(T_{i-1}, T_i], i = 1, 2, 3, \ldots, \mathbb{N}^*$ $(T_0 = 0, T_{\mathbb{N}^*} = T)$ can be written as:

$$\int_0^\zeta \frac{\varphi'(s)(\varphi(\zeta) - \varphi(s))^{1-\alpha_i}}{\Gamma(2 - \alpha_i)} x_\varphi^{[2]} ds = \mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_i, \varphi} x(\zeta)), \ \zeta \in (T_{i-1}, T_i]. \quad (31)$$

Remark 4. *From the above argument, we can state that the VOFBVP (8) has a (unique) solution if there exist (unique) functions $x_i(\zeta), i = 1, 2, 3, \ldots, N^*$, satisfying the following conditions:*

1. *$x_1 \in C^1[0, T_1]$ such that it satisfies Equation (29) with $x_1(\zeta)|_{\zeta=0} = u_0$ and $x_1(\zeta)|_{\zeta=T_1} = u_T$.*
2. *$x_i \in C^1[0, T_i]$ for $i = 2, 3, \ldots, N^*$, satisfying Equation (31) with $x_i(\zeta)|_{\zeta=0} = u_0$, $x_i(T_{i-1}) = x_{i-1}(T_{i-1})$, and $x_i(\zeta)|_{\zeta=T_i} = u_T$.*

Thus, we say that the VOFBVP (8) has a unique solution if the functions $x_i(\zeta)$ are unique for all $i = 1, 2, 3, \ldots, N^*$. For additional information and mathematical rigor, we refer to [12].

Lemma 5. *The solution of Equation (29) with $x(\zeta)|_{\zeta=0} = u_0$ and $x(\zeta)|_{\zeta=T_1} = u_T$ is the solution of the integral equation*

$$x(\zeta) = \mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta, s)u(s)ds, \ \zeta \in [0, T_1], \tag{32}$$

where u is the solution of the fractional order integral equation

$$u(\zeta) = \mathcal{F}(\zeta, \mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta, s)u(s)ds, \mathfrak{I}^{\alpha_1-\beta_1,\varphi}u(\zeta)), \tag{33}$$

and $\mathcal{G}(\zeta, s)$ is Green's function described by

$$\mathcal{G}_1(\zeta, s) = \begin{cases} \frac{1}{\Gamma(\alpha_1)}\left[\begin{array}{c} (\varphi(\zeta) - \varphi(s))^{\alpha_1-1} \\ -\frac{\varphi(\zeta)-\varphi(0)}{\varphi(T_1)-\varphi(0)}(\varphi(T_1) - \varphi(s))^{\alpha_1-1} \end{array}\right] & \text{if } \ 0 \le s \le \zeta \le T_1, \\ -\frac{(\varphi(\zeta)-\varphi(0))}{\Gamma(\alpha_1)(\varphi(T_1)-\varphi(0))}(\varphi(T_1) - \varphi(s))^{\alpha_1-1} & \text{if } \ 0 \le \zeta \le s \le T_1, \end{cases} \tag{34}$$

such that

$$\mathcal{G}_0 := \max\{|\mathcal{G}_1(\zeta, s)|, (\zeta, s) \in \mathcal{I}_1 \times \mathcal{I}_1\}, \text{ with } \mathcal{I}_1 = [0, T_1],$$

and

$$\mathfrak{h}_1(\zeta, x(\zeta)) = u_0 + \frac{(\varphi(\zeta) - \varphi(0))}{\varphi(T_1) - \varphi(0)}(u_T - u_0). \tag{35}$$

Proof. Let $x(\zeta)$ be a solution of Equation (29), then by applying the property that

$${}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi}x(\zeta) = \mathfrak{I}_{0^+}^{\alpha_1-\beta_1,\varphi}\, {}^c\mathfrak{D}_{0^+}^{\alpha_1,\varphi}x(\zeta), \quad \zeta \in [0, T_1], \tag{36}$$

we obtain that

$${}^c\mathfrak{D}_{0^+}^{\alpha_1,\varphi}x(\zeta) = \mathcal{F}(\zeta, \ x(\zeta), \ \mathfrak{I}_{0^+}^{\alpha_1-\beta_1,\varphi}\, {}^c\mathfrak{D}_{0^+}^{\alpha_1,\varphi}u(\zeta)), \ \zeta \in [0, T_1]. \tag{37}$$

If we take $u(\zeta) = {}^c\mathfrak{D}_{0^+}^{\alpha_1,\varphi}x(\zeta)$, then

$$u(\zeta) = \mathcal{F}(\zeta, \ x(\zeta), \ \mathfrak{I}_{0^+}^{\alpha_1-\beta_1,\varphi}\, u(\zeta)), \ \zeta \in [0, T_1]. \tag{38}$$

Applying Lemma 4, we obtain that

$$x(\zeta) = c_0 + c_1(\varphi(\zeta) - \varphi(0)) + \frac{1}{\Gamma(\alpha_1)}\int_0^{\zeta} \varphi'(s)(\varphi(\zeta) - \varphi(s))^{\alpha_1-1}u(s)ds. \tag{39}$$

Substituting the boundary conditions $x(\zeta)|_{\zeta=0} = u_0$ and $x(T)|_{\zeta=T_1} = u_T$, we get

$$c_0 = u_0, \tag{40}$$

and

$$c_1 = \frac{(u_T - u_0)}{(\varphi(T_1) - \varphi(0))} - \frac{1}{\Gamma(\alpha_1)(\varphi(T_1) - \varphi(0))}\int_0^{T_1} \varphi'(s)[\varphi(T_1) - \varphi(s)]^{\alpha_1-1}u(s)ds. \tag{41}$$

Hence, the solution of Equation (29) can be written as:

$$\begin{aligned} x(\zeta) = u_0 + \frac{(\varphi(\zeta) - \varphi(0))}{\varphi(T_1) - \varphi(0)}(u_T - u_0) + \frac{1}{\Gamma(\alpha_1)}\int_0^{\zeta} \varphi'(s)(\varphi(\zeta) - \varphi(s))^{\alpha_1-1}u(s)ds \\ - \frac{(\varphi(\zeta) - \varphi(0))}{\Gamma(\alpha_1)(\varphi(T_1) - \varphi(0))}\int_0^{T_1} \varphi'(s)[\varphi(T_1) - \varphi(s)]^{\alpha_1-1}u(s)ds, \end{aligned} \tag{42}$$

$$= u_0 + \frac{(\varphi(\zeta) - \varphi(0))}{\varphi(T_1) - \varphi(0)}(u_T - u_0)$$
$$- \frac{\varphi(\zeta) - \varphi(0)}{\Gamma(\alpha_1)(\varphi(T_1) - \varphi(0))} \int_\zeta^{T_1} \varphi'(s)[\varphi(T_1) - \varphi(s)]^{\alpha_1 - 1} u(s)ds$$
$$+ \frac{1}{\Gamma(\alpha_1)} \int_0^\zeta \varphi'(s) \left[[\varphi(\zeta) - \varphi(s)]^{\alpha_1 - 1} - \frac{\varphi(\zeta) - \varphi(0)}{\varphi(T_1) - \varphi(0)} [\varphi(T_1) - \varphi(s)]^{\alpha_1 - 1} \right] u(s)ds.$$

If $\mathfrak{h}_1(\zeta, x(\zeta)) = u_0 + \frac{(u_T - u_0)(\varphi(\zeta) - \varphi(0))}{\varphi(T_1) - \varphi(0)}$, then the solution of (29) is also a solution of (32). □

3.1. Existence and Uniqueness of Solutions

The following result is based on Banach's fixed point theorem to obtain the existence of a unique solution of the VOFBVP (8).

Theorem 2. *Suppose that assumptions (A_1)–(A_3) hold, then the VOFBVP (8) has a unique solution.*

Proof. The VOFBVP (8) can be written as Equation (27).
Now, by Lemma 5, Equation (27) can be written in the interval $[0, T_1]$ as Equation (29), which can also be written as:

$$x(\zeta) = \mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s) \mathcal{G}_1(\zeta, s) u(s) ds, 0 \le \zeta \le T_1. \tag{43}$$

Define operator $\wp : C[0, T_1] \to C[0, T_1]$ by

$$\wp x(\zeta) = \mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s) \mathcal{G}_1(\zeta, s) u(s) ds, 0 \le \zeta \le T_1. \tag{44}$$

In fact, $\wp x(\zeta) \in C[0, T_1]$, since $x(\zeta) \in C[0, T_1]$. Let

$$\mathfrak{g}(\zeta,\ x(\zeta),\ {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta)) = \varphi(\zeta)^r \mathcal{F}(\zeta,\ x(\zeta),\ {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta)), \tag{45}$$

then by assumption (A_3), we have function $\mathfrak{g} : [0, T_1] \times \mathcal{R}^2 \to \mathcal{R}$ is continuous. Thus, for $\zeta, \zeta_0 \in [0, T_1]$, we have

$$\begin{aligned}
& |\wp x(\zeta) - \wp x(\zeta_0)| \qquad (46)\\
= \ & |\mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s) \mathcal{G}_1(\zeta, s) u(s) ds - \mathfrak{h}_1(\zeta_0, x(\zeta_0)) - \int_0^{T_1} \varphi'(s) \mathcal{G}_1(\zeta_0, s) u(s) ds|, \\
\le \ & |\mathfrak{h}_1(\zeta, x(\zeta)) - \mathfrak{h}_1(\zeta_0, x(\zeta_0))| + |\int_0^{T_1} \varphi'(s) |\mathcal{G}_1(\zeta, s) - \mathcal{G}_1(\zeta_0, s)|\ |u(s)| ds, \\
\le \ & |\frac{(u_T - u_0)}{\varphi(T_1) - \varphi(0)}(\varphi(\zeta) - \varphi(\zeta_0)) \\
& - \frac{1}{(\varphi(T_1) - \varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1) - \varphi(s))^{\alpha_1 - 1} u(s) ds|\ |\varphi(\zeta) - \varphi(\zeta_0)| \\
& + \frac{1}{\Gamma(\alpha_1)} | \int_0^\zeta \varphi'(s)(\varphi(\zeta) - \varphi(s))^{\alpha_1 - 1} \mathcal{F}(s, x(s), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(s)) ds \\
& - \int_0^{\zeta_0} \varphi'(s)(\varphi(\zeta_0) - \varphi(s))^{\alpha_1 - 1} \mathcal{F}(s, x(s), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(s)) ds|,
\end{aligned}$$

By substituting the variables with $s = \zeta\tau$, using (45), we obtain the following inequality:

$$
\begin{aligned}
& |\wp x(\zeta) - \wp x(\zeta_0) \\
\leq \; & |\frac{(u_T - u_0)}{\varphi(T_1) - \varphi(0)}(\varphi(\zeta) - \varphi(\zeta_0)) \\
& - \frac{1}{(\varphi(T_1) - \varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1) - \varphi(s))^{\alpha_1 - 1} u(s) ds| \, |\varphi(\zeta) - \varphi(\zeta_0)| \\
& + |\frac{\varphi(\zeta)^{\alpha_1 - r}}{\Gamma(\alpha_1)} \int_0^1 \varphi'(\tau)(1 - \varphi(\tau))^{\alpha_1 - 1} \varphi(\tau)^{-r} \mathfrak{g}(\zeta\tau, x(\zeta\tau), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta\tau)) d\tau \\
& - \frac{\varphi(\zeta_0)^{\alpha_1 - r}}{\Gamma(\alpha_1)} \int_0^1 \varphi'(\tau)(1 - \varphi(\tau))^{\alpha_1 - 1} \varphi(\tau)^{-r} \mathfrak{g}(\zeta_0\tau, x(\zeta_0\tau), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta_0\tau)) d\tau|,
\end{aligned}
$$

$$
\begin{aligned}
\leq \; & |\frac{(u_T - u_0)}{\varphi(T_1) - \varphi(0)}(\varphi(\zeta) - \varphi(\zeta_0)) \\
& - \frac{1}{(\varphi(T_1) - \varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1) - \varphi(s))^{\alpha_1 - 1} u(s) ds| \, |\varphi(\zeta) - \varphi(\zeta_0)| \\
& + \frac{|\varphi(\zeta)^{\alpha_1 - r} - \varphi(\zeta_0)^{\alpha_1 - r}|}{\Gamma(\alpha_1)} \int_0^1 \varphi'(\tau)(1 - \varphi(\tau))^{\alpha_1 - 1} \varphi(\tau)^{-r} \mathfrak{g}(\zeta\tau, x(\zeta\tau), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta\tau)) d\tau \\
& + \frac{\varphi(\zeta_0)^{\alpha_1 - r}}{\Gamma(\alpha_1)} \int_0^1 \varphi'(\tau)(1 - \varphi(\tau))^{\alpha_1 - 1} \varphi(\tau)^{-r} \left| \begin{array}{c} \mathfrak{g}(\zeta\tau, x(\zeta\tau), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta\tau)) \\ -\mathfrak{g}(\zeta_0\tau, x(\zeta_0\tau), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta_0\tau)) \end{array} \right| d\tau.
\end{aligned}
$$

Thus, from the continuity of both $\varphi(\zeta)^{\alpha_1 - r}$ and $\mathfrak{g}$, we deduce that $\wp x(\zeta)$ is continuous on $[0, T_1]$, i.e., $\wp x(\zeta) \in C[0, T_1]$.

In addition, if $x(\zeta), y(\zeta) \in C[0, T_1]$, we have

$$
\begin{aligned}
& |\wp x(\zeta) - \wp y(\zeta)| \\
= \; & |\mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta, s) u(s) ds - \mathfrak{h}_1(\zeta, y(\zeta)) - \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta, s) v(s) ds|, \\
\leq \; & |\mathfrak{h}_1(\zeta, y(\zeta)) - \mathfrak{h}_1(\zeta, x(\zeta))| + \int_0^{T_1} \varphi'(s)|\mathcal{G}_1(\zeta, s)| \, |u(s) - v(s)| ds,
\end{aligned}
$$

$$
\begin{aligned}
\leq \; & \frac{1}{(\varphi(T_1) - \varphi(0))\Gamma(\alpha_1)} | \int_0^{T_1} \varphi'(s)(\varphi(T_1) - \varphi(s))^{\alpha_1 - 1} v(s) ds \\
& - \int_0^{T_1} \varphi'(s)(\varphi(T_1) - \varphi(s))^{\alpha_1 - 1} u(s) ds| \\
& + \int_0^{T_1} \varphi'(s)|\mathcal{G}_1(\zeta, s)| \, |u(s) - v(s)| ds,
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\leq \; & \frac{1}{(\varphi(T_1) - \varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1) - \varphi(s))^{\alpha_1 - 1} |v(s) - u(s)| ds \\
& + \int_0^{T_1} \varphi'(s)|\mathcal{G}_1(\zeta, s)| \, |u(s) - v(s)| ds.
\end{aligned}
$$

However, from assumption (A_1) and if we take the supremum for $\zeta \in [0, T_1]$, we get

$$\begin{aligned}
|u(s)-v(s)| &= |\mathcal{F}(\zeta, x(\zeta), {}^{c}\mathfrak{D}_{0^+}^{\beta_1,\varphi}x(\zeta)) - \mathcal{F}(\zeta, y(\zeta), {}^{c}\mathfrak{D}_{0^+}^{\beta_1,\varphi}y(\zeta))|, \\
&= |\mathcal{F}(\zeta, x(\zeta), \mathfrak{I}_{0^+}^{\alpha_1-\beta_1,\varphi}u(\zeta)) - \mathcal{F}(\zeta, y(\zeta), \mathfrak{I}_{0^+}^{\alpha_1-\beta_1,\varphi}v(\zeta))|, \\
&\leq \mu(\zeta)\left(|x(\zeta)-y(\zeta)| + \int_0^{\zeta}\varphi'(s)\frac{(\varphi(\zeta)-\varphi(s))^{\alpha_1-\beta_1-1}}{\Gamma(\alpha_1-\beta_1)}|u(s)-v(s)|ds\right), \\
&\leq \|\mu\|\left(\|x-y\| + \frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\|u-v\|\right).
\end{aligned} \tag{48}$$

Hence,

$$\|u-v\| \leq \frac{\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)}\|x-y\|. \tag{49}$$

Thus,

$$|\wp x(\zeta) - \wp y(\zeta)| \leq \frac{\left(\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-1}}{\Gamma(\alpha_1+1)} + \mathcal{G}_0(\varphi(T_1)-\varphi(0))\right)\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)}\|x-y\|. \tag{50}$$

Since $\frac{\left(\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-1}}{\Gamma(\alpha_1+1)} + \mathcal{G}_0(\varphi(T_1)-\varphi(0))\right)\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)} < 1$, from Banach's contraction principle, we obtain that operator $\wp$ has unique fixed point $x_1(\zeta) \in C[0, T_1]$ such that $x_1(0) = u_0$, and $x_1(T_1) = u_T$. Therefore, $x_1(\zeta)$ is a unique solution of Equation (29) with the boundary conditions $x_1(\zeta)|_{\zeta=0} = u_0$ and $x_1(\zeta)|_{\zeta=T_1} = u_T$.

In addition, Equation (27) in the interval $(T_1, T_2]$ can be written as Equation (30). So, in order to consider the existence result of the solution to Equation (30), we have to discuss its existence in the $(0, T_2]$.

Consider the following equation

$$\int_0^{\zeta}\varphi'(s)\frac{(\varphi(\zeta)-\varphi(s))^{1-\alpha_2}}{\Gamma(2-\alpha_2)}x_{\varphi}^{[2]}ds = \mathcal{F}(\zeta, x(\zeta), {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\zeta)),\ \zeta \in (0, T_2]. \tag{51}$$

It is clear that if $x \in C[0, T_2]$ satisfies Equation (51), then it also satisfies Equation (30). Hence, let $x^* \in C[0, T_2]$ be a solution of Equation (51) such that $x^*(\zeta)|_{\zeta=0} = u_0$, $x_2(T_1) = x_1(T_1)$, and $x^*(\zeta)|_{\zeta=T_2} = u_T$. That is,

$$\int_0^{\zeta}\varphi'(s)\frac{(\varphi(\zeta)-\varphi(s))^{1-\alpha_2}}{\Gamma(2-\alpha_2)}x^{*[2]}_{\varphi}(s)ds = \mathcal{F}(\zeta, x^*(\zeta), {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}x^*(\zeta)),\ \zeta \in (T_1, T_2]. \tag{52}$$

Hence, we deduce that if $x^* \in C[0, T_2]$ is a solution of Equation (51), then $x^* \in C(T_1, T_2]$ is also a solution of Equation (30).

Based on this result, we will consider the existence of the solution of Equation (51) instead of Equation (30).

In a similar manner as above, if we take the operator $\mathfrak{I}_{0^+}^{\alpha_2}$ on both sides of Equation (51) and use Lemma 4, we obtain for $0 \leq \zeta \leq T_2$ that

$$x(\zeta) = c_0 + c_1(\varphi(\zeta)-\varphi(0)) + \frac{1}{\Gamma(\alpha_2)}\int_0^{\zeta}\varphi'(s)(\varphi(\zeta)-\varphi(s))^{\alpha_2-1}\mathcal{F}(\zeta, x(\zeta), {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\zeta))ds. \tag{53}$$

Substituting the boundary conditions $x(\zeta)|_{\zeta=0} = u_0$ and $x(\zeta)|_{\zeta=T_2} = u_T$, we obtain $c_0 = u_0$, and $c_1 = \frac{(u_T-u_0)}{(\varphi(T_2)-\varphi(0))} - \frac{1}{\Gamma(\alpha_2)(\varphi(T_2)-\varphi(0))}\int_0^{T_2}\varphi'(s)[\varphi(T_2)-\varphi(s)]^{\alpha_2-1}u(s)ds$. Hence, the solution of Equation (51) can be written as:

$$\begin{aligned} x(\zeta) &= u_0 + (u_T - u_0)\frac{(\varphi(\zeta)-\varphi(0))}{(\varphi(T_2)-\varphi(0))} \\ &\quad - \frac{\zeta}{(\varphi(T_2)-\varphi(0))\Gamma(\alpha_2)}\int_0^{T_2}\varphi'(s)[\varphi(T_2)-\varphi(s)]^{\alpha_2-1}u(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha_2)}\int_0^{\zeta}\varphi'(s)(\varphi(\zeta)-\varphi(s))^{\alpha_2-1}u(s)ds, \\ &= \mathfrak{h}_2(\zeta, x(\zeta)) + \int_0^{T_2}\varphi'(s)\mathcal{G}_2(\zeta,s)u(s)ds. \end{aligned} \tag{54}$$

Define operator $\wp : C[0,T_2] \to C[0,T_2]$ by

$$\wp x(\zeta) = \mathfrak{h}_2(\zeta, x(\zeta)) + \int_0^{T_2}\varphi'(s)\mathcal{G}_2(\zeta,s)u(s)ds, 0 \le \zeta \le T_2. \tag{55}$$

In a similar argument as above, it follows from the continuity of the function

$$\mathfrak{g}(\zeta,\ x(\zeta),\ {}^c\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\zeta)) = \varphi(\zeta)^r\mathcal{F}(\zeta,\ x(\zeta),\ {}^c\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\zeta)), \tag{56}$$

that the operator $\wp : C[0,T_2] \to C[0,T_2]$ is continuous and well defined. In addition, if $u(\zeta), v(\zeta) \in C[0,T_2]$, we have

$$\begin{aligned} |\wp u(\zeta) - \wp v(\zeta)| &= |\mathfrak{h}_2(\zeta, y(\zeta)) - \mathfrak{h}_2(\zeta, x(\zeta))| \\ &\quad + \left|\int_0^{T_2}\varphi'(s)\mathcal{G}_2(\zeta,s)u(s)ds - \int_0^{T_2}\varphi'(s)\mathcal{G}_2(\zeta,s)v(s)ds\right|, \end{aligned} \tag{57}$$

$$\begin{aligned} &\le \frac{1}{(\varphi(T_2)-\varphi(0))\Gamma(\alpha_2)}\int_0^{T_2}\varphi'(s)[\varphi(T_2)-\varphi(s)]^{\alpha_2-1}|v(s)-u(s)|ds \\ &\quad + \left|\int_0^{\zeta}\varphi'(s)\frac{(\varphi(\zeta)-\varphi(s))^{\alpha_2-\beta_2-1}}{\Gamma(\alpha_2-\beta_2)}|u(s)-v(s)|ds\right|, \\ &\le \frac{\left(\frac{(\varphi(T_2)-\varphi(0))^{\alpha_2-1}}{\Gamma(\alpha_2+1)} + \mathcal{G}_0(\varphi(T_2)-\varphi(0))\right)\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_2)-\varphi(0))^{\alpha_2-\beta_2}}{\Gamma(\alpha_2-\beta_2+1)}\right)}\|u-v\|. \end{aligned}$$

Hence, from assumption (A_3), we obtain that Banach's contraction principle assures that the operator $\wp$ has unique fixed point $x_2(\zeta) \in C[0,T_2]$ such that $x_2(0) = u_0$, $x_2(T_1) = x_1(T_1)$ and $x_2(T_2) = u_T$. Therefore, $x_2(\zeta)$ is a unique solution of Equation (30) with the boundary conditions $x_2(\zeta)|_{\zeta=0} = u_0$ and $x_2(\zeta)|_{\zeta=T_2} = u_T$.

In a similar manner, we can prove that Equation (27) defined on $(T_{i-1}, T_i]$, for all $i = 3, 4, \ldots, \mathbb{N}^*$ with $T_{\mathbb{N}^*} = T$, has one unique solution $x_i(\zeta) \in C[0,T_i]$ such that $x_i(\zeta)|_{\zeta=0} = u_0$, $x_i(T_{i-1}) = x_{i-1}(T_{i-1})$, and $x_i(\zeta)|_{\zeta=T_i} = u_T$.

Therefore, we already proved that the VOFBVP (8) has one unique solution. Thus, the proof is completed. □

Now, we present our second existence result for the VOFBVP (8), which is based on Krasnoselskii's fixed point theorem.

Definition 9. *From a mild solution of the VOFBVP (8), we refer to a function $u \in C(\mathcal{I}_1, \mathcal{R})$, $\mathcal{I}_1 = [0, T_1]$, that satisfies the integral Equation (33), with u the solution of the following integral equation:*

$$u(\zeta) = \mathcal{F}(\zeta,\ \mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta, s)u(s)ds,\ \mathfrak{I}_{0^+}^{\alpha_1-\beta_1,\varphi}u(\zeta)), \tag{58}$$

for all $\zeta \in [0, T_1]$.

Lemma 6. *Let $x(\zeta), y(\zeta) \in C[0, T_1]$, and let $1 < \beta_1 < 2$, then*

$$|{}^{c}\mathfrak{D}_{0^+}^{\beta_1,\varphi}y(\zeta) - {}^{c}\mathfrak{D}_{0^+}^{\beta_1,\varphi}x(\zeta)| \leq \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1-\beta_1-2}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1 - 1)|}\|y - x\|, \tag{59}$$

where

$$\Delta = \left(1 + \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1 - \beta_1 + 1)}\left(1 + \frac{\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)}\right)\right). \tag{60}$$

Proof. It is clear that if we take supremum for all $\zeta \in [0, T_1]$ that

$$\begin{aligned} |{}^{c}\mathfrak{D}_{0^+}^{\beta_1,\varphi}y(\zeta) - {}^{c}\mathfrak{D}_{0^+}^{\beta_1,\varphi}x(\zeta)| &= \Big|\int_0^{\zeta} \varphi'(s)\frac{(\varphi(\zeta) - \varphi(s))^{1-\beta_1}}{\Gamma(2-\beta_1)} y_{\varphi}^{[2]}(s)ds \\ &\quad - \int_0^{\zeta} \varphi'(s)\frac{(\varphi(\zeta) - \varphi(s))^{1-\beta_1}}{\Gamma(2-\beta_1)} x_{\varphi}^{[2]}(s)ds\Big| \\ &\leq \int_0^{\zeta} \varphi'(s)\frac{(\varphi(\zeta) - \varphi(s))^{1-\beta_1}}{\Gamma(2-\beta_1)}\left|y_{\varphi}^{[2]}(s) - x_{\varphi}^{[2]}(s)\right|ds. \end{aligned} \tag{61}$$

However, from [26], we have

$$\begin{aligned} x_{\varphi}^{[n-1]}(s) &= \frac{\left(x_{\varphi}^{[n-2]}(s)\right)'}{\varphi'(s)} \\ &= \frac{1}{\Gamma(\alpha_1 - n + 1)}\int_a^s \varphi'(\tau)(\varphi(s) - \varphi(\tau))^{\alpha_1-n}\mathcal{F}(\tau, x(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\tau))d\tau + x_a^{n-1}. \end{aligned} \tag{62}$$

Since $\mathcal{F}(\tau, x(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\tau))$ is continuous on $\mathcal{I}$, then there exists a constant A such that

$$\left|\frac{1}{\Gamma(\alpha_1 - n + 1)}\int_a^s \varphi'(\tau)(\varphi(s) - \varphi(\tau))^{\alpha_1-n}\mathcal{F}(\tau, x(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\tau))d\tau\right| \leq A\frac{(\varphi(s) - \varphi(\tau))^{\alpha_1-n+1}}{\Gamma(\alpha_1 - n + 2)}, \tag{63}$$

which vanishes at the initial point $t = a$, and thus $x_{\varphi}^{[n-1]}(a) = x_a^{n-1}$.

Thus, we have

$$\begin{aligned} &\left|y_{\varphi}^{[2]}(s) - x_{\varphi}^{[2]}(s)\right| \\ &= \left|\frac{\left(y_{\varphi}^{[1]}(s)\right)'}{\varphi'(s)} - \frac{\left(x_{\varphi}^{[1]}(s)\right)'}{\varphi'(s)}\right|, \\ &= \left|\begin{array}{l} \frac{1}{\Gamma(\alpha_1-2)}\int_0^s \varphi'(\tau)(\varphi(s) - \varphi(\tau))^{\alpha_1-3}\mathcal{F}(\tau, y(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}y(\tau))d\tau + y_0^2 \\ -\frac{1}{\Gamma(\alpha_1-2)}\int_0^s \varphi'(\tau)(\varphi(s) - \varphi(\tau))^{\alpha_1-3}\mathcal{F}(\tau, x(\tau),\ {}^{c}\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\tau))d\tau - x_0^2 \end{array}\right|, \end{aligned} \tag{64}$$

$$\leq \frac{1}{\Gamma(\alpha_1-2)}\int_0^s \varphi'(\tau)(\varphi(s)-\varphi(\tau))^{\alpha_1-3}\left|\begin{array}{c}\mathcal{F}(\tau,y(\tau),\,{}^c\mathfrak{D}_{0^+}^{\beta_2,\varphi}y(\tau))\\ -\mathcal{F}(\tau,x(\tau),\,{}^c\mathfrak{D}_{0^+}^{\beta_2,\varphi}x(\tau))\end{array}\right|d\tau+\left|y_0^2-x_0^2\right|,$$

$$\begin{aligned}\leq\ & \frac{1}{\Gamma(\alpha_1-2)}\int_0^s \varphi'(\tau)(\varphi(s)-\varphi(\tau))^{\alpha_1-3}\|\mu\|\left(\|x-y\|+\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\|u-v\|\right)d\tau\\ & +\left|y_0^2-x_0^2\right|,\end{aligned}$$

$$\begin{aligned}\leq\ & \frac{1}{\Gamma(\alpha_1-2)}\int_0^s \varphi'(\tau)(\varphi(s)-\varphi(\tau))^{\alpha_1-3}\|\mu\|\\ & \times\left(\|x-y\|+\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\left(1+\frac{\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)}\right)\|x-y\|\right)d\tau\\ & +\left|y_0^2-x_0^2\right|,\end{aligned}$$

$$\begin{aligned}\leq\ & \Delta\|x-y\|\frac{\|\mu\|}{\Gamma(\alpha_1-2)}\int_0^s \varphi'(\tau)(\varphi(s)-\varphi(\tau))^{\alpha_1-3}d\tau+\left|y_0^2-x_0^2\right|,\\ \leq\ & \Delta\frac{\|\mu\|(\varphi(s)-\varphi(0))^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\|x-y\|,\end{aligned}$$

where

$$\Delta=\left(1+\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\left(1+\frac{\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)}\right)\right),\tag{65}$$

and $y_0^2-x_0^2=u_0^2-u_0^2=0$.

Thus,

$$\begin{aligned}|{}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi}y(\zeta)-{}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi}x(\zeta)| &= |\int_0^\zeta \varphi'(s)\frac{(\varphi(\zeta)-\varphi(s))^{1-\beta_1}}{\Gamma(2-\beta_1)}y_\varphi^{[2]}(s)ds\\ &\quad-\int_0^\zeta \varphi'(s)\frac{(\varphi(\zeta)-\varphi(s))^{1-\beta_1}}{\Gamma(2-\beta_1)}x_\varphi^{[2]}(s)ds|,\\ &\leq \int_0^\zeta \varphi'(s)\frac{(\varphi(\zeta)-\varphi(s))^{1-\beta_1}}{\Gamma(2-\beta_1)}\left|y_\varphi^{[2]}(s)-x_\varphi^{[2]}(s)\right|ds,\\ &\leq \Delta\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1-2}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|}\|y-x\|.\end{aligned}\tag{66}$$

□

Lemma 7. *The function $\mathfrak{h}_1{:}\mathcal{I}\times\mathcal{R}\to\mathcal{R}$ is a Lipschitzian function with a Lipschitz constant c such that*

$$\|\mathfrak{h}_1(\zeta,x(\zeta))-\mathfrak{h}_1(\zeta,y(\zeta))\|\leq c\ \|x-y\|,\tag{67}$$

with $c=\frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|}$.

Proof. Let $x(\zeta),y(\zeta)\in C[0,T_1]$. By applying assumption (A_1) and taking supremum for all $\zeta\in[0,T_1]$, we get

$$
\begin{aligned}
&|\mathfrak{h}_1(\zeta, x(\zeta)) - \mathfrak{h}_1(\zeta, y(\zeta))| \\
= \;& \Big| \frac{1}{(\varphi(T_1)-\varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1)-\varphi(0))^{\alpha_1-1} v(s)ds \\
&- \frac{1}{(\varphi(T_1)-\varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1)-\varphi(0))^{\alpha_1-1} u(s)ds \Big|, \\
\leq \;& \frac{1}{(\varphi(T_1)-\varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1)-\varphi(0))^{\alpha_1-1} |v(s)-u(s)|ds,
\end{aligned} \tag{68}
$$

$$
\leq \frac{1}{(\varphi(T_1)-\varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1)-\varphi(0))^{\alpha_1-1} \left| \begin{array}{c} \mathcal{F}(\zeta, y(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} y(\zeta)) \\ -\mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta)) \end{array} \right| ds,
$$

$$
\begin{aligned}
\leq \;& \frac{1}{(\varphi(T_1)-\varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1)-\varphi(0))^{\alpha_1-1} |\varphi(\zeta)| \\
& \times \left(\begin{array}{c} |y(\zeta)-x(\zeta)| \\ +|{}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} y(\zeta) - {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta)| \end{array} \right) ds,
\end{aligned}
$$

$$
\leq \frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|} \|x-y\|.
$$

Thus

$$
\|\mathfrak{h}_1(\zeta, x) - \mathfrak{h}_1(\zeta, y)\| \leq c\, \|x-y\|, \tag{69}
$$

where $c = \frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|}$. □

Theorem 3. *Suppose that assumption* (A_1) *holds. If*

$$
\frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|} < 1, \tag{70}
$$

then the VOFBVP (8) has at least one mild solution in $C[0, T_1]$.

Proof. By converting the VOFBVP (8) into a fixed point problem, define the operator $\wp : C(\mathcal{I}_1, \mathcal{R}) \to C(\mathcal{I}_1, \mathcal{R})$ with $\mathcal{I}_1 = [0, T_1]$ as:

$$
\wp x(\zeta) = \mathfrak{h}_1(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta, s)u(s)ds,\ \zeta \in [0, T_1], \tag{71}
$$

with

$$
u(\zeta) = \mathcal{F}(\zeta,\ x(\zeta), \mathfrak{I}_{0^+}^{\alpha_1-\beta_1,\varphi} u(\zeta)). \tag{72}
$$

In addition, let $B_{\varrho_1} = \{x \in C(\mathcal{I}_1, \mathcal{R}) : \|x\| \leq \varrho_1\}$ be a closed subset of $C[0, T_1]$, where ϱ_1 is a positive constant satisfying

$$
\varrho_1 \geq \max_{\zeta\in[0,T_1]} \left(\frac{\Im + \Re\mathfrak{F}}{1-\Re\aleph}, \frac{\mathfrak{F}\mathfrak{M}}{1-\mathfrak{M}\aleph} \right), \tag{73}
$$

where

$$
\Im = 2u_0 + u_T, \tag{74}
$$

$$
\aleph = \|\mu\| \left(1 + \Delta \frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1-2}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|} \right), \tag{75}
$$

$$\Re = \left(\frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - 1}}{\Gamma(\alpha_1 + 1)} + \mathcal{G}_0(\varphi(T_1) - \varphi(0)) \right), \tag{76}$$

and

$$\mathfrak{M} = \mathcal{G}_0(\varphi(T_1) - \varphi(0)). \tag{77}$$

Obviously, $B_{\varrho_1}$is a Banach space with a metric in $C[0, T_1]$.

Now, consider the operators $\wp_1$ and $\wp_2$ defined on B_{ϱ_1} as follows:

$$\wp_1 x(\zeta) = \mathfrak{h}_1(\zeta, x(\zeta)), \tag{78}$$

$$\wp_2 x(\zeta) = \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta, s)u(s)ds. \tag{79}$$

Then, for any $x \in C([0, T_1], \mathcal{R})$, we have

$$\wp x(\zeta) = \wp_1 x(\zeta) + \wp_2 x(\zeta), \quad \zeta \in [0, T_1]. \tag{80}$$

The proof is divided into three steps as follows:

Step 1: $\wp_1 x_1 + \wp_2 x_2 \in B_{\varrho_1}$ for every $x_1, x_2 \in B_{\varrho_1}$.

Let $x_1, x_2 \in B_{\varrho_1}$ and $\zeta \in \mathcal{I}$, and we have

$$\begin{aligned} &|\wp_1 x_1(\zeta) + \wp_2 x_2(\zeta)| \\ &\leq |\wp_1 x_1(\zeta)| + |\wp_2 x_2(\zeta)|, \\ &\leq |\mathfrak{h}_1(\zeta, x_1(\zeta))| + \int_0^{T_1} \varphi'(s)|\mathcal{G}_1(\zeta, s)||\mathcal{F}(\zeta,\ x(\zeta),\ {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta))|ds, \end{aligned} \tag{81}$$

$$\begin{aligned} &\leq |u_0| + |u_T - u_0| \frac{(\varphi(\zeta) - \varphi(0))}{\varphi(T_1) - \varphi(0)} \\ &+ \frac{1}{(\varphi(T_1) - \varphi(0))\Gamma(\alpha_1)} \int_0^{T_1} \varphi'(s)(\varphi(T_1) - \varphi(s))^{\alpha_1 - 1}|u(s)|ds \\ &+ \int_0^{T_1} \varphi'(s)|\mathcal{G}_1(\zeta, s)||u(s)|ds. \end{aligned}$$

By applying Lemma 6 and taking the supremum for $\zeta \in [0, T_1]$, we have

$$\begin{aligned} |u(\zeta)| &= |\mathcal{F}(\zeta,\ x(\zeta),\ {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta))|, \\ &\leq \|\mu\|(|x(\zeta)| + |{}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta))|) + \mathfrak{F},\ \text{where } \mathfrak{F} = \sup_{\zeta \in \mathcal{I}} |\mathcal{F}(\zeta, 0, 0)|, \\ &\leq \|\mu\| \left(1 + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2 - \beta_1)\Gamma(\alpha_1 - 1)|} \right) \|x\| + \mathfrak{F}. \end{aligned} \tag{82}$$

Thus, for each $\zeta \in [0, T_1]$

$$\begin{aligned} |\wp_1 x_1(\zeta) + \wp_2 x_2(\zeta)| \quad \leq \quad & 2u_0 + u_T \\ & + \left(\frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - 1}}{\Gamma(\alpha_1 + 1)} + \mathcal{G}_0(\varphi(T_1) - \varphi(0)) \right) \\ & \times \left(\|\mu\| \left(1 + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2 - \beta_1)\Gamma(\alpha_1 - 1)|} \right) \|x\| + \mathfrak{F} \right). \end{aligned} \tag{83}$$

Taking the supremum over $\zeta \in [0, T_1]$, we get

$$\|\wp_1 x_1 + \wp_2 x_2\| \leq\ \varrho_1, \tag{84}$$

for
$$\varrho_1 \geq \frac{\Im + \Re\mathfrak{F}}{1 - \Re\aleph}, \tag{85}$$
where
$$\Im = 2u_0 + u_T,$$
$$\aleph = \|\mu\| \left(1 + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2 - \beta_1)\Gamma(\alpha_1 - 1)|} \right), \tag{86}$$
and
$$\Re = \left(\frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - 1}}{\Gamma(\alpha_1 + 1)} + \mathcal{G}_0(\varphi(T_1) - \varphi(0)) \right). \tag{87}$$
This proves that $\wp_1 x_1 + \wp_2 x_2 \in B_{\varrho_1}$ for every $x_1, x_2 \in B_{\varrho_1}$.
Step 2: The operator $\wp_1$ is a contraction mapping on B_{ϱ_1}.
It is clear from Lemma 7 that $\wp_1$ is a contraction mapping with constant
$$c = \frac{\Delta \|\mu\| (\varphi(T_1) - \varphi(0))^{2\alpha_1 - \beta_1 - 3}}{|\Gamma(\alpha_1 + 1)\Gamma(2 - \beta_1)\Gamma(\alpha_1 - 1)|} < 1. \tag{88}$$
Step 3: The operator $\wp_2$ is completely continuous (compact and continuous) on B_{ϱ_1}.
First, we prove that operator $\wp_2$ is continuous.
Let $\{x_\mathfrak{n}\}_{\mathfrak{n} \in \mathbb{N}}$ be a sequence such that $x_\mathfrak{n} \to x$ as $\mathfrak{n} \to \infty$ in $C([0, T_1], \mathcal{R})$.
To show that $\wp_2$ is continuous, we have to prove that
$$\|\wp_2 x_\mathfrak{n} - \wp_2 x\| \to 0 \ \text{ as } \ \mathfrak{n} \to \infty. \tag{89}$$
For each $\zeta \in [0, T_1]$, we have
$$|\wp_2 x_\mathfrak{n} - \wp_2 x| \leq \int_0^{T_1} \varphi'(s) |\mathcal{G}_1(\zeta, s)| |u_\mathfrak{n}(s) - u(s)| ds, \tag{90}$$
where $u_\mathfrak{n}, u \in C([0, T_1], \mathcal{R})$, such that
$$u_\mathfrak{n}(\zeta) = \mathcal{F}(\zeta,\ x_\mathfrak{n}(\zeta),\ {}^c\mathfrak{D}_{0^+}^{1,\varphi} x_\mathfrak{n}(\zeta)) = \mathcal{F}(\zeta, x_\mathfrak{n}(\zeta), \mathfrak{I}^{\alpha_1 - \beta_1, \varphi}\ u_\mathfrak{n}(\zeta)), \tag{91}$$
$$u(\zeta) = \mathcal{F}(\zeta,\ x(\zeta),\ {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta)) = \mathcal{F}(\zeta, x(\zeta), \mathfrak{I}^{\alpha_1 - \beta_1, \varphi}\ u(\zeta)), \tag{92}$$
and by assumption (A_1), we obtain that
$$\begin{aligned} |u_\mathfrak{n}(\zeta) - u(\zeta)| &= |\mathcal{F}(\zeta, x_\mathfrak{n}(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x_\mathfrak{n}(\zeta)) - \mathcal{F}(\zeta, x(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta))|, \\ &\leq |\varphi(\zeta)| (|x_\mathfrak{n}(\zeta) - x(\zeta)| + |\ {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x_\mathfrak{n}(\zeta) -\ {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} x(\zeta)|), \end{aligned} \tag{93}$$
$$\leq \|\mu\| (\|x_\mathfrak{n} - x\| + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2 - \beta_1)\Gamma(\alpha_1 - 1)|} \|x_\mathfrak{n} - x\|),$$
$$\leq \|\mu\| (1 + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2 - \beta_1)\Gamma(\alpha_1 - 1)|}) \|x_\mathfrak{n} - x\|.$$
Thus, if we take the supremum for $\zeta \in [0, T_1]$, we get
$$\|u_\mathfrak{n} - u\| \leq \|\mu\| (1 + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2 - \beta_1)\Gamma(\alpha_1 - 1)|}) \|x_\mathfrak{n} - x\|. \tag{94}$$

Since $x_{\mathfrak{n}} \to x$, then we obtain $u_{\mathfrak{n}}(\zeta) \to u(\zeta)$ as $\mathfrak{n} \to \infty$ for each $\zeta \in [0, T_1]$. In addition, if we select a real number $\varepsilon > 0$ such that for each $\zeta \in [0, T_1]$, we have $|u_{\mathfrak{n}}(\zeta)| \leq \varepsilon/2$ and $|u(\zeta)| \leq \varepsilon/2$, then we can conclude that

$$\begin{aligned} |\mathcal{G}_1(\zeta,s)||u_{\mathfrak{n}}(s) - u(s)| &\leq |\mathcal{G}_1(\zeta,s)|(|u_{\mathfrak{n}}(s)| + |u(s)|), \\ &\leq \varepsilon|\mathcal{G}_1(\zeta,s)|. \end{aligned} \tag{95}$$

Now, for each $\zeta \in [0, T_1]$, the function $s \to \varepsilon|\mathcal{G}_1(\zeta,s)|$ is integrable on $\mathcal{I}$. Then, by applying the Lebesgue dominated convergence theorem, it is obtained that

$$\|\wp_2 x_{\mathfrak{n}} - \wp_2 x\| \to 0 \ \ as \ \ \mathfrak{n} \to \infty. \tag{96}$$

Consequently, $\wp_2$ is continuous.
In addition, we have

$$\|\wp_2 x\| \leq \mathcal{G}_0(\varphi(T_1) - \varphi(0)) \left[\|\mu\| \left(1 + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1 - 1)|} \right) \|x\| + \mathfrak{F} \right] \leq \varrho_1. \tag{97}$$

Thus, the proof that $\wp_2$ is uniformly bounded on B_{ϱ_1} is complete.

Finally, we prove that $\wp_2$ maps bounded sets into equicontinuous sets of $C(\mathcal{I}, \mathcal{R})$, i.e., B_{ϱ_1} is equicontinuous.

Suppose that for any real number $\epsilon > 0$, there exists $\delta > 0$ such that for every ζ_1 and ζ_2 in the interval $\mathcal{I}$, where $\zeta_1 < \zeta_2$, we have $|\zeta_2 - \zeta_1| < \delta$. Then, we can conclude that

$$\begin{aligned} |\wp_2 x(\zeta_2) - \wp_2 x(\zeta_1)| \ &\leq \ \int_0^{T_1} \varphi'(s)|\mathcal{G}_1(\zeta_2,s) - \mathcal{G}_1(\zeta_1,s)| \, |u(s)| ds, \\ &\leq \ \|u\| \int_0^{T_1} \varphi'(s)|\mathcal{G}_1(\zeta_2,s) - \mathcal{G}_1(\zeta_1,s)| \, ds, \\ &\leq \ \left[\|\mu\| \left(1 + \Delta \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1 - 2}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1 - 1)|} \right) + \mathfrak{F} \right] \\ &\qquad \times \int_0^{T_1} \varphi'(s)|\mathcal{G}(\zeta_2,s) - \mathcal{G}(\zeta_1,s)| \, ds. \end{aligned} \tag{98}$$

As ζ_1 approaches ζ_2, the right-hand side of the inequality above becomes independent of x and tends toward zero. Consequently,

$$|\wp_2 x(\zeta_2) - \wp_2 x(\zeta_1)| \to 0, \ \ \forall \, |\zeta_2 - \zeta_1| \to 0. \tag{99}$$

Thus, the set $\{\wp x\}$ is equicontinuous on $B\varrho_1$, and the operator $\wp$ is compact, as established by the Arzela-Ascoli Theorem [2]. Consequently, we can conclude that $\wp$: $C([0, T_1], \mathcal{R}) \to C([0, T_1], \mathcal{R})$ is both continuous and compact. Hence, all the hypotheses of Krasnoselskii's fixed point theorem are satisfied. This implies that the operator $\wp = \wp_1 + \wp_2$ possesses a fixed point $x_1(\zeta)$ within the set $C[0, T_1]$ on B_{ϱ_1}, with the conditions $x_1(0) = u_0$ and $x_1(T_1) = u_T$. Therefore, $x_1(\zeta)$ is a mild solution of Equation (29) with the boundary conditions $x_1(\zeta)|_{\zeta=0} = u_0$ and $x_1(\zeta)|_{\zeta=T_1} = u_T$.

Now, if we make the same argument as in Theorem 2, we have Equation (27) in the interval $(T_1, T_2]$, which is equivalent to Equation (30). So, considering the existence results of the solution for Equation (30) is equivalent to discussing its existence in the $(0, T_2]$. In addition, it is clear that if $x_2 \in C[0, T_2]$ satisfies Equation (51), then it also satisfies Equation (30) such that $x_2(\zeta)|_{\zeta=0} = u_0$, $x_2(T_1) = x_1(T_1)$, and $x_2(\zeta)|_{\zeta=T_2} = u_T$.

In a similar manner, and for $i = 3, \ldots, N^*$, we can conclude that Equation (30) defined on $(T_i - 1, T_i]$ has at least one mild solution $x_i(\zeta) \in C[0, T_i]$ with $x_i(\zeta)|_{\zeta=0} = u_0$, $x_i(T_{i-1}) = x_{i-1}(T_{i-1})$, and $x_i(\zeta)|_{\zeta=Ti} = u_T$ $(T_{\mathbb{N}^*} = T)$. Therefore, by applying Krasnoselskii's fixed point theorem, we deduce that the VOFBVP (8) has at least one mild solution in $C[0,T]$. The proof is completed. □

3.2. Ulam-Hyers Stability of Solutions

In the following analysis, we examine the stability of the VOFBVP (8) according to the Ulam-Hyers criteria. Let $\varepsilon > 0$, $\Phi : \mathcal{I} \to \mathcal{R}^+$ be a continuous function, and consider the following inequalities:

$$\left| {}^c\mathfrak{D}_{0^+}^{\alpha(\zeta),\varphi} y(\zeta) - \mathcal{F}(\zeta, y(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta(\zeta),\varphi} y(\zeta)) \right| \leq \varepsilon \text{ for every } \zeta \in \mathcal{I}, \tag{100}$$

and

$$\left| {}^c\mathfrak{D}_{0^+}^{\alpha(\zeta),\varphi} y(\zeta) - \mathcal{F}(\zeta, y(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta(\zeta),\varphi} y(\zeta)) \right| \leq \Phi(\zeta) \text{ for every } \zeta \in \mathcal{I}. \tag{101}$$

Definition 10 ([35]). *The VOFBVP (8) is Ulam-Hyers stable if there exists a real number $c_{\mathcal{F}} > 0$ so that there exists a solution $x \in C(\mathcal{I},\mathcal{R})$ of (8) such that $|y(\zeta) - x(\zeta)| \leq \varepsilon\ c_{\mathcal{F}}$ for every $\zeta \in \mathcal{I}$ and $y \in C(\mathcal{I},\mathcal{R})$ is a solution of the inequality (100).*

Definition 11 ([35]). *The VOFBVP (8) is said to be generalized Ulam-Hyers stable if there is $c_{\mathcal{F}} \in C(\mathcal{R}^+,\mathcal{R}^+)$ with $c_{\mathcal{F}}(0) = 0$ so that there exists a solution $x \in C(\mathcal{I},\mathcal{R})$ of the VOFBVP (8) such that $|y(\zeta) - x(\zeta)| \leq c_{\mathcal{F}}(\varepsilon)$ for every $\zeta \in \mathcal{I}$, $\epsilon > 0$, and $y \in C(\mathcal{I},\mathcal{R})$ is a solution of the inequality (101).*

Theorem 4. *Suppose that the assumptions of Theorem 3 are satisfied, then the VOFBVP (8) is Ulam−Hyers stable.*

Proof. Let $\varepsilon > 0$ and let $z \in C([0,T_1],\mathcal{R})$ be a function which satisfies the inequality (100), i.e.,

$$|{}^c\mathfrak{D}_{0^+}^{\alpha_1,\varphi} z(\zeta) - \mathcal{F}(\zeta, z(\zeta), {}^c\mathfrak{D}_{0^+}^{\beta_1,\varphi} z(\zeta))| \leq \varepsilon \text{ for every } \zeta \in [0,T_1], \tag{102}$$

and let $y \in C(I,\mathcal{R})$ be a solution of the VOFBVP (8) which by Lemma 5 is equivalent to the fractional order integral equation

$$y(\zeta) = \mathfrak{h}_1(\zeta, y(\zeta)) + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta,s)u(s)ds, \tag{103}$$

where u is the solution of the fractional order integral equation

$$u(\zeta) = \mathcal{F}(\zeta, \mathfrak{h}(\zeta, x(\zeta)) + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta,s)u(s)ds, \mathfrak{I}^{\alpha_1-\beta_1,\varphi}u(\zeta)). \tag{104}$$

Taking the left-sided φ-Riemann-Liouville fractional integral $\mathfrak{I}_{0^+}^{\alpha_1,\varphi}$ on both sides of inequality (102), we get

$$|z(\zeta) - \mathfrak{h}_1(\zeta, z(\zeta)) - \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta,s)v(s)ds| \leq \varepsilon \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}. \tag{105}$$

For each $\zeta \in [0,T_1]$ and from Lemma 2, we have

$$\begin{aligned} |z(\zeta) - y(\zeta)| &= |z(\zeta) - \mathfrak{h}_1(\zeta, y(\zeta)) - \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta,s)v(s)ds|, \\ &\leq |z(\zeta) - \mathfrak{h}_1(\zeta, z(\zeta)) - \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta,s)v(s)ds| \\ &\quad -\mathfrak{h}_1(\zeta, y(\zeta)) - \int_0^{T_1} \varphi'(\zeta)\mathcal{G}_1(\zeta,s)u(s)ds + \mathfrak{h}_1(\zeta, z(\zeta)) \\ &\quad + \int_0^{T_1} \varphi'(s)\mathcal{G}_1(\zeta,s)v(s)ds|, \end{aligned} \tag{106}$$

$$\leq \varepsilon\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + |\mathfrak{h}_1(\zeta,z(\zeta)) - \mathfrak{h}_1(\zeta,y(\zeta))| + \int_0^{T_1}\varphi'(\zeta)|\mathcal{G}_1(\zeta,s)|\,|v(s)-u(s)|ds,$$

$$\leq \frac{\varepsilon\,(\varphi(T_1)-\varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + \frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|}\|z-y\| + \mathcal{G}_0\|\mu\|(1+\Delta\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1-1}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|})\|z-y\|,$$

which implies that for each $\zeta \in [0,T_1]$

$$\|z-y\| \leq \frac{\varepsilon\,(\varphi(T_1)-\varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)} + \left[\begin{array}{c} \frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|} \\ +\mathcal{G}_0\|\mu\|(1+\Delta\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1-1}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|}) \end{array}\right]\|z-y\|. \tag{107}$$

Hence,

$$\|z-y\| \leq \varepsilon\,\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}\left[1-\left[\begin{array}{c} \frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|} \\ +\mathcal{G}_0\|\mu\|(1+\Delta\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1-1}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|}) \end{array}\right]\right]^{-1} = \varsigma\,\epsilon, \tag{108}$$

where

$$\varsigma = \frac{(\varphi(T_1)-\varphi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}\left[1-\left[\begin{array}{c} \frac{\Delta\|\mu\|(\varphi(T_1)-\varphi(0))^{2\alpha_1-\beta_1-3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|} \\ +\mathcal{G}_0\|\mu\|(1+\Delta\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1-1}}{|\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|}) \end{array}\right]\right]^{-1}. \tag{109}$$

Thus, the VOFBVP (8) is Ulam-Hyers stable for all $\zeta \in [0,T_1]$.

In a similar manner as Theorem 2, we obtain that Equation (27) in the interval $(T_1,T_2]$ is equivalent to Equation (30). So, considering the stability results of solutions for Equation (30) is equivalent to discussing stability results in the $(0,T_2]$. Hence, for $i = 3,\ldots,\mathbb{N}^*$, we obtain that Equation (30) defined on $(T_{i-1},T_i]$ is Ulam-Hyers stable on $[0,T_i]$ with $x_i(\zeta)|_{\zeta=0} = u_0$, $x_i(T_{i-1}) = x_{i-1}(T_{i-1})$, and $x_i(\zeta)|_{\zeta=Ti} = u_T$ $(T_{\mathbb{N}^*} = T)$. Therefore, the VOFBVP (8) is Ulam-Hyers stable on the whole interval $[0,T]$. □

Remark 5. *Indeed, if we put $\Phi(\varepsilon) = \varsigma\,\varepsilon$, then we get $\Phi(0) = 0$, which yields that the VOFBVP (8) is generalized Ulam-Hyers stable.*

4. Numerical Example

In the following, we present two numerical examples illustrating the obtained results and further enhance the comprehension of the theoretical findings.

Example 1. *Given the following VOFBVP:*

$$\begin{cases} {}^{c}\mathfrak{D}^{\alpha(\zeta),\zeta^2}y(\zeta) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}\left[\frac{5+y(\zeta)+\,{}^{c}\mathfrak{D}^{\beta(\zeta),\zeta^2}y(\zeta)}{1+y(\zeta)+\,{}^{c}\mathfrak{D}^{\beta(\zeta),\zeta^2}y(\zeta)}\right] \text{ for all } \zeta \in [0,3], \\ y(0) = 1, \text{ and } y(3) = 1, \end{cases} \tag{110}$$

where

$$\alpha(\zeta) = \begin{cases} \frac{7}{5} \text{ if } 0 \leq \zeta \leq 1 \\ \frac{6}{5} \text{ if } 1 < \zeta \leq 2 \\ \frac{9}{5} \text{ if } 2 < \zeta \leq 3 \end{cases}, \tag{111}$$

and

$$\beta(\zeta) = \begin{cases} \frac{6}{5} \text{ if } 0 \le \zeta \le 1 \\ \frac{11}{10} \text{ if } 1 < \zeta \le 2 \\ \frac{8}{5} \text{ if } 2 < \zeta \le 3 \end{cases}, \tag{112}$$

with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$, *and* $T_3 = 3$.

It is obvious that

$$\mathcal{F}(\zeta, u, v) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}\left[\frac{5+|u|+|v|}{1+|u|+|v|}\right] \tag{113}$$

is a mutually continuous function. In addition, for any $u_1, v_1, u_2, v_2 \in \mathcal{R}$, *and* $\zeta \in [0, T]$ *we have*

$$|\mathcal{F}(\zeta, u_1, v_1) - \mathcal{F}(\zeta, u_2, v_2)| \le \frac{1}{69e}(|u_1 - u_2| + |v_1 - v_2|). \tag{114}$$

Thus,

$$|\mathcal{F}(\zeta, u, v)| = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}(3+|u|+|v|), \text{ with } \mathfrak{F} = \frac{5}{69e}, \text{ and } \|\mu\| = \frac{1}{69e}. \tag{115}$$

Hence, assumptions $(A_1) - (A_3)$ *are satisfied with*

$$\mu(\zeta) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}, \text{ and } \|\mu\| = \frac{1}{69e}. \tag{116}$$

With (111) *and* (112), *we consider three BVPs for Caputo- fractional differential equations of constant order*

$$\begin{cases} {}^c\mathfrak{D}^{\frac{7}{5}, \zeta^2} y(\zeta) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}\left[\frac{5+y(\zeta)+ {}^c\mathfrak{D}^{\frac{6}{5}, \zeta^2} y(\zeta)}{1+y(\zeta)+ {}^c\mathfrak{D}^{\frac{6}{5}, \zeta^2} y(\zeta)}\right] \text{ for all } \zeta \in [0,1], \\ y(0) = 1, \text{ and } y(1) = 1, \end{cases} \tag{117}$$

$$\begin{cases} {}^c\mathfrak{D}^{\frac{6}{5}, \zeta^2} y(\zeta) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}\left[\frac{5+y(\zeta)+ {}^c\mathfrak{D}^{\frac{11}{10}, \zeta^2} y(\zeta)}{1+y(\zeta)+ {}^c\mathfrak{D}^{\frac{11}{10}, \zeta^2} y(\zeta)}\right] \text{ for all } \zeta \in (1,2], \\ y(1) = 1, \text{ and } y(2) = 1, \end{cases} \tag{118}$$

$$\begin{cases} {}^c\mathfrak{D}^{\frac{9}{5}, \zeta^2} y(\zeta) = \frac{\sqrt{2\zeta+1}}{69e^{2\zeta+1}}\left[\frac{5+y(\zeta)+ {}^c\mathfrak{D}^{\frac{8}{5}, \zeta^2} y(\zeta)}{1+y(\zeta)+ {}^c\mathfrak{D}^{\frac{8}{5}, \zeta^2} y(\zeta)}\right] \text{ for all } \zeta \in (2,3], \\ y(2) = 1, \text{ and } y(3) = 1, \end{cases} \tag{119}$$

Now, on the subinterval $[0,1]$, *we have* $T_1 = 1, \alpha_1 = \frac{7}{5}, \beta_1 = \frac{6}{5}$.

Condition (70) *is satisfied on* $[0,1]$ *since*

$$\Delta_1 = \left(1 + \frac{(\varphi(T_1) - \varphi(0))^{\alpha_1 - \beta_1}}{\Gamma(\alpha_1 - \beta_1 + 1)}\left(1 + \frac{\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)}\right)\right) = 2.09497, \tag{120}$$

and

$$c_1 = \frac{\Delta_1\|\mu\|(\varphi(T_1) - \varphi(0))^{2\alpha_1 - \beta_1 - 3}}{|\Gamma(\alpha_1+1)\Gamma(2-\beta_1)\Gamma(\alpha_1-1)|} = 0.00348194 < 1. \tag{121}$$

which implies that BVP (135) *has at least one mild solution* $x_1 \in C[0,1]$.

In addition, we have $\mathcal{G}_0 < 0.5$, *and condition* (50) *is satisfied on* $[0,1]$ *since*

$$\frac{\left(\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1-1}}{\Gamma(\alpha_1+1)} + \mathcal{G}_0(\varphi(T_1) - \varphi(0))\right)\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1-\beta_1}}{\Gamma(\alpha_1-\beta_1+1)}\right)} = 0.00699859 < 1, \tag{122}$$

which implies that BVP (135) one unique solution $x_1 \in C[0,1]$.

Similarly, on the interval $(1,2]$, we have $T_2 = 2, \alpha_2 = \frac{6}{5}, \beta_2 = \frac{11}{10}$.

Condition (70) is satisfied on $(1,2]$ since

$$\Delta_2 = \left(1 + \frac{(\varphi(T_2) - \varphi(0))^{\alpha_2 - \beta_2}}{\Gamma(\alpha_2 - \beta_2 + 1)}\left(1 + \frac{\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_2) - \varphi(0))^{\alpha_2 - \beta_2}}{\Gamma(\alpha_2 - \beta_2 + 1)}\right)}\right)\right) = 2.21392, \quad (123)$$

and

$$c_2 = \frac{\Delta_2 \|\mu\| (\varphi(T_2) - \varphi(0))^{2\alpha_2 - \beta_2 - 3}}{|\Gamma(\alpha_2 + 1)\Gamma(2 - \beta_2)\Gamma(\alpha_2 - 1)|} = 0.000206868 < 1. \quad (124)$$

which implies that BVP (136) has at least one mild solution $x_2 \in C(1,2]$.

In addition, we have $\mathcal{G}_0 < 0.5$ and condition (50) is satisfied on $(1,2]$ since

$$\frac{\left(\frac{(\varphi(T_2) - \varphi(0))^{\alpha_2 - 1}}{\Gamma(\alpha_2 + 1)} + \mathcal{G}_0(\varphi(T_2) - \varphi(0))\right)\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_2) - \varphi(0))^{\alpha_2 - \beta_2}}{\Gamma(\alpha_2 - \beta_2 + 1)}\right)} = 0.0171587 < 1, \quad (125)$$

which implies that BVP (135) one unique solution $x_2 \in C(1,2]$.

Finally, on the interval $(2,3]$, we have $T_3 = 3, \alpha_3 = \frac{9}{5}, \beta_3 = \frac{8}{5}$.

Condition (70) is satisfied on $(2,3]$ since

$$\Delta_3 = \left(1 + \frac{(\varphi(T_3) - \varphi(0))^{\alpha_3 - \beta_3}}{\Gamma(\alpha_3 - \beta_3 + 1)}\left(1 + \frac{\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_3) - \varphi(0))^{\alpha_3 - \beta_3}}{\Gamma(\alpha_3 - \beta_3 + 1)}\right)}\right)\right) = 2.69925, \quad (126)$$

and

$$c_3 = \frac{\Delta_3 \|\mu\| (\varphi(T_3) - \varphi(0))^{2\alpha_3 - \beta_3 - 3}}{|\Gamma(\alpha_3 + 1)\Gamma(2 - \beta_3)\Gamma(\alpha_3 - 1)|} = 0.000369338 < 1. \quad (127)$$

which implies that BVP (137) has at least one mild solution $x_2 \in C(2,3]$.

In addition, we have $\mathcal{G}_o < 1.5$ and condition (50) is satisfied on $(2,3]$ since

$$\frac{\left(\frac{(\varphi(T_3) - \varphi(0))^{\alpha_3 - 1}}{\Gamma(\alpha_3 + 1)} + \mathcal{G}_0(\varphi(T_3) - \varphi(0))\right)\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_3) - \varphi(0))^{\alpha_3 - \beta_3}}{\Gamma(\alpha_3 - \beta_3 + 1)}\right)} = \frac{\frac{1}{69e}\left(\frac{(3^2)^{\frac{1}{5}}}{\Gamma(\frac{11}{5})} + 1.5 \times 3^2\right)}{\left(1 - \frac{\frac{1}{69e}(3^2)^{\frac{1}{10}}}{\Gamma(\frac{11}{10})}\right)} = 0.0428218 < 1, \quad (128)$$

which implies that BVP (135) one unique solution $x_3 \in C(2,3]$.

It follows from Theorems 3 and 2 that problem (110) has a unique mild solution $x(\zeta) \in C[0,3]$ such that

$$x(\zeta) = \begin{cases} x_1(\zeta), \ \zeta \in [0,1], \\ x_2(\zeta), \ \zeta \in (1,2], \\ x_3(\zeta), \ \zeta \in (2,3]. \end{cases} \quad (129)$$

Example 2. *Given the following VOFBVP:*

$$\begin{cases} {}^c\mathfrak{D}^{\alpha(\zeta),\varphi}x(\zeta) = \frac{e^{-3\zeta}}{e^{\zeta}+75}\left[5 - \frac{|x(\zeta)|}{1+|x(\zeta)|} - \frac{|{}^c\mathfrak{D}^{\beta(\zeta),\varphi}x(\zeta)|}{1+|{}^c\mathfrak{D}^{\beta(\zeta),\varphi}x(\zeta)|}\right] \text{ for all } \zeta \in [0,3], \\ x(0) = 1, \ \text{ and } x(3) = 1, \end{cases} \quad (130)$$

where $\varphi(\zeta) = \sqrt{\zeta + 1}$, increasing function such that $\varphi'(\zeta) \neq 0, \forall \zeta \in [0,3]$, $\alpha(\zeta) = \frac{6}{5} + \frac{2}{9}\zeta$, $0 \leq \zeta \leq 3$, $\beta(\zeta) = 1 + \frac{1}{7}\zeta$, $0 \leq \zeta \leq 3$, with $T_0 = 0$, $T_1 = 1$, $T_2 = 2$, and $T_3 = 3$. It is clear that $1 < \beta(\zeta) < \alpha(\zeta) < 2$ for all $\zeta \in [0,3]$.

It is obvious that

$$\mathcal{F}(\zeta,u,v) = \frac{e^{-3\zeta}}{e^{\zeta}+75}\left[5 - \frac{|u|}{1+|u|} - \frac{|v|}{1+|v|}\right], \tag{131}$$

is a mutually continuous function. In addition, for any u_1,v_1,u_2,$v_2 \in \mathcal{R}$, *and* $\zeta \in [0,T]$ *we have*

$$|\mathcal{F}(\zeta,u_1,v_1) - \mathcal{F}(\zeta,u_2,v_2)| \leq \frac{1}{76}(|u_1-u_2| + |v_1-v_2|). \tag{132}$$

Thus,

$$|\mathcal{F}(\zeta,u,v)| = \frac{e^{-3\zeta}}{e^{\zeta}+75}\left(5 - \frac{|u|}{1+|u|} - \frac{|v|}{1+|v|}\right), \text{ with } \mathfrak{F} = \frac{5}{76}, \text{ and } \|\mu\| = \frac{1}{76}. \tag{133}$$

Hence, assumptions (A_1)–(A_3) *are satisfied with*

$$\mu(\zeta) = \frac{e^{-3\zeta}}{e^{\zeta}+75}, \text{ and } \|\mu\| = \frac{1}{76}. \tag{134}$$

From (111) *and* (112), *we consider three BVPs for Caputo- fractional differential equations of constant order*

$$\begin{cases} {}^{c}\mathfrak{D}^{\frac{6}{5}+\frac{2}{9}\zeta,\ \sqrt{\zeta+1}}x(\zeta) = \frac{e^{-3\zeta}}{e^{\zeta}+75}\left[5 - \frac{|x(\zeta)|}{1+|x(\zeta)|} - \frac{\left|{}^{c}\mathfrak{D}^{1+\frac{1}{7}\zeta,\ \sqrt{\zeta+1}}x(\zeta)\right|}{1+\left|{}^{c}\mathfrak{D}^{1+\frac{1}{7}\zeta,\ \sqrt{\zeta+1}}x(\zeta)\right|}\right] \text{ for all } \zeta \in [0,1], \\ x(0) = 1, \text{ and } x(1) = 1, \end{cases} \tag{135}$$

$$\begin{cases} {}^{c}\mathfrak{D}^{\frac{6}{5}+\frac{2}{9}\zeta,\ \sqrt{\zeta+1}}x(\zeta) = \frac{e^{-3t}}{e^{t}+75}\left[5 - \frac{|x(\zeta)|}{1+|x(\zeta)|} - \frac{\left|{}^{c}\mathfrak{D}^{1+\frac{1}{7}\zeta,\ \sqrt{\zeta+1}}x(\zeta)\right|}{1+\left|{}^{c}\mathfrak{D}^{1+\frac{1}{7}\zeta,\ \sqrt{\zeta+1}}x(\zeta)\right|}\right] \text{ for all } \zeta \in (1,2], \\ x(1) = 1, \text{ and } x(2) = 1, \end{cases} \tag{136}$$

$$\begin{cases} {}^{c}\mathfrak{D}^{\frac{6}{5}+\frac{2}{9}\zeta,\ \sqrt{\zeta+1}}x(\zeta) = \frac{e^{-3\zeta}}{e^{\zeta}+75}\left[5 - \frac{|x(\zeta)|}{1+|x(\zeta)|} - \frac{\left|{}^{c}\mathfrak{D}^{1+\frac{1}{7}\zeta,\ \sqrt{\zeta+1}}x(\zeta)\right|}{1+\left|{}^{c}\mathfrak{D}^{1+\frac{1}{7}\zeta,\ \sqrt{\zeta+1}}x(\zeta)\right|}\right] \text{ for all } \zeta \in (2,3], \\ x(2) = 1, \text{ and } x(3) = 1, \end{cases} \tag{137}$$

Now, on the subinterval $[0,1]$, *we have* $T_1 = 1, \alpha_1(\zeta) = \frac{6}{5} + \frac{2}{9}\zeta$, *and* $\beta_1(\zeta) = 1 + \frac{1}{7}\zeta$.

Condition (70) *is satisfied on* $[0,1]$ *since*

$$\begin{aligned} \Delta_1 &= 1 + \frac{(\varphi(T_1)-\varphi(0))^{\alpha_1(\zeta)-\beta_1(\zeta)}}{\Gamma(\alpha_1(\zeta)-\beta_1(\zeta)+1)}\left(1 + \frac{\|\mu\|}{\left(1 - \frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1(\zeta)-\beta_1(\zeta)}}{\Gamma(\alpha_1(\zeta)-\beta_1(\zeta)+1)}\right)}\right), \\ &= 1 + \frac{\left(\sqrt{2}-1\right)^{\frac{1}{5}+\frac{5}{63}\zeta}}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\left(1 + \frac{\frac{1}{76}}{\left(1 - \frac{\frac{1}{76}\left(\sqrt{2}-1\right)^{\frac{1}{5}+\frac{5}{63}\zeta}}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\right)}\right), \\ &\leq 1 + \frac{\left(\sqrt{2}-1\right)^{\frac{1}{5}}}{\Gamma(\frac{6}{5})}\left(1 + \frac{\frac{1}{76}}{\left(1 - \frac{\frac{1}{76}\left(\sqrt{2}-1\right)^{\frac{1}{5}}}{\Gamma(\frac{6}{5})}\right)}\right) \leq 1.92527, \end{aligned} \tag{138}$$

and

$$c_1 = \frac{\Delta_1 \|\mu\| (\varphi(T_1) - \varphi(0))^{2\alpha_1(\zeta) - \beta_1(\zeta) - 3}}{|\Gamma(\alpha_1(\zeta)+1)\Gamma(2-\beta_1(\zeta))\Gamma(\alpha_1(\zeta)-1)|},$$
$$\leq \frac{\frac{1.92527}{76}(\sqrt{2}-1)^{-4+2\left(\frac{6}{5}+\frac{2\zeta}{9}\right)-\frac{\zeta}{7}}}{|\Gamma\left(1-\frac{1}{7}\zeta\right)\Gamma(\frac{1}{5}+\frac{2}{9}\zeta)\Gamma\left(\frac{11}{5}+\frac{2}{9}\zeta\right)|}, \tag{139}$$
$$\leq \frac{\frac{1.86478}{76}(\sqrt{2}-1)^{-\frac{409}{315}}}{|\Gamma(\frac{6}{7})\Gamma(\frac{19}{45})\Gamma\left(\frac{109}{45}\right)|} \leq 0.0271925 < 1,$$

which implies that BVP (135) has at least one mild solution $x_1 \in C[0,1]$.

In addition, we have

$$\mathcal{G}_1(\zeta,s) = \begin{cases} \frac{1}{\Gamma(\alpha_1(\zeta))}\left[\begin{array}{c}(\varphi(\zeta)-\varphi(s))^{\alpha_1(\zeta)-1} \\ -\frac{\varphi(\zeta)-\varphi(0)}{\varphi(T_1)-\varphi(0)}(\varphi(T_1)-\varphi(s))^{\alpha_1(\zeta)-1}\end{array}\right] & \text{if } \ 0 \leq s \leq \zeta \leq T_1, \\ -\frac{(\varphi(\zeta)-\varphi(0))}{\Gamma(\alpha_1(\zeta))(\varphi(T_1)-\varphi(0))}(\varphi(T_1)-\varphi(s))^{\alpha_1(\zeta)-1} & \text{if } \ 0 \leq \zeta \leq s \leq T_1, \end{cases} \tag{140}$$
$$= \begin{cases} \frac{1}{\Gamma(\frac{6}{5}+\frac{2}{9}\zeta)}\left[\begin{array}{c}(\sqrt{\zeta+1}-\sqrt{s+1})^{\frac{1}{5}+\frac{2}{9}\zeta} \\ -\frac{\sqrt{\zeta+1}-1}{\sqrt{2}-1}(\sqrt{2}-\sqrt{s+1})^{\frac{1}{5}+\frac{2}{9}\zeta}\end{array}\right] & \text{if } \ 0 \leq s \leq \zeta \leq T_1, \\ -\frac{1}{\Gamma(\frac{6}{5}+\frac{2}{9}\zeta)}\left[\frac{\sqrt{\zeta+1}-1}{\sqrt{2}-1}(\sqrt{2}-\sqrt{s+1})^{\frac{1}{5}+\frac{2}{9}\zeta}\right] & \text{if } \ 0 \leq \zeta \leq s \leq T_1 \end{cases}$$

which implies that $\mathcal{G}_0 \leq 0.913108$ *and condition (50) is satisfied on* $[0,1]$ *since*

$$\frac{\left(\frac{(\varphi(T_1)-\varphi(0))^{\alpha_1(\zeta)-1}}{\Gamma(\alpha_1(\zeta)+1)} + \mathcal{G}_0(\varphi(T_1)-\varphi(0))\right)\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_1)-\varphi(0))^{\alpha_1(\zeta)-\beta_1(\zeta)}}{\Gamma(\alpha_1(\zeta)-\beta_1(\zeta)+1)}\right)} \leq \frac{\frac{1}{76}\left(\frac{(\sqrt{2}-1)^{\frac{1}{5}+\frac{2}{9}\zeta}}{\Gamma(\frac{11}{5}+\frac{2}{9}\zeta)} + 0.913108(\sqrt{2}-1)\right)}{\left(1-\frac{\frac{1}{76}(\sqrt{2}-1)^{\frac{1}{5}+\frac{5}{63}\zeta}}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\right)},$$
$$\leq \frac{\frac{1}{76}\left(\frac{(\sqrt{2}-1)^{\frac{1}{5}}}{\Gamma(\frac{11}{5})} + 0.913108(\sqrt{2}-1)\right)}{\left(1-\frac{\frac{1}{76}(\sqrt{2}-1)^{\frac{1}{5}}}{\Gamma(\frac{6}{5})}\right)},$$
$$\approx 0.015171 < 1, \tag{141}$$

which implies that BVP (135) one unique solution $x_1 \in C[0,1]$.

Similarly, on the interval $(1,2]$*, we have* $T_2 = 2, \alpha_2(\zeta) = \frac{6}{5} + \frac{2}{9}\zeta$*, and* $\beta_2(\zeta) = 1 + \frac{1}{7}\zeta$.

Condition (70) is satisfied on $(1,2]$ *since*

$$\Delta_2 = 1 + \frac{(\varphi(T_2)-\varphi(0))^{\alpha_2(\zeta)-\beta_2(\zeta)}}{\Gamma(\alpha_2(\zeta)-\beta_2(\zeta)+1)}\left(1+\frac{\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_2)-\varphi(0))^{\alpha_2(\zeta)-\beta_2(\zeta)}}{\Gamma(\alpha_2(\zeta)-\beta_2(\zeta)+1)}\right)}\right),$$
$$= 1 + \frac{\left(\sqrt{3}-1\right)^{\frac{1}{5}+\frac{5}{63}\zeta}}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\left(1+\frac{\frac{1}{76}}{\left(1-\frac{\frac{1}{76}(\sqrt{3}-1)^{\frac{1}{5}+\frac{5}{63}\zeta}}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\right)}\right), \tag{142}$$
$$= 1 + \frac{\left(\sqrt{3}-1\right)^{\frac{1}{5}}}{\Gamma(\frac{6}{5})}\left(1+\frac{\frac{1}{76}}{\left(1-\frac{\frac{1}{76}(\sqrt{3}-1)^{\frac{1}{5}}}{\Gamma(\frac{6}{5})}\right)}\right) \leq 2.01771,$$

and

$$\begin{aligned} c_2 &= \frac{\Delta_2\|\mu\|(\varphi(T_2)-\varphi(0))^{2\alpha_2(\zeta)-\beta_2(\zeta)-3}}{|\Gamma(\alpha_2(\zeta)+1)\Gamma(2-\beta_2(\zeta))\Gamma(\alpha_2(\zeta)-1)|}, \\ &\le \frac{\frac{2.01771}{76}(\sqrt{3}-1)^{-4+2\left(\frac{6}{5}+\frac{2\zeta}{9}\right)-\frac{\zeta}{7}}}{|\Gamma\left(1-\frac{1}{7}\zeta\right)\Gamma(\frac{1}{5}+\frac{2}{9}\zeta)\Gamma\left(\frac{11}{5}+\frac{2}{9}\zeta\right)|}, \\ &\le \frac{\frac{2.01771}{76}(\sqrt{3}-1)^{-1.14762}}{|\Gamma(\frac{6}{7})\Gamma(\frac{19}{45})\Gamma\left(\frac{109}{45}\right)|} \le 0.0141952 < 1, \end{aligned} \tag{143}$$

which implies that BVP (136) has at least one mild solution $x_2 \in C(1,2]$.

In addition, we have $\mathcal{G}_0 < 0.695425$, *and condition (50) is satisfied on* $(1,2]$ *since*

$$\begin{aligned} \frac{\left(\frac{(\varphi(T_2)-\varphi(0))^{\alpha_1(\zeta)-1}}{\Gamma(\alpha_2(\zeta)+1)}+\mathcal{G}_0(\varphi(T_2)-\varphi(0))\right)\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_2)-\varphi(0))^{\alpha_2(\zeta)-\beta_2(\zeta)}}{\Gamma(\alpha_2(\zeta)-\beta_2(\zeta)+1)}\right)} &\le \frac{\frac{1}{76}\left(\frac{(\sqrt{3}-1)^{\frac{1}{5}+\frac{2}{9}\zeta}}{\Gamma(\frac{11}{5}+\frac{2}{9}\zeta)}+0.695425(\sqrt{3}-1)\right)}{\left(1-\frac{\frac{1}{76}(\sqrt{3}-1)^{\frac{1}{5}+\frac{5}{63}\zeta}}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\right)}, \\ &\le \frac{\frac{1}{76}\left(\frac{(\sqrt{3}-1)^{\frac{1}{5}}}{\Gamma(\frac{11}{5})}+0.695425(\sqrt{3}-1)\right)}{\left(1-\frac{\frac{1}{76}(\sqrt{3}-1)^{\frac{1}{5}}}{\Gamma(\frac{6}{5})}\right)}, \\ &\approx 0.0160643 < 1, \end{aligned} \tag{144}$$

which implies that BVP (135) one unique solution $x_2 \in C(1,2]$.

Finally, on the interval $(2,3]$, *we have* $T_3 = 3, \alpha_3(\zeta) = \frac{6}{5}+\frac{2}{9}\zeta$, *and* $\beta_3(\zeta) = 1+\frac{1}{7}\zeta$. *Condition (70) is satisfied on* $(2,3]$ *since*

$$\begin{aligned} \Delta_3 &= 1+\frac{(\varphi(T_3)-\varphi(0))^{\alpha_3(\zeta)-\beta_3(\zeta)}}{\Gamma(\alpha_3(\zeta)-\beta_3(\zeta)+1)}\left(1+\frac{\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_3)-\varphi(0))^{\alpha_3(\zeta)-\beta_3(\zeta)}}{\Gamma(\alpha_3(\zeta)-\beta_3(\zeta)+1)}\right)}\right), \\ &= 1+\frac{1}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\left(1+\frac{\frac{1}{76}}{\left(1-\frac{1}{76\,\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\right)}\right), \\ &= 1+\frac{1}{\Gamma(\frac{151}{105})}\left(1+\frac{\frac{1}{76}}{\left(1-\frac{\frac{1}{76}}{\Gamma(\frac{151}{105})}\right)}\right) \le 2.14395, \end{aligned} \tag{145}$$

and

$$\begin{aligned} c_3 &= \frac{\Delta_3\|\mu\|(\varphi(T_3)-\varphi(0))^{2\alpha_3(\zeta)-\beta_3(\zeta)-3}}{|\Gamma(\alpha_3(\zeta)+1)\Gamma(2-\beta_3(\zeta))\Gamma(\alpha_3(\zeta)-1)|}, \\ &\le \frac{\frac{2.14395}{76}}{|\Gamma\left(1-\frac{1}{7}\zeta\right)\Gamma(\frac{1}{5}+\frac{2}{9}\zeta)\Gamma\left(\frac{11}{5}+\frac{2}{9}\zeta\right)|}, \\ &\le \frac{\frac{2.14395}{76}}{|\Gamma(\frac{5}{7})\Gamma(\frac{29}{45})\Gamma\left(\frac{119}{45}\right)|} \le 0.0107133 < 1, \end{aligned} \tag{146}$$

which implies that BVP (137) has at least one mild solution $x_3 \in C(2,3]$.

In addition, we have $\mathcal{G}_o < 0.475897$*, and condition (50) is satisfied on* $(2,3]$ *since*

$$\begin{aligned} \frac{\left(\frac{(\varphi(T_2)-\varphi(0))^{\alpha_1(\zeta)-1}}{\Gamma(\alpha_2(\zeta)+1)} + \mathcal{G}_0(\varphi(T_2)-\varphi(0))\right)\|\mu\|}{\left(1-\frac{\|\mu\|(\varphi(T_2)-\varphi(0))^{\alpha_2(\zeta)-\beta_2(\zeta)}}{\Gamma(\alpha_2(\zeta)-\beta_2(\zeta)+1)}\right)} &\leq \frac{\frac{1}{76}\left(\frac{1}{\Gamma(\frac{11}{5}+\frac{2}{9}\zeta)}+0.475897\right)}{\left(1-\frac{\frac{1}{76}}{\Gamma(\frac{6}{5}+\frac{5}{63}\zeta)}\right)}, \\ &\leq \frac{\frac{1}{76}\left(\frac{1}{\Gamma(\frac{119}{45})}+0.475897\right)}{\left(1-\frac{\frac{1}{76}}{\Gamma(\frac{428}{315})}\right)}, \\ &\approx 0.0153867 < 1, \end{aligned} \tag{147}$$

which implies that BVP (135) has one unique solution $x_3 \in C(2,3]$.

It follows from Theorems 3 and 2 that problem (110) has a unique mild solution $x(\zeta) \in C[0,3]$ *such that*

$$x(\zeta) = \begin{cases} x_1(\zeta), \; \zeta \in [0,1], \\ x_2(\zeta), \; \zeta \in (1,2], \\ x_3(\zeta), \; \zeta \in (2,3]. \end{cases} \tag{148}$$

5. Conclusions

In conclusion, this study on the two-point boundary value problem of φ-Caputo fractional differential equations with variable order has provided insightful results regarding the existence, uniqueness, and Ulam-Hyers stability of solutions. The application of Banach's and Krasnoselskii's fixed point theorems has established a robust mathematical framework for analyzing such problems. Furthermore, the inclusion of a numerical example has illustrated the practical implications of the obtained results.

Looking ahead, there are several promising avenues for future research in this field:

1. *Extension to higher dimensions:* Future investigations can explore the behavior of variable-order fractional differential equations in higher-dimensional spaces, providing deeper insights into complex systems.
2. *Sensitivity analysis*: Conducting sensitivity analysis to assess how variations in parameters impact the solutions of variable-order fractional differential equations can offer valuable insights into system behavior under different conditions.
3. *Applications in engineering and science*: Applying the findings of this study to real-world problems in engineering, physics, and biology can lead to practical insights and applications with substantial impact.
4. *Numerical methods*: Further development and refinement of numerical methods for solving variable-order fractional differential equations can enhance computational efficiency and accuracy, making these equations more accessible for practical use.
5. *Comparative studies*: Conducting comparative studies with classical integer order differential equations and exploring when fractional order solutions converge or diverge from integer order solutions can provide a comprehensive understanding of their relationships.

These recommended directions for future research will contribute to a deeper understanding of Caputo fractional differential equations with variable order and pave the way for further advancements in theoretical and applied mathematics.

Author Contributions: Conceptualization, Y.A. (Yahia Awad); methodology, Y.A. (Yahia Awad) and H.F.; formal analysis, Y.A. (Yahia Awad), H.F. and Y.A. (Yousuf Alkhezi); investigation, Y.A. (Yahia Awad) and H.F.; data curation, Y.A. (Yahia Awad) and H.F.; writing—original draft preparation, Y.A. (Yahia Awad), H.F. and Y.A. (Yousuf Alkhezi); writing—review and editing, Y.A. (Yahia Awad), H.F. and Y.A. (Yousuf Alkhezi); visualization, Y.A. (Yahia Awad) and H.F. ; supervision, Y.A. (Yahia Awad); project administration, Y.A. (Yahia Awad); funding acquisition, Y.A. (Yousuf Alkhezi). All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors express deep gratitude to the editor and referees for their invaluable input and contributions that have significantly enhanced the manuscript's quality and impact.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Burton, T.A.; Kirk, C. A fixed point theorem of Krasnoselskii-Schaefer type. *Math. Nachrichten* **1998** , *189*, 23–31. [CrossRef]
2. Curtain, R.F.; Pritchard, A.J. *Functional Analysis in Modern Applied Mathematics*; Academic Press: Cambridge, MA, USA, 1997.
3. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science Publishers: Yverdon-les-Bains, Switzerland, 1993; p. 1.
4. Wang, F.; Khan, M.N.; Ahmad, I.; Ahmad, H.; Abu-Zinadah, H.; Chu, Y.M. Numerical solution of traveling waves in chemical kinetics: time-fractional fishers equations. *Fractals* **2022**, *30*, 2240051. [CrossRef]
5. Ahmad, Z.; Bonanomi, G.; di Serafino, D.; Giannino, F. Transmission dynamics and sensitivity analysis of pine wilt disease with asymptomatic carriers via fractal-fractional differential operator of Mittag–Leffler kernel. *Appl. Numer. Math.* **2023**, *185*, 446–65. [CrossRef]
6. Ahmad, Z.; El-Kafrawy, S.A.; Alandijany, T.A.; Giannino, F.; Mirza, A.A.; El-Daly, M.M.; Faizo, A.A.; Bajrai, L.H.; Kamal, M.A.; Azhar, E.I. A global report on the dynamics of COVID-19 with quarantine and hospitalization: A fractional order model with non-local kernel. *Comput. Biol. Chem.* **2022**, *98*, 107645. [CrossRef]
7. Ahmad, Z.; Ali, F.; Khan, N.; Khan, I. Dynamics of fractal-fractional model of a new chaotic system of integrated circuit with Mittag–Leffler kernel. *Chaos Solitons Fractals* **2021**, *153*, 111602. [CrossRef]
8. Ahmad, H.; Khan, T.A.; Ahmad, I.; Stanimirović, P. S.; Chu, Y.M. A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations. *Results Phys.* **2020**, *19*, 103462. [CrossRef]
9. Baleanu, D.; Jajarmi, A.; Mohammadi, H.; Rezapour, S. A new study on the mathematical modelling of human liver with Caputo–Fabrizio fractional derivative. *Chaos Solitons Fractals* **2020**, *134*, 109705. [CrossRef]
10. Agarwal, R.P.; Benchohra, M.; Hamani, S. Boundary value problems for fractional differential equations. *Georgian Math. J.* **2009** , *16*, 401–411. [CrossRef]
11. Agrawal, O.P. Some generalized fractional calculus operators and their applications in integral equations. *Fract. Calc. Appl. Anal.* **2012**, *15*, 700–711. [CrossRef]
12. An, J.; Chen, P. Uniqueness of solutions to initial value problem of fractional differential equations of variable-order. *Dyn. Syst*. **2019**, *28*, 607–623.
13. Awad, Y.; Fakih, H. Existence and Uniqueness Results for a two-point Nonlinear Boundary Value Problem of Caputo Fractional Differential Equations of Variable Order. *TWMS J. Appl. Eng. Math.* **2022**, *accepted*.
14. Benkerrouche, A.; Souid, M.S.; Jarad, F.; Hakem, A. On boundary value problems of Caputo fractional differential equation of variable order via Kuratowski MNC technique. *Adv. Contin. Discret. Model.* **2022**, *2022*, 43. [CrossRef]
15. Hristova, S.; Benkerrouche, A.; Souid, M.S.; Hakem, A. Boundary value problems of Hadamard fractional differential equations of variable order. *Symmetry* **2021**, *13*, 896. [CrossRef]
16. Sarwar, S. On the Existence and Stability of Variable Order Caputo Type Fractional Differential Equations. *Fractal Fract.* **2022**, *6*, 51. [CrossRef]
17. Sun, H.; Chang, A.; Zhang, Y.; Chen, W. A review on variable-order fractional differential equations: Mathematical foundations, physical models, numerical methods and applications. *Fract. Calc. Appl. Anal.* **2019**, *22*, 27–59. [CrossRef]
18. Tavares, D.; Almeida, R.; Torres, D.F. Caputo derivatives of fractional variable order: Numerical approximations. *Commun. Nonlinear Sci. Numer. Simul.* **2016**, *35*, 69–87. [CrossRef]
19. Zhang, S. The uniqueness result of solutions to initial value problems of differential equations of variable-order. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A. Mat.* **2018**, *112*, 407–423. [CrossRef]
20. Sene, N. Fundamental Results about the Fractional Integro-Differential Equation Described with Caputo Derivative. *J. Funct. Spaces* **2022**, *2022*, 9174488. [CrossRef]
21. Sousa, J.V.; de Oliveira, E.C. Two new fractional derivatives of variable order with non-singular kernel and fractional differential equation. *Comput. Appl. Math.* **2018**, *37*, 5375–5394. [CrossRef]
22. Zhang, S. Existence and uniqueness result of solutions to initial value problems of fractional differential equations of variable-order. *J. Frac. Calc. Anal* **2013**, *4*, 82–98.
23. Zhang, S.; Hu, L. Unique existence result of approximate solution to initial value problem for fractional differential equation of variable order involving the derivative arguments on the half-axis. *Mathematics* **2019**, *7*, 286. [CrossRef]
24. Abdo, M.S.; Panchal, S.K.; Saeed, A.M. Fractional boundary value problem with ψ-Caputo fractional derivative. *Proc.-Math. Sci.* **2019**, *129*, 65. [CrossRef]
25. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]

26. Almeida, R.; Malinowska, A.B.; Monteiro, M.T.T. Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. *Math. Methods Appl. Sci.* **2018**, *41*, 336–352. [CrossRef]
27. Awad, Y.; Kaddoura, I. On the Ulam-Hyers-Rassias stability for a boundary value problem of implicit ψ-Caputo fractional integro-differential equation. *TWMS J. Appl. Eng. Math.* **2021**, *accepted*.
28. Awad, Y. Well Posedness and Stability for the Nonlinear φ-Caputo Hybrid Fractional boundary value problems with two-point hybrid boundary conditions. *Jordan J. Math. Stat.* **2022**, *accepted*.
29. Awad, Y.; Alkhezi, Y. Analysis of Implicit Solutions for a Coupled System of Hy-brid Fractional Order Differential Equations with Hybrid Integral Boundary Conditions in Banach Algebras. *Symmetry* **2023**, *15*, 1758. [CrossRef]
30. Awad, Y. On the Existence and Stability of Positive Solutions of Eigenvalue Problems for a Class of p-Laplacian ψ-Caputo Fractional Integro-Differential Equations. *J. Math.* **2023**, *accepted*.
31. El Mfadel, A.; Melliani, S.; Elomari, M.H. Existence and uniqueness results of boundary value problems for nonlinear fractional differential equations involving ψ-Caputo-type fractional derivatives. *Acta Math. Univ. Comen.* **2023**, *92*, 23–33.
32. Wahash, H.A.; Mohammed, A.B. D. O.; Panchal, S.K. Existence and stability of a nonlinear fractional differential equation involving a ψ-Caputo operator. *Adv. Theory Nonlinear Anal. Its Appl.* **2020**, *4*, 266–278. [CrossRef]
33. Lakshmikantham, V.; Vatsala, A.S. Basic theory of fractional differential equations. *Nonlinear Anal. Theory Methods Appl.* **2008**, *69*, 2677–2682. [CrossRef]
34. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; p. 204.
35. Rus, I.A. Ulam stabilities of ordinary differential equations in a Banach space. *Carpathian J. Math.* **2010** , *26*, 103–107.