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Analytical Solutions of Temperature Distribution in a Rectangular Parallelepiped

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Abstract: In the present article, we give analytical solutions for temperature distribution in a rectangular parallelepiped with the help of a multivariable I-function. The results established in this paper are of a general character from which several known and new results can be deduced. We also give the special and particular cases of our main findings.

Keywords: multivariable I-function; multivariable H-function; temperature distribution

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1. Introduction and Preliminaries

Fractional calculus is three centuries old—as old as conventional calculus. Its importance has been highlighted by many researchers in recent years.

Fractional calculus is based on integrals and the derivatives of non-integer arbitrary order, fractional differential equations and methods of their solution, approximations and implementation techniques. The concept of differentiation and integration to non-integer order is by no means new. Interest in this subject was evident almost as soon as the ideas of classical calculus were known. For the past three centuries, this subject was considered by mathematicians, and only in the last few years has it been applied to the fields of engineering, science and economics. As is well known, several physical phenomena are often better described by fractional derivatives. However, recent attempts have been made to define the fractional derivative as a local operator in fractal science theory.

In recent years, several authors have studied the functions of two or more variables, for example, see [1–5]. Recent expansion in the theory of I-functions has become important due to the introduction of the multivariable I-function which has been studied by many authors (for recent work, see [6,7]). Recently, Kumar & Ayant [8] provided an application of the Jacobi polynomial and multivariable Aleph-function in heat conduction in a non-homogeneous moving rectangular parallelepiped. Prasad & Pati [9] used the modified multivariable H-function and provided the temperature distribution in a rectangular parallelepiped. In the present paper, we provide an application of the multivariable I-function for temperature distribution in a rectangular parallelepiped.

The multivariable I-function is defined in terms of the multiple Mellin–Barnes-type integral, and is given in the following manner [10]:

$$I(z_1,\cdots,z_r)=I^{0,n_2;0,n_3;\cdots;0,n_r:m',n';\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p',q';\cdots;p^{(r)},q^{(r)}}\left(\begin{array}{c|c} z_1 & \left(a_{2j};\alpha^{(1)}_{2j},\alpha^{(2)}_{2j}\right)_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ z_r & \left(b_{2j};\beta^{(1)}_{2j},\beta^{(2)}_{2j}\right)_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} \left(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj}\right)_{1,p_r}:\left(a^{(1)}_j,\alpha^{(1)}_j\right)_{1,p^{(1)}};\cdots;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \\ \left(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj}\right)_{1,q_r}:\left(b^{(1)}_j,\beta^{(1)}_j\right)_{1,q^{(1)}};\cdots;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \end{array}\right),$$

$$=\frac{1}{(2\pi\omega)^r}\int_{\mathcal{L}_1}\cdots\int_{\mathcal{L}_r}\xi(s_1,\cdots,s_r)\left\{\prod_{i=1}^{r}\phi_i(s_i)\,z_i^{s_i}\right\}\mathrm{d}s_1\cdots\mathrm{d}s_r, \tag{1}$$

where $z_i\neq 0$, $\omega=\sqrt{-1}$, and

$$\phi_i(s_i)=\frac{\left\{\prod_{j=1}^{m^{(i)}}\Gamma\left(b^{(i)}_j-\beta^{(i)}_j s_i\right)\right\}\left\{\prod_{j=1}^{n^{(i)}}\Gamma\left(1-a^{(i)}_j+\alpha^{(i)}_j s_i\right)\right\}}{\left\{\prod_{j=m^{(i)}+1}^{q^{(i)}}\Gamma\left(1-b^{(i)}_j+\beta^{(i)}_j s_i\right)\right\}\left\{\prod_{j=n^{(i)}+1}^{p^{(i)}}\Gamma\left(a^{(i)}_j-\alpha^{(i)}_j s_i\right)\right\}}\ (\text{for all } i\in\{1,\cdots,r\}), \tag{2}$$

$$\begin{aligned}\xi(s_1,\cdots,s_r)&=\frac{\prod_{k=2}^{r}\left\{\prod_{j=1}^{n_k}\Gamma\left(1-a_{kj}+\sum_{i=1}^{k}\alpha^{(i)}_{kj}s_i\right)\right\}}{\prod_{k=2}^{r}\left\{\prod_{j=n_k+1}^{p_k}\Gamma\left(a_{kj}-\sum_{i=1}^{k}\alpha^{(i)}_{kj}s_i\right)\right\}}\\ &\times\frac{1}{\prod_{k=2}^{r}\left\{\prod_{j=1}^{q_k}\Gamma\left(1-b_{kj}+\sum_{i=1}^{k}\beta^{(i)}_{kj}s_i\right)\right\}}.\end{aligned} \tag{3}$$

For the existence and convergence conditions of (1) (the reader may wish to refer to work by Prasad [10]).

The absolute convergence condition of the multiple Mellin–Barnes-type contour (1) can be obtained by extension of the corresponding conditions for the multivariable H-function, given by

$$|\arg z_i|<\frac{1}{2}\Omega_i\,\pi,$$

where

$$\begin{aligned}\Omega_i&=\sum_{k=1}^{n^{(i)}}\alpha^{(i)}_k-\sum_{k=n^{(i)}+1}^{p^{(i)}}\alpha^{(i)}_k+\sum_{k=1}^{m^{(i)}}\beta^{(i)}_k-\sum_{k=m^{(i)}+1}^{q^{(i)}}\beta^{(i)}_k+\left(\sum_{k=1}^{n_2}\alpha^{(i)}_{2k}-\sum_{k=n_2+1}^{p_2}\alpha^{(i)}_{2k}\right)\\ &+\left(\sum_{k=1}^{n_r}\alpha^{(i)}_{rk}-\sum_{k=n_r+1}^{p_r}\alpha^{(i)}_{rk}\right)-\left(\sum_{k=1}^{q_2}\beta^{(i)}_{2k}+\sum_{k=1}^{q_3}\beta^{(i)}_{3k}+\cdots+\sum_{k=1}^{q_r}\beta^{(i)}_{rk}\right),\end{aligned} \tag{4}$$

where $i=1,\cdots,r$.

Throughout the present paper, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form:

$$I(z_1,\cdots,z_r)=O\left(|z_1|^{\alpha'_1},\cdots,|z_r|^{\alpha'_r}\right),\ \max(|z_1|,\cdots,|z_r|)\to 0$$

$$I(z_1,\cdots,z_r)=O\left(|z_1|^{\beta'_1},\cdots,|z_r|^{\beta'_s}\right),\ \min(|z_1|,\cdots,|z_r|)\to +\infty$$

where $k=1,\cdots,z;\ \alpha'_k=\min\left[\Re\left(b^{(k)}_j/\beta^{(k)}_j\right)\right],\ j=1,\cdots,m^{(k)}$ and

$$\beta'_k=\max\left[\Re\left(\left(a^{(k)}_j-1\right)/\alpha^{(k)}_j\right)\right],\ j=1,\cdots,n^{(k)}.$$

We use the following notations in this paper:

$$U = p_2, q_2; p_3, q_3; \cdots; p_{r-1}, q_{r-1}; \;\; V = 0, n_2; 0, n_3; \cdots; 0, n_{r-1}, \tag{5}$$

$$W = \left(p^{(1)}, q^{(1)}\right); \cdots; \left(p^{(r)}, q^{(r)}\right); \;\; X = \left(m^{(1)}, n^{(1)}\right); \cdots; \left(m^{(r)}, n^{(r)}\right), \tag{6}$$

$$A = \left(a_{2k}, \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}\right)_{1,p_2}; \cdots; \left(a_{(r-1)k}, \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \cdots, \alpha_{(r-1)k}^{(r-1)}\right)_{1,p_{r-1}}, \tag{7}$$

$$B = \left(b_{2k}, \beta_{2k}^{(1)}, \beta_{2k}^{(2)}\right)_{1;q_2}; \cdots; \left(b_{(r-1)k}, \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \cdots, \beta_{(r-1)k}^{(r-1)}\right)_{1,q_{r-1}}, \tag{8}$$

$$\mathbb{A} = \left(a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots, \alpha_{rk}^{(r)}\right)_{1,p_r} : \left(a_k^{(1)}, \alpha_k^{(1)}\right)_{1,p'}; \cdots; \left(a_k^{(r)}, \alpha_k^{(r)}\right)_{1,p^{(r)}}, \tag{9}$$

$$\mathbb{B} = \left(b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \cdots, \beta_{rk}^{(r)}\right)_{1,q_r} : \left(b_k^{(1)}, \beta_k^{(1)}\right)_{1,q'}; \cdots; \left(b_k^{(r)}, \beta_k^{(r)}\right)_{1,q^{(r)}}. \tag{10}$$

2. Formulation of the Problem

The temperature $\theta(x, y, z, t)$ at any point of a rectangular parallelepiped of edges a, b, c, can be represented by the following partial differential equation:

$$\frac{\partial\theta}{\partial t} = K_1\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right) + \psi(x, y, z, t) + c_0\,\theta(x, y, z, t), \tag{11}$$

where t is the time, $K_1 = \frac{K}{\rho_c}$, in which K is the thermal conductivity of the rectangular parallelepiped, ρ is the density, c is the specific heat and ψ is the heat source within it; K, ρ, c and c_0 are constants.

The initial and boundary conditions are taken as

$$\theta(x, y, z, 0) = f(x, y, z), \tag{12}$$

$$\theta(a, y, z, t) = g_1(y, z), \tag{13}$$

$$\theta(x, b, z, t) = h_1(x, z), \tag{14}$$

$$\theta(x, y, c, t) = r_1(x, y), \tag{15}$$

$$\theta(0, y, z, t) = g_2(y, z), \tag{16}$$

$$\theta(x, 0, z, t) = h_2(x, z), \tag{17}$$

$$\theta(x, y, 0, t) = r_2(x, y), \tag{18}$$

3. Solution of the Problem

Required Integral

We will need the following result:

Lemma 1.
$$\int_0^b \sin\left(\frac{n\pi y}{b}\right)e^{-\mu y}\mathrm{d}y = \frac{\pi n b}{(\mu^2 b^2 + n^2\pi^2)}\left[(-)^{n+1}e^{-\mu b} + 1\right]. \tag{19}$$

For the solution of (11) under the conditions (12)–(18), we take the triple finite Fourier transform which is represented as follows:

$$\bar{\theta}(m, n, q, t) = \int_0^a \int_0^b \int_0^c \theta(x, y, z, t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{20}$$

Now, multiplying both sides of (11) by $\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)$, and integrating over the whole rectangular parallelepiped, we get

$$\begin{aligned}
&\int_0^a\int_0^b\int_0^c \frac{\partial\theta}{\partial t}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&= K_1\int_0^a\int_0^b\int_0^c\left[\frac{\partial\theta^2}{\partial x^2}+\frac{\partial\theta^2}{\partial y^2}+\frac{\partial\theta^2}{\partial z^2}\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&+\int_0^a\int_0^b\int_0^c \psi(x,y,z,t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&+c_0\int_0^a\int_0^b\int_0^c \theta(x,y,z,t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.
\end{aligned}\tag{21}$$

By the application of results of (20), (13)–(18) and Sneddon [11], the equation (21) is transformed to

$$\begin{aligned}
&\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}t}+K_1\pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{q^2}{c^2}-\frac{c_0}{K_1}\right]\bar{\theta}=K_1[K_2\bar{F}_1(n,q)+K_3\bar{F}_2(m,q)+K_4\bar{F}_3(m,n)]\\
&+K_1[K_5\bar{F}_4(n,q)+K_6\bar{F}_5(m,q)+K_7\bar{F}_6(m,n)]+\bar{\psi}(m,n,q,t),
\end{aligned}\tag{22}$$

where,

$$\bar{F}_1(n,q)=\int_0^b\int_0^c g_1(y,z)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}y\,\mathrm{d}z,\tag{23}$$

$$\bar{F}_2(m,q)=\int_0^a\int_0^c h_1(x,z)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}x\,\mathrm{d}z,\tag{24}$$

$$\bar{F}_3(m,n)=\int_0^a\int_0^b r_1(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\,\mathrm{d}y,\tag{25}$$

$$\bar{F}_4(n,q)=\int_0^b\int_0^c g_2(y,z)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}y\,\mathrm{d}z,\tag{26}$$

$$\bar{F}_5(m,q)=\int_0^a\int_0^c h_2(x,z)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{q\pi z}{c}\right)\mathrm{d}x\,\mathrm{d}z,\tag{27}$$

$$\bar{F}_6(m,n)=\int_0^a\int_0^b r_2(x,y)\sin\left(\frac{m\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}x\,\mathrm{d}y,\tag{28}$$

$$K_2=(-)^{m+1}\frac{m\pi}{a},\tag{29}$$

$$K_3=(-)^{n+1}\frac{n\pi}{b},\tag{30}$$

$$K_4=(-)^{q+1}\frac{q\pi}{c},\tag{31}$$

$$K_5=\frac{m\pi}{a},\tag{32}$$

$$K_6=\frac{n\pi}{b},\tag{33}$$

$$K_7=\frac{n\pi}{c}.\tag{34}$$

The Equation (22) can be written as

$$\begin{aligned}
&\frac{d\bar{\theta}}{dt}+K_1B\bar{\theta}=K_1[K_2\bar{F}_1(n,q)+K_3\bar{F}_2(m,q)+K_4\bar{F}_3(m,n)]\\
&+K_1[K_5\bar{F}_4(n,q)+K_6\bar{F}_5(m,q)+K_7\bar{F}_6(m,n)]+\bar{\psi}(m,n,q,t),
\end{aligned}\tag{35}$$

where,

$$B=\pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{q^2}{c^2}-\frac{c_0}{K_1}\right],\tag{36}$$

here c_0 is chosen that $B > 0$.

Applying the boundary condition (12) on the linear differential equation (35), we get the following result:

$$\begin{aligned}\bar{\theta}(m,n,q,t) &= \bar{f}(m,n,q)\, e^{-K_1 B t} + \frac{1}{B}[K_2 \bar{F}_1(n,q) + K_3 \bar{F}_2(m,q) + K_4 \bar{F}_3(m,n)] \\ &+ K_1[K_5 \bar{F}_4(n,q) + K_6 \bar{F}_5(m,q) + K_7 \bar{F}_6(m,n)]\left(1 - e^{-K_1 B t}\right) \\ &+ \int_0^t e^{-K_1 B(t-\tau)} \bar{\psi}(m,n,q,\tau)\, \mathrm{d}\tau, \end{aligned} \tag{37}$$

where,

$$\bar{f}(m,n,q) = \int_0^a \int_0^b \int_0^c f(x,y,z) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z. \tag{38}$$

Using the theorem for the finite sine transform and the result of Sneddon [11], we get the following solution:

$$\begin{aligned}\theta(x,y,z,t) &= \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \bar{f}(m,n,q)\, e^{-K_1 B t} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \frac{K_2}{B} \bar{F}_1(n,q)\left(1 - e^{-K_1 B t}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \frac{K_3}{B} \bar{F}_2(m,q)\left(1 - e^{-K_1 B t}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \frac{K_4}{B} \bar{F}_3(m,n)\left(1 - e^{-K_1 B t}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \frac{K_5}{B} \bar{F}_4(n,q)\left(1 - e^{-K_1 B t}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \frac{K_6}{B} \bar{F}_5(m,q)\left(1 - e^{-K_1 B t}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \frac{K_7}{B} \bar{F}_6(m,n)\left(1 - e^{-K_1 B t}\right) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \int_0^t e^{-K_1 B(t-\tau)}\, \bar{\psi}(m,n,q,\tau)\, \mathrm{d}\tau. \end{aligned} \tag{39}$$

4. Particular Case

On taking $g_1(y,z) = g_2(y,z) = h_1(x,z) = h_2(x,z) = r_1(x,y) = r_2(x,y) = 0$, the six faces of the rectangular parallelepiped are kept at zero temperature, the solution (39) reduces to

$$\begin{aligned}\theta(x,y,z,t) &= \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \bar{f}(m,n,q) e^{-K_1 B t} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \\ &+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \int_0^t e^{-K_1 B(t-\tau)}\, \bar{\psi}(m,n,q,\tau)\, \mathrm{d}\tau. \end{aligned} \tag{40}$$

Example

Since the multivariable I-function defined by Prasad [10] is the generalized function in the field of special functions, we are interested in obtaining a particular solution of the

Equation (40) by assuming both the initial temperature distribution at any point (x, y, z) and the heat source of general character in terms of the multivariable I-function.

For the first attempt, let us take (variables separation method)

$$f(x,y,z) = f_1(x) f_2(y) f_3(z), \tag{41}$$

where, $f_2(y) = e^{-\mu y}$, $f_3(z) = e^{-\delta z}$ and

$$f_1(x) = I^{V;0,n_r:X}_{U;p_r,q_r:W}\left(\begin{array}{c|c} c_1 x^{m_1} & A; \mathbb{A} \\ \vdots & \\ c_r x^{m_r} & B; \mathbb{B} \end{array}\right). \tag{42}$$

We obtain

$$\begin{aligned} &\bar{f}(m,n,p) \\ &= \sum_{r_1=0}^{+\infty} \frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!} \frac{\pi^2 nqbc}{(\mu^2 b^2 + n^2\pi^2)(\delta^2 c^2 + q^2\pi^2)} \left[(-)^{n+1} e^{-\mu b} + 1\right]\left[(-)^{q+1} e^{-\delta c} + 1\right] \\ &\times I^{V;0,n_r+1:X}_{U;p_r+1,q_r+1:W}\left(\begin{array}{c|c} c_1 a^{m_1} & A; (-2r_1-2; m_1, \cdots, m_r), \mathbb{A} \\ \vdots & \\ c_r a^{m_r} & B; (-2r_1-1; m_1, \cdots, m_r), \mathbb{B} \end{array}\right), \end{aligned} \tag{43}$$

provided that $\min\{\mu, \delta, m_i\} > 0 \ (i = 1, \cdots, r)$, $2 + \sum_{i=1}^{r} m_i \min_{1 \leqslant j \leqslant m^{(i)}} \Re\left(\frac{b_j^{(i)}}{\beta_j^{(i)}}\right) > 0$, and $|\arg c_i| < \frac{1}{2}\Omega_i \pi$, where Ω_i is defined by (4).

Proof of (43)**.** Considering the relation (41) and applying Lemma 19, according to (39), we have

$$\begin{aligned} \bar{f}(m,n,q) &= \int_0^a \int_0^b \int_0^c f(x,y,z) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{q\pi z}{c}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \frac{\pi^2 nqbc}{(\mu^2 b^2 + n^2\pi^2)(\delta^2 c^2 + q^2\pi^2)} \left[(-)^{n+1} e^{-\mu b} + 1\right]\left[(-)^{q+1} e^{-\delta c} + 1\right] \\ &\times \int_0^a \sin\left(\frac{m\pi x}{a}\right) f_1(x)\, \mathrm{d}x. \end{aligned} \tag{44}$$

Now, replacing $f_1(x)$ by the multivariable I-function with the help of (42), we have

$$\begin{aligned} \bar{f}(m,n,q) &= \frac{\pi^2 nqbc}{(\mu^2 b^2 + n^2\pi^2)(\delta^2 c^2 + q^2\pi^2)} \left[(-)^{n+1} e^{-\mu b} + 1\right]\left[(-)^{q+1} e^{-\delta c} + 1\right] \\ &\times \int_0^a \sin\left(\frac{m\pi x}{a}\right) I^{V;0,n_r:X}_{U;p_r,q_r:W}\left(\begin{array}{c|c} c_1 x^{m_1} & A; \mathbb{A} \\ \vdots & \vdots \\ c_r x^{m_r} & B; \mathbb{B} \end{array}\right) \mathrm{d}z, \end{aligned} \tag{45}$$

using the integrals representation of the multivariable I-function with the help of (1), and interchanging the order of integrations, which is justified under the conditions mentioned above, then we arrive at

$$\begin{aligned} \bar{f}(m,n,q) &= \frac{\pi^2 nqbc}{(\mu^2 b^2 + n^2\pi^2)(\delta^2 c^2 + q^2\pi^2)} \left[(-)^{n+1} e^{-\mu b} + 1\right]\left[(-)^{q+1} e^{-\delta c} + 1\right] \\ &\times \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \xi(s_1, \cdots, s_r) \prod_{i=1}^{r} \phi_i(s_i) z_i^{s_i} c_i^{s_i} \int_0^a \sin\left(\frac{m\pi x}{a}\right) x^{\sum_{i=1}^{r} c_i s_i}\, \mathrm{d}x\, \mathrm{d}s_1 \cdots \mathrm{d}s_r. \end{aligned} \tag{46}$$

On the other hand, we have the following relation:

$$\sin\left(\frac{m\pi x}{a}\right) = \sum_{r_1=0}^{+\infty} \frac{(-1)^{2r_1+1}}{(2r_1+1)!}\left(\frac{m\pi x}{a}\right)^{2r_1+1}, \tag{47}$$

By using the above relation and interchanging the order of integration and summation, which is permissible under the stated validity conditions, then we get

$$\begin{aligned}\bar{f}(m,n,q) &= \frac{\pi^2 nqbc}{(\mu^2b^2+n^2\pi^2)(\delta^2c^2+q^2\pi^2)}\left[(-)^{n+1}e^{-\mu b}+1\right]\left[(-)^{q+1}e^{-\delta c}+1\right] \\ &\times \sum_{r_1=0}^{+\infty} \frac{(m\pi)^{2r_1+1}(-1)^{2r_1+1}}{a^{2r_1+1}(2r_1+1)!}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(s_1,\cdots,s_r)\prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}c_i^{s_i} \\ &\times \int_0^a x^{\sum_{i=1}^r c_is_i+2r_1+1}\,\mathrm{d}x\,\mathrm{d}s_1\cdots\mathrm{d}s_r.\end{aligned} \tag{48}$$

Evaluating the inner integral and using the relation $\frac{1}{a} = \frac{\Gamma(a)}{\Gamma(a+1)}$, then

$$\begin{aligned}\bar{f}(m,n,q) &= \frac{\pi^2 nqbc}{(\mu^2b^2+n^2\pi^2)(\delta^2c^2+q^2\pi^2)}\left[(-)^{n+1}e^{-\mu b}+1\right]\left[(-)^{q+1}e^{-\delta c}+1\right] \\ &\times \sum_{r_1=0}^{+\infty} \frac{(-1)^{2r_1+1}}{(2r_1+1)!}(m\pi)^{2r_1+1}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\xi(s_1,\cdots,s_r) \\ &\times \prod_{i=1}^{r}\phi_i(s_i)z_i^{s_i}c_i^{s_i}\,\frac{\Gamma(\sum_{i=1}^r c_is_i+2r_1+2)}{\Gamma(\sum_{i=1}^r c_is_i+2r_1+3)}\,a^{\sum_{i=1}^r c_is_i}\,\mathrm{d}s_1\cdots\mathrm{d}s_r.\end{aligned} \tag{49}$$

Now, interpreting the multiple integrals (49) in terms of the I-function of r-variables, we obtain the required result (43). □

Again, for the heat source, let

$$\psi(x,y,z,t) = e^{-\alpha t}\psi'(x,y,z). \tag{50}$$

For the first attempt, let us take (variables separation method)

$$\psi'(x,y,z) = \psi_1(x)\psi_2(y)\psi_3(z), \tag{51}$$

where, $\psi_2(y) = e^{-\mu' y}$, $\psi_3(z) = e^{-\delta' z}$ and

$$\psi_1(x) = I^{V';0,n'_r:X'}_{U';p'_r,q'_r:W'}\left(\begin{array}{c|c} c'_1x^{m'_1} & A';\mathbb{A}' \\ \vdots & \\ c'_rx^{m'_r} & B';\mathbb{B}' \end{array}\right), \tag{52}$$

where,

$$U' = p'_2,q'_2;p'_3,q'_3;\cdots;p'_{r-1},q'_{r-1};\quad V' = 0,n'_2;0,n'_3;\cdots;0,n'_{r-1}. \tag{53}$$

$$W' = \left(p'^{(1)},q'^{(1)}\right);\cdots;\left(p'^{(r)},q'^{(r)}\right);\quad X = \left(m'^{(1)},n'^{(1)}\right);\cdots;\left(m'^{(r)},n'^{(r)}\right). \tag{54}$$

$$A' = \left(a'_{2k};\alpha'^{(1)}_{2k},\alpha'^{(2)}_{2k}\right)_{1,p'_2};\cdots;\left(a'^{(1)}_{(r-1)};\alpha'^{(1)}_{(r-1)k},\alpha'^{(2)}_{(r-1)k},\cdots,\alpha'^{(r-1)}_{(r-1)k}\right)_{1,p'_{r-1}}. \tag{55}$$

$$B' = \left(b'_{2k};\beta'^{(1)}_{2k},\beta'^{(2)}_{2k}\right)_{1,q'_2};\cdots;\left(b'^{(1)}_{(r-1)};\beta'^{(1)}_{(r-1)k},\beta''^{(2)}_{(r-1)k},\cdots,\beta'^{(r-1)}_{(r-1)k}\right)_{1,q'_{r-1}}. \tag{56}$$

$$\mathbb{A}' = \left(a'_{rk};\alpha'^{(1)}_{rk},\alpha'^{(2)}_{rk},\cdots,\alpha'^{(r)}_{rk}\right)_{1,p'_r} : \left(a'^{(1)}_k;\alpha'^{(1)}_k\right)_{1,p'^{(1)}},\cdots;\left(a'^{(r)}_k;\alpha'^{(r)}_k\right)_{1,p'^{(r)}}. \tag{57}$$

$$\mathbb{B}' = \left(b'_{rk}; \beta'^{(1)}_{rk}, \beta'^{(2)}_{rk}, \cdots, \beta'^{(r)}_{rk}\right)_{1,q'_r} : \left(b'^{(1)}_k; \beta'^{(1)}_k\right)_{1,q'^{(1)}}, \cdots ; \left(\beta'^{(r)}_k; \beta'^{(r)}_k\right)_{1,q'^{(r)}}. \tag{58}$$

Using the value of $\psi(x, y, z, t)$ in the Equation (20) and integrating ψ with respect to τ between the limits 0 and t, then we obtain

$$\begin{aligned}
&\int_0^t e^{-K_1B(t-\tau)}\,\bar{\psi}(m, n, q, \tau)\,\mathrm{d}\tau = \sum_{r_1=0}^{+\infty} \frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!} \frac{\pi^2 nqbc}{(\mu'^2b^2+n^2\pi^2)(\delta'^2c^2+q^2\pi^2)} \\
&\times \left[(-)^{n+1}e^{-\mu'b}+1\right]\left[(-)^{q+1}e^{-\delta'c}+1\right] \frac{e^{-K_1Bt}}{K_1B-\alpha}\left(e^{(K_1B-\alpha)t}-1\right) \\
&\times I^{V';0,n'_r+1:X'}_{U';p'_r+1,q'_r+1:W'}\left(\begin{array}{c|c} c'_1a^{m_1} & A'; (-2r_1-2; m'_1, \cdots, m'_r), \mathbb{A}' \\ \vdots & \vdots \\ c'_ra^{m_r} & B'; (-2r_1-1; m'_1, \cdots, m'_r), \mathbb{B}' \end{array}\right),
\end{aligned} \tag{59}$$

provided that $\min\{\alpha, \mu', \delta', m'_i\} > 0$ $(i = 1, \cdots, r)$, $2 + \sum_{i=1}^r m'_i \min\limits_{1\leqslant j\leqslant m'^{(i)}} \Re\left(\frac{b'^{(i)}_j}{\beta'^{(i)}_j}\right) > 0$, and $\left|\arg c'_i\right| < \frac{1}{2}\Omega'_i\pi$, where

$$\begin{aligned}
\Omega'_i = &\sum_{k=1}^{n^{(i)}} \alpha'^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha'^{(i)}_k + \sum_{k=1}^{m^{(i)}} \beta'^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta'^{(i)}_k + \left(\sum_{k=1}^{n_2} \alpha'^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2} \alpha'^{(i)}_{2k}\right) \\
&+ \left(\sum_{k=1}^{n_r} \alpha'^{(i)}_{rk} - \sum_{k=n_r+1}^{p_r} \alpha'^{(i)}_{rk}\right) - \left(\sum_{k=1}^{q_2} \beta'^{(i)}_{2k} + \sum_{k=1}^{q_3} \beta'^{(i)}_{3k} + \cdots + \sum_{k=1}^{q_r} \beta'^{(i)}_{rk}\right).
\end{aligned} \tag{60}$$

The proof of (59) is similar to (43).

Now, putting the known values of $\bar{f}(m, n, p)$ and $\int_0^t e^{-K_1B(t-\tau)}\,\bar{\psi}(m, n, q, \tau)\,\mathrm{d}\tau$ in Equation (40), we obtain the solution of our problem, defined as

$$\begin{aligned}
\theta(x, y, z, t) = &\frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \sum_{r_1=0}^{+\infty} \frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!} \frac{\pi^2 nqbc}{(\mu^2b^2+n^2\pi^2)(\delta^2c^2+q^2\pi^2)} \\
&\times \left[(-)^{n+1}e^{-\mu b}+1\right]\left[(-)^{q+1}e^{-\delta c}+1\right]e^{-K_1Bt} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right) \\
&\times I^{V;0,n_r+1:X}_{U;p_r+1,q_r+1:W}\left(\begin{array}{c|c} c_1a^{m_1} & A; (-2r_1-2; m_1, \cdots, m_r), \mathbb{A} \\ \vdots & \\ c_ra^{m_r} & B; (-2r_1-1; m_1, \cdots, m_r), \mathbb{B} \end{array}\right) \\
&+ \frac{8}{abc} \sum_{m,n,q=1}^{+\infty} \sum_{r_1=0}^{+\infty} \frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!} \frac{\pi^2 nqbc}{(\mu'^2b^2+n^2\pi^2)(\delta'^2c^2+q^2\pi^2)} \\
&\times \left[(-)^{n+1}e^{-\mu'b}+1\right]\left[(-)^{q+1}e^{-\delta'c}+1\right] \\
&\times \frac{e^{-K_1Bt}}{K_1B-\alpha}\left(e^{(K_1B-\alpha)t}-1\right) \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right) \\
&\times I^{V';0,n'_r+1:X'}_{U';p'_r+1,q'_r+1:W'}\left(\begin{array}{c|c} c'_1a^{m_1} & A'; (-2r_1-2; m'_1, \cdots, m'_r), \mathbb{A}' \\ \vdots & \\ c'_ra^{m_r} & B'; (-2r_1-1; m'_1, \cdots, m'_r), \mathbb{B}' \end{array}\right),
\end{aligned} \tag{61}$$

provided that $\min\{\mu, \delta, m_i\} > 0$, $\min\{\alpha, \mu', \delta', m'_i\} > 0$ for $i = 1, \cdots, r$, $2 + \sum_{i=1}^r m_i \min\limits_{1\leqslant j\leqslant m'^{(i)}} \Re\left(\frac{b^{(i)}_j}{\beta^{(i)}_j}\right) > 0$, $2 + \sum_{i=1}^r m'_i \min\limits_{1\leqslant j\leqslant m'^{(i)}} \Re\left(\frac{b'^{(i)}_j}{\beta'^{(i)}_j}\right) > 0$, $|\arg c_i| < \frac{1}{2}\Omega_i\pi$, and $\left|\arg c'_i\right| < \frac{1}{2}\Omega'_i\pi$.

5. Special Cases

If $U_r = V_r = A = B = U'_r = V'_r = A' = B' = 0$, then the multivariable I-functions reduce to multivariable H-functions as defined by Srivastava et al. [12–15]. We have the following result:

Corollary 1.

$$\theta(x,y,z,t) = \frac{8}{abc}\sum_{m,n,q=1}^{+\infty}\sum_{r_1=0}^{+\infty}\frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!}\frac{\pi^2 nqbc}{(\mu^2b^2+n^2\pi^2)(\delta^2c^2+q^2\pi^2)}$$
$$\times \left[(-)^{n+1}e^{-\mu b}+1\right]\left[(-)^{q+1}e^{-\delta c}+1\right]e^{-K_1Bt}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)$$
$$\times H^{0,n_r+1:X}_{p_r+1,q_r+1:W}\left(\begin{array}{c|c} c_1a^{m_1} & (-2r_1-2;m_1,\cdots,m_r),\mathbb{A} \\ \vdots\, c_ra^{m_r} & (-2r_1-1;m_1,\cdots,m_r),\mathbb{B}\end{array}\right)$$

$$+\frac{8}{abc}\sum_{m,n,q=1}^{+\infty}\sum_{r_1=0}^{+\infty}\frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!}\frac{\pi^2 nqbc}{(\mu'^2b^2+n^2\pi^2)(\delta'^2c^2+q^2\pi^2)}$$
$$\times \left[(-)^{n+1}e^{-\mu' b}+1\right]\left[(-)^{q+1}e^{-\delta' c}+1\right]$$
$$\times \frac{e^{-K_1Bt}}{K_1B-\alpha}\left(e^{(K_1B-\alpha)t}-1\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)$$
$$\times H^{0,n'_r+1:X'}_{p'_r+1,q'_r+1:W'}\left(\begin{array}{c|c} c'_1a^{m_1} & (-2r_1-2;m'_1,\cdots,m'_r),\mathbb{A}' \\ \vdots & \\ c'_ra^{m_r} & (-2r_1-1;m'_1,\cdots,m'_r),\mathbb{B}'\end{array}\right), \tag{62}$$

under the same conditions that (61) *with* $U_r = V_r = A = B = U'_r = V'_r = A' = B' = 0$.

Corollary 2. *The heat source* $\psi(x,y,z,t)$ *vanishes, and the formal solution is given by*

$$\theta(x,y,z,t) = \frac{8}{abc}\sum_{m,n,q=1}^{+\infty}\sum_{r_1=0}^{+\infty}\frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!}\frac{\pi^2 nqbc}{(\mu^2b^2+n^2\pi^2)(\delta^2c^2+q^2\pi^2)}$$
$$\times \left[(-)^{n+1}e^{-\mu b}+1\right]\left[(-)^{q+1}e^{-\delta c}+1\right]e^{-K_1Bt}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)$$
$$\times I^{V;0,n_r+1:X}_{U;p_r+1,q_r+1:W}\left(\begin{array}{c|c} c_1a^{m_1} & A;(-2r_1-1;m_1,\cdots,m_r),\mathbb{A} \\ \vdots & \vdots \\ c_ra^{m_r} & B;(-2r_1-2;m_1,\cdots,m_r),\mathbb{B}\end{array}\right), \tag{63}$$

under the conditions (43).

Corollary 3. *Consider the above formula, if* $U_r = V_r = A = B = 0$, *then we have*

$$\theta(x,y,z,t) = \frac{8}{abc}\sum_{m,n,q=1}^{+\infty}\sum_{r_1=0}^{+\infty}\frac{(-)^{r_1}(m\pi)^{2r_1+1}}{(2r_1+1)!}\frac{\pi^2 nqbc}{(\mu^2b^2+n^2\pi^2)(\delta^2c^2+q^2\pi^2)}$$
$$\times \left[(-)^{n+1}e^{-\mu b}+1\right]\left[(-)^{q+1}e^{-\delta c}+1\right]e^{-K_1Bt}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)$$
$$\times H^{0,n+1:X}_{p+1,q+1:W}\left(\begin{array}{c|c} c_1a^{m_1} & (-2r_1-1;m_1,\cdots,m_r),\mathbb{A} \\ \vdots & \vdots \\ c_ra^{m_r} & (-2r_1-2;m_1,\cdots,m_r),\mathbb{B}\end{array}\right), \tag{64}$$

under the conditions (43) *and* $U_r = V_r = A = B = 0$.

6. Conclusions

The significance of our findings lies in its generality. By specializing the various parameters and variables of the multivariable I-function in our results, we can obtain new results in the form of various special functions of one and several variables. Thus, the result obtained in this paper can yield a large number of results, involving a large variety of special functions and polynomials, concerning the problem of temperature distribution in a rectangular parallelepiped.

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