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On New Estimates of q-Hermite–Hadamard Inequalities with Applications in Quantum Calculus

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Abstract: In this paper, we first establish two quantum integral (q-integral) identities with the help of derivatives and integrals of the quantum types. Then, we prove some new q-midpoint and q-trapezoidal estimates for the newly established q-Hermite-Hadamard inequality (involving left and right integrals proved by Bermudo et al.) under q-differentiable convex functions. Finally, we provide some examples to illustrate the validity of newly obtained quantum inequalities.

Keywords: Hermite-Hadamard inequality; q-integral; quantum calculus; convex function

MSC: 26D10; 26D15; 26A51

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1. Introduction

In recent studies, fractional calculus has proved to be the among of the most widely used areas of mathematical science. This is because, we can see the activities of researchers in this field. Besides, there have been published papers in which fixed point theorems play a key role in existence results for given fractional differential equations [1–3]. Due to the expansion of this branch of mathematics, mathematicians studied a new field in which the concept of limit has no role in the definitions of operators. Also, because of the fundamental role of the quantum parameter q, they called it the theory of quantum fractional calculus. The initial steps in this field were taken by Jackson [4,5] and then, it was extended to more practical fields such as combinatorics, quantum mechanics, discrete mathematics, hypergeometric series, particle physics, and theory of relativity. To remember and fully understand the concepts of q-calculus, one can mention the sources [6–8].

Recently, different quantum initial value problems (IVPs) and boundary value problems (BVPs) have been given and discussed by some methods including the fixed-point theorems, lower-upper solutions, or iteration techniques. To demonstrate such applications, we can mention oscillation on q-difference inclusions [9], multi-order q-BVPs [10], p-Laplacian q-difference equations [11], q-symmetric problems [12], singular q-problems [13], q-integro-equations [14], q-delay equations [15], q-intego-equations on time scales [16], and so on [17–19].

Consider the function $\rho : I \to \mathbb{R}$ so that I is a real interval. Then, ρ is called a convex function if

$$\rho(tx + (1-t)y) \leq t\rho(x) + (1-t)\rho(y)$$

holds for each $t \in [0,1]$ and $x, y \in I$.

From [20], it is established that ρ is convex if and only if ρ satisfies the Hermite-Hadamard inequality, formulated as

$$\rho\left(\frac{\nu+\sigma}{2}\right) \leq \frac{1}{\sigma-\nu}\int_{\nu}^{\sigma}\rho(x)dx \leq \frac{\rho(\nu)+\rho(\sigma)}{2}, \tag{1}$$

for each $\nu, \sigma \in I$ with $\nu < \sigma$.

On the other hand, Alp et al. [21] proved a new structure of quantum type of the Hermite-Hadamard inequality for convex mappings via the left q-integrals, and stated it as follows

$$\rho\left(\frac{q\nu+\sigma}{[2]_q}\right) \leq \frac{1}{\sigma-\nu}\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_q x \leq \frac{q\rho(\nu)+\rho(\sigma)}{[2]_q}. \tag{2}$$

In 2020, Bermudo, Kórus and Valdés [22] applied the right q-integral to derive the right variant of the above inequality; i.e.,

$$\rho\left(\frac{\nu+q\sigma}{[2]_q}\right) \leq \frac{1}{\sigma-\nu}\int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_q x \leq \frac{\rho(\nu)+q\rho(\sigma)}{[2]_q}. \tag{3}$$

Remark 1. *From inequalities* (2) *and* (3)*, the following two-sided inequality of Hermite–Hadamard type is obtained (see, [22]):*

$$\begin{aligned}\rho\left(\frac{\nu+\sigma}{2}\right) &\leq \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_q x + \int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_q x\right] \\ &\leq \frac{\rho(\nu)+\rho(\sigma)}{2}.\end{aligned} \tag{4}$$

About the left and right inequalities (2) and (3), one can consult [23–30]. In [31], Noor et al. established an extended version of (2). In [32–35], the authors got help from two families of the convex and coordinated convex mappings for proving the Newton and Simpson's type inequalities in the context of quantum calculus. Moreover, to investigate different versions of the Ostrowski's inequalities, see [36,37].

Motivated by the ongoing research, we obtain another version of q-Hermite–Hadamard inequality in consideration of convex mappings, and prove some new q-midpoint type inequalities for convex mappings of the q-differentiable type. Also, in some examples, we show that the newly obtained inequalities are the generalizations of the existing Hermite-Hadamard inequality and midpoint inequalities. These new results can be used for finding some error bounds for the midpoint and trapezoidal rules in q-integration formulas that are very important in the field of numerical analysis.

This paper is organized as follows: The basics of quantum calculus along with other topics in the present area are addressed briefly in the next section. In Sections 3 and 4, some q-midpoint and q-trapezoid type estimates are studied for the inequality (4) under the q-differentiable functions. The connection between our results and other results in the literature are also stated. We provide some mathematical examples in Section 5 to demonstrate the validity of the newly developed inequalities. Section 6 concludes the paper by giving some ideas for the future.

2. Preliminaries of q-Calculus

In the preliminaries, we collect the definitions and several properties of quantum operators. Along with these, some famous inequalities are restated with respect to quantum integrals. In the whole of the article, $0 < q < 1$ is constant.

The q-analogue of $n \in \mathbb{N}$ is a special sum of q-powers. It is defined as

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}. \tag{5}$$

The q-Jackson integral for the function ρ on $[0, \sigma]$ is given by [4]

$$\int_0^\sigma \rho(x) d_q x = (1-q)\sigma \sum_{n=0}^{\infty} q^n \rho(\sigma q^n), \tag{6}$$

and q-Jackson integral for a function ρ defined on $[\nu, \sigma]$ is given as [4]

$$\int_\nu^\sigma \rho(x) d_q x = \int_0^\sigma \rho(x) d_q x - \int_0^\nu \rho(x) d_q x. \tag{7}$$

Definition 1 ([38]). *Let $\rho : [\nu, \sigma] \to \mathbb{R}$ be continuous. The left q-derivative of ρ at $x \in [\nu, \sigma]$ is defined by*

$${}_\nu D_q \rho(x) = \frac{\rho(x) - \rho(qx + (1-q)\nu)}{(1-q)(x-\nu)}, \ x \neq \nu. \tag{8}$$

If $\nu = 0$ and ${}_0 D_q \rho(x) = D_q \rho(x)$, then (8) becomes

$$D_q \rho(x) = \frac{\rho(x) - \rho(qx)}{(1-q)x}, \ x \neq 0.$$

It is the same q-Jackson derivative [4,38,39].

Definition 2 ([38]). *Let $\rho : [\nu, \sigma] \to \mathbb{R}$ be continuous. The left q-integral of ρ at $z \in [\nu, \sigma]$ is defined by*

$$\int_\nu^z \rho(x) {}_\nu d_q x = (1-q)(z-\nu) \sum_{n=0}^{\infty} q^n \rho(q^n z + (1-q^n)\nu). \tag{9}$$

If $\nu = 0$, then (9) becomes

$$\int_0^z \rho(x) {}_0 d_q x = \int_0^z \rho(x) d_q x = (1-q)z \sum_{n=0}^{\infty} q^n \rho(q^n z).$$

It is the same q-Jackson integral [4,38,39].

Later, Bermudo et al. extended the following new quantum operators, which are introduced as the right q-operators.

Definition 3 ([22]). *The right q-derivative of $\rho : [\nu, \sigma] \to \mathbb{R}$ is given by*

$${}^\sigma D_q \rho(x) = \frac{\rho(qx + (1-q)\sigma) - \rho(x)}{(1-q)(\sigma - x)}, \quad x \neq \sigma.$$

Definition 4 ([22]). *The right q-definite integral of $\rho : [\nu, \sigma] \to \mathbb{R}$ on $[\nu, \sigma]$ is given by*

$$\int\limits_\nu^\sigma \rho(x) {}^\sigma d_q x = (1-q)(\sigma - \nu) \sum_{k=0}^{\infty} q^k \rho\left(q^k \nu + \left(1 - q^k\right)\sigma\right).$$

Lemma 1 ([40]). *The equality*

$$\begin{aligned}
&\int_0^c \kappa(t)\ {}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)d_qt \\
=\ & \frac{1}{\sigma-\nu}\int_0^c D_q\kappa(t)\rho(qt\nu+(1-qt)\sigma)d_qt - \left.\frac{\kappa(t)\rho(t\nu+(1-t)\sigma)}{\sigma-\nu}\right|_0^c,
\end{aligned}$$

holds if $\rho,\kappa : [\nu,\sigma] \to \mathbb{R}$ *are continuous.*

Lemma 2 ([41]). *The equality*

$$\begin{aligned}
&\int_0^c \kappa(t)_{\nu}D_q\rho(t\sigma+(1-t)\nu)d_qt \\
=\ & \left.\frac{\kappa(t)\rho(t\sigma+(1-t)\nu)}{\sigma-\nu}\right|_0^c - \frac{1}{\sigma-\nu}\int_0^c D_q\kappa(t)\rho(qt\sigma+(1-qt)\nu)d_qt.
\end{aligned}$$

holds if $\rho,\kappa : [\nu,\sigma] \to \mathbb{R}$ *are continuous.*

3. *q*-Trapezoidal Inequalities

In this section, we establish some right estimates of the inequality (4) using differentiable convex functions.

Lemma 3. *If* $\rho : [\nu,\sigma] \subset \mathbb{R} \to \mathbb{R}$ *is q-differentiable such that* ${}_{\nu}D_q\rho$ *and* ${}^{\sigma}D_q\rho$ *are integrable and continuous on* $[\nu,\sigma]$, *then*

$$\begin{aligned}
&\frac{\rho(\nu)+\rho(\sigma)}{2} - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx + \int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_qx\right] \qquad (10)\\
=\ & \frac{\sigma-\nu}{2}\left[\int_0^1 qt_{\nu}D_q\rho(t\sigma+(1-t)\nu)d_qt + \int_0^1 qt\ {}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)d_qt\right].
\end{aligned}$$

Proof. By Lemma 2, we compute

$$\begin{aligned}
I_1 &= \int_0^1 qt\ {}_{\nu}D_q\rho(t\sigma+(1-t)\nu)d_qt \\
&= qt\left.\frac{\rho(t\sigma+(1-t)\nu)}{\sigma-\nu}\right|_0^1 - \frac{q}{\sigma-\nu}\int_0^1\rho(t\sigma+(1-t)\nu)d_qt \\
&= q\frac{\rho(\sigma)}{\sigma-\nu} - \frac{q}{\sigma-\nu}\left\{\frac{1-q}{q}\sum_{n=0}^{\infty}q^n\rho(q^n\sigma+(1-q^n)\nu) - \frac{(1-q)}{q}\rho(\sigma)\right\} \\
(\sigma-\nu)I_1 &= \rho(\sigma) - \frac{1}{\sigma-\nu}\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx. \qquad (11)
\end{aligned}$$

Similarly, from Lemma 1, we get

$$\begin{aligned}
I_2 &= \int_0^1 qt\ {}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)d_qt \\
&= -qt\left.\frac{\rho(t\nu+(1-t)\sigma)}{\sigma-\nu}\right|_0^1 + \frac{q}{\sigma-\nu}\int_0^1\rho(t\nu+(1-t)\sigma)d_qt
\end{aligned}$$

$$
\begin{aligned}
&= -q\frac{\rho(\nu)}{\sigma-\nu} + \frac{q}{\sigma-\nu}\left\{\frac{1-q}{q}\sum_{n=0}^{\infty} q^n \rho(q^n\nu + (1-q^n)\sigma) - \frac{(1-q)}{q}\rho(\nu)\right\} \\
(\sigma-\nu)I_2 &= \frac{1}{\sigma-\nu}\int_{\nu}^{\sigma} \rho(x)\ {}^{\sigma}d_q x - \rho(\nu). \qquad (12)
\end{aligned}
$$

Thus, we obtain the desired identity by combining (11) and (12). □

Theorem 1. *Under the hypotheses of Lemma 3, we have the following inequality if* $\left|{}^{\sigma}D_q\rho\right|$ *and* $\left|{}_{\nu}D_q\rho\right|$ *are convex:*

$$
\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2} - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_q x + \int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_q x\right]\right| \qquad (13) \\
\leq\ & \frac{q(\sigma-\nu)}{2[3]_q}\left[\left|{}_{\nu}D_q\rho(\sigma)\right| + \left|{}^{\sigma}D_q\rho(\nu)\right| + \frac{q^2\left(\left|{}_{\nu}D_q\rho(\nu)\right| + \left|{}^{\sigma}D_q\rho(\sigma)\right|\right)}{[2]_q}\right].
\end{aligned}
$$

Proof. From Lemma 3 and using the convexity of $\left|{}_{\nu}D_q\rho\right|$ and $\left|{}^{\sigma}D_q\rho\right|$, we obtain

$$
\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2} - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_q x + \int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_q x\right]\right| \\
\leq\ & \frac{\sigma-\nu}{2}\left[\int_0^1 qt\left|{}_{\nu}D_q\rho(t\sigma+(1-t)\nu)d_qt\right| + \int_0^1 qt\ \left|{}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)d_qt\right|\right] \\
\leq\ & \frac{\sigma-\nu}{2}\left[\int_0^1 qt\left\{t\left|{}_{\nu}D_q\rho(\sigma)\right| + (1-t)\left|{}_{\nu}D_q\rho(\nu)\right|\right\}d_qt\right. \\
&\left. + \int_0^1 qt\left\{t\left|{}^{\sigma}D_q\rho(\nu)\right| + (1-t)\left|{}^{\sigma}D_q\rho(\sigma)\right|\right\}d_qt\right] \\
=\ & \frac{\sigma-\nu}{2}\left[\left|{}_{\nu}D_q\rho(\sigma)\right|\int_0^1 qt^2 d_qt + \left|{}_{\nu}D_q\rho(\nu)\right|\int_0^1 qt(1-t)d_qt\right. \\
&\left. + \left|{}^{\sigma}D_q\rho(\nu)\right|\int_0^1 qt^2 d_qt + \left|{}^{\sigma}D_q\rho(\sigma)\right|\int_0^1 qt(1-t)d_qt\right] \\
=\ & \frac{\sigma-\nu}{2}\left[\left|{}_{\nu}D_q\rho(\sigma)\right|\frac{q}{[3]_q} + \left|{}_{\nu}D_q\rho(\nu)\right|\frac{q^3}{[2]_q[3]_q}\right. \\
&\left. + \left|{}^{\sigma}D_q\rho(\nu)\right|\frac{q}{[3]_q} + \left|{}^{\sigma}D_q\rho(\sigma)\right|\frac{q^3}{[2]_q[3]_q}\right] \\
=\ & \frac{\sigma-\nu}{2}\left[\left\{\left|{}_{\nu}D_q\rho(\sigma)\right| + \left|{}^{\sigma}D_q\rho(\nu)\right|\right\}\left\{\frac{q}{[3]_q}\right\}\right. \\
&\left. + \left\{\left|{}_{\nu}D_q\rho(\nu)\right| + \left|{}^{\sigma}D_q\rho(\sigma)\right|\right\}\left\{\frac{q^3}{[2]_q[3]_q}\right\}\right],
\end{aligned}
$$

which completes the proof. □

Theorem 2. *Under all the hypotheses of Lemma 3, we have the following inequality if* $\left|{}^{\sigma}D_q\rho\right|^{p_1}$ *and* $\left|{}_{\nu}D_q\rho\right|^{p_1}$*,* $p_1 \geq 1$ *are convex:*

$$\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2} - \frac{1}{\sigma-\nu}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx + \int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_qx\right]\right| \\
\leq\ & \frac{q(\sigma-\nu)}{2[2]_q}\left[\left(\frac{[2]_q\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1} + q^2\left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\frac{[2]_q\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1} + q^2\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right].
\end{aligned} \tag{14}$$

Proof. The power mean inequality and Lemma 3 give

$$\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2} - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx + \int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_qx\right]\right| \\
\leq\ & \frac{\sigma-\nu}{2}\left[\int_0^1 qt\left|{}_{\nu}D_q\rho(t\sigma+(1-t)\nu)\right|d_qt + \int_0^1 qt\ \left|{}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)\right|d_qt\right] \\
\leq\ & \frac{(\sigma-\nu)}{2}\left[\left(\int_0^1 qtd_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 qt\left|{}_{\nu}D_q\rho(t\sigma+(1-t)\nu)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_0^1 qtd_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 qt\ \left|{}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

By the convexity of $\left|{}_{\nu}D_q\rho\right|^{p_1}$ and $\left|{}^{\sigma}D_q\rho\right|^{p_1}$, we have

$$\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2} - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx + \int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_qx\right]\right| \\
\leq\ & \frac{(\sigma-\nu)}{2}\left[\left(\int_0^1 qtd_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 qt\left\{t\left|{}_{\nu}D_q\rho(\sigma)\right| + (1-t)\left|{}_{\nu}D_q\rho(\nu)\right|\right\}d_qt\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_0^1 qtd_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 qt\left\{t\left|{}^{\sigma}D_q\rho(\nu)\right| + (1-t)\left|{}^{\sigma}D_q\rho(\sigma)\right|\right\}d_qt\right)^{\frac{1}{p_1}}\right] \qquad (15) \\
=\ & \frac{q(\sigma-\nu)}{2[2]_q}\left[\left(\frac{[2]_q\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1} + q^2\left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\frac{[2]_q\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1} + q^2\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Thus, the proof is completed. □

Theorem 3. *Under the hypotheses of Lemma 3, the following inequality is satisfied if* $\left|{}^{\sigma}D_q\rho\right|^{p_1}$ *and* $\left|{}_{\nu}D_q\rho\right|^{p_1}$, $p_1>1$ *are convex:*

$$\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2}-\frac{1}{2(\sigma-\nu)}\left[\int\limits_{\nu}^{\sigma}\rho(x)\;{}_{\nu}d_qx+\int\limits_{\nu}^{\sigma}\rho(x)\;{}^{\sigma}d_qx\right]\right|\\
\leq\;&\frac{q(\sigma-\nu)}{2}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left[\left(\frac{\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1}+q\left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\frac{\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}+q\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right],
\end{aligned}\tag{16}$$

where $r_1^{-1}+p_1^{-1}=1$.

Proof. The Hölder inequality and Lemma 3 give

$$\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2}-\frac{1}{2(\sigma-\nu)}\left[\int\limits_{\nu}^{\sigma}\rho(x)\;{}_{\nu}d_qx+\int\limits_{\nu}^{\sigma}\rho(x)\;{}^{\sigma}d_qx\right]\right|\\
\leq\;&\frac{\sigma-\nu}{2}\left[\int\limits_{0}^{1}qt\left|{}_{\nu}D_q\rho(t\sigma+(1-t)\nu)\right|d_qt+\int\limits_{0}^{1}qt\;\left|{}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)\right|d_qt\right]\\
\leq\;&\frac{(\sigma-\nu)}{2}\left[\left(\int_0^1(qt)^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int\limits_{0}^{1}\left|{}_{\nu}D_q\rho(t\sigma+(1-t)\nu)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\int_0^1(qt)^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int\limits_{0}^{1}\;\left|{}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

By the convexity of $\left|{}_{\nu}D_q\rho\right|^{p_1}$ and $\left|{}^{\sigma}D_q\rho\right|^{p_1}$, we have

$$\begin{aligned}
&\left|\frac{\rho(\nu)+\rho(\sigma)}{2}-\frac{1}{2(\sigma-\nu)}\left[\int\limits_{\nu}^{\sigma}\rho(x)\;{}_{\nu}d_qx+\int\limits_{\nu}^{\sigma}\rho(x)\;{}^{\sigma}d_qx\right]\right|\\
\leq\;&\frac{q(\sigma-\nu)}{2}\left[\left(\int_0^1t^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int\limits_{0}^{1}\left\{t\left|{}_{\nu}D_q\rho(\sigma)\right|+(1-t)\left|{}_{\nu}D_q\rho(\nu)\right|\right\}d_qt\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\int_0^1t^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int\limits_{0}^{1}\left\{t\left|{}^{\sigma}D_q\rho(\nu)\right|+(1-t)\left|{}^{\sigma}D_q\rho(\nu)\right|\right\}d_qt\right)^{\frac{1}{p_1}}\right]\\
=\;&\frac{q(\sigma-\nu)}{2}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left[\left(\frac{\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1}+q\left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right.\\
&\left.+\left(\frac{\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}+q\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Thus, the proof is completed. □

4. *q*-Midpoint Inequalities

In this section, we establish some right estimates of inequality (4) for differentiable convex functions.

Lemma 4. *If $\rho : [\nu, \sigma] \subset \mathbb{R} \to \mathbb{R}$ is q-differentiable such that ${}_{\nu}D_q\rho$ and ${}^{\sigma}D_q\rho$ are integrable and continuous on $[\nu, \sigma]$, then*

$$\begin{aligned} &\rho\left(\frac{\nu+\sigma}{2}\right) - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma} \rho(x)\ {}_{\nu}d_q x + \int_{\nu}^{\sigma} \rho(x)\ {}^{\sigma}d_q x\right] \\ = &\frac{\sigma-\nu}{2}\left[\int_0^{\frac{1}{2}} qt\ {}_{\nu}D_q\rho(t\sigma+(1-t)\nu)d_q t + \int_{\frac{1}{2}}^{1} (qt-1)\ {}_{\nu}D_q\rho(t\sigma+(1-t)\nu)d_q t \right. \\ &\left. + \int_0^{\frac{1}{2}} (-qt)\ {}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)d_q t + \int_{\frac{1}{2}}^{1} (1-qt)\ {}^{\sigma}D_q\rho(t\nu+(1-t)\sigma)d_q t\right]. \end{aligned}$$

Proof. It can be easily proved by following the procedure used in Lemma 3. □

Theorem 4. *Under the hypotheses of Lemma 4, the following inequality holds if $|{}^{\sigma}D_q\rho|$ and $|{}_{\nu}D_q\rho|$ are convex:*

$$\begin{aligned} &\left|\rho\left(\frac{\nu+\sigma}{2}\right) - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma} \rho(x){}_{\nu}d_q x + \int_{\nu}^{\sigma} \rho(x){}^{\sigma}d_q x\right]\right| \\ \leq &\frac{(\sigma-\nu)}{2}\left[|{}_{\nu}D_q\rho(\sigma)|\frac{3}{4\left([4]_q+q[2]_q\right)} + |{}_{\nu}D_q\rho(\nu)|\frac{5q^2+4q-2q^3-1}{4\left([4]_q+q[2]_q\right)}\right. \\ &\left.+|{}^{\sigma}D_q\rho(\nu)|\frac{3}{4\left([4]_q+q[2]_q\right)} + |{}^{\sigma}D_q\rho(\sigma)|\frac{5q^2+4q-2q^3-1}{8\left([4]_q+q[2]_q\right)}\right]. \end{aligned} \tag{17}$$

Proof. It can be easily proved by following the procedure used in Theorem 1. □

Theorem 5. *Under the hypotheses of Lemma 4, this inequality is satisfied if $|{}^{\sigma}D_q\rho|^{p_1}$ and $|{}_{\nu}D_q\rho|^{p_1}$, $p_1 \geq 1$ are convex:*

$$\begin{aligned} &\left|\rho\left(\frac{\nu+\sigma}{2}\right) - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma} \rho(x){}_{\nu}d_q x + \int_{\nu}^{\sigma} \rho(x){}^{\sigma}d_q x\right]\right| \\ \leq &\frac{(\sigma-\nu)}{2}\left[\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\right. \\ &\times\left(|{}_{\nu}D_q\rho(\sigma)|^{p_1}\frac{q}{8[3]_q} + |{}_{\nu}D_q\rho(\nu)|^{p_1}\frac{[3]_q+q^2}{8\left([4]_q+q[2]_q\right)}\right)^{\frac{1}{p_1}} \\ &+\left(\frac{2-q}{4[2]_q}\right)^{1-\frac{1}{p_1}} \\ &\times\left(|{}_{\nu}D_q\rho(\sigma)|^{p_1}\frac{6-q[2]_q}{8\left([4]_q+q[2]_q\right)} + |{}_{\nu}D_q\rho(\nu)|^{p_1}\frac{5q-2q^2-2}{8[3]_q}\right)^{\frac{1}{p_1}} \end{aligned} \tag{18}$$

$$
\begin{aligned}
&+\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}} \\
&\times\left(\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}\frac{q}{8[3]_q}+\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}\frac{[3]_q+q^2}{8\left([4]_q+q[2]_q\right)}\right)^{\frac{1}{p_1}} \\
&+\left(\frac{2-q}{4[2]_q}\right)^{1-\frac{1}{p_1}} \\
&\times\left(\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}\frac{6-q[2]_q}{8\left([4]_q+q[2]_q\right)}+\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}\frac{5q-2q^2-2}{8[3]_q}\right)^{\frac{1}{p_1}}\Bigg].
\end{aligned}
$$

Proof. It can be easily proved by following the procedure used in Theorem 2. □

Theorem 6. *Under the hypotheses of Lemma 4, we have the following inequality if* $\left|{}^{\sigma}D_q\rho\right|^{p_1}$ *and* $\left|{}_{\nu}D_q\rho\right|^{p_1}$, $p_1>1$ *are convex:*

$$
\begin{aligned}
&\left|\rho\left(\frac{\nu+\sigma}{2}\right)-\frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\,{}_{\nu}d_qx+\int_{\nu}^{\sigma}\rho(x)\,{}^{\sigma}d_qx\right]\right| \\
\leq\ &\frac{(\sigma-\nu)}{2}\left[q\left(\frac{1}{2^{r_1+1}[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\left|{}_{\nu}D_q\rho(\sigma)\right|\frac{1}{4[2]_q}+\left|{}_{\nu}D_q\rho(\nu)\right|\frac{1+2q}{4[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&+\left(\int_{\frac{1}{2}}^{1}(1-qt)^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\left|{}_{\nu}D_q\rho(\sigma)\right|\frac{3}{4[2]_q}+\left|{}_{\nu}D_q\rho(\nu)\right|\frac{6q-1}{4[2]_q}\right)^{\frac{1}{p_1}} \\
&+q\left(\frac{1}{2^{r_1+1}[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\left|{}^{\sigma}D_q\rho(\nu)\right|\frac{1}{4[2]_q}+\left|{}^{\sigma}D_q\rho(\sigma)\right|\frac{1+2q}{4[2]_q}\right)^{\frac{1}{p_1}} \\
&\left.+\left(\int_{\frac{1}{2}}^{1}(1-qt)^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\left|{}^{\sigma}D_q\rho(\nu)\right|\frac{3}{4[2]_q}+\left|{}^{\sigma}D_q\rho(\sigma)\right|\frac{6q-1}{4[2]_q}\right)^{\frac{1}{p_1}}\right],
\end{aligned} \tag{19}
$$

where $p_1^{-1}+r_1^{-1}=1$.

Proof. It can be easily proved by following the procedure used in Theorem 3. □

5. Examples

In this section, we show the validity of the established inequalities using some examples.

Example 1. *For a convex function* $\rho:[0,1]\to\mathbb{R}$ *given as* $\rho(x)=x^2+2$, *by (13) with* $q=\frac{1}{2}$, *the left side of the inequality*

$$
\begin{aligned}
&\left|\frac{f(\nu)+\rho(\sigma)}{2}-\frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx+\int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_qx\right]\right| \\
=\ &\left|\frac{5}{2}-\frac{1}{2}\left[\int_{0}^{1}\left(x^2+2\right)_0d_{\frac{1}{2}}x+\int_{0}^{1}\left(x^2+2\right)^1d_{\frac{1}{2}}x\right]\right| \\
=\ &0.09
\end{aligned}
$$

and the right side of it becomes

$$\begin{aligned} & \frac{q(\sigma-\nu)}{2[3]_q}\left[\left|{}_{\nu}D_q\rho(\sigma)\right|+\left|{}^{\sigma}D_q\rho(\nu)\right|+\frac{q^2\left(\left|{}_{\nu}D_q\rho(\nu)\right|+\left|{}^{\sigma}D_q\rho(\sigma)\right|\right)}{[2]_q}\right] \\ = \ & 0.33. \end{aligned}$$

It is clear that

$$0.09 < 0.33.$$

Example 2. *For a convex function $\rho : [0,1] \to \mathbb{R}$ given by $\rho(x) = x^2 + 2$, by (14) with $q = \frac{1}{2}$ and $p_1 = 2$, the left side of the inequality*

$$\begin{aligned} & \left|\frac{\rho(\nu)+\rho(\sigma)}{2}-\frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx+\int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_qx\right]\right| \\ = \ & \left|\frac{5}{2}-\frac{1}{2}\left[\int_0^1\left(x^2+2\right){}_0d_{\frac{1}{2}}x+\int_0^1\left(x^2+2\right){}^1d_{\frac{1}{2}}x\right]\right| \\ = \ & 0.09 \end{aligned}$$

and the right side of it becomes

$$\begin{aligned} & \frac{q(\sigma-\nu)}{2[2]_q}\left[\left(\frac{[2]_q\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1}+q^2\left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right. \\ & \left.+\left(\frac{[2]_q\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}+q^2\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right] \\ = \ & 0.35. \end{aligned}$$

It is clear that

$$0.09 < 0.35.$$

Example 3. *For a convex function $\rho : [0,1] \to \mathbb{R}$ given by $\rho(x) = x^2 + 2$, from (16) with $q = \frac{1}{2}$ and $p_1 = r_1 = 2$, the left side of the inequality*

$$\begin{aligned} & \left|\frac{\rho(\nu)+\rho(\sigma)}{2}-\frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\ {}_{\nu}d_qx+\int_{\nu}^{\sigma}\rho(x)\ {}^{\sigma}d_qx\right]\right| \\ = \ & \left|\frac{5}{2}-\frac{1}{2}\left[\int_0^1\left(x^2+2\right){}_0d_{\frac{1}{2}}x+\int_0^1\left(x^2+2\right){}^1d_{\frac{1}{2}}x\right]\right| \\ = \ & 0.09 \end{aligned}$$

and the right side of it becomes

$$\begin{aligned} & \frac{q(\sigma-\nu)}{2}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left[\left(\frac{\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1}+q\left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right. \\ & \left.+\left(\frac{\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}+q\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right] \\ = \ & 0.45. \end{aligned}$$

It is clear that

$$0.09 < 0.45.$$

Example 4. *For a convex function* $\rho : [0,1] \to \mathbb{R}$ *given by* $\rho(x) = x^2 + 2$, *from (17) with* $q = \frac{1}{2}$, *the left side of the inequality*

$$\begin{aligned}
&\left|\rho\left(\frac{\nu+\sigma}{2}\right) - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma} \rho(x)\,_{\nu}d_q x + \int_{\nu}^{\sigma} \rho(x)\,^{\sigma}d_q x\right]\right| \\
= &\left|\frac{9}{4} - \frac{1}{2}\left[\int_0^1 \left(x^2+2\right)\,_0 d_{\frac{1}{2}} x + \int_0^1 \left(x^2+2\right)\,^1 d_{\frac{1}{2}} x\right]\right| \\
= &\ 0.15
\end{aligned}$$

and the right side becomes

$$\begin{aligned}
&\frac{(\sigma-\nu)}{2}\left[\left|{}_{\nu}D_q\rho(\sigma)\right|\frac{3}{4\left([4]_q + q[2]_q\right)} + \left|{}_{\nu}D_q\rho(\nu)\right|\frac{5q^2+4q-2q^3-1}{4\left([4]_q + q[2]_q\right)}\right. \\
&\left. + \left|{}^{\sigma}D_q\rho(\nu)\right|\frac{3}{4\left([4]_q + q[2]_q\right)} + \left|{}^{\sigma}D_q\rho(\sigma)\right|\frac{5q^2+4q-2q^3-1}{8\left([4]_q + q[2]_q\right)}\right] \\
= &\ 0.38.
\end{aligned}$$

It is clear that

$$0.15 < 0.38.$$

Example 5. *For a convex function* $\rho : [0,1] \to \mathbb{R}$ *given by* $\rho(x) = x^2 + 2$, *by (18) with* $q = \frac{1}{2}$, *the left side of the inequality*

$$\begin{aligned}
&\left|\rho\left(\frac{\nu+\sigma}{2}\right) - \frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma} \rho(x)\,_{\nu}d_q x + \int_{\nu}^{\sigma} \rho(x)\,^{\sigma}d_q x\right]\right| \\
= &\left|\frac{9}{4} - \frac{1}{2}\left[\int_0^1 \left(x^2+2\right)\,_0 d_{\frac{1}{2}} x + \int_0^1 \left(x^2+2\right)\,^1 d_{\frac{1}{2}} x\right]\right| \\
= &\ 0.15
\end{aligned}$$

and the right side of it becomes

$$\begin{aligned}
&\frac{(\sigma-\nu)}{2}\left[\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1}\frac{q}{8[3]_q} + \left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}\frac{[3]_q + q^2}{8\left([4]_q + q[2]_q\right)}\right)^{\frac{1}{p_1}} \\
&+\left(\frac{2-q}{4[2]_q}\right)^{1-\frac{1}{p_1}} \\
&\times\left(\left|{}_{\nu}D_q\rho(\sigma)\right|^{p_1}\frac{6-q[2]_q}{8\left([4]_q + q[2]_q\right)} + \left|{}_{\nu}D_q\rho(\nu)\right|^{p_1}\frac{5q-2q^2-2}{8[3]_q}\right)^{\frac{1}{p_1}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}} \\
&\times\left(\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}\frac{q}{8[3]_q}+\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}\frac{[3]_q+q^2}{8\left([4]_q+q[2]_q\right)}\right)^{\frac{1}{p_1}} \\
&+\left(\frac{2-q}{4[2]_q}\right)^{1-\frac{1}{p_1}} \\
&\left.\times\left(\left|{}^{\sigma}D_q\rho(\nu)\right|^{p_1}\frac{6-q[2]_q}{8\left([4]_q+q[2]_q\right)}+\left|{}^{\sigma}D_q\rho(\sigma)\right|^{p_1}\frac{5q-2q^2-2}{8[3]_q}\right)^{\frac{1}{p_1}}\right] \\
= \quad & 0.50.
\end{aligned}
$$

where $p_1 = 2$, it is clear that

$$0.15 < 0.50.$$

Example 6. *For a convex function $\rho : [0,1] \to \mathbb{R}$ given by $\rho(x) = x^2 + 2$, from ([19]) with $q = \frac{1}{2}$and $p_1 = r_1 = 2$, the left side of the inequality*

$$
\begin{aligned}
&\left|\rho\left(\frac{\nu+\sigma}{2}\right)-\frac{1}{2(\sigma-\nu)}\left[\int_{\nu}^{\sigma}\rho(x)\,{}_{\nu}d_qx+\int_{\nu}^{\sigma}\rho(x)\,{}^{\sigma}d_qx\right]\right| \\
= \quad &\left|\frac{9}{4}-\frac{1}{2}\left[\int_0^1\left(x^2+2\right){}_0d_{\frac{1}{2}}x+\int_0^1\left(x^2+2\right){}^1d_{\frac{1}{2}}x\right]\right| \\
= \quad & 0.15
\end{aligned}
$$

and the right side of it becomes

$$
\begin{aligned}
&\frac{(\sigma-\nu)}{2}\left[q\left(\frac{1}{2^{r_1+1}[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\left|{}_{\nu}D_q\rho(\sigma)\right|\frac{1}{4[2]_q}+\left|{}_{\nu}D_q\rho(\nu)\right|\frac{1+2q}{4[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&+\left(\int_{\frac{1}{2}}^{1}(1-qt)^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\left|{}_{\nu}D_q\rho(\sigma)\right|\frac{3}{4[2]_q}+\left|{}_{\nu}D_q\rho(\nu)\right|\frac{6q-1}{4[2]_q}\right)^{\frac{1}{p_1}} \\
&+q\left(\frac{1}{2^{r_1+1}[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\left|{}^{\sigma}D_q\rho(\nu)\right|\frac{1}{4[2]_q}+\left|{}^{\sigma}D_q\rho(\sigma)\right|\frac{1+2q}{4[2]_q}\right)^{\frac{1}{p_1}} \\
&\left.+\left(\int_{\frac{1}{2}}^{1}(1-qt)^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\left|{}^{\sigma}D_q\rho(\nu)\right|\frac{3}{4[2]_q}+\left|{}^{\sigma}D_q\rho(\sigma)\right|\frac{6q-1}{4[2]_q}\right)^{\frac{1}{p_1}}\right] \\
= \quad & 0.39.
\end{aligned}
$$

It is clear that

$$0.15 < 0.39.$$

6. Conclusions

In this paper, new variants of midpoint and trapezoidal inequalities for differentiable convex functions in the framework of q-calculus are established. We also used well-known power mean and Hölder inequalities to find q-type of trapezoidal and midpoint inequalities in consideration of q-differentiable convex mappings. These new results can be used for finding some error bounds for the midpoint and trapezoidal rules in q-integration formulas that are very important in the field of numerical analysis. It is an interesting idea that other mathematicians in this field can derive new inequalities for quantum coordinated convex mappings.

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