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Existence Results for Generalized Vector Quasi-Equilibrium Problems in Hadamard Manifolds †

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Abstract: The purpose of this article was to establish manifold versions of existence theorems for generalized vector quasi-equilibrium problems in locally compact and σ-compact spaces without any continuity assumption. The fixed-point theorem in a product Hadamard manifold is the key focus of our discussion. We further applied our theorems to saddle point and minimax problems.

Keywords: generalized vector quasi-equilibrium problem; fixed point; locally compact; σ-compact; Hadamard manifold; cone-convexity; minimax problem

MSC: 90C33; 58C30

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1. Introduction

Equilibrium theory is as an essential branch of applied mathematics and has become a main inspiration for research in diverse research, such as fixed point theory, variational inequalities, complementarity problems, convex optimization problems, saddle point problems, and metric fixed-point applications; for example, see Aussel et al. [1], Blum and Oettli [2], Bueno et al. [3], Cotrina and García [4], Debnath et al. [5], Noor and Oettli [6], Park [7], and Todorčević [8].

Recently, numerous researchers have investigated vector equilibrium problems, such as Ansari [9] and Balaj [10]. Given two nonempty sets X and Y and a Hausdorff topological vector space E, let $F : X \times Y \times X \to 2^E$, $Q : X \to 2^X$ and $S : X \to 2^Y$ be multifunctions. Suppose that $C : X \to 2^E$ is a multifunction such that for each $x \in X$, $C(x)$ is a closed pointed convex cone in E with $\operatorname{int} C(x) \neq \emptyset$. We are most interested in the following two types of generalized vector quasi-equilibrium problems (GVQEPs):

1. GVQEP-I: find $(x_0, y_0) \in X \times Y$ such that $x_0 \in Q(x_0)$, $y_0 \in S(x_0)$ and
$$F(x_0, y_0, z) \nsubseteq -\operatorname{int} C(x_0), \quad \text{for all } z \in Q(x_0).$$
2. GVQEP-II: find $(x_0, y_0) \in X \times Y$ such that $x_0 \in Q(x_0)$, $y_0 \in S(x_0)$ and
$$F(x_0, y_0, z) \subseteq C(x_0) \quad \text{for all } z \in Q(x_0).$$

Every solution of GVQEP-II must be a solution of GVQEP-I. Suppose that $E = \mathbb{R}$, $C(x) = [0, \infty)$ for all $x \in \mathbb{R}$, and $\psi : X \times Y \times X \to \mathbb{R}$ is a real-valued function. In this case, we let $F(x, y, z) = \psi(x, y, z)$ for $(x, y, z) \in X \times Y \times X$. Then, both problems (I) and (II) are reduced to the same generalized quasi-equilibrium problem (GQEP): find $(x_0, y_0) \in X \times Y$ such that $x_0 \in Q(x_0)$, $y_0 \in S(x_0)$ and
$$\psi(x_0, y_0, z) \geq 0 \quad \text{for all } z \in Q(x_0).$$
In 1987, Parida and Sen [11] first considered GQEPs in finite-dimensional spaces.

Our primary goal is to develop new existence theorems for GVQEPs of types I and II in Hadamard manifolds without any continuity or monotonicity assumption on F. Hadamard manifolds are introduced in Section 2. We confine our attention to the case where each of X and Y is a locally compact and σ-compact set in a Hadamard manifold. Because mathematicians typically work with multifunctions from a compact space to a topological vector space, this setting will be technically challenging. Nevertheless, we intend to generalize the notion of cone-convexities for multifunctions in topological vector spaces to manifolds (see Definition 1) and pursue a fixed-point theorem approach in a product Hadamard manifold (see Theorem 3) to overcome these topological and geometric difficulties. To the best of our knowledge, no previous research has investigated existence theorems for GVQEPs associated to locally compact and σ-compact sets in Hadamard manifolds.

Many real-life problems can be equivalently formulated as variational inequalities or boundary value problems on Riemannian manifolds. It turns out that the generalization of topological concepts and techniques on topological vector spaces to Riemannian manifolds is extremely important in the theory of variational inequalities. The main reference for this material is Németh [12]. Numerous recent works have sought to determine which Riemannian manifolds provide a useful framework for research on related optimization and equilibrium problems. Moreover, many concepts and techniques regarding fitting in Euclidean spaces have been extended to Riemannian manifolds. Most of these generalized methods require the sectional curvature of a Riemannian manifold to be nonpositive. Hence, Hadamard manifolds, which have these characteristics, have drawn attention from researchers as a suitable framework for problems in diverse disciplines. Examples include Ansari and Babu [13], Huang [14,15], Iusem and Mohebbi [16], Park [17], and Upadhyay et al. [18].

This paper is organized as follows. In Section 2 we define the notation and provide some background information. Sections 3 and 4 outline existence results for GVQEP-I and GVQEP-II, respectively, in a Hadamard manifold; see Theorems 4, 5, 9, and 10. Finally, in Section 5 the results established in Sections 3 and 4 are applied to study the GQEP (Theorem 13) and minimax problems (Theorems 14 and 15 and Corollary 3).

A still unsolved question is whether we can extend our results for multifunctions with topological vector range space to a Riemannian manifold. One of the tough challenges for all researchers in this domain is there is no natural way to define a cone on a Riemannian manifold. Our results can be regarded as a crucial first step to the study of this topic.

2. Preliminaries

In the rest of this article, unless otherwise specified, $\mathbb{R}^+$ denotes the set of all nonnegative real numbers and E denotes a Hausdorff topological vector space. Let X and Y be topological spaces. We denote the family of all subsets of X as 2^X, and the interior of a subset E of X is denoted as $\operatorname{int} E$. Let $S : X \to 2^Y$ be a multifunction. The image of a set $A \subseteq X$ under S is the set $S(A) = \bigcup\{S(x) : x \in A\}$; S is compact if its range $S(X)$ is contained in a compact subset of Y; $S|_A$ is the restriction of S to A; the graph of S is the set $\mathrm{Gr}(S) = \{(x, y) \in X \times Y : y \in S(x)\}$. The (lower) inverse of S is the multifunction $S^- : Y \to 2^X$ defined by $S^-(y) = \{x \in X : y \in S(x)\}$; the inverse image of a set $B \subseteq Y$ under S is the set $S^-(B) = \{x \in X : S(x) \cap B \neq \emptyset\}$. The values $S^-(y)$ for $y \in Y$ are also called the fibers of S. The multifunction S is upper semicontinuous (u.s.c.) if $S^-(B)$ is closed for each closed subset B of Y, and S is closed if its graph $\mathrm{Gr}\, S = \{(x, y) \in X \times Y : y \in Sx\}$ is a closed subset of $X \times Y$. When $Y = X$, an element $x \in X$ is a fixed point of S if $x \in S(x)$; the set of all fixed points of S is denoted $\mathrm{Fix}(S)$.

The next result is a basic property of closed multifunctions, which can be found in [19] (Chap. 17).

Theorem 1. *Let A be a subset of a topological space X. If $S : A \to 2^X$ is a closed multifunction, then* $\mathrm{Fix}(S)$ *is closed in A.*

For a multifunction with a compact Hausdorff range space, the question of whether the multifunction is u.s.c. is equivalent to the question of whether its graph is closed [19] (Theorem 17.11).

Theorem 2. (closed graph theorem) *A multifunction with a compact Hausdorff range space is closed if and only if it is a closed-valued u.s.c. multifunction.*

In this paper, the term "smooth" always means "of class C^∞". Let M be a connected Riemannian manifold endowed with a Riemannian metric g, let T_xM be the tangent space of M at x, and let $TM = \bigcup_{x\in M} T_xM$ be the tangent bundle of M. The distance between x and y, denoted by $d(x,y)$, is defined as the infimum of the lengths of all piecewise smooth curves from x to y. With the distance function d, M is a metric space with the same metric topology as the original manifold topology.

Let ∇ be the Riemannian connection associated with (M,g). A smooth curve $\gamma : I \to M$ is a geodesic if $\nabla_{\dot\gamma}\dot\gamma = 0$ for all $t \in I$, where I is an open interval. If γ is a geodesic, then $\|\dot\gamma\|$ is constant; we say that γ is normalized when $\|\dot\gamma\| = 1$. Recall that for every $x \in M$ and $v \in T_xM$, there exists a unique maximal geodesic $\gamma_v : I \to M$ satisfying $\gamma_v(0) = x$ and $\dot\gamma_v(0) = v$. Furthermore, the geodesic $\gamma_v(t)$ depends smoothly on t, x, and $v \in T_xM$; see [20] (Chap. 3, Proposition 2.7). Let

$$\mathcal{E} = \{(x,v) \in TM : \gamma_v(t) \text{ is defined on an interval containing } [0,1]\}.$$

The exponential map $\exp : \mathcal{E} \to M$ is given by $\exp(x,v) = \gamma_v(1)$, $(x,v) \in \mathcal{E}$. Evidently, exp is smooth. For each $x \in M$, the restriction $\exp_x$ of exp to the set $\mathcal{E}_x = \mathcal{E} \cap T_xM$ is defined as $\exp_x v = \exp(x,v)$. The manifold (M,g) is geodesically complete if any geodesic of M can be extended to a geodesic defined on all $\mathbb{R}$. A geodesic joining x and y in M is minimizing if its length is equal to $d(x,y)$. The Hopf–Rinow theorem [20] (Chap. 7, Theorem 2.8) asserts that if a connected Riemannian manifold M is complete, then any pair of points in M can be joined by a (not necessarily unique) minimizing geodesic.

A Hadamard manifold is a complete simply connected Riemannian manifold with a nonpositive sectional curvature. For the remainder of this article, M denotes a Hadamard manifold. The Cartan–Hadamard theorem [20] (Chap. 7, Theorem 3.1) states that $\exp_x : T_xM \to M$ is a diffeomorphism at any point $x \in M$, and any pair of points in M can be joined by a unique normalized minimizing geodesic. Geodesics in this article are unique normalized minimizing geodesics unless stated otherwise. A subset A of M is convex if, for any two points $x, y \in A$, the geodesic joining x and y is contained in A.

A topological space is paracompact if every open covering of the space admits a locally finite open refinement. Every locally compact and σ-compact Hausdorff space is paracompact. A Hadamard manifold is σ-compact (i.e., a countable union of compact subsets) and hence is paracompact.

The notions of cone-convexities for multifunctions in [21] can be defined in manifold settings.

Definition 1. *Let $X \subseteq M$ be a nonempty convex set, and C be a closed pointed convex cone in E with* $\operatorname{int} C \neq \emptyset$. *A multifunction $F : X \to 2^E$ is said to be*

1. *above-properly C-quasiconvex on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,*

$$\begin{array}{ll} \textit{either} & F(\exp_{x_1}(\lambda \exp_{x_1}^{-1} x_2)) \subseteq F(x_1) - C, \\ \textit{or} & F(\exp_{x_1}(\lambda \exp_{x_1}^{-1} x_2)) \subseteq F(x_2) - C; \end{array}$$

2. *below-properly C-quasiconvex on X if for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$,*

$$\begin{array}{ll} \textit{either} & F(x_1) \subseteq F(\exp_{x_1}(\lambda \exp_{x_1}^{-1} x_2)) + C, \\ \textit{or} & F(x_2) \subseteq F(\exp_{x_1}(\lambda \exp_{x_1}^{-1} x_2)) + C. \end{array}$$

Consider $E = \mathbb{R}$ and $C = \mathbb{R}^+$ in Definition 1. When F is a single-valued function, both (i) and (ii) reduce to the usual definition of convexity for real-valued functions.

Unless otherwise stated, $C : X \to 2^E$ denotes a multifunction such that for each $x \in X$, $C(x)$ is a closed pointed convex cone in E with $\operatorname{int} C(x) \neq \emptyset$. The following proposition is the basic result concerning convexity for multifunctions.

Proposition 1. *Let X be a nonempty set, let $Y \subseteq M$ be a nonempty convex set, and let $F : X \times Y \to 2^E$ be a multifunction. Define $A, B : X \to 2^Y$ by*

$$A(x) = \{z \in Y : F(x,z) \subseteq -\operatorname{int} C(x)\},$$
$$B(x) = \{z \in Y : F(x,z) \nsubseteq C(x)\}.$$

For each $x \in X$, we have the following:

1. *If $F(x,z)$ is above-properly $C(x)$-quasiconvex in z, then $A(x)$ is convex.*
2. *If $F(x,z)$ is below-properly $C(x)$-quasiconvex in z, then $B(x)$ is convex.*

Proof. (i) Let $z_1, z_2 \in A(x)$ and $\lambda \in [0,1]$. Then $F(x,z_1)$ and $F(x,z_2)$ are contained in $-\operatorname{int} C(x)$. Because $F(x,z)$ is above-properly $C(x)$-quasiconvex in z, $F(x, \exp_{z_1}(\lambda \exp_{z_1}^{-1} z_2))$ is contained in either $F(x,z_1) - C(x)$ or $F(x,z_2) - C(x)$ and hence is contained in $-\operatorname{int} C(x) - C(x) = -\operatorname{int} C(x)$. Thus $\exp_{z_1}(\lambda \exp_{z_1}^{-1}) \in A(x)$, and $A(x)$ is convex.

(ii) Let $z_1, z_2 \in B(x)$ and $\lambda \in [0,1]$. Because $F(x,z)$ is below-properly $C(x)$-quasiconvex in z, we have

$$\text{either} \quad F(x,z_1) \subseteq F(x, \exp_{z_1}(\lambda \exp_{z_1}^{-1} z_2)) + C(x), \tag{1}$$
$$\text{or} \quad F(x,z_2) \subseteq F(x, \exp_{x_1}(\lambda \exp_{x_1}^{-1} x_2)) + C(x). \tag{2}$$

The inclusion (1) implies that $F(x, \exp_{z_1}(\lambda \exp_{z_1}^{-1} z_2)) \nsubseteq C(x)$; otherwise

$$F(x, \exp_{z_1}(\lambda \exp_{z_1}^{-1} z_2)) \subseteq C(x)$$

yields $F(x,z_1)$ is contained in $C(x) + C(x) = C(x)$. Similarly, the inclusion (2) implies that $F(x, \exp_{z_1}(\lambda \exp_{z_1}^{-1} z_2)) \nsubseteq C(x)$. In either case we have $\exp_{z_1}(\lambda \exp_{z_1}^{-1}) \in B(x)$, so $B(x)$ is convex. □

The next theorem [22], (Theorem 4.1) provides the topological conditions that guarantee the existence of fixed points for a multifunction in a product Hadamard manifold.

Theorem 3. *Let $X_1, \ldots, X_k$ be locally compact and σ-compact convex sets, each in a Hadamard manifold, and let $X = \prod_{i=1}^k X_i$. For $i = 1, \ldots, k$, let $T_i : X \to 2^{X_i}$ be a compact multifunction with nonempty convex values such that $X = \bigcup_{y \in X_i} \operatorname{int} T_i^-(y)$. Then, there exists a point $\hat{x} = \prod_{i=1}^k \hat{x}_i \in X$ such that $\hat{x}_i \in T_i(\hat{x})$ for $i = 1, \ldots, k$, that is, $\hat{x} \in T(\hat{x}) = \prod_{i=1}^k T_i(\hat{x})$.*

3. Existence Results for (GVQEP-I)

This section outlines the development of existence theorems for GVQEP-I under various topological conditions (without continuity). Theorem 3 plays a key role in this discussion.

Theorem 4. *Let X and Y be locally compact and σ-compact convex sets, each in a Hadamard manifold. Let $F : X \times Y \times X \to 2^E$, $Q : X \to 2^X$ and $S : X \to 2^Y$ be three multifunctions satisfying the following conditions:*

1. *Q is compact with convex values such that $X = \bigcup_{z \in X} \operatorname{int} Q^-(z)$;*
2. *S is compact with convex values such that $X = \bigcup_{u \in Y} \operatorname{int} S^-(u)$;*
3. *$S|_{\mathrm{Fix}(Q)}$ is closed;*
4. *for each $x \in X$, $F(z,u,z) \nsubseteq -\operatorname{int} C(x)$ for all $(z,u) \in Q(x) \times S(x)$;*
5. *for each $(x,y) \in X \times Y$, the set $\{z \in X : F(x,y,z) \subseteq -\operatorname{int} C(x)\}$ is convex;*

6. $\bigcup_{z\in X}\{(x,y)\in Q^-(z)\times Y : F(x,y,z)\subseteq -\operatorname{int}C(x)\}$
$= \bigcup_{z\in X}\operatorname{int}\{(x,y)\in Q^-(z)\times Y : F(x,y,z)\subseteq -\operatorname{int}C(x)\}$.

Then, there exists $(x_0,y_0)\in X\times Y$ such that $x_0\in Q(x_0)$, $y_0\in S(x_0)$ and

$$F(x_0,y_0,z)\nsubseteq -\operatorname{int}C(x_0),\quad \textit{for all } z\in Q(x_0).$$

Proof. Consider the multifunctions $T_1 : X\times Y\to 2^X$ and $T_2 : X\times Y\to 2^Y$ defined by

$$\begin{aligned} &T_1(x,y)=\{z\in Q(x) : F(x,y,z)\subseteq -\operatorname{int}C(x)\},\ \text{for } (x,y)\in X\times Y,\\ &T_2(x,y)=S(x),\ \text{for } (x,y)\in X\times Y. \end{aligned}$$

For each $(x,y)\in X\times Y$, the set

$$T_1(x,y)=Q(x)\cap\{z\in X : F(x,y,z)\subseteq -\operatorname{int}C(x)\}$$

is convex. Because

$$\begin{aligned} &T_1^-(z)=\{(x,y)\in Q^-(z)\times Y : F(x,y,z)\subseteq -\operatorname{int}C(x)\},\ \text{for } z\in X,\\ &T_2^-(u)=S^-(u)\times Y,\ \text{for } u\in Y, \end{aligned}$$

applying conditions (ii) and (vi) reveals that

$$\begin{aligned} &\bigcup_{z\in X}T_1^-(z)=\bigcup_{z\in X}\operatorname{int}T_1^-(z),\\ &\bigcup_{u\in Y}T_2^-(u)=\bigcup_{u\in Y}\operatorname{int}T_2^-(u)=X\times Y. \end{aligned}\tag{3}$$

Let $H : X\times Y\to 2^{X\times Y}$ be the product multifunction of T_1 and T_2 defined by $H(x,y)=T_1(x,y)\times T_2(x,y)$. Because T_1 and T_2 are compact with convex values, so too is H.

Let $\Omega=\operatorname{Gr}\left(S|_{\operatorname{Fix}(Q)}\right)=\{(x,y)\in\operatorname{Fix}(Q)\times Y : y\in S(x)\}$; Ω is rendered nonempty by Theorem 3 and is closed in $X\times Y$ by condition (iii). We assume that $H(x_0,y_0)=\emptyset$ for some $(x_0,y_0)\in\Omega$; it then follows that $T_1(x_0,y_0)=\emptyset$ or $T_2(x_0,y_0)=\emptyset$, and from hypothesis $S(x_0)\neq\emptyset$, we have $x_0\in Q(x_0)$, $y_0\in S(x_0)$ and

$$F(x_0,y_0,z)\nsubseteq -\operatorname{int}C(x_0),\quad \text{for all } z\in Q(x_0).$$

To prove this statement, assume by way of contradiction that $H(x,y)\neq\emptyset$ for all $(x,y)\in\Omega$. Define a multifunction $P : X\times Y\to 2^{X\times Y}$ by

$$P(x,y)=\begin{cases} H(x,y) & \text{if } (x,y)\in\Omega,\\ Q(x)\times S(x) & \text{if } (x,y)\in(X\times Y)\setminus\Omega; \end{cases}$$

it is compact with nonempty convex values. The multifunction P can be viewed as the product of $P_1 : X\times Y\to 2^X$ and $T_2 : X\times Y\to 2^Y$, where P_1 is given by

$$P_1(x,y)=\begin{cases} T_1(x,y) & \text{if } (x,y)\in\Omega,\\ Q(x) & \text{if } (x,y)\in(X\times Y)\setminus\Omega. \end{cases}$$

Observe that

$$X\times Y=\bigcup_{z\in X}P_1^-(z)=\bigcup_{z\in X}\operatorname{int}P_1^-(z).\tag{4}$$

We first note that P_1 has nonempty values; hence, the first equality holds. It remains to show that the left-hand side of the second equality is contained in the right-hand side. Fix an arbitrary point $z_0 \in X$. Because $T_1^-(z_0) \subseteq Q^-(z_0) \times Y$, we have

$$\begin{aligned} P_1^-(z_0) &= [T_1^-(z_0) \cap \Omega] \cup \{[Q^-(z_0) \times Y] \cap [(X \times Y) \setminus \Omega]\} \\ &= T_1^-(z_0) \cup \{[Q^-(z_0) \times Y] \cap [(X \times Y) \setminus \Omega]\}. \end{aligned}$$

By (3) we infer that

$$T_1^-(z_0) \subseteq \bigcup_{z \in X} T_1^-(z) = \bigcup_{z \in X} \operatorname{int} T_1^-(z) \subseteq \bigcup_{z \in X} \operatorname{int} P_1^-(z).$$

Since $(X \times Y) \setminus \Omega$ is open, it follows from condition (i) that

$$\begin{aligned} [Q^-(z_0) \times Y] \cap [(X \times Y) \setminus \Omega] &\subseteq \left[\bigcup_{z \in X} Q^-(z)\right] \times Y \cap [(X \times Y) \setminus \Omega] \\ &= \bigcup_{z \in X} \operatorname{int} \{[Q^-(z) \times Y] \cap [(X \times Y) \setminus \Omega]\} \\ &\subseteq \bigcup_{z \in X} \operatorname{int} P_1^-(z). \end{aligned}$$

Hence, (4) holds. This result, together with (3) and Theorem 3, yields a fixed point (x_0, y_0) of P. Indeed, (x_0, y_0) must be in Ω; thus, $(x_0, y_0) \in Q(x_0) \times S(x_0)$ and

$$F(x_0, y_0, x_0) \subseteq -\operatorname{int} C(x_0),$$

contradicting condition (iv). This concludes the proof. □

As an application of the preceding result, we present some notable cases of vector equilibrium (or quasi-equilibrium) problems. The following result is a stronger version of Theorem 4 for closed multifunctions in a compact Hausdorff space (hence, it is equivalent to upper semicontinuity).

Theorem 5. *Let X and Y be locally compact and σ-compact convex sets, each in a Hadamard manifold. Let $F : X \times Y \times X \to 2^E$, $Q : X \to 2^X$ and $S : X \to 2^Y$ be three multifunctions satisfying the following conditions:*

1. *Q is closed and compact with convex values such that $X = \bigcup_{z \in X} \operatorname{int} Q^-(z)$;*
2. *S is closed and compact with convex values such that $X = \bigcup_{u \in Y} \operatorname{int} S^-(u)$;*
3. *For each $x \in X$, $F(z, u, z) \nsubseteq -\operatorname{int} C(x)$ for all $(z, u) \in Q(x) \times S(x)$;*
4. *For each $(x, y) \in X \times Y$, $F(x, y, z)$ is above-properly $C(x)$-quasiconvex in z;*
5. *$\bigcup_{z \in X} \{(x, y) \in Q^-(z) \times Y : F(x, y, z) \subseteq -\operatorname{int} C(x)\}$ $= \bigcup_{z \in X} \operatorname{int} \{(x, y) \in Q^-(z) \times Y : F(x, y, z) \subseteq -\operatorname{int} C(x)\}$.*

Then, there exists a solution $(x_0, y_0) \in X \times Y$ of GVQEP-I.

Proof. By condition (iv) and Proposition 1, for each $(x, y) \in X \times Y$, the set $\{z \in X : F(x, y, z) \subseteq -\operatorname{int} C(x)\}$ is convex. Theorem 1 ensures that the set $\operatorname{Fix}(Q)$ is closed. Hence, the closed graph Theorem 2 states that Q and S are closed-valued u.s.c. multifunctions. Therefore, $S|_{\operatorname{Fix}(Q)}$ is u.s.c. with closed values and is closed (again, this accords with Theorem 2). Theorem 4 is now applied. □

The following explicit example in the hyperbolic upper-half plane $\mathbb{H}^2$ (with constant Gaussian curvature -1) illustrates the practical application of Theorem 4.

Example 1. *We consider the hyperbolic upper-half plane $\mathbb{H}^2$. Let $(p, q]$ denote the geodesic connecting two points $p, q \in \mathbb{H}^2$; we use a square bracket when the geodesic includes the endpoint and a parenthesis to indicate that the endpoint is not included. Let $X = Y = (e^{\pi i/4}, e^{3\pi i/4})$; they are* *locally compact and σ-compact convex sets in $\mathbb{H}^2$. Let $I_1 = (e^{\pi i/4}, e^{\pi i/2}]$ and $I_2 = (e^{\pi i/2}, e^{3\pi i/4})$; so $X = I_1 \cup I_2$. Let $Q : X \to 2^X$ and $S : X \to 2^Y$ be defined by*

$$Q(e^{it}) = S(e^{it}) = \begin{cases} I_1 & \text{if } e^{it} \in I_1, \\ \{e^{\pi i/2}\} & \text{if } e^{it} \in I_2. \end{cases}$$

Take E to be the complex plane $\mathbb{C}$ and let $\Lambda(w_1, w_2)$ denote a closed pointed convex cone in $\mathbb{C}$ spanned by two vectors w_1 and w_2; in particular, it is a ray issuing from the origin when $w_1 = w_2$. Define $F : X \times Y \times X \to 2^{\mathbb{C}}$ by

$$F(e^{it}, e^{iu}, e^{iv}) = \Lambda(e^{i(t+v)}, e^{i(u+v)});$$

it is obtained by rotating the cone $\Lambda(e^{it}, e^{iu})$ about the origin through an angle v. Let $\Lambda_1 = \Lambda(e^{\pi i}, e^{3\pi i/2})$ and $\Lambda_2 = \Lambda(e^{-\pi i/2}, 1)$, and define $C : X \to 2^{\mathbb{C}}$ by

$$C(e^{it}) = \begin{cases} \Lambda_1 & \text{if } e^{it} \in I_1, \\ \Lambda_2 & \text{if } e^{it} \in I_2. \end{cases}$$

Notice that $-\operatorname{int} C(e^{it})$ is the first quadrant $-\operatorname{int} \Lambda_1$ for $e^{it} \in I_1$ and is the second quadrant $-\operatorname{int} \Lambda_2$ for $e^{it} \in I_2$, and $F(e^{it}, e^{iu}, e^{iv})$ is disjoint from $-\operatorname{int} \Lambda_1$ for all $(e^{it}, e^{iu}, e^{iv}) \in X \times Y \times X$.

In order to apply Theorem 4, we need to verify the following facts concerning the characterizations of the multifunctions Q (or S) and F.

(1) Q and S are compact with convex values, and

$$X = Q^-(e^{\pi i/2}) = \bigcup_{v \in X} \operatorname{int} Q^-(v) = \bigcup_{v \in Y} \operatorname{int} S^-(v).$$

(2) $\operatorname{Fix}(Q) = I_1$ is closed and hence $S|_{\operatorname{Fix}(Q)} = I_1 \times I_1$ is closed.

(3) Fix any $e^{it} \in X$ and let $(e^{iv}, e^{iu}) \in Q(e^{it}) \times S(e^{it})$. Observe that

$$F(e^{iv}, e^{iu}, e^{iv}) \nsubseteq -\operatorname{int} C(e^{it}).$$

To see this, we consider two cases. First, if $e^{it} \in I_1$, then $(e^{iv}, e^{iu}) \in I_1 \times I_1$ and

$$F(e^{iv}, e^{iu}, e^{iv}) = \Lambda(e^{2iv}, e^{i(u+v)}) \subseteq \Lambda(e^{\pi i/2}, e^{\pi i}) \nsubseteq -\operatorname{int} \Lambda_1.$$

Second, if $e^{it} \in I_2$, then $(e^{iv}, e^{iu}) \in \{e^{\pi i/2}\} \times \{e^{\pi i/2}\}$ and therefore

$$F(e^{iv}, e^{iu}, e^{iv}) = \Lambda(e^{\pi i}, e^{\pi i}) \nsubseteq -\operatorname{int} \Lambda_2.$$

(4) To prove the validity of condition (v) in Theorem 4, let $(e^{it}, e^{iu}) \in X \times Y$. As we remarked earlier, it suffices to assume $e^{it} \in I_2$. Then choose $\alpha = \max\{t, u\}$ so that

$$F(e^{it}, e^{iu}, e^{iv}) \subseteq -\operatorname{int} \Lambda_2 \quad \text{for all } v \in (\pi/4, \pi - \alpha);$$

this proves condition (v).

(5) To prove condition (vi) in Theorem 4 is satisfied, we observe that for any $v \in X$,

$$\{(e^{it}, e^{iu}) \in Q^-(e^{iv}) \times Y : F(e^{it}, e^{iu}, e^{iv}) \subseteq -\operatorname{int} C(e^{it})\} = \emptyset. \tag{5}$$

To see this, note that $Q^-(e^{iv}) = \emptyset$ for all $e^{iv} \in I_2$. Thus we need only consider the case where $e^{iv} \in I_1$. Let $e^{it} \in Q^-(e^{iv})$. If $e^{iv} \in I_1 \setminus \{e^{\pi i/2}\}$, then $Q^-(e^{iv}) = I_1 \setminus \{e^{\pi i/2}\}$, and since $t + v > \pi/2$, the Equation (5) holds. Now we assume that $e^{iv} = e^{\pi i/2}$, so $Q^-(e^{iv}) = I_1 \cup I_2$. Similarly, if $e^{it} \in I_1$, (5) is true. If $e^{it} \in I_2$, it follows that $t + (\pi/2) > \pi$, and again (5) is true.

Consequently, Theorem 4 yields a solution $(e^{it_0}, e^{iu_0}) \in Q(e^{it_0}) \times S(e^{it_0})$ to GVQEP-I such that $F(e^{it_0}, e^{iu_0}, e^{iv}) \nsubseteq -\operatorname{int} C(e^{it_0})$ for all $e^{iv} \in Q(e^{it_0})$. In particular, we can choose points $e^{it_0} = e^{\pi i/3}$ and $e^{iu_0} = e^{5\pi i/12}$ so that $(e^{\pi i/3}, e^{5\pi i/12}) \in Q(e^{\pi i/3}) \times S(e^{\pi i/3})$ and

$F(e^{\pi i/3}, e^{5\pi i/12}, e^{iv}) \nsubseteq -\operatorname{int}\Lambda_1$ for all $e^{iv} \in Q(e^{\pi i/3})$. Moreover, it is also worth noting that $(e^{\pi i/3}, e^{5\pi i/12})$ is a solution to GVQEP of type I but not of type II.

As a corollary of Theorem 4, we offer an existence theorem for VQEPs in Hadamard manifolds.

Theorem 6. *Let X be a locally compact and σ-compact convex set in M. Let $F : X \times X \to 2^E$ and $Q : X \to 2^X$ be two multifunctions satisfying the following conditions:*

1. *Q is compact with convex values such that $X = \bigcup_{z\in X} \operatorname{int} Q^-(z)$;*
2. *$Q|_{\operatorname{Fix}(Q)}$ is closed;*
3. *for each $x \in X$, $F(z,z) \nsubseteq -\operatorname{int} C(x)$ for all $z \in Q(x)$;*
4. *for each $x \in X$, the set $\{z \in X : F(x,z) \subseteq -\operatorname{int} C(x)\}$ is convex;*
5. *$\bigcup_{z\in X}\{x \in Q^-(z) : F(x,z) \subseteq -\operatorname{int} C(x)\}$
$= \bigcup_{z\in X} \operatorname{int}\{x \in Q^-(z) : F(x,z) \subseteq -\operatorname{int} C(x)\}$.*

Then, there exists $x_0 \in Q(x_0)$ such that $F(x_0,z) \nsubseteq -\operatorname{int} C(x_0)$ for all $z \in Q(x_0)$.

Proof. If $Y = X$, $S = Q$, and $G : X \times Y \times X \to 2^E$ defined by $G(x,y,z) = F(x,z)$, the multifunctions G, Q, and S satisfy all conditions in Theorem 4. Therefore, we can apply Theorem 4. □

The following fact is revealed using Proposition 1 and Theorem 6.

Theorem 7. *Let X be a locally compact and σ-compact convex set in M. Let $F : X \times X \to 2^E$ and $Q : X \to 2^X$ be two multifunctions satisfying the following conditions:*

1. *Q is compact with convex values such that $X = \bigcup_{z\in X} \operatorname{int} Q^-(z)$;*
2. *$Q|_{\operatorname{Fix}(Q)}$ is closed;*
3. *for each $x \in X$, $F(z,z) \nsubseteq -\operatorname{int} C(x)$ for all $z \in Q(x)$;*
4. *for each $x \in X$, $F(x,z)$ is above-properly $C(x)$-quasiconvex in z;*
5. *$\bigcup_{z\in X}\{x \in Q^-(z) : F(x,z) \subseteq -\operatorname{int} C(x)\}$
$= \bigcup_{z\in X} \operatorname{int}\{x \in Q^-(z) : F(x,z) \subseteq -\operatorname{int} C(x)\}$.*

Then, there exists $x_0 \in Q(x_0)$ such that $F(x_0,z) \nsubseteq -\operatorname{int} C(x_0)$ for all $z \in Q(x_0)$.

We consider the case where X is compact (hence paracompact) in Theorem 7.

Theorem 8. *Let X be a compact convex set in M. Let $F : X \times X \to 2^E$ and $Q : X \to 2^X$ be two multifunctions satisfying the following conditions:*

1. *Q is closed with convex values such that $X = \bigcup_{z\in X} \operatorname{int} Q^-(z)$;*
2. *for each $x \in X$, $F(z,z) \nsubseteq -\operatorname{int} C(x)$ for all $z \in Q(x)$;*
3. *for each $x \in X$, $F(x,z)$ is above-properly $C(x)$-quasiconvex in z;*
4. *$\bigcup_{z\in X}\{x \in Q^-(z) : F(x,z) \subseteq -\operatorname{int} C(x)\}$
$= \bigcup_{z\in X} \operatorname{int}\{x \in Q^-(z) : F(x,z) \subseteq -\operatorname{int} C(x)\}$.*

Then, there exists $x_0 \in Q(x_0)$ such that $F(x_0,z) \nsubseteq -\operatorname{int} C(x_0)$ for all $z \in Q(x_0)$.

Proof. According to Theorem 1, $\operatorname{Fix}(Q) \times Y$ is closed, and hence $Q|_{\operatorname{Fix}(Q)}$ is closed. The conclusion follows immediately from Theorem 7. □

If Q is identically equal to X in Theorem 8, then condition (i) is automatically satisfied and can be removed. In this case, the VQEP is reduced to a vector equilibrium problem (VEP). We can obtain the following result under a stronger hypothesis than (iv) in the preceding theorem.

Corollary 1. *Let X be a compact convex set in M and C be a closed pointed convex cone in E with $\operatorname{int} C \neq \emptyset$. Suppose that $F : X \times X \to 2^E$ is a multifunction satisfying the following conditions:*

1. *for each $x \in X$, $F(x,x) \nsubseteq -\operatorname{int} C$;*

2. *for each $x \in X$, $F(x,z)$ is above-properly C-quasiconvex in z;*
3. *for each $z \in X$, $F(x,z)$ is u.s.c. in x.*

Then, there exists $x_0 \in X$ such that $F(x_0,z) \nsubseteq -\operatorname{int} C$ for all $z \in X$.

Proof. Fix any $z \in X$. Because $F(x,z)$ is u.s.c. in x and $E \setminus -\operatorname{int} C$ is closed, the set

$$\{x \in X : F(x,z) \subseteq -\operatorname{int} C\} = X \setminus \{x \in X : F(x,z) \cap (E \setminus -\operatorname{int} C) \neq \emptyset\}$$

is open. Therefore, condition (iv) of Theorem 8 is satisfied, and the conclusion follows. □

4. Existence Results for (GVQEP-II)

The main objective of this part of the study was to investigate GVQEP-II by using the techniques developed in establishing the various existence results for GVQEP-I and present the corresponding theorems for GVQEP-II. Our treatment of GVQEP-II is similar to the approach to GVQEP-I; therefore, we omit the relevant details and summarize several of the existence theorems. Recall that a solution to GVQEP-II is also a solution to GVQEP-I.

Theorem 9. *Let X and Y be locally compact and σ-compact convex sets, each in a Hadamard manifold. Let $F : X \times Y \times X \to 2^E$, $Q : X \to 2^X$ and $S : X \to 2^Y$ be three multifunctions satisfying the following conditions:*

1. *Q is compact with convex values such that $X = \bigcup_{z \in X} \operatorname{int} Q^-(z)$;*
2. *S is compact with convex values such that $X = \bigcup_{u \in Y} \operatorname{int} S^-(u)$;*
3. *$S|_{\mathrm{Fix}(Q)}$ is closed;*
4. *for each $x \in X$, $F(z,u,z) \subseteq C(x)$ for all $(z,u) \in Q(x) \times S(x)$;*
5. *for each $(x,y) \in X \times Y$, the set $\{z \in X : F(x,y,z) \nsubseteq C(x)\}$ is convex;*
6. *$\bigcup_{z \in X}\{(x,y) \in Q^-(z) \times Y : F(x,y,z) \nsubseteq C(x)\}$*
   $= \bigcup_{z \in X} \operatorname{int}\{(x,y) \in Q^-(z) \times Y : F(x,y,z) \nsubseteq C(x)\}$.

Then, there exists $(x_0,y_0) \in X \times Y$ such that $x_0 \in Q(x_0)$, $y_0 \in S(x_0)$ and

$$F(x_0,y_0,z) \subseteq C(x_0), \quad \text{for all } z \in Q(x_0).$$

Proof. The method used for this proof is similar to that used for Theorem 4. Let $T_1 : X \times Y \to 2^X$ and $T_2 : X \times Y \to 2^Y$ be defined by

$$T_1(x,y) = \{z \in Q(x) : F(x,y,z) \nsubseteq C(x)\}, \text{ for } (x,y) \in X \times Y,$$
$$T_2(x,y) = S(x), \text{ for } (x,y) \in X \times Y;$$

both T_1 and T_2 have convex values. The argument given in Theorem 4 is repeated to conclude that there exists $(x_0,y_0) \in X \times Y$ such that $x_0 \in Q(x_0)$, $y_0 \in S(x_0)$ and $F(x_0,y_0,z) \subseteq C(x_0)$ for all $z \in Q(x_0)$. □

The preceding theorem has the following two useful corollaries for GVQEP-II and VQEP.

Theorem 10. *Let X and Y be locally compact and σ-compact convex sets, each in a Hadamard manifold. Let $F : X \times Y \times X \to 2^E$, $Q : X \to 2^X$ and $S : X \to 2^Y$ be three multifunctions satisfying the following conditions:*

1. *Q is closed and compact with convex values such that $X = \bigcup_{z \in X} \operatorname{int} Q^-(z)$;*
2. *S is closed and compact with convex values such that $X = \bigcup_{u \in Y} \operatorname{int} S^-(u)$;*
3. *For each $x \in X$, $F(z,u,z) \subseteq C(x)$ for all $(z,u) \in Q(x) \times S(x)$;*
4. *For each $(x,y) \in X \times Y$, $F(x,y,z)$ is below-properly $C(x)$-quasiconvex in z;*
5. *$\bigcup_{z \in X}\{(x,y) \in Q^-(z) \times Y : F(x,y,z) \nsubseteq C(x)\}$*
   $= \bigcup_{z \in X} \operatorname{int}\{(x,y) \in Q^-(z) \times Y : F(x,y,z) \nsubseteq C(x)\}$.

Then, there exists a solution $(x_0,y_0) \in X \times Y$ of GVQEP-II.

Proof. First, observe from condition (iv) and Proposition 1 that for each $(x, y) \in X \times Y$, the set $\{z \in X : F(x, y, z) \nsubseteq C(x)\}$ is convex; this guarantees condition (v) in Theorem 9 is satisfied. Now, we apply Theorems 1, 2, and 9. □

Theorem 11. *Let X be a locally compact and σ-compact convex set in M. Let $F : X \times X \to 2^E$ and $Q : X \to 2^X$ be two multifunctions satisfying the following conditions:*

1. *Q is compact with convex values such that $X = \bigcup_{z \in X} \operatorname{int} Q^-(z)$;*
2. *$Q|_{\mathrm{Fix}(Q)}$ is closed;*
3. *For each $x \in X$, $F(z, z) \subseteq C(x)$ for all $z \in Q(x)$;*
4. *For each $x \in X$, the set $\{z \in X : F(x, z) \nsubseteq C(x)\}$ is convex;*
5. $\displaystyle\bigcup_{z \in X} \{x \in Q^-(z) : F(x, z) \nsubseteq C(x)\} = \bigcup_{z \in X} \operatorname{int}\{x \in Q^-(z) : F(x, z) \nsubseteq C(x)\}.$

Then, there exists $x_0 \in Q(x_0)$ such that $F(x_0, z) \subseteq C(x_0)$ for all $z \in Q(x_0)$.

Proof. This corollary follows from Theorem 9 if we let $Y = X$, $S = Q$, and $G : X \times Y \times X \to 2^E$ be defined by $G(x, y, z) = F(x, z)$. □

The following result is a special case of Theorem 11 if X is compact.

Theorem 12. *Let X be a compact convex set in M. Let $F : X \times X \to 2^E$ and $Q : X \to 2^X$ be two multifunctions satisfying the following conditions:*

1. *Q is closed with convex values such that $X = \bigcup_{z \in X} \operatorname{int} Q^-(z)$;*
2. *For each $x \in X$, $F(z, z) \subseteq C(x)$ for all $z \in Q(x)$;*
3. *For each $x \in X$, the set $\{z \in X : F(x, z) \nsubseteq C(x)\}$ is convex;*
4. $\displaystyle\bigcup_{z \in X} \{x \in Q^-(z) : F(x, z) \nsubseteq C(x)\} = \bigcup_{z \in X} \operatorname{int}\{x \in Q^-(z) : F(x, z) \nsubseteq C(x)\}.$

Then, there exists $x_0 \in Q(x_0)$ such that $F(x_0, z) \subseteq C(x_0)$ for all $z \in Q(x_0)$.

Proof. According to Theorem 1 and the closedness of Q, condition (ii) in Theorem 11 is satisfied; therefore, we apply Theorem 11. □

If Q is identically equal to X, a corollary of this theorem for the VEP follows immediately.

Corollary 2. *Let X be a compact convex set in M and let $F : X \times X \to 2^E$ be a multifunction satisfying the following conditions:*

1. *For each $x \in X$, $F(x, x) \subseteq C(x)$;*
2. *For each $x \in X$, the set $\{z \in X : F(x, z) \nsubseteq C(x)\}$ is convex;*
3. $\displaystyle\bigcup_{z \in X} \{x \in X : F(x, z) \nsubseteq C(x)\} = \bigcup_{z \in X} \operatorname{int}\{x \in X : F(x, z) \nsubseteq C(x)\}.$

Then, there exists $x_0 \in X$ such that $F(x_0, z) \subseteq C(x_0)$ for all $z \in X$.

5. Applications

As stated previously, if $E = \mathbb{R}$, $C(x) = \mathbb{R}^+$ for all $x \in \mathbb{R}$ and $F : X \times Y \times X \to \mathbb{R}$ is a real-valued function, both GVQEP-I and GVQEP-II are reduced to the same GQEP. The next result is an immediate outcome of applying Theorem 4 (or Theorem 9).

Theorem 13. *Let X and Y be locally compact and σ-compact convex sets, each in a Hadamard manifold. Let $\psi : X \times Y \times X \to \mathbb{R}$ be a function and $Q : X \to 2^X$ and $S : X \to 2^Y$ be two multifunctions satisfying the following conditions:*

1. *Q is compact with convex values such that $X = \bigcup_{z \in X} \operatorname{int} Q^-(z)$;*
2. *S is compact with convex values such that $X = \bigcup_{u \in Y} \operatorname{int} S^-(u)$;*
3. *$S|_{\mathrm{Fix}(Q)}$ is closed;*
4. *For each $x \in X$, $\psi(z, u, z) \geq 0$ for all $(z, u) \in Q(x) \times S(x)$;*
5. *For each $(x, y) \in X \times Y$, the set $\{z \in X : \psi(x, y, z) < 0\}$ is convex;*

6. $\bigcup_{z\in X}\{(x,y)\in Q^{-}(z)\times Y:\psi(x,y,z)<0\}$
$=\bigcup_{z\in X}\operatorname{int}\{(x,y)\in Q^{-}(z)\times Y:\psi(x,y,z)<0\}$.

Then, there exists $(x_0,y_0)\in X\times Y$ such that $x_0\in Q(x_0)$, $y_0\in S(x_0)$ and

$$\psi(x_0,y_0,z)\geq 0,\quad \text{for all } z\in Q(x_0).$$

The following result is a special case of Theorem 13 that has applicability to minimax problems.

Theorem 14. *Let X be a compact convex set and Y a locally compact and σ-compact convex set, each in a Hadamard manifold, and $\alpha\in\mathbb{R}$. Let $\varphi: X\times Y\to\mathbb{R}$ be a function and $S: X\to 2^Y$ a multifunction satisfying the following conditions:*

1. *S is closed and compact with convex values such that $X=\bigcup_{u\in Y}\operatorname{int}S^{-}(u)$;*
2. *For each $x\in X$, $\varphi(z,u)\geq\alpha$ for all $(z,u)\in X\times S(x)$;*
3. *For each $y\in Y$, the set $\{z\in X:\varphi(z,y)<\alpha\}$ is convex;*
4. $\bigcup_{z\in X}\{y\in Y:\varphi(z,y)<\alpha\}=\bigcup_{z\in X}\operatorname{int}\{y\in Y:\varphi(z,y)<\alpha\}$.

Then, there exists a point $y_0\in S(X)$ such that $\varphi(z,y_0)\geq\alpha$ for all $z\in X$.

Proof. We can apply Theorem 13 with $Q(x)=X$ for all $x\in X$, $\psi(x,y,z)=\varphi(z,y)-\alpha$ for all $(x,y,z)\in X\times Y\times X$; $\operatorname{Fix}(Q)=X$; hence, condition (iii) in Theorem 13 is satisfied. □

We close this section on equilibrium problems with a discussion of minimax problems.

Theorem 15. *Let X and Y be compact convex sets, each in a Hadamard manifold, and $\alpha\in\mathbb{R}$. Let $\varphi: X\times Y\to\mathbb{R}$ be a function and $S: X\to 2^Y$ and $T: Y\to 2^X$ two multifunctions satisfying the following conditions:*

1. *S is closed with convex values such that $X=\bigcup_{u\in Y}\operatorname{int}S^{-}(u)$;*
2. *T is closed with convex values such that $Y=\bigcup_{z\in X}\operatorname{int}T^{-}(z)$;*
3. *For each $x\in X$, $\varphi(z,u)\geq\alpha$ for all $(z,u)\in X\times S(x)$;*
4. *For each $y\in Y$, $\varphi(z,u)\leq\alpha$ for all $(z,u)\in T(y)\times Y$;*
5. *For each $y\in Y$, the set $\{x\in X:\varphi(x,y)<\alpha\}$ is convex;*
6. *For each $x\in X$, the set $\{y\in Y:\varphi(x,y)>\alpha\}$ is convex;*
7. $\bigcup_{x\in X}\{y\in Y:\varphi(x,y)<\alpha\}=\bigcup_{x\in X}\operatorname{int}\{y\in Y:\varphi(x,y)<\alpha\}$;
8. $\bigcup_{y\in Y}\{x\in X:\varphi(x,y)>\alpha\}=\bigcup_{y\in Y}\operatorname{int}\{x\in X:\varphi(x,y)>\alpha\}$.

Then, there exists a point $(x_0,y_0)\in T(Y)\times S(X)$ such that

$$\varphi(x_0,y)\leq\varphi(x_0,y_0)=\alpha\leq\varphi(x,y_0),\quad \text{for all } (x,y)\in T(Y)\times S(X).$$

Moreover, $\inf_{x\in T(Y)}\sup_{y\in S(X)}\varphi(x,y)=\sup_{y\in S(X)}\inf_{x\in T(Y)}\varphi(x,y)$.

Proof. It follows from Theorem 14 together with conditions (i), (iii), (v), and (vii) that there exists a point $y_0\in S(X)$ such that $\varphi(x,y_0)\geq\alpha$ for all $x\in X$. Again, using Theorem 14 and conditions (ii), (iv), (vi), and (viii), we obtain a point $x_0\in T(Y)$ such that $\varphi(x_0,y)\leq\alpha$ for all $y\in Y$. In particular, $\varphi(x_0,y_0)=\alpha$. From these inequalities, we have

$$\varphi(x_0,y)\leq\varphi(x_0,y_0)=\alpha\leq\varphi(x,y_0),\quad \text{for all } (x,y)\in T(Y)\times S(X);$$

hence,

$$\inf_{x\in T(Y)}\sup_{y\in S(X)}\varphi(x,y)\leq\sup_{y\in S(X)}\inf_{x\in T(Y)}\varphi(x,y).$$

The reverse inequality is always true; therefore,

$$\inf_{x\in T(Y)}\sup_{y\in S(X)}\varphi(x,y)=\sup_{y\in S(X)}\inf_{x\in T(Y)}\varphi(x,y),$$

as desired. □

The point (x_0, y_0) in Theorem 15 is called a saddle point of φ in $T(Y) \times S(X)$. Theorem 15 includes a standard manifold version of minimax theorem as a corollary if $S(x) = Y$ for all $x \in X$ and $T(y) = X$ for all $y \in Y$. We recall that a function $f : X \to \mathbb{R}$ on a topological space X is lower semicontinuous (l.s.c.) if for each $\lambda \in \mathbb{R}$ the set $\{x \in X : f(x) \leq \lambda\}$ is closed.

Corollary 3. *Let X and Y be compact convex sets, each in a Hadamard manifold. Let $\varphi : X \times Y \to \mathbb{R}$ be a function satisfying the following conditions:*

1. *For each $x \in X$, $\varphi(x, y) \geq 0$ for all $y \in Y$;*
2. *For each $y \in Y$, $\varphi(x, y) \leq 0$ for all $x \in X$;*
3. *$\varphi(x, y)$ is quasiconvex in x and u.s.c. in y;*
4. *$\varphi(x, y)$ is quasiconcave in y and l.s.c. in x.*

Then, there exists a saddle point $(x_0, y_0) \in X \times Y$ such that

$$\varphi(x_0, y) \leq \varphi(x_0, y_0) = 0 \leq \varphi(x, y_0), \quad \text{for all } (x, y) \in X \times Y.$$

Moreover, $\inf_{x \in X} \sup_{y \in Y} \varphi(x, y) = \sup_{y \in Y} \inf_{x \in X} \varphi(x, y)$.

6. Conclusions

The vector quasi-equilibrium problems have been extensively studied by many researchers in compact or noncompact topological vector spaces. However, the traditional techniques in the literature cannot be easily extended to the manifold settings because geometric intuition is more difficult to rigorize in such cases. This paper presents a new fixed-point method for formulating existence theorems for generalized vector quasi-equilibrium problems in locally compact and σ-compact spaces (not necessarily compact) without any continuity or coercivity assumptions. This approach is achieved by using the key fixed-point theorem on product Hadamard manifolds. The main results of this paper can provide useful material for future research on minimax problems in Hadamard manifolds.

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References

1. Aussel, D.; Cotrina, J.; Iusem, A. An existence result for quasi-equilibrium problems. *J. Convex Anal.* **2017**, *24*, 55–66.
2. Blum, E.; Oettli, W. From optimization and variational inequalities to equilibrium problems. *Math. Stud.* **1993**, *63*, 123–145.
3. Bueno, L.F.; Haeser, G.; Lara, F.; Rojas, F.N. An augmented Lagrangian method for quasi-equilibrium problems. *Comput. Optim. Appl.* **2020**, *76*, 737–766. [CrossRef]
4. Cotrina, J.; García, Y. Equilibrium Problems: Existence Results and Applications. *Set-Valued Var. Anal.* **2018**, *26*, 159–177. [CrossRef]
5. Debnath, P.; Konwar, N.; Radenović, S. *Metric Fixed-Point Theory: Applications in Science, Engineering and Behavioural Sciences*; Springer: Berlin/Heidelberg, Germany, 2021.
6. Noor, M.A.; Oettli, W. On general nonlinear complementarity problems and quasi-equilibria. *Mathmatiche* **1994**, *49*, 313–331.
7. Park, S. Fixed points and quasi-equilibrium problems. *Math. Comput. Model.* **2004**, *34*, 947–954. [CrossRef]
8. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics;* Springer: Berlin/Heidelberg, Germany, 2019.
9. Ansari, Q.H. Vectorial form of the Ekeland-type variational principle with applications to vector equilibrium problems and fixed point theory. *J. Math. Anal. Appl.* **2007**, *334*, 561–575. [CrossRef]
10. Balaj, M. Scalar and vector equilibrium problems with pairs of bifunctions. *J. Glob. Optim.* **2022**, *84*, 739–753. [CrossRef]

11. Parida, J.; Sen, A. A variational-like inequality for multifunctions with applications. *J. Math. Anal. Appl.* **1987**, *124*, 73–78. [CrossRef]
12. Németh, S.Z. Variational inequalities on Hadamard manifolds. *Nonlinear Anal.* **2003**, *52*, 1491–1498. [CrossRef]
13. Ansari, Q.H.; Babu, F. Proximal point algorithm for inclusion problems in Hadamard manifolds with applications. *Optim. Lett.* **2021**, *15*, 901–921. [CrossRef]
14. Huang, S. KKM property in Riemannian manifolds. *J. Nonlinear Convex Anal.* **2018**, *19*, 313–322.
15. Huang, S. Section theorems in Hadamard manifolds. *J. Nonlinear Convex Anal.* **2021**, *22*, 1189–1203.
16. Iusem, A.N.; Mohebbi, V. An extragradient method for vector equilibrium problems on Hadamard manifolds. *J. Nonlinear Var. Anal.* **2021**, *5*, 459–476.
17. Park, S. Revisit to Generalized KKM maps. *J. Nonlinear Convex Anal.* **2021**, *22*, 2405–2412.
18. Upadhyay, B.B.; Treanţă, S.; Mishra, P. On Minty variational principle for nonsmooth multiobjective optimization problems on Hadamard manifolds. *Optimization* **2022**, 1–20. [CrossRef]
19. Aliprantis, C.D.; Border, K.C. *Infinite Dimensional Analysis*; Springer: New York, NY, USA, 1999.
20. do Carmo, M.P. *Riemannian Geometry*; Birkhäuser: Boston, MA, USA, 1992.
21. Lin, Y.C.; Ansari, Q.H.; Lai, H.C. Minimax theorems for set-valued mappings under cone-convexities. *Abstr. Appl. Anal.* **2012**, *2012*, 310818. [CrossRef]
22. Huang, S. Fixed point theorems and intersection theorems in product Hadamard manifolds. *J. Nonlinear Convex Anal.* **2021**, *22*, 2553–2563.