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Infinitely Many Solutions for a Perturbed Partial Discrete Dirichlet Problem Involving ϕ_c-Laplacian

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Abstract: In this paper, by using critical point theory, the existence of infinitely many small solutions for a perturbed partial discrete Dirichlet problems including the mean curvature operator is investigated. Moreover, the present study first attempts to address discrete Dirichlet problems with ϕ_c-Laplacian operator in relative to some relative existing references. Based on our knowledge, this is the research of perturbed partial discrete bvp with ϕ_c-Laplacian operator for the first time. At last, two examples are used to examplify the results.

Keywords: BVP; small solutions; partial difference equation; ϕ_c-Laplacian; critical point theory

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1. Introduction

We focus on the problem below, namely $(f^{\lambda,\mu})$

$$-[\Delta_1(\phi_c(\Delta_1 s(c-1,d)))+\Delta_2(\phi_c(\Delta_2 s(c,d-1)))]=\lambda f((c,d),s(c,d))+\mu g((c,d),s(c,d)),$$
$$(c,d)\in\mathbb{Z}(1,x)\times\mathbb{Z}(1,y),$$

based on boundary conditions

$$\begin{aligned} s(c,0)=s(c,y+1)=0, &\quad c\in\mathbb{Z}(0,x+1),\\ s(0,d)=s(x+1,d)=0, &\quad d\in\mathbb{Z}(0,y+1), \end{aligned} \tag{1}$$

where x and y indicate given positive integers, λ and μ represent positive real parameter, Δ_1 and Δ_2 denote the forward difference operators set by $\Delta_1 s(c,d)=s(c+1,d)-s(c,d)$ and $\Delta_2 s(c,d)=s(c,d+1)-s(c,d)$, $\Delta_1^2 s(c,d)=\Delta_1(\Delta_1 s(c,d))$ and $\Delta_2^2 s(c,d)=\Delta_2(\Delta_2 s(c,d)), \phi_c$ refers to a special ϕ-Laplacian operator [1] defined by $\phi_c(s)=\frac{s}{\sqrt{1+s^2}}$, and $f((c,d),\cdot),g((c,d),\cdot)\in C(\mathbb{R},\mathbb{R})$ for each $(c,d)\in\mathbb{Z}(1,x)\times\mathbb{Z}(1,y)$.

Difference equations have been extensively used in various field, involving natural science, as shown in [2–5]. In 2003, Yu and Guo [6] made the first attempt to investigate a class of second order difference equations. Next, many scholars attempted to investigate difference equations and made significant achievements, involving the obtained findings of periodic solutions [6,7], homoclinic solutions [8,9], as well as boundary value problems [10–18].

In 2016, Bonanno et al. [12] considered the following problem

$$\begin{cases} \Delta^2 s(\alpha-1)+\lambda f(\alpha,s(\alpha))=0, & \alpha\in\mathbb{Z}(1,M),\\ s(0)=s(M+1)=0, \end{cases} \tag{2}$$

and acquired the presence of positive solutions of (2).

In 2017, Mawhin et al. [13] explored the following problem

$$\begin{cases} -\Delta(\phi_p(\Delta s(\alpha-1))) + h(\alpha)\phi_p(s(\alpha)) = \lambda f(\alpha, s(\alpha)), & \alpha \in \mathbb{Z}(1,M), \\ s(0) = s(M+1) = 0. \end{cases} \tag{3}$$

In 2021, Ling and Zhou [16] investigated the following Robin problem

$$\begin{cases} -\Delta(\varphi_p(\Delta s(\alpha-1)) + q(\alpha)\varphi_p(s(\alpha)) = \lambda f(\alpha, s(\alpha)), & \alpha \in \mathbb{Z}(1,M), \\ \Delta s(0) = s(M+1) = 0, \end{cases} \tag{4}$$

and acquired the presence of positive solutions of (4).

The above difference equations investigated concern merely one variable. Nevertheless, the difference equations including two variables are less explored. Known as partial difference equations, the difference equations are denoted as PDE. PDE have been broadly applied in numerous domains. Boundary value problems of PDE appear to remain a difficult problem drawing the attention from many mathematical researchers [19–24], and other meaningful results [25–27].

In 2015, Heidarkhani and Imbesi [19] considered the problem below

$$\Delta_1^2 s(c-1,d) + \Delta_2^2 s(c,d-1) + \lambda f((c,d), s(c,d)) = 0, \quad (c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y), \tag{5}$$

with (1), and obtained that there are three solutions of (5) at minimum.

In 2021, Du and Zhou [23] studied the problem below

$$\Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1))) + \lambda f((c,d), s(c,d)) = 0, \quad (c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y), \tag{6}$$

with (1), and acquired the presence of multiple solutions of (6).

In 2023, Xiong [24] studied the following problem, namely (DKP)

$$-(a + b\|s\|^p)(\Delta_1(\phi_p(\Delta_1 s(c-1,d) + \Delta_2(\phi_p(\Delta_2 s(c,d-1)))) = \lambda f((c,d), s(c,d)), \quad (c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y),$$

with (1), and acquired the presence of positive solutions of problem (DKP).

In 2023, Xiong studied the presence of infinitely many solutions for the partial discrete Kirchhoff type problems which involves p-Laplacian, and obtained an unbounded sequence of solutions of problem (DKP). However, in the present work, the presence of small solutions for a perturbed partial discrete Dirichlet problems including the mean curvature operator is investigated. Thus, different problems are considered in the above two papers. Meanwhile, different results are obtained.

When compared to the results of the PDE with p-Laplacian, it was discovered that the perturbed PDE with ϕ_c-Laplacian had seldom been investigated; this can be primarily due to that dealing with the latter is more difficult, and ϕ_c-Laplacian operator has a very strong practical value [28,29]. Based on our knowledge, we first attempt to handle the problem $(f^{\lambda,\mu})$ in comparison with some relative existing references. Here, the parameter μ in the problem $(f^{\lambda,\mu})$ is extremely small. When the norm $\|s\|$ is small, a solution of the problem $(f^{\lambda,\mu})$ is a small solution. According to our knowledge, the current work is the first attempt to demonstrate the presence of small solutions for a partial difference equation with ϕ_c-Laplacian operator. The contributions and novelty of the present study are summarized:

(1) This study is the first attempt to demonstrate the presence of infinitely many small solutions for a PDE with ϕ_c-Laplacian operator.

(2) The difficulty to be overcomed in this paper is how to determine r in Theorem 1 .

(3) We demonstrate the presence of infinitely many small solutions for a perturbed PDE including ϕ_c-Laplacian by adopting the critical point theory.

(4) We present two examples to show our conclusion.

The remain of this study is presented as follows. The variational framework in association with $(f^{\lambda,\mu})$ is established, as shown in Section 2. Section 3 gives the main results. As shown in Section 4, our major results are explained with two examples. As shown in Section 5, we give the discussion. Finally, in Section 6, we give the conclusion.

2. Preliminaries

The present section makes the first attempt to build the variational framework connected to $(f^{\lambda,\mu})$. In addition, the following xy-dimensional Banach space is considered.

$S = \{s : \mathbb{Z}(0, x+1) \times \mathbb{Z}(0, y+1) \to \mathbb{R} : s(c, 0) = s(c, y+1) = 0, c \in \mathbb{Z}(0, x+1)$ and $s(0, d) = s(x+1, d) = 0, d \in \mathbb{Z}(0, y+1)\}$, which is endowed with the norm:

$$\|s\| = \left(\sum_{d=1}^{y}\sum_{c=1}^{x+1}(\Delta_1 s(c-1,d))^2 + \sum_{c=1}^{x}\sum_{d=1}^{y+1}(\Delta_2 s(c,d-1))^2\right)^{\frac{1}{2}}, \quad s \in S.$$

Define

$$\begin{cases} \Phi(s) = \sum\limits_{d=1}^{y}\sum\limits_{c=1}^{x+1}(\sqrt{1+(\Delta_1 s(c-1,d))^2}-1) + \sum\limits_{c=1}^{x}\sum\limits_{d=1}^{y+1}(\sqrt{1+(\Delta_2 s(c,d-1))^2}-1), \\ \Psi(s) = \sum\limits_{d=1}^{y}\sum\limits_{c=1}^{x}\big(F((c,d),s(c,d)) + \frac{\mu}{\lambda}G((c,d),s(c,d))\big), \end{cases} \tag{7}$$

for every $s \in S$, where $F((c,d),s) = \int_0^s f((c,d),\tau)d\tau$, $G((c,d),s) = \int_0^s g((c,d),\tau)d\tau$ for every $((c,d),s) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y) \times \mathbb{R}$. Let

$$I_\lambda(s) = \Phi(s) - \lambda\Psi(s)$$

for any $s \in S$. Obviously, $\Phi, \Psi \in C^1(S, R)$.

$$\begin{aligned} \Phi'(s)(v) &= \lim_{t\to 0}\frac{\Phi(s+tv)-\Phi(s)}{t} \\ &= \sum_{d=1}^{y}\sum_{c=1}^{x+1}\phi_c(\Delta_1 s(c-1,d))\Delta_1 v(c-1,d) + \sum_{c=1}^{x}\sum_{d=1}^{y+1}\phi_c(\Delta_2 s(c,d-1))\Delta_2 v(c,d-1) \\ &= -\sum_{d=1}^{y}\sum_{c=1}^{x}\Delta_1\phi_c(\Delta_1 s(c-1,d))v(c,d) - \sum_{c=1}^{x}\sum_{d=1}^{y}\Delta_2\phi_c(\Delta_2 s(c,d-1))v(c,d) \end{aligned}$$

and

$$\Psi'(s)(v) = \lim_{t\to 0}\frac{\Psi(s+tv)-\Psi(s)}{t} = \sum_{d=1}^{y}\sum_{c=1}^{x}\Big(f((c,d),s(c,d)) + \frac{\mu}{\lambda}g((c,d),s(c,d))\Big)v(c,d),$$

for $\forall s, v \in S$. Obviously, for any $s, v \in S$,

$$\begin{aligned} (\Phi-\lambda\Psi)'(s)(v) = -\sum_{d=1}^{y}\sum_{c=1}^{x}[&\Delta_1\phi_c(\Delta_1 s(c-1,d)) + \Delta_2\phi_c(\Delta_2 s(c,d-1)) \\ &+ \lambda f((c,d),s(c,d)) + \mu g((c,d),s(c,d))]v(c,d). \end{aligned} \tag{8}$$

Obviously, s represents a critical point of the functional $\Phi - \lambda\Psi$ in S if and only if it is demonstrated as a solution of the problem $(f^{\lambda,\mu})$. Thus, we reduce the existence of the solutions of $(f^{\lambda,\mu})$ to the existence of the critical points of $\Phi - \lambda\Psi$ on S.

Lemma 1 (Proposition 1 of [20])**.** *For each $s \in S$, $p > 1$, the following inequality can be obtained:*

$$\max_{(c,d)\in\mathbb{Z}(1,x)\times\mathbb{Z}(1,y)}\{|s(c,d)|\} \le \frac{(x+y+2)^{\frac{p-1}{p}}}{4}\|s\|. \tag{9}$$

Remark 1. *In particular, when $p = 2$, the following can be obtained:*

$$\max_{(c,d)\in\mathbb{Z}(1,x)\times\mathbb{Z}(1,y)}\{|s(c,d)|\} \leq \frac{(x+y+2)^{\frac{1}{2}}}{4}\|s\|, \quad s \in S. \tag{10}$$

Lemma 2 (Proposition 2.1 of [23])**.** *Assume that there is s: $\mathbb{Z}(0, x+1) \times \mathbb{Z}(0, y+1) \to \mathbb{R}$ and thus the following remains true:*

$$s(c,d) > 0 \ \text{ or } \ \Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1))) \leq 0, \tag{11}$$

for all $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$. Therefore, either $s(c,d) > 0$ for all $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$ or $s \equiv 0$.

The following can be obtained from Lemma 2:

Corollary 1. *Assume that there is s: $\mathbb{Z}(0, x+1) \times \mathbb{Z}(0, y+1) \to \mathbb{R}$ such that*

$$s(c,d) < 0 \ \text{ or } \ \Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1))) \geq 0, \tag{12}$$

for all $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$. Therefore, either $s(c,d) < 0$ for all $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$ or $s \equiv 0$.

Truncation techniques are adopted for discussing the presence of constant-sign solutions. We describe the truncations of the functions $f((c,d),t)$ and $g((c,d),t)$ for each $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$ as described in [22].

3. Main Results

We attempt the application of Theorem 4.3 of [30] into the function $I_\lambda^\pm : X \to \mathbb{R}$,

$$I_\lambda^\pm(s) := \Phi(s) - \lambda\Psi^\pm(s),$$

where

$$\Psi^\pm(s) = \sum_{d=1}^{y}\sum_{c=1}^{x}\left(F^\pm((c,d),s(c,d)) + \frac{\mu}{\lambda}G^\pm((c,d),s(c,d))\right),$$

$$F^\pm((c,d),s) := \int_0^s f^\pm((c,d),\tau)d\tau, \ \ G^\pm((c,d),s) := \int_0^s g^\pm((c,d),\tau)d\tau,$$

for each $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$. Later, we use Lemma 2 or Corollary 1 to acquire our results.

Let

$$A_{0^\pm} = \liminf_{t\to 0^+} \frac{\sum\limits_{d=1}^{y}\sum\limits_{c=1}^{x}\max\limits_{0\leq\xi\leq t} F((c,d),\pm\xi)}{t^2}, \quad B^{0^\pm} = \limsup_{t\to 0^\pm} \frac{\sum\limits_{d=1}^{y}\sum\limits_{c=1}^{x} F((c,d),t)}{t^2}.$$

$$C^{0^\pm} = \limsup_{t\to 0^+} \frac{\sum\limits_{d=1}^{y}\sum\limits_{c=1}^{x}\max\limits_{0\leq\xi\leq t} G((c,d),\pm\xi)}{t^2}.$$

$$\bar{\mu}_\lambda^\pm := \frac{1}{C^{0^\pm}}\left(\frac{8}{x+y+2} - \lambda A_{0^\pm}\right).$$

Theorem 1. *Define $f((c,d),s)$ as a continuous function of s, and $f((c,d),0) \geq 0$, $g((c,d),\cdot) \in C(\mathbb{R},\mathbb{R})$ for every $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$. Assume that*

(i_1) $A_{0^+} < \frac{8}{(x+y)(x+y+2)}B^{0^+}$,

(g_1) there is $\delta > 0$ such that at $[0,\delta]$, $G((c,d),s) \geq 0$ and $C^{0^+} < +\infty$.

Then, in terms of each $\lambda \in \left(\frac{x+y}{B^{0^+}}, \frac{8}{(x+y+2)A_{0^+}}\right)$ *and* $\mu \in [0, \bar{\mu}_\lambda^+)$, *problem* $(f^{\lambda,\mu})$ *possesses a sequence of positive solutions converging to zero.*

Proof. Clearly, for each $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$, $g((c,d),0) \geq 0$.

At present, this study explores the auxiliary problem (f^{λ,μ^+}).

$$-[\Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1)))] = \lambda f^+((c,d),s(c,d)) + \mu g^+((c,d),s(c,d)),\\ (c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y),$$

based on boundary conditions (1). For $t > 0$, let

$$r = -1 + \sqrt{1 + \frac{(4t)^2}{x+y+2}}.$$

Assume $s \in S$ and the below:

$$\Phi(s) = \sum_{d=1}^{y}\sum_{c=1}^{x+1}(\sqrt{1+(\Delta_1 s(c-1,d))^2} - 1) + \sum_{c=1}^{x}\sum_{d=1}^{y+1}(\sqrt{1+(\Delta_2 s(c,d-1))^2} - 1) \leq r.$$

Let $v_1(c-1,d) = \sqrt{1+(\Delta_1 s(c-1,d))^2} - 1 \geq 0, v_2(c,d-1) = \sqrt{1+(\Delta_2 s(c,d-1))^2} - 1 \geq 0$ for $(c,d) \in \mathbb{Z}(0,x) \times \mathbb{Z}(0,y)$, then $\sum\limits_{d=1}^{y}\sum\limits_{c=1}^{x+1} v_1(c-1,d) + \sum\limits_{c=1}^{x}\sum\limits_{d=1}^{y+1} v_2(c,d-1) < r$ and

$$\begin{aligned} \|s\|^2 &= \sum_{d=1}^{y}\sum_{c=1}^{x+1}(\Delta_1 s(c-1,d))^2 + \sum_{c=1}^{x}\sum_{d=1}^{y+1}(\Delta_2 s(c,d-1))^2 \\ &\leq \left[\left(\sum_{d=1}^{y}\sum_{c=1}^{x+1} v_1(c,d)\right)^2 + \left(\sum_{c=1}^{x}\sum_{d=1}^{y+1} v_2(c,d)\right)^2\right] + 2\left[\sum_{d=1}^{y}\sum_{c=1}^{x+1} v_1(c,d) + \sum_{c=1}^{x}\sum_{d=1}^{y+1} v_2(c,d)\right] \\ &= (\Phi(s))^2 + 2\Phi(s). \end{aligned} \tag{13}$$

So

$$(\Phi(s))^2 + 2\Phi(s) \leq r^2 + 2r.$$

According to (10), we have

$$\|s\|_\infty \leq \frac{(x+y+2)^{\frac{1}{2}}}{4}\left(\sum_{d=1}^{y}\sum_{c=1}^{x+1}(\Delta_1 s(c-1,d))^2 + \sum_{c=1}^{x}\sum_{d=1}^{y+1}(\Delta_2 s(c,d-1))^2\right)^{\frac{1}{2}} \leq t.$$

Therefore, we have $\Phi^{-1}[0,r] \subseteq \{s \in S : \|s\|_\infty \leq t\}$.

In line with the definition of φ, the following can be obtained.

$$\begin{aligned} \varphi(r) &= \frac{\sup\limits_{v \in \Phi^{-1}[0,r]} \Psi^+(v)}{r} \\ &\leq \frac{t^2}{r}\left(\frac{\sum\limits_{d=0}^{y}\sum\limits_{c=0}^{x}\max\limits_{0\leq\xi\leq t} F((c,d),\xi)}{t^2} + \frac{\mu}{\lambda}\frac{\sum\limits_{d=0}^{y}\sum\limits_{c=0}^{x}\max\limits_{0\leq\xi\leq t} G((c,d),\xi)}{t^2}\right). \end{aligned}$$

According to condition (i_1), (g_1), and $\lim\limits_{t\to 0^+}\frac{t^2}{r} = \lim\limits_{t\to 0^+}\frac{t^2}{-1+\sqrt{1+\frac{(4t)^2}{x+y+2}}}$, we have

$$\varphi_0 \leq \frac{x+y+2}{8}(A_{0^+} + \frac{\mu}{\lambda}C^{0^+}) < +\infty.$$

We assert that if $\lambda \in \left(\frac{x+y}{B^{0^+}}, \frac{8}{(x+y+2)A_{0^+}}\right)$, and $\mu \in [0, \bar{\mu}_\lambda^+)$, then $\lambda \in (0, \frac{1}{\varphi_0})$.
When $C^{0^+} = 0$, then

$$\varphi_0 \leq \frac{x+y+2}{8} A_{0^+} < \frac{1}{\lambda},$$

when $C^{0^+} > 0$, then

$$\begin{aligned}\varphi_0 &\leq \frac{x+y+2}{8}\left(A_{0^+} + \frac{\bar{\mu}_\lambda^+}{\lambda} C^{0^+}\right) \\ &= \frac{x+y+2}{8}\left(A_{0^+} + \frac{1}{\lambda}\frac{1}{C^{0^+}}\left(\frac{8}{x+y+2} - \lambda A_{0^+}\right) C^{0^+}\right) \\ &= \frac{1}{\lambda}.\end{aligned}$$

Obviously, $(0, 0, \cdots, 0) \in S$ is a global minimum of Φ.

Then, it is necessary to demonstrate that $(0, 0, \cdots, 0)$ is not a local minima of I_λ^+.

When $B^{0^+} = +\infty$, we find $\{l_t\} \subset (0, \delta)$ such that $\lim_{t \to +\infty} l_t = 0$, and

$$\sum_{d=1}^{y}\sum_{c=1}^{x} F^+((c,d), l_t) = \sum_{d=1}^{y}\sum_{c=1}^{x} F((c,d), l_t) \geq \frac{2(x+y)l_t^2}{\lambda}, \quad \text{for } t \in \mathbb{Z}(1).$$

Define a sequence $\{\eta_t\}$ in S with

$$\eta_t(c,d) = \begin{cases} l_t, & (c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y), \\ 0, & c = 0, d \in \mathbb{Z}(0, y+1) \text{ or } c = x+1, d \in \mathbb{Z}(0, y+1), \\ 0, & d = 0, c \in \mathbb{Z}(0, x+1) \text{ or } d = y+1, c \in \mathbb{Z}(0, x+1). \end{cases}$$

Based on $G^+((c,d), \eta_t(c,d)) = G((c,d), \eta_t(c,d)) \geq 0$, $\forall (c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$, we can acquire:

$$\begin{aligned}I_\lambda^+(\eta_t) &\leq 2(x+y)\left(\sqrt{1+l_t^2} - 1\right) - \lambda\left(\sum_{d=1}^{y}\sum_{c=1}^{x} F((c,d), l_t)\right) \\ &\leq -(x+y)l_t^2 \\ &< 0.\end{aligned}$$

When $B^{0^+} < +\infty$, let $\lambda \in \left(\frac{x+y}{B^{0^+}}, \frac{8}{(x+y+2)A_{0^+}}\right)$, choose $\varepsilon_0 > 0$ such that

$$x + y - \lambda(B^{0^+} - \varepsilon_0) < 0.$$

Next, there exists $\{l_t\} \subset (0, \delta)$ such that $\lim_{t \to +\infty} l_t = 0$ and

$$(B^{0^+} - \varepsilon_0)l_t^2 \leq \sum_{d=1}^{y}\sum_{c=1}^{x} F^+((c,d), l_t) = \sum_{d=1}^{y}\sum_{c=1}^{x} F((c,d), l_t) \leq (B^{0^+} + \varepsilon_0)l_t^2.$$

Based on the definition of the sequence $\{\eta_t\}$ in S being the same as the case where $B^{0^+} = +\infty$, we hold:

$$\begin{aligned}I_\lambda^+(\eta_t) &\leq 2(x+y)\left(\sqrt{1+l_t^2} - 1\right) - \lambda\left(\sum_{d=1}^{y}\sum_{c=1}^{x} F((c,d), l_t)\right) \\ &\leq (x+y)l_t^2 - \lambda(B^{0^+} - \varepsilon_0)l_t^2 \\ &= (x+y - \lambda(B^{0^+} - \varepsilon_0))l_t^2 \\ &< 0.\end{aligned}$$

According to the above discussion, we have $I_\lambda^+(\eta_t) < 0$.

According to $I_\lambda^+(0,0,\cdots,0) = 0$ and $\lim_{t\to\infty} \eta_t = (0,0,\cdots,0)$, we obtain that $(0,0,\cdots,0)$ does not indicate a local minima of I_λ^+.

So, the entire conditions of Theorem 4.3 of [30] are obtained. According to Theorem 4.3 of [30], $\forall(c,d) \in \mathbb{Z}(1,x)\times\mathbb{Z}(1,y)$, problem (f^{λ,μ^+}) has a non-zero solution $s(c,d)$, and according to Lemma 2, problem $(f^{\lambda,\mu})$ possesses a positive solution $s(c,d)$. $\square$

Under the condition that $\lambda = 1$, in accordance with Theorem 1, we have

Corollary 2. *Let $f((c,d),s)$ be the same as defined in Theorem 1, and $f((c,d),0) \geq 0$, $g((c,d),\cdot)$ is defined like in Theorem 1. Therefore, it can be assumed that*
(i_2) $\frac{(x+y)(x+y+2)}{8} A_{0^+} < 1 < B^{0^+}$,
(g_1) *such as the condition of* (g_1) *in Theorem 1.*
Then, in terms of each $\mu \in [0, \bar{\mu}_1^+)$, the problem below (f^μ)

$$-[\Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1)))] = f((c,d),s(c,d)) + \mu g((c,d),s(c,d)),$$
$$(c,d) \in \mathbb{Z}(1,x)\times\mathbb{Z}(1,y),$$

with (1), obtains the same result as Theorem 1.

Similarly, the following results can be obtained.

Theorem 2. *Let $f((c,d),s)$ be the same as defined in Theorem 1, and $f((c,d),0) \leq 0$, $g((c,d),\cdot)$ is defined like in Theorem 1. Assume that*
(i_3) $A_{0^-} < \frac{8}{(x+y)(x+y+2)} B^{0^-}$,
(g_2) *there is $\delta > 0$ such that at $[-\delta, 0]$, $G((c,d),s) \geq 0$ and $C^{0^-} < +\infty$.*
Then, in terms of each $\lambda \in \left(\frac{x+y}{B^{0^-}}, \frac{8}{(x+y+2)A_{0^-}}\right)$, and $\mu \in [0, \bar{\mu}_\lambda^-)$, problem $(f^{\lambda,\mu})$ possesses a sequence of negative solutions converging to zero.

With $\lambda = 1$, in accordance with Theorem 2, we have

Corollary 3. *Let $f((c,d),s)$ be the same as defined in Theorem 1, and $f((c,d),0) \leq 0$, $g((c,d),\cdot)$ is defined like in Theorem 1. Assume that*
(i_4) $\frac{(x+y)(x+y+2)}{8} A_{0^-} < 1 < B^{0^-}$,
(g_2) *such as the condition of* (g_2) *in Theorem 2.*
In terms of each $\mu \in [0, \bar{\mu}_1^-)$, the problem considered has the same conclusion as Theorem 2.

By the combination of Theorem 1 with Theorem 2, the following can be obtained:

Theorem 3. *Let $f((c,d),s)$ be the same as defined in Theorem 1, and $f((c,d),0) = 0$, $g((c,d),\cdot)$ is defined like in Theorem 1. Assume that*
(i_5) $\max\{A_{0^+}, A_{0^-}\} < \frac{8}{(x+y)(x+y+2)} \min\{B^{0^+}, B^{0^-}\}$,
(g_3) *there is $\delta > 0$ such that at $[-\delta, \delta]$, $G((c,d),s) \geq 0$ and $C^{0^\pm} < +\infty$.*
Subsequently, for every $\lambda \in \left(\frac{x+y}{\min\{B^{0^+}, B^{0^-}\}}, \frac{8}{(x+y+2)\max\{A_{0^+}, A_{0^-}\}}\right)$, and $\mu \in [0, \min\{\bar{\mu}_\lambda^+, \bar{\mu}_\lambda^-\})$, problem $(f^{\lambda,\mu})$ possesses two sequences of constant-sign solutions converging to zero (one positive and one negative).

Under the condition that $\lambda = 1$, in accordance with Theorem 3, we have

Corollary 4. *Let $f((c,d),s)$ be the same as defined in Theorem 1, and $f((c,d),0) = 0$, $g((c,d),\cdot)$ is defined like in Theorem 1. Assume that*
(i_6) $\frac{(x+y)(x+y+2)}{8} A_{0^\pm} < 1 < B^{0^\pm}$,

(g_3) *such as the condition of* (g_3) *in Theorem 3.*
In terms of each $\mu \in [0, \min\{\bar{\mu}_1^+, \bar{\mu}_1^-\})$, *the problem considered has the same conclusion as Theorem 3.*

Remark 2. *Consider the problem* (f^λ)

$$-[\Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1)))] = \lambda f((c,d), s(c,d)), \ \ (c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y),$$

with (1).

Theorem 4. *Let* $f((c,d),s)$ *be the same as defined in Theorem 1, and* $f((c,d),0) \geq 0$ *for every* $(c,d) \in \mathbb{Z}(1,x) \times \mathbb{Z}(1,y)$. *Assume that*
(i_7) $A_{0^+} < \frac{8}{(x+y)(x+y+2)} B^{0^+}$.
Then, for each $\lambda \in \left(\frac{x+y}{B^{0^+}}, \frac{8}{(x+y+2)A_{0^+}}\right)$, *the problem considered has the same conclusion as Theorem 1.*

4. Examples

Let us explain Theorem 1 with one example.

Example 1. *Suppose* $x = 2$, $y = 2$, $\varepsilon = 0.01$ *and the definitions of functions* f *and* g *are given as shown below.*

$$f((c,d),t) = f(t) = \begin{cases} t(2 + 4\varepsilon + 2\cos(\varepsilon \ln t) - \varepsilon \sin(\varepsilon \ln t)), & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{14}$$

and

$$g((c,d),t) = g(t) = 10\varepsilon t. \tag{15}$$

Then, for each $\lambda_1 \in (0.495, 16.666)$ *and* $\mu_1 \in [0, (\frac{20}{3} - \frac{2}{5}\lambda_1))$, *the following problem, namely* (f^{λ_1,μ_1})

$$\begin{aligned} -[\Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1)))] = \lambda_1 f((c,d), s(c,d)) + \mu_1 g((c,d), s(c,d)), \\ (c,d) \in \mathbb{Z}(1,2) \times \mathbb{Z}(1,2) \end{aligned}$$

based on boundary conditions

$$s(c,0) = s(c,3) = 0, \quad c \in \mathbb{Z}(0,3),$$

$$s(0,d) = s(3,d) = 0, \quad d \in \mathbb{Z}(0,3),$$

obtains the same result as Theorem 1.

In fact,

$$F((c,d),t) = \int_0^t f((c,d),\tau)d\tau = \begin{cases} t^2(1 + 2\varepsilon + \cos(\varepsilon \ln t)), & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{16}$$

$$G((c,d),t) = \int_0^t g((c,d),\tau)d\tau = 5\varepsilon t^2. \tag{17}$$

Since $f((c,d),t) > 0, g((c,d),t) > 0$ *for* $t > 0$, *it can be known* $F((c,d),t)$ *and* $G((c,d),t)$ *are increasing in* $t \in (0, +\infty)$. *Thus,* $\max_{0 \leq \xi \leq t} F((c,d), \pm\xi) = F((c,d),t)$ *and* $\max_{0 \leq \xi \leq t} G((c,d), \pm\xi) = G((c,d),t)$, *for every* $t \geq 0$. *Obviously,*

$$A_{0^\pm} = \liminf_{t \to 0^+} \frac{xyF((c,d),t)}{t^2} = \liminf_{t \to 0^+} \frac{4t^2(1 + 2\varepsilon + \cos(\varepsilon \ln t))}{t^2} = 0.08,$$

$$B^{0^\pm} = \limsup_{t\to 0^\pm} \frac{xyF((c,d),t)}{t^2} = \limsup_{t\to 0^\pm} \frac{4t^2(1+2\varepsilon+\cos(\varepsilon \ln t))}{t^2} = 8.08.$$

The condition (i_1) of Theorem 1 can be confirmed, since

$$A_{0^+} = 0.08 < \frac{8}{(x+y)(x+y+2)} B^{0^+} \approx 2.69.$$

Subsequently, the condition (g_1) of Theorem 1 can be further verified, since

$$C^{0^\pm} = \limsup_{t\to 0^+} \frac{\sum_{d=1}^{y}\sum_{c=1}^{x} G((c,d),t)}{t^2} = \limsup_{t\to 0^+} \frac{5xyt^2\varepsilon}{t^2} = 0.2 < +\infty.$$

To sum up, the entire conditions of Theorem 1 are obtained.

Therefore, for each $\lambda_1 \in (0.495, 16.666)$ and $\mu_1 \in [0, (\frac{20}{3} - \frac{2}{5}\lambda_1))$, problem (f^{λ_1,μ_1}) has the same conclusion as Theorem 1.

Let us explain Theorem 3 with another example.

Example 2. *Suppose $x = 2$, $y = 2$, and the definitions of functions f and g are given as shown below.*

$$f((c,d),t) = f(t) = \begin{cases} \frac{5}{2}t + 2t\sin(\frac{1}{5}\ln t^2) + \frac{2}{5}t\cos(\frac{1}{5}\ln t^2), & t > 0, \\ 0, & t \le 0, \end{cases} \tag{18}$$

and

$$g((c,d),t) = g(t) = 6t. \tag{19}$$

Then, for each $\lambda_2 \in (\frac{4}{9}, \frac{4}{3})$ and $\mu_2 \in [0, (\frac{1}{9} - \frac{1}{12}\lambda_2))$, the following problem, namely (f^{λ_2,μ_2})

$$-[\Delta_1(\phi_c(\Delta_1 s(c-1,d))) + \Delta_2(\phi_c(\Delta_2 s(c,d-1)))] = \lambda_2 f((c,d),s(c,d)) + \mu_2 g((c,d),s(c,d)),$$
$$(c,d) \in \mathbb{Z}(1,2) \times \mathbb{Z}(1,2)$$

based on boundary conditions

$$s(c,0) = s(c,3) = 0, \quad c \in \mathbb{Z}(0,3),$$

$$s(0,d) = s(3,d) = 0, \quad d \in \mathbb{Z}(0,3),$$

obtains the same conclusion as Theorem 3.

Actually,

$$F((c,d),t) = \int_0^t f((c,d),\tau)d\tau = \begin{cases} \frac{5}{4}t^2 + t^2\sin(\frac{1}{5}\ln t^2), & t > 0, \\ 0, & t \le 0, \end{cases} \tag{20}$$

$$G((c,d),t) = \int_0^t g((c,d),\tau)d\tau = 3t^2. \tag{21}$$

Since $f((c,d),t) > 0, g((c,d),t) > 0$ for $t > 0$, it can be known $F((c,d),t)$ and $G((c,d),t)$ are increasing in $t \in (0,+\infty)$. Thus, $\max_{0\le\xi\le t} F((c,d),\pm\xi) = F((c,d),t)$ and $\max_{0\le\xi\le t} G((c,d),\pm\xi) = G((c,d),t)$, for every $t \ge 0$. Obviously,

$$A_{0^\pm} = \liminf_{t\to 0^+} \frac{xyF((c,d),t)}{t^2} = \liminf_{t\to 0^+} \frac{4(\frac{5}{4}t^2 + t^2\sin(\frac{1}{5}\ln t^2))}{t^2} = 1,$$

$$B^{0^{\pm}} = \limsup_{t\to 0^{\pm}} \frac{xyF((c,d),t)}{t^2} = \limsup_{t\to 0^{\pm}} \frac{4(\frac{5}{4}t^2 + t^2\sin(\frac{1}{5}\ln t^2))}{t^2} = 9.$$

The condition (i_5) of Theorem 3 can be verified, since

$$\max\{A_{0^+}, A_{0^-}\} = 1 < \frac{8}{(x+y)(x+y+2)} \min\{B^{0^+}, B^{0^-}\} = 3.$$

Subsequently, the condition (g_3) of Theorem 3.3 can be further verified, since

$$C^{0^{\pm}} = \limsup_{t\to 0^{+}} \frac{\sum_{d=1}^{y}\sum_{c=1}^{x} G((c,d),t)}{t^2} = \limsup_{t\to 0^{+}} \frac{3xyt^2}{t^2} = 12 < +\infty.$$

To sum up, the entire conditions of Theorem 3 are obtained.

Therefore, for each $\lambda_2 \in (\frac{4}{9}, \frac{4}{3})$ and $\mu_2 \in [0, (\frac{1}{9} - \frac{1}{12}\lambda_2))$, problem (f^{λ_2,μ_2}) has the same conclusion as Theorem 3.

5. Discussion

In [16], the problem considered by the authors contains only one discrete variable. Unlike [16], the present study considers the partial difference equations with ϕ_c-Laplacian and the equations possess two discrete variables. In [21], the author focused on the three solutions of the PDE, with the primary tool being to refer to Theorem 2.1 in [31]. As a result, the method and the findings show difference from those in the previous studies. This study is the initial attempt to focus on the infinitely many solutions of the partial difference equations with ϕ_c-Laplacian, which is more complex to address. It is known that the establishment of variational structures is more complicated when considering the PDE including the mean curvature operator.

6. Conclusions

To conclude, the current work focuses on investigating the presence of small solutions of the perturbed PDE with ϕ_c-Laplacian. Different from the findings presented in [18], the present study obtains the presence of infinitely many solutions, which can be found in Theorems 1–3. Based on Theorem 4.3 of [30] and Lemma 2 of the present study, this study obtains a sequence of positive solutions converging to zero, as presented in Theorem 1. Moreover, with the application of truncation techniques, this work acquires two sequences of C-S solutions converging to zero. We find that one is positive while the other is negative. This work solves the presence of infinitely many small solutions to the boundary value problem of the PDE, and the presence of large C-S solutions of PDE with ϕ_c-Laplacian can be studied as future research problems.

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