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Article

Asymptotic Behavior of Solutions of the Cauchy Problem for a Hyperbolic Equation with Periodic Coefficients II

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Abstract: The paper is devoted to studying the behavior of solutions of the Cauchy problem for large values of time—more precisely, obtaining an asymptotic expansion characterizing the behavior of the solution of the Cauchy problem for a one-dimensional second-order hyperbolic equation with periodic coefficients for large values of the time parameter t. To obtain this asymptotic expansion as $t \to \infty$, methods of the spectral theory of differential operators are used, as well as the properties of the spectrum of a non-positive Hill operator with periodic coefficients.

Keywords: asymptotic behavior of solutions; second-order hyperbolic equation; periodic coefficients; Cauchy problem; Hill operator

MSC: 35B10; 35B40; 35C20; 35L10; 35Q41

Citation: Matevossian, H.A.; Korovina, M.V.; Vestyak, V.A. Asymptotic Behavior of Solutions of the Cauchy Problem for a Hyperbolic Equation with Periodic Coefficients II. *Axioms* **2022**, *11*, 473. https://doi.org/10.3390/axioms11090473

Academic Editor: Hans J. Haubold

Received: 28 July 2022
Accepted: 13 September 2022
Published: 16 September 2022

1. Introduction

The paper is a continuation of the paper by the authors [1], where to obtain the asymptotics of the solution of the Cauchy problem (1) and (2), the positive Hill operator ($H_0 > 0$) and the properties of the spectrum $\sigma(H_0)$ of this operator were studied.

Here, we will consider the case when the Hill operator is non-positive ($H_0 \leq 0$), which means that the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 on the complex plane of the variable λ coincides with zero or negative.

This paper also studies the behavior of the solution for $|x| < b$ and $t \to \infty$ of the following Cauchy:

$$u_{tt}(x,t) - (p(x)\,u_x(x,t))_x + q(x)\,u(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \{t > 0\}, \tag{1}$$

$$u(x,t)|_{t=0} = 0, \quad u_t(x,t)|_{t=0} = f(x), \quad x \in \mathbb{R}, \tag{2}$$

where the functions $p(x)$ and $q(x)$ are periodic with period 1,

$$p(x+1) = p(x) \geq \text{const} > 0, \quad q(x+1) = q(x) \geq 0.$$

We also assume that the functions $p(x)$ and $q(x)$ are continuous or have a finite number of discontinuities of the first kind on the period, $f \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} f \subset [0,1]$; b is an arbitrary fixed constant.

For the completeness of the description of these problems, we note that the study of the behavior as $t \to \infty$ as a solution of the problems (1) and (2), and the corresponding multidimensional problems, provided that the potential differs from a constant by a finite function or is sufficient, rapidly tends to a constant at infinity; many papers and books are devoted to this area (see, for ex. [1–9]).

Let us point out some papers in which the problems are studied, similarly to the problems (1) and (2), with different conditions on the potential and coefficients.

The behavior at large time t of the solution of the Cauchy problem for the hyperbolic equation

$$u_{xx} - q(x)u_{tt} = 0, \quad 0 < q_0 \leq q(x) \leq q_\infty < +\infty, \quad (x,t) \in \mathbb{R} \times \{t > 0\},$$

$$u(x,t)|_{t=0} = \varphi(x), \quad u_t(x,t)|_{t=0} = \psi(x), \quad x \in \mathbb{R}.$$

was studied in [3,4]. In [4], under the assumption that the function $q(x)$ tends in a certain way to the limit as $x \to \pm\infty$, some estimates for the rate of decay of the solution as $t \to \infty$, related to the stabilization rate $q(x)$ at $x \to \pm\infty$ were obtained; that is, the estimates $xq'(x) \geq 0$ and $\sup |x|^{s+1}|q'(x)| < \infty$ are satisfied for some $s > 1$ as $x \to \pm\infty$.

Paper [5] studies that a perturbed Hill operator with an exponentially decreasing impurity potential has a resonance (or an odd number of resonances) at each sufficiently distant lacuna on the second ("non-physical") sheet.

In [6], for the one-dimensional perturbed Hill operator H, whose impurity potential has a finite first moment, the "Levinson series" is obtained. This series of relations generalize the well-known Levinson formula to the case when there is a periodic potential. The "Levinson series" is an effective tool for studying the discrete spectrum in lacunae (gap bands). In particular, it is shown that in the case of a reflectionless impurity potential with a finite second moment, there are no eigenvalues of the operator H in the distant lacunae of the spectrum.

In [7], for the Hill operator with a 1-periodic potential $q(x)$ with the condition $\int_0^1 q(x)\,dx = 0$, estimates of the periodic potentials are established for gap lengths.

We also note paper [8], in which the one-dimensional stationary Schrödinger equation with a quasi-periodic potential $u(\omega\, t)$ is studied. It is shown that if the frequency vector ω is large enough, the Schrödinger equation admits two linear independent Floquet solutions for a set of positive energy measure.

Note that in [9], the behavior of the solution of the Cauchy problem for a hyperbolic equation with a periodic potential $q(x)$ is also studied, that is, a problem similar to the problems (1) and (2), which are considered in this paper with $p(x) = 1$.

In the case of periodic coefficients $p(x)$ and $q(x)$, the first results on the behavior of solutions of the Cauchy problem and the initial-boundary value problem of both homogeneous and inhomogeneous hyperbolic equations were obtained in [10,11].

In addition to the papers and books mentioned above, we also note more important literature, such as [12–16], which reflect the spectral properties of the Hill operator from different points of view. In particular, the papers [12,13] were also devoted to the definition of the Hill equation from its spectrum.

Let us present a scheme for studying the Cauchy problem (1) and (2), for the case of a non-positive Hill operator. To solve the Cauchy problem (1) and (2), in the case when the left end of the spectrum of the Hill operator coincides with zero or is negative, it is necessary to make a Fourier transform to reduce the Cauchy problem to a stationary problem. Then, we write the solution of this problem in terms of the resolvent of the Hill operator and apply the inverse Fourier transform. When the left end of the spectrum coincides with zero, at the point $k = \lambda_0 = 0$, the vertical cut in the lower half-plane of the variable k is not made. At the negative left end of the spectrum, the approach to solving the problem is the same as in [1].

Notations: $L^2(\Omega)$ is the space of measurable functions in Ω for which

$$||u; L^2(\Omega)|| = \left(\int_\Omega |u|^2 dx \right)^{1/2} < \infty.$$

The Sobolev space $H^1(\Omega)$ in Ω is defined as:

$$H^1(\Omega) = \{u : u \in L^2(\Omega), \nabla u \in L^2(\Omega)\},$$

provided with the norm

$$||u; H^1(\Omega)||^2 = ||u; L^2(\Omega)||^2 + ||\nabla u; L^2(\Omega)||^2.$$

2. Preliminaries

Definition 1. *A function $u \in C^2(\mathbb{R} \times \{t \geq 0\})$ is called a periodic (anti-periodic) solution of the Cauchy problems (1) and (2) if it satisfies the relation*

$$u(x+1,t) = (-1)^j u(x,t)$$

for any $x \in \mathbb{R}$ and $t \geq 0$, with $j = 0$ and $j = 1$ in the case of periodic and anti-periodic solutions, respectively.

2.1. Spectrum and Green's Function of the Hill Operator

Continuing the function $u(x,t)$ by zero in the region $t < 0$, and applying the Fourier transform with respect to the variable t in the Cauchy problem (1) and (2), for the function

$$y(x,k) = \int_0^\infty u(x,t)\, e^{ikt} dt$$

we obtain the equation

$$(p(x)\, y'(x,k))' + (k^2 - q(x))\, y(x,k) = -f(x) \tag{3}$$

For any function $g(x)$ from $L^2(-\infty,+\infty)$, we define its norm in the same space

$$||g; L^2|| = ||g; L^2(-\infty,+\infty)||.$$

If the function $g(x)$ is defined on the entire axis $(-\infty,+\infty)$, then by $\hat{g}(x)$, we denote the restriction of this function on the segment $[0,1]$.

Let us present some necessary facts from the spectral theory of differential equations. For any function $g(x,k)$, we denote by g' the derivative with respect to x and by g_k the derivative with respect to k

Let $\{y = \theta(x,k),\, y = \varphi(x,k)\}$ be the fundamental system of solutions of the homogeneous (for $f(x) \equiv 0$) Equation (3) such that

$$\begin{cases} \theta(0,k) = 1, & \theta'(0,k) = 0, \\ \varphi(0,k) = 0, & \varphi'(0,k) = 1. \end{cases}$$

It is known [17] that $\theta(x,k)$ and $\varphi(x,k)$ are entire functions in k real on the real axis, and for $|k| \to \infty$, we have the form

$$\begin{cases} \theta(x,k) = & \cos kx + O(|k|^{-1} e^{|\tau|x}), \\ \varphi(x,k) = & \frac{1}{k}\sin kx + O(|k|^{-2} e^{|\tau|x}), \quad \tau = Im\, k. \end{cases} \tag{4}$$

uniformly in $x \in [-b,b]$. These expansions can be differentiated with respect to x and with respect to k.

Let us denote $\theta(k) = \theta(1,k)$, $\theta'(k) = \theta'(1,k)$, $\varphi(k) = \varphi(1,k)$, $\varphi'(k) = \varphi'(1,k)$ and $F(k) \equiv \theta(k) + \varphi'(k)$. The functions $\theta(k)$, $\theta'(k)$, $\varphi(k)$, $\varphi'(k)$ and $F(k)$ are even on the real axis of the complex plane of the variable k.

The Hill operator is the differential operator

$$H_0 := -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

generated in the Hilbert space $L^2(\mathbb{R})$ by the operation

$$\Lambda_0 y := -(p(x)\,y')' + q(x)\,y,$$

where the functions $p(x)$ and $q(x)$ are periodic with period 1.

The spectrum $\sigma(H_0)$ of the Hill operator H_0 is absolutely continuous and is a finite or infinite sequence of isolated segments (zones) separated by lacunae going to infinity.

Note that the Hill operator has only a continuous spectrum, which lies on the real axis and is left semi-bounded [17]. Let us replace the spectral parameter λ by k^2 so that the spectrum $\sigma(H_0)$ of the operator H_0 on the complex plane of the variable k consists of points for which $H_0 - k^2$ does not have a bounded inverse on an everywhere dense set in $L^2(\mathbb{R})$.

For a more detailed characterization of the spectrum $\sigma(H_0)$ of the Hill operator H_0, consider the following periodic (anti-periodic) Sturm–Liouville problems.

Let $\hat{v}(x, \lambda_n)$ be an eigenfunction of the periodic Sturm–Liouville problem:

$$\begin{gathered} -(p(x)y')' + q(x)y = \lambda_n y, \quad x \in [0,1], \\ y(0) = y(1), \quad y'(0) = y'(1), \end{gathered} \tag{5}$$

and $\hat{v}(x, \mu_n)$ is the eigenfunction of the anti-periodic Sturm–Liouville problem:

$$\begin{gathered} -(p(x)y')' + q(x)y = \mu_n y, \quad x \in [0,1], \\ y(0) = -y(1), \quad y'(0) = -y'(1). \end{gathered} \tag{6}$$

normalized by the condition $||\hat{v}; L^2([0,1])|| = 1$, where $\lambda_n = \lambda_n^2$ and $\mu_n = \mu_n^2$, $n = 0, 1, 2, \ldots$, is the set of all eigenvalues of problems (5) and (6), respectively, numbered in ascending order, taking into account the multiplicity.

Continuing the function $\hat{v}(x, \lambda_n)$ (or $\hat{v}(x, \mu_n)$) to the entire real axis, in a periodic (or anti-periodic) way, we obtain a function, which we denote by $v(x, \lambda_n)$ (or $v(x, \mu_n)$).

Since Equation (3) contains the parameter k^2, by replacing the spectral parameter λ by k^2, we can use the expression "complex plane of the variable k" instead of the expression "complex plane of the variable λ".

Denote by $\mathbb{C}'$ the complex plane, as in [1], in which vertical cuts were made from the points $\pm\lambda_0$ in the lower half-plane of the variable k.

In this article, the complex plane $\mathbb{C}'$ will be defined separately, depending on the cases $\lambda_0 = 0$ and $\lambda_0 < 0$.

2.2. Auxiliary Statements

For the convenience of reading this article, we present the formulations of some Propositions and Lemmas from [1].

Proposition 1 ([1])**.** *For the solution of the problems* (1) *and* (2)*, the following representation is valid:*

$$u(x,t) = \frac{1}{2\pi} J_L + v_1(x,t),$$

where the function $v_1(x,t)$ for $x \in [-b, b]$ and $t > 0$ satisfies the estimate

$$|v_1(x,t)| \le C(b)\, e^{-td} ||f; L^2||.$$

Proposition 2 ([1])**.** *For any $t > 0$ and $x \in [-b, b]$, we have the estimate*

$$|J_{L_3}| \le C(b)\, e^{-td} ||f; L^2||.$$

As is known [17], if λ_n and μ_n are the ends of a lacuna, then $\lambda_n = \lambda_n^2$ is a simple eigenvalue of the periodic Sturm–Liouville problem (5), and $\mu_n = \mu_n^2$ is a simple eigenvalue of the anti-periodic Sturm–Liouville problem (6).

For the eigenfunction of problem (5), corresponding to the eigenvalue $\lambda_n = \lambda_n^2$, we will search in the form

$$\hat{v}(x,\lambda_n) = A\,\hat{\theta}(x,\lambda_n) + B\,\hat{\varphi}(x,\lambda_n).$$

Therefore, we obtain the following system

$$\begin{cases} A\,(\theta(\lambda_n) - 1) + B\,\varphi(\lambda_n) = 0, \\ A\,\theta'(\lambda_n) + B\,(\varphi'(\lambda_n) - 1) = 0. \end{cases} \tag{7}$$

Since $\lambda_n = \lambda_n^2$ are simple eigenvalues of problem (5), then the determinant of the system (7) is equal to zero, and all coefficients of the system do not vanish simultaneously. Together with the equality $F(k) \equiv \theta(k) + \varphi'(k) = 2$ for $k = \lambda_n$ (this results from the fact that $k^2 = \lambda_n^2$), which served as the definition of the numbers λ_n, this leads to the fact that at the points, λ_n satisfies one of the following relations:

$$(A_1)\ \theta(\lambda_n) \neq 1,\ \theta'(\lambda_n) \neq 0,\ \varphi(\lambda_n) \neq 0,\ \varphi'(\lambda_n) \neq 1;$$

$$(A_2)\ \theta(\lambda_n) = 1,\ \theta'(\lambda_n) \neq 0,\ \varphi(\lambda_n) = 0,\ \varphi'(\lambda_n) = 1;$$

$$(A_3)\ \theta(\lambda_n) = 1,\ \theta'(\lambda_n) = 0,\ \varphi(\lambda_n) \neq 0,\ \varphi'(\lambda_n) = 1.$$

Note that for any $x, \xi \in \mathbb{R}$, the functions $\theta(x,k)$, $\varphi(x,k)$, $h(x,\xi,k)$ and $F^2(k) - 4$ are even on the real axis of the complex plane of variable k.

Lemma 1 ([1]). *For points $\pm\lambda_n$, $n = 0,1,2,\ldots$, if λ_n are the ends of lacunae (that is, simple zeros of the function $F(k) - 2$), then the equalities*

$$h(x,\xi,\pm\lambda_n) = C_{\lambda_n} v(x,\lambda_n)\, v(\xi,\lambda_n), \quad -b \leq x,\ \xi \leq b,$$

are satisfied, where the function $v(x,\lambda_n) = v(x,-\lambda_n)$ is the eigenfunction of the periodic Sturm–Liouville problem, and the numbers C_{λ_n} depending on the cases (A_1)–(A_3) have the form

$$\begin{aligned} A_1 : C_{\lambda_n} &= \varphi(\lambda_n) \int_0^1 \left(\theta(x,\lambda_n) + \frac{1-\theta(\lambda_n)}{\varphi(\lambda_n)}\,\varphi(x,\lambda_n)\right)^2 dx; \\ A_2 : C_{\lambda_n} &= -\theta'(\lambda_n) \int_0^1 (\varphi(x,\lambda_n))^2 dx; \\ A_3 : C_{\lambda_n} &= \varphi(\lambda_n) \int_0^1 (\theta(x,\lambda_n))^2 dx. \end{aligned}$$

3. Main Results

3.1. The Case When the Left End of the Spectrum $\sigma(H_0)$ of the Hill Operator H_0 Coincides with zero: $\lambda_0 = 0$.

If on the complex plane of the variable λ, the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 coincides with zero, then on the complex plane of the variable k, the spectrum $\sigma(H_0)$ of the Hill operator H_0 is merging segments:

$$[-\mu_{2n+2}, -\lambda_{2n+2}], [-\lambda_{2n+1}, -\mu_{2n+1}], [-\mu_0, \mu_0], [\mu_{2n+1}, \lambda_{2n+1}], [\lambda_{2n+2}, \mu_{2n+2}], n = 0,1,2\ldots.$$

The point $k = 0$ is a two-fold zero of the function $F(k)^2 - 4$, and this means that the simple zero of the function $\sqrt{G(k)} = \sqrt{F(k)^2 - 4}$.

Denote by $\mathbb{C}'$ and L the complex plane and contour, as in [1], for which vertical cuts were made from the points $\pm\lambda_0$ in the lower half-plane of the variable k . In this case, no vertical cut is drawn from the point $k = \lambda_0 = 0$.

On the complex plane $\mathbb{C}'$, we consider the contour L, which can be represented as

$$L = L_1 \cup L_2 \cup L_3, \quad L_3 = L_- \cap \mathbb{C}',$$

where L_1 and L_2 are defined in the same way as in [1], $L_- = \{k : Im\,k = -d, d > 0\}$.

For the integrals J_1 and J_2, which are defined in [1], estimates (17) and (18) remain valid. The function

$$\frac{h(x,\xi,k)\,e^{-ikt}}{\sqrt{(\theta(k)+\varphi'(k))^2-4}}$$

for $x \in [-b,b]$ and $t > 0$ at the point $k = 0$ can have a first-order pole, since $k = 0$ is a simple zero of the function $\sqrt{F(k)^2-4}$.

Let δ be some finite contour in $\mathbb{C}'$. Denote by J_δ the integral

$$J_\delta = \int_\delta \int_0^1 \frac{h(x,\xi,k)}{\sqrt{(\theta(k)+\varphi'(k))^2-4}} f(\xi)\,e^{-ikt} d\xi dk, \quad x \in [-b,b].$$

Proposition 3. *For the solution of problems* (1) *and* (2)*, the following representation is*

$$u(x,t) = \frac{1}{2\pi} J_L + U(x) + v_1(x,t),$$

where the function $v_1(x,t)$ for $x \in [-b,b]$ and $t > 0$ satisfies the estimate:

$$|v_1(x,t)| \le C(b)\,e^{-td}||f;L^2||,$$

and the function $U(x)$ has the form

$$U(x) = -i\int_0^1 \left(\lim_{k\to 0} \frac{k\,h(x,\xi,k)\,e^{-ikt}}{\sqrt{(\theta(k)+\varphi'(k))^2-4}} \right) f(\xi)\,d\,\xi. \tag{8}$$

Proof. The proof of this statement is similar to the proof of Proposition 1 from [1]. Due to the fact that in this case, no vertical cut is made from the point $k = \lambda_0$ into the lower half-plane of the variable k, the term $U(x)$ is separated in the expansion of the solution of the Cauchy problem (1) and (2). Furthermore, this term is also separated in the asymptotic representation of the periodic solution of the Cauchy problem. □

Theorem 1. *If the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 on the complex plane of the variable λ coincides with zero, $p(x) \ge \text{const} > 0$, $q(x) \ge 0, q \not\equiv 0$, then the solution of the Cauchy problem* (1) *and* (2) *for $x \in [-b,b]$ and $t > 0$ has the form*

$$u(x,t) = U(x) + \frac{1}{\sqrt{t}}\{u_1(x,t) + u_2(x,t)\} + v(x,t),$$

where $u_1(x,t)$ is a periodic solution of the Cauchy problem, for which

$$u_1(x,t) = \sum_{n=1}^{\infty} b_{\lambda_n} a_{\lambda_n} v(x,\lambda_n)\sin(\lambda_n t + (-1)^n\frac{\pi}{4}),$$

$u_2(x,t)$ is a anti-periodic solution of the Cauchy problem, for which

$$u_2(x,t) = \sum_{n=0}^{\infty} b_{\mu_n} a_{\mu_n} v(x,\mu_n)\sin(\mu_n t + (-1)^{n+1}\frac{\pi}{4}),$$

the function $U(x)$ is defined as

$$U(x) = b_0 f_0\, v(x,0) \quad \textit{with} \quad f_0 = \int_0^1 v(\xi,0)\,f(\xi)\,d\,\xi,$$

while the function $v(x,t)$ for $|x| < b$ and $t > 0$ satisfies the estimate

$$|v(x,t)| \le \frac{C(b)}{t}||f;L^2(\mathbb{R})||.$$

The function $v(x,0)$ is obtained from the normalized eigenfunction $\hat{v}(x,0)$ of the periodic Sturm–Liouville problem (5) corresponding to the eigenvalue $\lambda_0 = \lambda_0^2 = 0$, if it is continued along the entire axis in a periodic way, b_0 is some constant defined by the formula (10) below.

Here, the summation is carried out only over those n for which $\lambda_n = \lambda_n^2$ (or $\mu_n = \mu_n^2$) are simple eigenvalues of the periodic (or anti-periodic) Sturm–Liouville problem.

Proof. Denote by $B(a)$ the circle $B(a) = \{k : |k - \pi a| \le \frac{\pi}{4}\}$ and $B(0) = \{k : |k| \le \frac{\pi}{4}\}$.

Since the function $\sqrt{(\theta(k) + \varphi'(k))^2 - 4}$ has a simple zero at the point $k = 0$, and in a small neighborhood, $B(0)$ of this point has no other zeros, then for $k \in B(0)$, the following equality holds:

$$\sqrt{(\theta(k) + \varphi'(k))^2 - 4} = k\, G_0(k),$$

and

$$|G_0(k)| \ge C > 0 \quad \text{for} \quad k \in B(0).$$

Taking into account the formula (8), we obtain

$$U(x) = -i \int_0^1 \frac{h(x,\xi,0)}{G_0(0)} f(\xi)\, d\,\xi.$$

By Lemma 1 from [1] for $x, \xi \in [-b, b]$

$$h(x,\xi,0) = C_0 v(x,0)\, v(\xi,0).$$

Consequently,

$$U(x) = -i\frac{C_0\, v(x,0)}{G_0(0)} \int_0^1 v(\xi,0)\, f(\xi)\, d\,\xi = b_0 f_0 v(x,0), \tag{9}$$

where

$$b_0 = -i\frac{C_0}{G_0(0)}. \tag{10}$$

Proposition 3 together with the Formulas (9) and (10) implies the validity of the theorem. □

3.2. The Case When the Left End of the Spectrum $\sigma(H_0)$ of the Hill Operator H_0 Is Negative

Let us now consider the case when the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 is negative on the complex plane of the variable λ and coincides with the point $(-\lambda_0)$.

It is known [17] that $(-\lambda_0) = (-\lambda_0^2)$ is the smallest and simple eigenvalue of the Sturm–Liouville problem (5).

If the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 coincides with the point $(-\lambda_0)$, then a part of the spectrum of the Hill operator H_0 on the complex plane of the variable k is located on the imaginary axis. Then, the highest point of the spectrum on the imaginary axis will be $i\lambda_0$, $\lambda_0 > 0$. At the point $i\lambda_0$, the function $\sqrt{G(k)} = \sqrt{(\theta(k) + \varphi'(k))^2 - 4}$ has a branch point, since the point $i\lambda_0$ is a simple zero of the function $G(k)$ [17].

Let us cut the complex plane of the variable k along the vertical ray $\{k : Re\, k = 0,$ $Im\, k \le \alpha_0\}$ and denote the resulting domain by $\mathbb{C}'$.

Let us put

$$m_1(k) = \frac{\varphi'(k) - \theta(k)}{2\varphi(k)} + \frac{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}}{2\varphi(k)}, \quad k \in \mathbb{C}',$$

$$m_2(k) = \frac{\varphi'(k) - \theta(k)}{2\varphi(k)} - \frac{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}}{2\varphi(k)}, \quad k \in \mathbb{C}',$$

where the branch of the root is determined by the condition $\sqrt{F(k)^2 - 4} > 0$ for $k = 0$.

Note that the function $\sqrt{F(k)^2 - 4}$ has branching only at the ends of the lacuna [17], so $m_1(k)$ and $m_2(k)$ are single-valued in $\mathbb{C}'$. Then, for any k, $Im\, k > 0$

$$\begin{aligned} \psi_1(x,k) \equiv &\quad \theta(x,k) + m_1(k)\, \varphi(x,k) \in L^2(-\infty, 0), \\ \psi_2(x,k) \equiv &\quad \theta(x,k) + m_2(k)\, \varphi(x,k) \in L^2(0, +\infty). \end{aligned} \tag{11}$$

We define Green's function of Equation (3) for k from the upper half-plane

$$\Gamma(x,\xi,k) = \begin{cases} \frac{\psi_1(\xi,k)\psi_2(x,k)}{m_2(k) - m_1(k)} & \text{for} \quad \xi < x, \\ \frac{\psi_1(x,k)\psi_2(\xi,k)}{m_2(k) - m_1(k)} & \text{for} \quad \xi > x, \end{cases}$$

and, taking into account the identities (11) and the equality

$$\theta(x,k)\, \varphi'(x,k) - \theta'(x,k)\, \varphi(x,k) = 1, \quad x \in \mathbb{R},$$

we obtain

$$\Gamma(x,\xi,k) = \begin{cases} -\frac{h(x,\xi,k)}{\sqrt{F(k)^2-4}} + \frac{1}{2}(\theta(\xi,k)\, \varphi(x,k) - \theta(x,k)\, \varphi(\xi,k)) & \text{for} \quad \xi < x, \\ -\frac{h(x,\xi,k)}{\sqrt{F(k)^2-4}} + \frac{1}{2}(\theta(x,k)\, \varphi(\xi,k) - \theta(\xi,k)\, \varphi(x,k)) & \text{for} \quad \xi > x, \end{cases} \tag{12}$$

where

$$h(x,\xi,k) = \varphi(k)\, \theta(x,k)\, \theta(\xi,k) - \theta'(k)\, \varphi(\xi,k)\varphi(x,k) +$$
$$+ \tfrac{\varphi'(k) - \theta(k)}{2} (\theta(\xi,k)\, \varphi(x,k) + \theta(x,k)\, \varphi(\xi,k)).$$

The solution to problems (1) and (2) has the form

$$u(x,t) = -\frac{1}{2\pi} \int_{Im\, k = a} \int_0^1 \Gamma(x,\xi,k)\, f(\xi)\, e^{-ikt} d\xi\, dk, \tag{13}$$

where a is some positive constant.

Note that Green's function $\Gamma(x,\xi,k)$ for every $x, \xi \in [-b,b]$ continues analytically to $\mathbb{C}'$.

To study the properties of the (13) integral, we introduce the following notation: $L_+ = \{k : Im\, k = a, a > \lambda_0\}$, $L_{\lambda_0 - \varepsilon} = \{k : Im\, k = \lambda_0 - \varepsilon\} \bigcap \mathbb{C}'$, and q_l is the segment $Re\, k = l\pi + \frac{\pi}{3}$, $\lambda_0 - \varepsilon \le Im\, k \le a$, l is any real number.

Consider the integral

$$J_1 \equiv -\int_{L+} \int_0^x (\theta(\xi,k)\, \varphi(x,k) - \theta(x,k)\, \varphi(\xi,k))\, f(\xi)\, e^{-ikt} d\xi\, dk, \quad x \in [-b,b]. \tag{14}$$

From the relations (4), it follows that

$$\int_{q_l} \int_0^x (\theta(\xi,k)\, \varphi(x,k) - \theta(x,k)\, \varphi(\xi,k))\, f(\xi)\, e^{-ikt} d\xi\, dk \to 0 \quad \text{as} \quad |l| \to \infty,$$

moreover, $|l|$ can tend to infinity in any way, so in (14), one can replace the line L_+ by $L_{\lambda_0 - \varepsilon}$. In addition, according to (4), we have

$$\theta(\xi,k)\, \varphi(x,k) - \theta(x,k)\, \varphi(\xi,k) = S_1(x,\xi,k) + S_2(x,\xi,k),$$

where

$$S_1(x,\xi,k) = \frac{1}{k} \cos k\xi \, \sin kx - \frac{1}{k} \cos kx \, \sin k\xi$$

is an entire function $k \in \mathbb{C}'$ for each $x, \xi \in [-b, b]$, and the function $S_2(x, \xi, k)$ for $k \to \infty$ uniformly in $x, \xi \in [-b, b]$ has the form

$$S_2(x, \xi, k) = O\Big(|k|^{-2} e^{|\tau|(x+\xi)}\Big).$$

Thus,

$$J_1 = J_1^{(1)} + J_1^{(2)} + J_1^{(3)},$$

where

$$J_1^{(1)} = -\int_{L_{\lambda_0-\varepsilon}} \int_0^x \frac{1}{k} \cos k\xi \, \sin kx \, f(\xi) \, e^{-ikt} d\xi \, dk,$$

$$J_1^{(2)} = \int_{L_{\lambda_0-\varepsilon}} \int_0^x \frac{1}{k} \cos kx \, \sin k\xi \, f(\xi) \, e^{-ikt} d\xi \, dk,$$

$$J_1^{(3)} = -\int_{L_{\lambda_0-\varepsilon}} \int_0^x S_2(x, \xi, k) \, f(\xi) \, e^{-ikt} d\xi \, dk.$$

Let us explore these integrals. Putting $k = \sigma + i(\lambda_0 - \varepsilon)$ with $k \in L_{\lambda_0-\varepsilon}$, we obtain

$$J_1^{(1)} = -\int_{-\infty}^{+\infty} \frac{1}{\sigma + i(\lambda_0 - \varepsilon)} \, \sin(\sigma + i(\lambda_0 - \varepsilon)) x \, e^{-i\sigma t} e^{(\lambda_0-\varepsilon)t} \Phi(\sigma, x) \, d\sigma, \quad x \in [-b, b], \tag{15}$$

where

$$\Phi(\sigma, x) \equiv \int_0^x \cos(\sigma + i(\lambda_0 - \varepsilon))\xi \, f(\xi) \, d\xi = \frac{1}{2} \int_0^x e^{i\sigma\xi} e^{-(\lambda_0-\varepsilon)\xi} f(\xi) \, d\xi + \frac{1}{2} \int_0^x e^{-i\sigma\xi} e^{(\lambda_0-\varepsilon)\xi} f(\xi) \, d\xi. \tag{16}$$

Let us examine the first term in (16). Consider the function

$$w(x, \xi) = \begin{cases} e^{-(\lambda_0-\varepsilon)\xi} f(\xi) & \text{for} \quad \xi < x, \\ 0 & \text{for} \quad \xi > x. \end{cases}$$

For any fixed $x \in [-b, b]$, we have $w \in L^2(-\infty, +\infty)$ and

$$||w; L^2|| = \left(\int_0^x e^{2(\lambda_0-\varepsilon)\xi} f^2(\xi) \, d\xi \right)^{1/2} \le \left(\int_0^1 e^{2(\lambda_0-\varepsilon)\xi} f^2(\xi) \, d\xi \right)^{1/2} \le C_1 ||f; L^2||,$$

where C_1 does not depend on f and x.

For all $x \in [-b, b]$, due to the Parseval equality for the Fourier transform, we have

$$|| \int_0^x e^{i\sigma\xi} e^{-(\lambda_0-\varepsilon)\xi} f(\xi) \, d\xi; L^2(\mathbb{R}_\sigma) || = \sqrt{2\pi} ||w; L^2(\mathbb{R}_\xi)|| \le C_1 \sqrt{2\pi} ||f; L^2||.$$

The second term of the equality (16) is studied in a similar way. Therefore, for any fixed $x \in [-b, b]$,

$$||\Phi(\sigma, x); L^2(\mathbb{R}_\sigma)|| \le C_2 ||f; L^2||,$$

where C_2 does not depend on f and x.

In article [18], an elegant method is presented that allows, in the case of a periodic potential $q(x)$, to obtain a very simple Parseval formula for the problem in the entire space.

By the Cauchy–Schwartz inequality and the last inequality, from (15), we obtain

$$|J_1^{(1)}| \le C_3 \, e^{(\lambda_0-\varepsilon)t} ||f; L^2||,$$

where C_3 depends only on b.

In the same way, we obtain

$$|J_1^{(2)}| \le C_4 \, e^{(\lambda_0-\varepsilon)t} ||f; L^2||,$$

where C_4 depends only on b.

To investigate $J_1^{(3)}$, we note that

$$J_1^{(3)} = -\int_{L_-}\int_0^x S_2(x,\xi,k)\, f(\xi)\, e^{-ikt} d\xi\, dk =$$

$$= -\int_{-\infty}^{+\infty} \frac{1}{\sigma + i(\lambda_0-\varepsilon)}\, e^{-i\sigma t} e^{(\lambda_0-\varepsilon)t} \left(\int_0^x f(\xi) O\left(\frac{e^{(\lambda_0-\varepsilon)(x+\xi)}}{|\sigma + i(\lambda_0-\varepsilon)|} \right) d\xi \right) d\sigma.$$

It is easy to show that

$$\left| \int_0^x f(\xi) O\left(\frac{e^{(\lambda_0-\varepsilon)(x+\xi)}}{|\sigma + i(\lambda_0-\varepsilon)|} \right) (\lambda_0-\varepsilon)\xi \right|^2 \le \frac{C}{|\sigma + i(\lambda_0-\varepsilon)|^2} ||f;L^2||.$$

By the Cauchy–Schwartz inequality, we obtain the estimate

$$|J_1^{(3)}| \le C_5\, e^{(\lambda_0-\varepsilon)t} ||f;L^2||,$$

where C_5 depends only on b.

From the estimates for $J_1^{(1)}$, $J_1^{(2)}$, and $J_1^{(3)}$, it follows that

$$|J_1| \le C(b)\, e^{(\lambda_0-\varepsilon)t} ||f;L^2||.$$

Likewise, for the integral

$$J_2 \equiv -\int_{L+}\int_x^1 (\theta(x,k)\, \varphi(\xi,k) - \theta(\xi,k)\, \varphi(x,k))\, f(\xi)\, e^{-ikt} d\xi\, dk, \quad x \in [-b,b],$$

we obtain the estimate

$$|J_2| \le C(b)\, e^{(\lambda_0-\varepsilon)t} ||f;L^2||.$$

Thus, we learn that the integrals J_1 and J_2 decrease exponentially as $t \to \infty$.

Let us choose the number $\varepsilon > 0$ in two ways:

(i) $\varepsilon = \frac{\lambda_0}{2}$, if on the imaginary axis, except for the point $i\lambda_0$, there are no other branch points of the function $\sqrt{G(k)} = \sqrt{(\theta(k) + \varphi'(k))^2 - 4}$;

(ii) $\varepsilon = \frac{d}{2}$, where $d > 0$ is the distance from the point $i\lambda_0$ to the nearest branch point of the function $\sqrt{G(k)} = \sqrt{(\theta(k) + \varphi'(k))^2 - 4}$ located on the imaginary axis.

Denote by l_{λ_0} the contour going from the point $i(\lambda_0 - \varepsilon)$ along the left edge of this cut to the point $i\lambda_0$, and then from the point $i\lambda_0$ along to the right edge of the cut up to the point $i(\lambda_0 - \varepsilon)$.

Green's function is defined by formula (12), where the single-valued branch of the root $\sqrt{G(k)} = \sqrt{(\theta(k) + \varphi'(k))^2 - 4}$ is determined by the condition $\sqrt{G(k)}\Big|_{i(\lambda_0+\varepsilon)} > 0$.

Let $M \equiv L_{\lambda_0-\varepsilon} \bigcup l_{\lambda_0}$.

Proposition 4. *For the solution of the problems (1) and (2), the following representation is valid:*

$$u(x,t) = \frac{1}{2\pi} J_M + v_2(x,t),$$

where the function $v_2(x,t)$ for $x \in [-b,b]$ and $t > 0$ satisfies the estimate

$$|v_2(x,t)| \le C(b)\, e^{(\lambda_0-\varepsilon)t} ||f;L^2||.$$

Proof. This statement is proved similarly to the proof of Proposition 1 from [1], where J_M is considered instead of J_L from [1], and $k = \sigma + i(\lambda_0 - \varepsilon)$ with $k \in L_{\lambda_0-\varepsilon}$. □

Proposition 5. *For any $t > 0$ and $x \in [-b,b]$ we have the estimate*

$$|J_{L_{\lambda_0-\varepsilon}}| \le C(b)\, e^{(\lambda_0-\varepsilon)t} ||f;L^2||.$$

Proof. The proof of this statement is similar to the proof of Proposition 2 from [1], where $J_{L_{\lambda_0-\varepsilon}}$ is considered instead of J_{L_3} from [1], and $k = \sigma + i(\lambda_0 - \varepsilon)$ with $k \in L_{\lambda_0-\varepsilon}$. □

Theorem 2. *If the left end of the spectrum $\sigma(H_0)$ of the Hill operator H_0 is negative on the complex plane of the variable λ and coincides with the point $(-\lambda_0)$, then the solution of the Cauchy problems (1) and (2) for $x \in [-b, b]$ and $t \to \infty$ has the form*

$$u(x,t) = \frac{e^{\lambda_0 t}}{\sqrt{t}} \cdot \big(b_0 f_{\lambda_0} v(x,0) + v(x,t)\big),$$

where

$$f_{\lambda_0} = \int_0^1 v(\xi,0)\, f(\xi)\, d\,\xi \quad \text{and} \quad b_0 = ie^{-\frac{3}{4}i\pi} \frac{C_0}{\pi\, g(i\lambda_0)} \cdot \Gamma\left(\frac{1}{2}\right),$$

while the function $v(x,t)$ for $|x| < b$ and $t \to \infty$ satisfies the estimate

$$|v(x,t)| \le C(b)\, t^{-1} ||f; L^2(\mathbb{R})||.$$

The function $v(x,0)$ is obtained from the normalized eigenfunction $\hat{v}(x,0)$ of the periodic Sturm–Liouville problem (5) corresponding to the eigenvalue $\lambda_0 = \lambda_0^2$, if it is continued along the entire axis in a periodic way.

Proof. To complete the proof of the theorem, by virtue of Propositions 2 and 3, it remains to study the following integral:

$$J_{l_{\lambda_0}} = \int_{l_{\lambda_0}} \int_0^1 \frac{h(x,\xi,k)}{\sqrt{(\theta(k) + \varphi'(k))^2 - 4}} f(\xi)\, e^{-ikt} d\xi dk.$$

Let $B(\lambda_0) = \{k : |k - i\lambda_0| \le \frac{3}{2}\varepsilon\}$. It is easy to show that for $k \in B(\lambda_0) \cap \mathbb{C}'$, the following representation holds:

$$\sqrt{G(k)} = \sqrt{k - i\lambda_0} \cdot g_0(k), \quad |g_0(k)| \ge c > 0,$$

and the branch of the root $\sqrt{k - i\lambda_0}$ is selected from the condition of its positivity for positive values of $k - i\lambda_0$.

For $k \in l_{\lambda_0}$ we set $k = i\tau$, where $\lambda_0 - \varepsilon \le \tau \le \lambda_0$. It is easy to see that if k belongs to the left side of the contour l_{λ_0}, then the following equality holds

$$\sqrt{k - i\lambda_0} = \sqrt{i(\tau - \lambda_0)} = e^{\frac{3}{4}i\pi} \sqrt{\lambda_0 - \tau},$$

and if k belongs to the right side of the contour l_{λ_0}, then the root has the opposite sign. The values of the function $g_0(k)$ coincide at the corresponding points of the left and right parts of the contour l_{λ_0}.

Considering that $\tau \in [\lambda_0 - \varepsilon, \lambda_0]$ is on the left side of the contour, and $\tau \in [\lambda_0, \lambda_0 - \varepsilon]$ on the right side of the contour, we learn that

$$\begin{aligned} J_{l_{\lambda_0}} &= 2ie^{-\frac{3}{4}i\pi} \int_0^1 \int_{\lambda_0-\varepsilon}^{\lambda_0} \frac{h(x,\xi,i\tau)}{\sqrt{\lambda_0-\tau}\, g_0(i\tau)} f(\xi)\, e^{t\tau}\, d\tau\, d\xi \\ &= 2ie^{-\frac{3}{4}i\pi} \int_0^1 f(\xi) \left(\int_{\lambda_0-\varepsilon}^{\lambda_0} \frac{h(x,\xi,\tau)}{\sqrt{\lambda_0-\tau}\, g_0(i\tau)}\, e^{t\tau}\, d\tau \right) d\xi. \end{aligned} \tag{17}$$

Making the change of variables $\lambda_0 - \tau = y$ into the integral (17), we obtain

$$J_{l_{\lambda_0}} = 2ie^{-\frac{3}{4}i\pi} e^{\lambda_0 t} \int_0^1 f(\xi) \left(\int_0^{\varepsilon} \frac{h(x,\xi,i(\lambda_0 - y))}{\sqrt{y}\, g_0(i(\lambda_0 - y))}\, e^{-ty}\, dy \right) d\xi. \tag{18}$$

To study the inner integral in (18), we use Watson's lemma [19], then

$$\int\limits_0^{\varepsilon} \frac{h(x,\xi,i(\lambda_0-y))}{\sqrt{y}\,g_0(i(\lambda_0-y))}\,e^{-ty}\,dy = \\ = \frac{1}{\sqrt{t}}\Gamma\left(\frac{1}{2}\right)\frac{h(x,\xi,i\lambda_0)}{g_0(i\lambda_0)} + O\left(\frac{1}{t^{3/2}}\Gamma\left(\frac{3}{2}\right)\left(\frac{h(x,\xi,i(\lambda_0-y))}{g_0(i(\lambda_0-y))}\right)'_y\Bigg|_{y=0}\right) \text{ as } t\to\infty. \tag{19}$$

Now, substituting (19) into (18), we obtain

$$J_{l_{\lambda_0}} = 2ie^{-\frac{3}{4}i\pi}e^{\lambda_0 t}\left(\frac{1}{\sqrt{t}}\Gamma\left(\frac{1}{2}\right)\frac{1}{g_0(i\lambda_0)}\int\limits_0^1 f(\xi)h(x,\xi,i\lambda_0)\,d\xi + \int\limits_0^1 O\left(t^{-3/2}\right)f(\xi)\,d\xi\right) \text{ as } t\to\infty. \tag{20}$$

Arguing in the same way as in the proof of Lemma 1 from [1], we can show that

$$h(x,\xi,i\lambda_0) = C_0 v(x,0)\,v(\xi,0), \quad -b\le x,\xi\le b, \tag{21}$$

where the constant C_0 is determined depending on which of the conditions A_1, A_2, A_3 is satisfied at the point $i\lambda_0$.

From (20) and (21), it follows

$$J_{l_{\lambda_0}} = 2ie^{-\frac{3}{4}i\pi}e^{\lambda_0 t}\frac{1}{\sqrt{t}}\Gamma\left(\frac{1}{2}\right)\frac{C_0}{g_0(i\lambda_0)}\,f_{\lambda_0}v(x,0) + \int\limits_0^1 O\left(t^{-3/2}\right)f(\xi)\,d\xi \text{ as } t\to\infty. \tag{22}$$

From Propositions 2 and 3, as well as from (22), the validity of Theorem 2 follows. □

4. Applications

In [20], a numerical study of the one-dimensional Schrödinger operator with the potential $q(x) = \cos(x) + \varepsilon\cos(kx)$ is considered, where $\varepsilon > 0$ and k are irrational. This governs the quantum wave function of an independent electron within a crystalline lattice perturbed by some impurities whose dissemination induces long-range order only, which is rendered by means of the quasi-periodic potential q. In the paper [21], a simple one-dimensional model of an incommensurable "harmonic crystal" is studied in terms of the spectrum of the corresponding Schrödinger equation.

We also note papers [22,23], in which the necessity of solving the equations of mathematical physics with variable coefficients is due to the applied problems leading to them. Such problems lead to topical issues of studying the nonstationary interaction of fields of various nature, in which one-dimensional problems of the nonstationary interaction of mechanical and electromagnetic fields are solved.

5. Conclusions

Boundary problems for the second-order hyperbolic equation with an irregular singular point at infinity were considered in papers [24,25].

Note paper [25], in which the problem of obtaining the asymptotics of solutions of differential operators in a neighborhood of an irregular singular point is considered, where we constructed a uniform asymptotics of solutions of linear differential equations with second-order meromorphic coefficients in a neighborhood of a singular point. There, we apply the obtained results to the equations of mathematical physics.

In what follows, it is in our interest to generalize the problems (1) and (2), where instead of Equation (1), we consider the equation

$$a^0(t)\,u_{tt}(x,t) - (p(x)\,u_x(x,t))_x + q(x)\,u(x,t) = 0, \quad (x,t)\in\mathbb{R}\times[0;\infty),$$

while imposing some conditions on the coefficient $a^0(t)$.

In [24], a condition on the coefficient $a^0(t)$ is formulated, which is sufficient for the convergence of power series entering into the quasi-classical asymptotics of solutions. To construct the asymptotics of this equation, we can use the method given in this article and the methods of resurgent analysis, similar to how it was conducted in papers [24,25].

Author Contributions: H.A.M. is responsible for the formulation of the problem and the method of its solution. M.V.K. is responsible for the generalization of the method for studying such problems using, among other things, the method of resurgent analysis. V.A.V. is responsible for the application of these problems in mechanics, aerospace and technical physics. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank three anonymous reviewers for their valuable comments, which greatly improved the final expression of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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