

Article

Inferences of a Mixture Bivariate Alpha Power Exponential Model with Engineering Application

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Abstract: The univariate alpha power exponential (APE) distribution has several appealing characteristics. It behaves similarly to Weibull, Gamma, and generalized exponential distributions with two parameters. In this paper, we consider different bivariate mixture models starting with two independent univariate APE models, and, in the latter case, starting from two dependent APE models. Several useful structural properties of such a mixture model (under the assumption of two independent APE distribution) are discussed. Bivariate APE (BAPE), in short, modelled under the dependent set up are also discussed in the context of a copula-based construction. Inferential aspects under the classical and under the Bayesian paradigm are considered to estimate the model parameters, and a simulation study is conducted for this purpose. For illustrative purposes, a well-known motor data is re-analyzed to exhibit the flexibility of the proposed bivariate mixture model.

Keywords: alpha power exponential mixture model; Bayesian approach; Gaussian copula; bivariate model; EM algorithm; simulation study; motor data

MSC: 65C20; 60E05; 62P30; 62L15



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1. Introduction

Univariate continuous alpha power exponential distribution with two parameters has received a considerable amount of attention among researchers working in the area of distribution theory. Since the introduction of the model, an appreciable amount of work has taken place on the development of bivariate APE distribution and its different generalizations; for more information, see Ref. [1] and the references cited therein. In this paper, we discuss several different bivariate APE models with various mixing mechanisms. Noticeably, mixture distribution plays an important role in various data-analysis purposes; see [2]. Precisely, if the data are coming from different subpopulations, and their identifications are unknown, then the mixture distribution can be used quite effectively to analyze the data set. Unlike any other symmetric distribution (for example, normal) APE distribution is a skewed distribution. Consequently, if the subpopulations do not have symmetric distribution, the mixture of APE resulting in a bivariate APE distribution may be used to analyze these data. Furthermore, a multi-model distribution can be well approximated by a mixture distribution. Not much work has been carried out on a bivariate mixture of APE distribution.

This is an effort towards that direction. We consider two different mixing strategies to construct the bivariate mixture APE (BMAPE type I). In both cases, we start with the following: let $X \sim APE(\alpha_1, \theta_1)$ and $Y \sim APE(\alpha_2, \theta_2)$, and let them be independent. In

the first, we consider a finite mixing strategy as described above and call it a BMAPE distribution. While, in the second scenario, we consider a copula-based construction approach. We consider a bivariate Gaussian copula to construct a copula-based bivariate distribution which we call (henceforth) BMAPE (type I). One of the major objectives for a copula-based construction is that we can better discuss the dependence structure between two components. For a copula-based bivariate and multivariate distribution, see [3]. In the next, we provide a non-exhaustive list of pertinent references that serves as motivation to carry out the present work.

Statisticians and practitioners have been particularly interested in studying groups that demonstrate similar behavior in response to pre-determined criteria. Refs. [4,5] developed and utilized a method known as finite mixture distributions, and provided the majority of the early evidence for the study of heterogeneous populations as a possible application of finite mixture models. Studies focusing on heterogeneous populations became more popular in the modern era as data-related computation facilities improved. For some useful references, see [6–10] and the references cited therein. Starting with one parameter exponential baseline distribution, Ref. [1] developed the alpha power exponential (APE) distribution, discussed its several useful properties, and studied the associated statistical inference. Ref. [11] introduced the alpha power Weibull distribution and demonstrated that it provides superior modelling to several other generalizations of the Weibull distribution, in particular, when modeling heterogeneous population. To introduce the alpha power generalized exponential (APGE) distribution, Ref. [12] employed the generalized exponential baseline distribution and the alpha power transformation (APT) technique. Ref. [13] developed closed-form expressions for the alpha power generalized exponential (APGE) distribution's moment properties. Ref. [14] presented a new family of APT-based distributions by inverting the quantile function of any distribution as a function of the APT cumulative distribution function (CDF). In the next, we provide a non-exhaustive list of references related to APT-based mixture distributions that are developed in the univariate domain. The alpha power exponentiated Weibull distribution was suggested by Ref. [3], who investigated the general mathematical features of the APT family. Ref. [15] introduced the Marshall Olkin alpha power family as a new generalization of the APT and a novel approach to studying the mathematical properties of the APT model was introduced by Ref. [16]. The probability density function (PDF) and the hazard functions of the APE distribution are similar to those of the Weibull, Gamma, and generalized exponential (GE) distributions under certain specific choices of the model parameters. As a result, it can be used as a variable alternative for the widely used Weibull, Gamma, and GE distributions. Specially, when, component-wise, each one of them cannot provide best fit, such a mixture mode will be a good fit. As the CDF of the APE distribution has a closed form which is analytically tractable, it can be used to analyze censored data sets as well.

Mixture distributions are made up of a finite or an infinite number of components, which may be comprised of several different distributions, and can be used to represent various aspects of data for example, two of the most notable studies that explore different sorts of combinations of distributions can be found in Refs. [17,18]. Discussion on a finite three-component mixture of exponential distribution can be found in Ref. [19]. In the literature, there are not many references of finite and/or infinite mixtures of probability models. A non-exhaustive list of references can be cited as follows. A mixture of bivariate exponential distributions (MBE), in which the authors predicted the elements of MBE are given in Ref. [20]. Discussion on a new bivariate distribution whose both marginals are finite mixtures of gamma distribution are nicely explored in Ref. [21]. For a review of known bivariate distributions, we refer the readers to Refs. [22–24]. The univariate exponential distribution is one of the most celebrated distributions because of its importance in many lifetime applications and its structural properties. Consequently, if we consider a simple bivariate expansion, there might be limitations due to properties, especially for modeling heterogeneous data. This motivates us to consider a mixture form that will have many desirable properties. In this article, we propose and study a finite mixture of a bivariate

continuous probability model arising from two independent univariate APE distributions, which we call BMAPE, in short. In this paper, we consider various mechanisms of mixing univariate APE distributions to construct a bivariate mixture of APE models, which are listed below:

- (i) In the first case, we start with two independent and univariate APE with parameters $(\alpha_1, \alpha_2, \theta)$ and (μ_1, μ_2, λ) with $\alpha_1 \neq \alpha_2, \mu_1 \neq \mu_2$. Further, we assume that $\alpha_1, \alpha_2, \mu_1, \mu_2$ are random variables. Then, using the methodology of compounding, we develop the bivariate APE distribution, and, for the sake of notational simplicity, call it the BMAPE (type-I) distribution.
- (ii) In the second scenario, we consider a copula-based construction approach. We consider a bivariate Gaussian copula in which the margins are two univariate and independent APE distributions with the coupling parameter ρ . For notational simplicity, we call this the BMAPE (type-II) distribution, for more details of copula based on construction and associated properties, see [25].

The remainder of the paper is laid out as follows. In Section 2, we describe the BMAPE (type-I) distribution and discuss its properties. In Section 3, we provide some useful properties of the BMAPE (type-I) distribution, including the survival function, hazard rate function and conditional moments. In Section 4, we look at how the expectation-maximization (EM) technique can be used to estimate model parameters for BMAPE (type-I) distribution. The estimation approach for the BMAPE (type-II) distribution formed using the bivariate Gaussian copula is discussed in Section 5. In Section 6, we investigate how to estimate model parameters under the Bayesian paradigm. In Section 7, the simulation results are reported. A well-known Motor data set was re-analyzed in Section 8 to demonstrate the efficacy of the suggested BMAPE type models. Finally, in Section 9, we wrap up the paper with some concluding remarks.

2. The APE Model with a Bivariate Mixture

We will start with two independent univariate APE distributions with parameters of (α, θ) and (μ, λ) , respectively. We employ here the strategy of compounding by assuming that α and μ are random variables; then, the marginal joint distribution of X_{in} and Y_{in} may be derived from the joint distribution of α and μ respectively, which is given as follows

$$h1_{x,y}(x, y) = \iint h1(x, y, \alpha, \mu) d\alpha d\mu, \tag{1}$$

where X_{in} and Y_{in} are independent APE distributions. Next, we consider two different scenarios in the context of the nature of the distributions of X_{in} and Y_{in} . From (1), one may obtain the desired BMAPE (type-I) distribution.

We generate a bivariate mixture of the APE distribution in two scenarios. In the first scenario, X_{in} and Y_{in} are independent APE distributions with a generalized bivariate Bernoulli distribution for the shape parameters as

$$P = \begin{matrix} \mu_1 & \mu_2 \\ \alpha_1 \begin{bmatrix} P_{\alpha_1\mu_1} & P_{\alpha_1\mu_2} \\ P_{\alpha_2\mu_1} & P_{\alpha_2\mu_2} \end{bmatrix} \end{matrix}.$$

A random variable with an APE distribution has a CDF and a PDF for $X_{in} > 0$, given by, respectively,

$$F(x_{in}, \alpha, \theta) = \frac{\alpha^{1-e^{-\theta x_{in}}} - 1}{\alpha - 1}, \quad x_{in} > 0, \theta, \alpha > 0, i = 1, 2, \tag{2}$$

$$f(x_{in}, \alpha, \theta) = \frac{\theta \log(\alpha) e^{-\theta x_{in}} \alpha^{1-e^{-\theta x_{in}}}}{\alpha - 1}, \quad x_{in} > 0, \tag{3}$$

where α and θ are the shape and scale parameters.

Next, in order to construct a finite mixture of bivariate alpha power exponential model, we proceed as follows:

- (i) We consider $X_i \sim APE(\alpha_i, \theta) \forall i = 1, 2$, where α_1 and α_2 are random and θ is fixed, and each are independent for fixed choices of α_1 and α_2 . Likewise, $Y_i \sim APE(\mu_i, \lambda) \forall i = 1, 2$, where μ_1 and μ_2 are random and λ is fixed.
- (ii) Next, we further assume that (X_1, X_2) and (Y_1, Y_2) are independent but $(\alpha_i, \mu_i), \forall i = 1, 2$, are correlated through their bivariate generalized Bernoulli distribution with the shape parameter (α_i, μ_i) , assuming the following probability matrix.

$$P = \begin{matrix} & \mu_1 & \mu_2 \\ \alpha_1 & P_{\alpha_1\mu_1} & P_{\alpha_1\mu_2} \\ \alpha_2 & P_{\alpha_2\mu_1} & P_{\alpha_2\mu_2} \end{matrix},$$

where P is the mixing component with the condition $\sum_{i=1}^2 P_{\alpha_i\mu_i} = 1$.

Let $h1(x, y)$ be the joint PDF created from the marginal densities of $(x_i, y_i), i = 1, 2$, given earlier.

Then, the joint density of (x, y) can be written as

$$h1_{x,y}(x, y) = f(x_1|\alpha_1, \theta)g(y_1|\mu_1, \lambda)P_{\alpha_1\mu_1} + f(x_2|\alpha_1, \theta)g(y_2|\mu_2, \lambda)P_{\alpha_1\mu_2} + f(x_2|\alpha_2, \theta)g(y_1|\mu_1, \lambda)P_{\alpha_2\mu_1} + f(x_2|\alpha_2, \theta)g(y_2|\mu_2, \lambda)P_{\alpha_2\mu_2}, \tag{4}$$

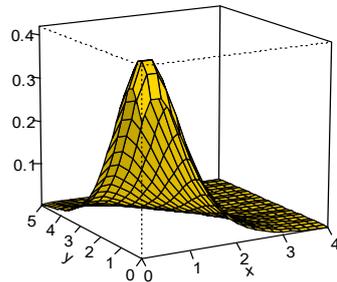
Then,

$$h1(x, y) = P_{\alpha_1\mu_1} \left(\frac{\theta \log(\alpha_1) e^{-\theta x_1} \alpha_1^{1-e^{-\theta x_1}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda y_1} \mu_1^{1-e^{-\lambda y_1}}}{\mu_1 - 1} \right) + P_{\alpha_1\mu_2} \left(\frac{\theta \log(\alpha_1) e^{-\theta x_1} \alpha_1^{1-e^{-\theta x_1}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda y_2} \mu_2^{1-e^{-\lambda y_2}}}{\mu_2 - 1} \right) + P_{\alpha_2\mu_1} \left(\frac{\theta \log(\alpha_2) e^{-\theta x_2} \alpha_2^{1-e^{-\theta x_2}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda y_1} \mu_1^{1-e^{-\lambda y_1}}}{\mu_1 - 1} \right) + P_{\alpha_2\mu_2} \left(\frac{\theta \log(\alpha_2) e^{-\theta x_2} \alpha_2^{1-e^{-\theta x_2}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda y_2} \mu_2^{1-e^{-\lambda y_2}}}{\mu_2 - 1} \right), \quad x_i > 0, y_i > 0. \tag{5}$$

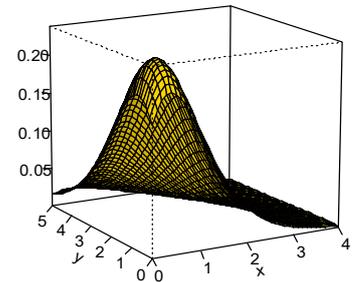
For this model, it appears that, for a valid PDF, we must have this constraint $\alpha, \mu > 0$ and $\alpha, \mu \neq 1$.

To make things easier in terms of writing and visualization, let $a = P_{\alpha_1\mu_1}, b = P_{\alpha_1\mu_2}, c = P_{\alpha_2\mu_1}$ and $d = P_{\alpha_2\mu_2}$. Depending on a variety of choices for the mixing parameters $a, b, c, d, \theta, \lambda, \alpha_i$ and μ_i for $i = 1, 2$, several shapes of the joint PDF in (5) are provided in Figure 1. Moreover, several contour plots for various values of a, b, c and d when $(\alpha_1, \alpha_2, \mu_1, \mu_2, \theta, \lambda) = (9, 7, 6, 5, 1.2, 0.6)$ are displayed in Figure 2. As one might expect, not all four components may be required in applications to real-world data sets. Consequently, some constraints can be applied, such as $b = c = 0, a = d = 0$ or $a = b = c = 0$, as appropriate. These constraints result in correlation values of +1, -1, and 0 among the shape parameter, respectively.

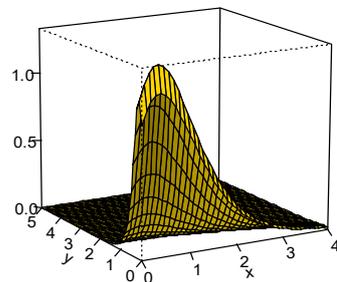
$$\alpha_1 = 8 \quad \alpha_2 = 4 \quad \mu_1 = 7 \quad \mu_2 = 5 \quad \theta = 1.8 \quad \lambda = 1.2$$



$$\alpha_1 = 7 \quad \alpha_2 = 5 \quad \mu_1 = 8 \quad \mu_2 = 4 \quad \theta = 1.5 \quad \lambda = 0.8$$



$$\alpha_1 = 5 \quad \alpha_2 = 7 \quad \mu_1 = 4 \quad \mu_2 = 8 \quad \theta = 1.5 \quad \lambda = 4.5$$



$$\alpha_1 = 4 \quad \alpha_2 = 8 \quad \mu_1 = 5 \quad \mu_2 = 7 \quad \theta = 4.5 \quad \lambda = 5.5$$

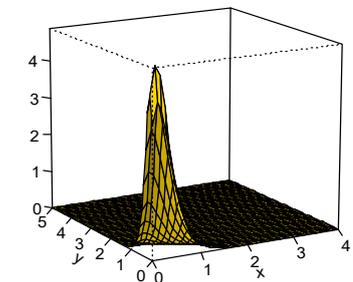
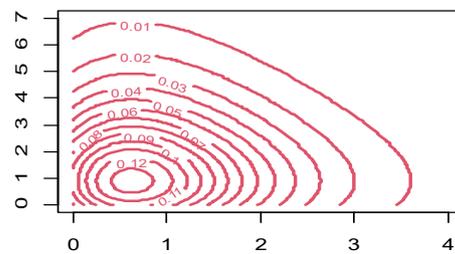
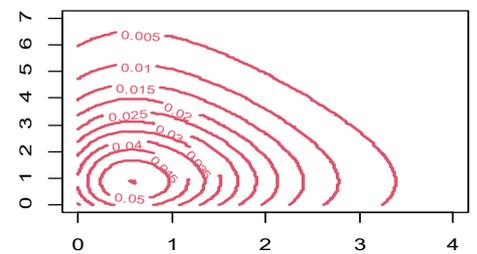


Figure 1. Joint density plot of the BMAPE (type-I) distribution for the selected parameter choices.

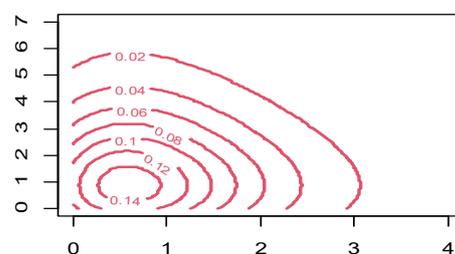
$$a = 0.4 \quad b = 0.3 \quad c = 0.2 \quad d = 0.1$$



$$a = 0.1 \quad b = 0.1 \quad c = 0.1 \quad d = 0.7$$



$$a = 0.1 \quad b = 0 \quad c = 0.1 \quad d = 0.8$$



$$a = 0 \quad b = 0.2 \quad c = 0 \quad d = 0.8$$

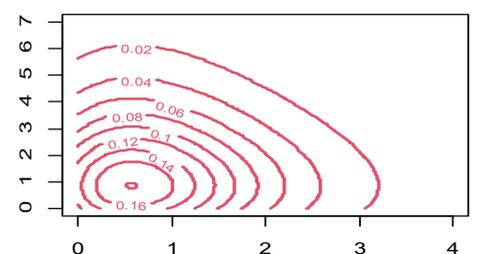


Figure 2. Contour plot of the BMAPE (type-I) distribution for the selected parameter choices.

The marginal densities of X and Y , respectively, from the joint PDF in (5) are not available in closed form; one may consider some numerical methods to obtain an approximate expression. The CDF for the joint model will be

$$\begin{aligned}
 F(x, y) &= \int_0^y \int_0^x h_1(t, s) dt ds = \int_0^y \int_0^x \left[a \left(\frac{\theta \log(\alpha_1) e^{-\theta t} \alpha_1^{1-e^{-\theta t}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda s} \mu_1^{1-e^{-\lambda s}}}{\mu_1 - 1} \right) \right. \\
 &\quad + b \left(\frac{\theta \log(\alpha_1) e^{-\theta t} \alpha_1^{1-e^{-\theta t}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda s} \mu_2^{1-e^{-\lambda s}}}{\mu_2 - 1} \right) \\
 &\quad + c \left(\frac{\theta \log(\alpha_2) e^{-\theta t} \alpha_2^{1-e^{-\theta t}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda s} \mu_1^{1-e^{-\lambda s}}}{\mu_1 - 1} \right) \\
 &\quad \left. + d \left(\frac{\theta \log(\alpha_2) e^{-\theta t} \alpha_2^{1-e^{-\theta t}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda s} \mu_2^{1-e^{-\lambda s}}}{\mu_2 - 1} \right) \right] dt ds \\
 &= a \frac{(\alpha_1^{1-e^{-\theta x}} - 1)(\mu_1^{1-e^{-\lambda y}} - 1)}{(\alpha_1 - 1)(\mu_1 - 1)} + b \frac{(\alpha_1^{1-e^{-\theta x}} - 1)(\mu_2^{1-e^{-\lambda y}} - 1)}{(\alpha_1 - 1)(\mu_2 - 1)} + c \frac{(\alpha_2^{1-e^{-\theta x}} - 1)(\mu_1^{1-e^{-\lambda y}} - 1)}{(\alpha_2 - 1)(\mu_1 - 1)} + d \frac{(\alpha_2^{1-e^{-\theta x}} - 1)(\mu_2^{1-e^{-\lambda y}} - 1)}{(\alpha_2 - 1)(\mu_2 - 1)}.
 \end{aligned} \tag{6}$$

For (6), to be a valid CDF, we must have

- (i) $F(x, y) \geq 0, \forall (x, y) \in \mathbb{R}^+ \iff \alpha_1, \mu_1, \alpha_2, \mu_2 > 0$ and $\alpha_1, \mu_1, \alpha_2, \mu_2 \neq 0$.
- (ii) $\lim_{x, y \rightarrow 0} F(x, y) = 0$.
- (iii) $\lim_{x, y \rightarrow \infty} F(x, y) = 1$.

Next, one may obtain the marginal survival functions for the smallest order statistics for a random sample of size two. For example, let $w_1 = \min\{X, Y\}$, then, $\bar{F}_{W_1}(w_1) = P\{\min(X, Y) > w_1\} = P\{X > w_1, Y > w_1\}$, while

$$\bar{F}_{W_2}(W_2) = P\{\max(X, Y) < w_2\} = P\{X \leq w_2, Y \leq w_2\} \tag{7}$$

From (7), one can obtain the associated PDF for W_2 . We discuss some useful structural properties in the next section for the BMAPE (type-I) distribution.

3. Structural Properties

Theorem: The BMAPE (type I) distribution is log-concave. Taking the negative logarithm of (5), for $0 < x < y < \infty$ (or for $0 < y < x < \infty$), we obtain

$$-\log[h_1(x, y)] = -\log \left\{ \begin{aligned} &P_{\alpha_1 \mu_1} \left(\frac{\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta x}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}}}{\mu_1 - 1} \right) + P_{\alpha_1 \mu_2} \left(\frac{\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta x}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}}}{\mu_2 - 1} \right) \\ &+ P_{\alpha_2 \mu_1} \left(\frac{\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}}}{\mu_1 - 1} \right) + P_{\alpha_2 \mu_2} \left(\frac{\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}}}{\mu_2 - 1} \right) \end{aligned} \right\}. \tag{8}$$

Next, taking partial derivatives w.r.t. x and y of (8), respectively, we obtain

$$\begin{aligned}
 D_y[D_x(-\log(h_1(x, y)))] &= \left\{ (P_{\alpha_1 \mu_2} P_{\alpha_2 \mu_1} - P_{\alpha_1 \mu_1} P_{\alpha_2 \mu_2}) e^{-\theta x - \lambda y} \theta \lambda (\alpha_1 - 1) \alpha_1^{e^{-\theta x} + 1} (\alpha_2 - 1) \alpha_2^{e^{-\theta x} + 1} \right\} \\
 &\times \left\{ (\mu_1 - 1)(\mu_2 - 1) \mu_1^{1+e^{-\lambda y}} \mu_2^{1+e^{-\lambda y}} \log(\alpha_1) [\log(\alpha_1) - \log(\alpha_2)] \right\} \{ \log(\alpha_2) \log(\alpha_1) [\log(\mu_1) - \log(\mu_2)] \} \\
 &\times \left\{ \alpha_1 (\alpha_2 - 1) \alpha_2^{e^{-\theta x}} \log(\alpha_1) \left[P_{\alpha_1 \mu_1} \mu_1 \log(\mu_1) (\mu_2 - 1) \mu_2^{e^{-\lambda y}} + P_{\alpha_1 \mu_2} \mu_2 \log(\mu_2) (\mu_1 - 1) \mu_1^{e^{-\lambda y}} \right] \right\} \\
 &\times \left\{ +(\alpha_1 - 1) \alpha_1^{e^{-\theta x}} \alpha_2 \log(\alpha_2) \left[P_{\alpha_2 \mu_1} \mu_1 \log(\mu_1) (\mu_2 - 1) \mu_2^{e^{-\lambda y}} + P_{\alpha_2 \mu_2} \mu_2 \log(\mu_2) (\mu_1 - 1) \mu_1^{e^{-\lambda y}} \right] \right\}^{-2}.
 \end{aligned} \tag{9}$$

It is obvious from (9) that, provided $\alpha_1 > 1, \mu_1 > 1, \alpha_2 > 1, \mu_2 > 1$, the expression in (9) is greater than zero with the assumption that $P_{\alpha_1 \mu_2} P_{\alpha_2 \mu_1} > P_{\alpha_1 \mu_1} P_{\alpha_2 \mu_2}$, i.e., the joint density in (5) is log-concave.

The BMAPE (type-I) distribution’s survival function will be

$$R(x, y) = P(X > x, Y > y) = 1 - P(X < x) - P(Y < y) + P(X < x, Y < y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y) \tag{10}$$

The marginal distribution functions of X and Y are $F_X(x)$ and $F_Y(y)$, respectively. The following is the hazard rate function (hrf):

$$hrf(x, y) = (1 - F_X(x) - F_Y(y) + F_{X,Y}(x, y))^{-1} \times \left\{ \begin{aligned} &a \left(\frac{\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta x}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}}}{\mu_1 - 1} \right) \\ &+ b \left(\frac{\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta x}}}{\alpha_1 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}}}{\mu_2 - 1} \right) \\ &+ c \left(\frac{\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}}}{\mu_1 - 1} \right) \\ &+ d \left(\frac{\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}}}{\alpha_2 - 1} \times \frac{\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}}}{\mu_2 - 1} \right) \end{aligned} \right\}.$$

For each fixed $Y = y$, the conditional PDF of X given Y as well as Y given X are given, respectively, by

$$f(x|y) = \frac{h1(x, y)}{h1_y(y)} \text{ and } f(y|x) = \frac{h1(x, y)}{h1_x(x)}.$$

Proposition 1. Let $(X, Y) \sim BMAPE$ (type I) $(a, b, c, d, \alpha_1, \alpha_2, \mu_1, \mu_2, \theta, \lambda)$. Then, (X, Y) has a positive association with a total positivity of order 2 (TP_2), provided $(\alpha_1, \alpha_2, \mu_1, \mu_2) > 1$. Observe that an absolute continuous bivariate random vector, such as (U_1, U_2) , has the TP_2 property if and only if for each u_{11}, u_{12}, u_{21} , and u_{22} , whenever $u_{11} < u_{12}$ and $u_{21} > u_{22}$, it satisfies $f_{U_1, U_2}(u_{11}, u_{21})f_{U_1, U_2}(u_{12}, u_{22}) - f_{U_1, U_2}(u_{12}, u_{21})f_{U_1, U_2}(u_{11}, u_{22}) \geq 0$, where $f_{U_1, U_2}(u_1, u_2)$ is the joint PDF. Take distinct ordered u_{11}, u_{12}, u_{21} and u_{22} such that $u_{11} < u_{12}$ and $u_{21} > u_{22}$, and our result follows instantly. Consequently, the positive quadrant dependence property (also known as the TP_2 property) would reveal a number of non-increasing features relating to conditional survival, the conditional CDF of X given Y , and Y given X , as well as the following finding: $g_1(\cdot)$ and $g_2(\cdot)$, $Cov(g_1(X), g_2(Y)) \geq 0$.

Proposition 2. Shape of the distribution

A critical point in a two-variable function is where the first-order partial derivatives are equal to zero. The following are the two most compelling reasons to investigate the critical points of a bivariate distribution:

- (1) A real-life data set (or sets) can assume a variety of shapes. A study such as this can help determine the adaptability of this bivariate distribution.
- (2) When working with bivariate distributions, it is essential to examine the joint PDF's tails as well as the point of inflection. This study is useful in the context of data fitting, as data sets having heavier tails and/or which are small can be adequately modeled by a distribution exhibiting one of these features, as appropriate. Critical-point(s) knowledge will aid in a better understanding of these features, as will be illustrated in Appendix A.

4. Inference

In this section, we consider the estimation of the model parameters of the BMAPE (type-I) distribution using the EM algorithm. The EM algorithm, originally proposed by [2], is an iterative approach for determining the maximum likelihood estimator of a parameter in a parametric probability distribution. To use the EM technique, we combine the data $(x_k, y_k), k = 1, \dots, n$ (to be obtained from a random sample of size n obtained from the associated joint density) with the group membership variables $\Phi_k = (a_k, b_k, c_k), k = 1, \dots, n$, where a_k is an indicator variable that is one if the k^{th} observation is in $f(x, \alpha_1, \mu_1)$ and zero otherwise. Similarly, we have four groups $f_{ij}, i, j = 1, 2$, for b_k, c_k , with the following densities of

$$f_{ij}(x, y) = f_i(x)f_j(y) = \frac{\theta_1 \log(\alpha) e^{-\theta_1 x} \alpha^{1-e^{-\theta_1 x}}}{\alpha - 1} \frac{\lambda_1 \log(\mu) e^{-\lambda_1 y} \mu^{1-e^{-\lambda_1 y}}}{\mu - 1},$$

where $P(f_{11}) = a, P(f_{12}) = b, P(f_{21}) = c$, and $P(f_{22}) = 1 - a - b - c$ are the equivalent mixing proportions.

The associated log likelihood function for the complete sample $\ell_{ij}(x, y) = \log f_{ij}(x, y)$, is symbolized by:

$$\ell = \sum_{k=1}^n a_k \ell_{11}(x_k, y_k) + \sum_{k=1}^n b_k \ell_{12}(x_k, y_k) + \sum_{k=1}^n c_k \ell_{21}(x_k, y_k) + \sum_{k=1}^n (1 - a_k - b_k - c_k) \ell_{22}(x_k, y_k). \tag{11}$$

The EM method has two steps: Step-E (expectation) and Step-M (maximization) in each iteration. As the group membership variables represented as Φ_k are linear, we replace the related expected values in (11) with the current estimates ($\hat{\alpha}_1, \hat{\mu}_1, \hat{\alpha}_2, \hat{\mu}_2, \hat{\lambda}, \hat{\theta}, \hat{a}, \hat{b}, \hat{c}$) of the parameters determined and, subsequently, an update occurs. For more information on this topic, see [26] as:

$$\hat{a}_k = \frac{\hat{a} f_{11}(x_k, y_k)}{\hat{a} f_{11}(x_k, y_k) + \hat{b} f_{12}(x_k, y_k) + \hat{c} f_{21}(x_k, y_k) + (1 - \hat{a} - \hat{b} - \hat{c}) f_{22}(x_k, y_k)}.$$

Similarly, for b_k and c_k , apply the same technique, as follows:

$$\hat{b}_k = \frac{\hat{b} f_{12}(x_k, y_k)}{\hat{a} f_{11}(x_k, y_k) + \hat{b} f_{12}(x_k, y_k) + \hat{c} f_{21}(x_k, y_k) + (1 - \hat{a} - \hat{b} - \hat{c}) f_{22}(x_k, y_k)}$$

$$\text{and } \hat{c}_k = \frac{\hat{c} f_{21}(x_k, y_k)}{\hat{a} f_{11}(x_k, y_k) + \hat{b} f_{12}(x_k, y_k) + \hat{c} f_{21}(x_k, y_k) + (1 - \hat{a} - \hat{b} - \hat{c}) f_{22}(x_k, y_k)}.$$

Following that, in the M-step, we maximize (11) across $(\lambda, \theta, \alpha_1, \alpha_2, \mu_1, \mu_2)$ for fixed values of Φ_k . The conditional independence of X and Y given the group membership achieves maximization by differentiating (11), which are:

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{-n}{\alpha_1 - 1} \sum_{k=1}^n (a_k + b_k) + \frac{n}{\alpha_1 \log(\alpha_1)} \sum_{k=1}^n (a_k + b_k) + \frac{1}{\alpha_1} \sum_{k=1}^n (a_k + b_k) (1 - e^{-\theta x}), \tag{12}$$

$$\frac{\partial \ell}{\partial \mu_1} = \frac{-n}{\mu_1 - 1} \sum_{k=1}^n (a_k + c_k) + \frac{n}{\mu_1 \log(\mu_1)} \sum_{k=1}^n (a_k + c_k) + \frac{1}{\mu_1} \sum_{k=1}^n (a_k + c_k) (1 - e^{-\lambda y}), \tag{13}$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{-n}{\alpha_2 - 1} \sum_{k=1}^n (c_k + d_k) + \frac{n}{\alpha_2 \log(\alpha_2)} \sum_{k=1}^n (c_k + d_k) + \frac{1}{\alpha_2} \sum_{k=1}^n (c_k + d_k) (1 - e^{-\theta x}), \tag{14}$$

$$\frac{\partial \ell}{\partial \mu_2} = \frac{-n}{\mu_2 - 1} \sum_{k=1}^n (b_k + d_k) + \frac{n}{\mu_2 \log(\mu_2)} \sum_{k=1}^n (b_k + d_k) + \frac{1}{\mu_2} \sum_{k=1}^n (b_k + d_k) (1 - e^{-\lambda y}), \tag{15}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} \sum_{k=1}^n (a_k + b_k + c_k + d_k) - \sum_{k=1}^n (a_k + b_k + c_k + d_k) x_k - \log(\alpha_1) \sum_{k=1}^n x_k (a_k + b_k) e^{-\theta x_k} - \log(\alpha_2) \sum_{k=1}^n x_k (c_k + d_k) e^{-\theta x_k}, \tag{16}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} \sum_{k=1}^n (a_k + b_k + c_k + d_k) - \sum_{k=1}^n (a_k + b_k + c_k + d_k) y_k - \log(\mu_1) \sum_{k=1}^n y_k (a_k + c_k) e^{-\lambda y_k} - \log(\mu_2) \sum_{k=1}^n y_k (b_k + d_k) e^{-\lambda y_k}. \tag{17}$$

The M-step is then completed by setting

$$\hat{a} = n^{-1} \sum_{k=1}^n a_k, \text{ etc.} \tag{18}$$

The model parameters must have a starting value to initiate the process, which is denoted by the symbol $\Phi_{(0)}$. To ensure that the proposed EM algorithm’s rate of convergence does not become significantly sluggish, careful selection of these initial values is required. Another factor to consider is that the maximum likelihood equation may have several solutions corresponding to local maxima, necessitating careful selection of starting values. In the literature, some discussion on the identification of starting value(s) have been discussed and the references cited therein. The “coda” and “maxLik” of R package were used to quantitatively solve these equations. After obtaining the maximum likelihood estimators (MLEs) for $\alpha_1, \mu_1, \alpha_2, \mu_2, \theta, \lambda, \delta, \rho_{11}, \rho_{12}, \rho_{21}$ and ρ_{22} , we replace these estimates in (a_k, b_k, c_k) . Then, the M-step is completed by setting $\hat{a} = n^{-1} \sum_{k=1}^n a_k$, and the process continues. To produce initial values for the mixing proportions individually, the matching moment’s method for the marginal univariate APE parameter is employed. The BMAPE

(type-I) parameter estimates are used as the starting points for the EM technique. Then, assuming that the two variables X and Y are independent, we combine the marginal mixing parameter moment estimators to obtain initial values for the bivariate mixing parameters. Specific details are provided in Section 7, and, more specifically, in Tables 2–4. By assuming that the two variables X and Y are independent, as in (3) and (4), we combine the marginal mixing parameter of moment estimators to obtain initial values for the bivariate mixing parameters. We equate (12)–(17) to zero to obtain the estimates of the BMAPE (type-I) distribution’s parameter. Approximate confidence intervals (ACIs) of the unknown parameters are also obtained. This part is not investigated here but its results are presented numerically in Sections 7 and 8.

5. BMAPE (Type-II) Model Gaussian Copula Distribution

In this scenario, the proposed BMAPE model uses APE marginals and provides a reasonable flexibility in terms of marginals and the associated correlation structure. A copula is a useful tool for constructing bivariate and multivariate distributions, see [24,25] and the references cited therein. Copulas are extensively employed and play a vital part in the study of dependency. Several bivariate and multivariate lifetime distributions have been proposed by Refs. [27,28] on using a wide variety of couples. In this paper, we discuss the construction of a BMAPE (type-II) distribution via bivariate Gaussian copula with two APE marginals.

5.1. BMAPE (Type-II) Distribution Based on a Gaussian Copula

Any multivariate distribution may be decomposed into a copula and its continuous marginal, according to Ref. [29]. Copulas are useful tools in a bivariate scenario to combine two marginal distributions such that, for every bivariate distribution function $F(x,y)$, with continuous marginal $F(x)$ and $F(y)$, one can write

$$F(x,y) = C\{F(x), F(y)\}, \quad (x,y) \in \mathbb{R}. \tag{19}$$

The density function associated with the bivariate distribution will be

$$f(x,y) = f(x)f(y)C(F(x), F(y)), \tag{20}$$

where $C\{F(x), F(y)\} = \frac{\partial^2}{\partial u \partial v}[C(u,v)]$ is the density function of a copula; see [30].

As explained, there are a variety of options for building BMAPE (type-II) distributions using copulas and APE marginals (1). A bivariate Gaussian copula is defined as

$$C_{Normal}(u,v) = \Phi^2\left(\Phi^{-1}(u), \Phi^{-1}(v); \rho\right), \tag{21}$$

where Φ^2 is the joint CDF of a bivariate normal vector with zero means, unit variates, and correlation coefficient ρ .

Here, we denote the joint distribution of the two variables as X_d and Y_d where d symbolizes the phrase dependence. The joint PDF of X_d and Y_d based on Gaussian copula becomes

$$f(x_d,y_d) = f(x_d)f(y_d) \left\{ \frac{1}{\sqrt{1-\rho^2}} \left(\exp\left[\frac{-\rho}{2(1-\rho^2)} \left\{ \rho(z_1^2 + z_2^2) - 2z_1z_2 \right\}\right] \right) \right\}, \tag{22}$$

where Z_1 and Z_2 are latent random variable as well as $f(x_d)$ and $f(y_d)$ being the density functions of APE distributions provided in (3) and (4), and $\rho \in [-1, 1]$ the dependence

parameter. The BAPE (type-II) distribution PDF is generated if the marginals are from an APE distribution.

$$f(x_d, y_d, \alpha, \mu, \theta, \lambda) = \left(\frac{\theta \log(\alpha) e^{-\theta x_d} \alpha^{1-e^{-\theta x_d}}}{\alpha - 1} \right) \left(\frac{\lambda \log(\mu) e^{-\lambda y_d} \mu^{1-e^{-\lambda y_d}}}{\mu - 1} \right) \times \left\{ \frac{1}{\sqrt{1-\rho^2}} (\exp[\frac{-\rho}{2(1-\rho^2)} \{\rho(z_1^2 + z_2^2) - 2z_1 z_2\}]) \right\}, I(x_d, y_d),$$

and $(\alpha, \mu, \theta, \lambda) > 0$. Assuming that X_d and Y_d are dependent APE distributions with a generalized bivariate Bernoulli distribution for the shape parameters α_i, μ_i and, hence, the PDF of the BMAPE (type-II) distribution is defined as

$$f(x_d, \alpha_i, \theta) = \sum_{i=1}^2 m_i f((x_d)_i, \alpha_i, \theta)_i,$$

where the mixing proportions are m_i , and it must satisfy $\sum_{i=1}^2 m_i = 1$, and $m_i \geq 0$, all of which are unknown. The first component of APE’s density is provided by (3). The PDF of the first component of APE is given by

$$f(x_d, \alpha, \theta) = \frac{\theta \log(\alpha) e^{-\theta x_d} \alpha^{1-e^{-\theta x_d}}}{\alpha - 1}, x_d > 0, \tag{23}$$

where θ is a fixed-scale parameter, $\theta > 0$, and α is a random-shape parameter, $\alpha > 0$, which takes two distinct values of α_1 and α_2 . Let Y_d have an APE mixing density and the PDF of the second component APE be determined by Y_d for a fixed-scale parameter λ

$$g(y_d, \mu, \lambda) = \frac{\lambda \log(\mu) e^{-\lambda y_d} \mu^{1-e^{-\lambda y_d}}}{\mu - 1}, y_d > 0, \tag{24}$$

with the values μ_1 and μ_2 being a random-shape parameter. We suppose that X_d and Y_d are dependent for given values (α, μ) and that α and μ are associated by their generalized bivariate distribution with the following probability matrix provided by

$$m_d = \begin{matrix} & \mu_1 & \mu_2 \\ \alpha_1 & m_{\alpha_1 \mu_1} & m_{\alpha_1 \mu_2} \\ \alpha_2 & m_{\alpha_2 \mu_1} & m_{\alpha_2 \mu_2} \end{matrix}.$$

If $f(x_d, y_d)$ is the combined PDF of (X_d, Y_d) , then

$$\begin{aligned} f(x_d, y_d) &= f(x_d|\alpha_1, \theta)g(y_d|\mu_1, \lambda)c(F(x_d|\alpha_1, \theta), G(y_d|\mu_1, \lambda))m_{\alpha_1 \mu_1} \\ &+ f(x_d|\alpha_1, \theta)g(y_d|\mu_2, \lambda)c(F(x_d|\alpha_1, \theta), G(y_d|\mu_2, \lambda))m_{\alpha_1 \mu_2} \\ &+ f(x_d|\alpha_2, \theta)g(y_d|\mu_1, \lambda)c(F(x_d|\alpha_2, \theta), G(y_d|\mu_1, \lambda))m_{\alpha_2 \mu_1} \\ &+ f(x_d|\alpha_2, \theta)g(y_d|\mu_2, \lambda)c(F(x_d|\alpha_2, \theta), G(y_d|\mu_2, \lambda))m_{\alpha_2 \mu_2}. \end{aligned} \tag{25}$$

The joint PDF in (25) can take on a variety of shapes, similar to the joint PDF in (4). For simplicity, let $x_d = x, y_d = y, a = m_{\alpha_1 \mu_1}, b = m_{\alpha_1 \mu_2}, c = m_{\alpha_2 \mu_1}$ and $m_{\alpha_2 \mu_2}$; as a result, the BMAPE (type-II) distribution PDF will be

$$\begin{aligned}
 f(x, y, \alpha_1, \alpha_2, \theta, \mu_1, \mu_2, \lambda) = & \\
 & \left\{ \frac{a}{(\alpha_1-1)(\mu_1-1)} \left(\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta x}} \right) \left(\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}} \right) \left\{ \frac{1}{\sqrt{1-\rho_{11}^2}} \left(\exp\left[\frac{-\rho_{11}}{2(1-\rho_{11}^2)} \{ \rho_{11}(z_1^2 + z_2^2) - 2z_1z_2 \} \right] \right) \right\} + \right. \\
 & \frac{b}{(\alpha_1-1)(\mu_2-1)} \left(\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta_1 x}} \right) \left(\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}} \right) \left\{ \frac{1}{\sqrt{1-\rho_{12}^2}} \left(\exp\left[\frac{-\rho_{12}}{2(1-\rho_{12}^2)} \{ \rho_{12}(z_1^2 + z_2^2) - 2z_1z_2 \} \right] \right) \right\} + \\
 & \frac{c}{(\alpha_2-1)(\mu_1-1)} \left(\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}} \right) \left(\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}} \right) \left\{ \frac{1}{\sqrt{1-\rho_{21}^2}} \left(\exp\left[\frac{-\rho_{21}}{2(1-\rho_{21}^2)} \{ \rho_{21}(z_1^2 + z_2^2) - 2z_1z_2 \} \right] \right) \right\} + \\
 & \left. \frac{d}{(\alpha_2-1)(\mu_2-1)} \left(\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}} \right) \left(\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}} \right) \left\{ \frac{1}{\sqrt{1-\rho_{22}^2}} \left(\exp\left[\frac{-\rho_{22}}{2(1-\rho_{22}^2)} \{ \rho_{22}(z_1^2 + z_2^2) - 2z_1z_2 \} \right] \right) \right\} \right\}, \\
 & I(x_d, y_d), (\alpha, \mu, \theta, \lambda) > 0, \text{ and } \rho_{ij} \in [-1, 1] \text{ denotes the dependence parameter.}
 \end{aligned} \tag{26}$$

5.2. Gaussian Copula and the EM Algorithm

Here again, the EM algorithm is introduced. To use the EM technique, we supplement the data $(x_k, y_k); k = 1, \dots, n$, with the group membership variables $(a_{dk}, b_{dk}, c_{dk}), k = 1, \dots, n$, where a_{dk} is one if the k^{th} observation is in $f_{ij}(x, y, \alpha_1, \mu_1, \theta, \lambda, \rho_{11})$, and zero otherwise. Similarly, we have four groups $f_{ij}, i, j = 1, 2$ for b_{dk}, c_{dk} , with densities as

$$f_{ij}(x, y, \alpha_i, \mu_i, \theta, \lambda, \rho_{ij}) = \left(\theta \log(\alpha_i) e^{-\theta x} \alpha_i^{1-e^{-\theta x}} \right) \left(\lambda \log(\mu_i) e^{-\lambda y} \mu_i^{1-e^{-\lambda y}} \right) \left\{ \frac{\exp\left(\frac{-\rho_{ij}}{2(1-\rho_{ij}^2)} \{ \rho_{ij}(z_1^2 + z_2^2) - 2z_1z_2 \} \right)}{\sqrt{1-\rho_{ij}^2}} \right\}, x, y, \alpha_i, \mu_i, \theta, \lambda > 0, \tag{27}$$

where $\rho_{ij} \in [-1, 1]$ denotes the dependence parameter. Let $P(f_{11}) = a_d, P(f_{12}) = b_d, P(f_{21}) = c_d$ and $P(f_{22}) = 1 - a_d - b_d - c_d$ be the mixing proportions. We define $\ell_{ij}(x, y) = \log f_{ij}(x, y, \alpha_i, \mu_i, \theta, \lambda, \rho_{ij})$ and then the EM algorithm as a technique of estimate by finding the complete log likelihood, say ℓ_d , is given as follows:

$$\ell_d = \sum_{k=1}^n a_{dk} \ell_{11}(x_k, y_k) + \sum_{k=1}^n b_{dk} \ell_{12}(x_k, y_k) + \sum_{k=1}^n c_{dk} \ell_{21}(x_k, y_k) + \sum_{k=1}^n (1 - a_{dk} - b_{dk} - c_{dk}) \ell_{22}(x_k, y_k). \tag{28}$$

As the group membership variables (a_{dk}, b_{dk}, c_{dk}) are linear, we enter their predicted values into (28) in the E-step, given the current estimates $(\hat{\alpha}_1, \hat{\mu}_1, \hat{\alpha}_2, \hat{\mu}_2, \hat{\theta}, \hat{\theta}, \hat{\lambda}, \hat{\lambda}, \hat{\rho}_{11}, \hat{\rho}_{12}, \hat{\rho}_{21}, \hat{\rho}_{22}, \hat{a}_d, \hat{b}_d, \hat{c}_d)$ of the parameter, computed as:

$$\hat{a}_{dk} = \frac{\hat{a}_d f_{11}(x_k, y_k)}{\hat{a}_d f_{11}(x_k, y_k) + \hat{b}_d f_{12}(x_k, y_k) + \hat{c}_d f_{21}(x_k, y_k) + \left(1 - \hat{a}_d - \hat{b}_d - \hat{c}_d\right) f_{22}(x_k, y_k)},$$

Similarly, we use the same approach for b_{dk} and c_{dk} . It is worth noting that the above algebraic simplification may be required to obtain numerical solution. For fixed values of (a_{dk}, b_{dk}, c_{dk}) , we maximize (28) over $(\alpha_1, \mu_1, \alpha_2, \mu_2, \theta, \lambda, \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22})$. The univariates and the Gaussian copula parameter can be dealt with separately.

Then, differentiating (28) results in

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{n \sum_{k=1}^n (a_k + b_k)}{\alpha_1 \log(\alpha_1)} + \frac{1}{\alpha_1} \sum_{k=1}^n (a_k + b_k) (1 - e^{-\theta x}), \tag{29}$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{n \sum_{k=1}^n (c_k + d_k)}{\alpha_2 \log(\alpha_2)} + \frac{1}{\alpha_2} \sum_{k=1}^n (c_k + d_k) (1 - e^{-\theta x}), \tag{30}$$

$$\frac{\partial \ell}{\partial \mu_1} = \frac{n \sum_{k=1}^n (a_k + c_k)}{\mu_1 \log(\mu_1)} + \frac{1}{\mu_1} \sum_{k=1}^n (a_k + c_k) (1 - e^{-\lambda y}), \tag{31}$$

$$\frac{\partial \ell}{\partial \mu_2} = \frac{n \sum_{k=1}^n (b_k + d_k)}{\mu_2 \log(\mu_2)} + \frac{1}{\mu_2} \sum_{k=1}^n (b_k + d_k) (1 - e^{-\lambda y}), \tag{32}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} \sum_{k=1}^n (a_k + b_k + c_k + d_k) - \sum_{k=1}^n x(a_k + b_k + c_k + d_k) + \sum_{k=1}^n x \log(\alpha_1)(a_k + c_k)e^{-\theta x} + \sum_{k=1}^n x \log(\alpha_2)(b_k + d_k)e^{-\theta x}, \quad (33)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} \sum_{k=1}^n (a_k + b_k + c_k + d_k) - \sum_{k=1}^n y(a_k + b_k + c_k + d_k) + \sum_{k=1}^n y \log(\mu)(a_k + b_k + c_k + d_k)e^{-\lambda y}, \quad (34)$$

$$\begin{aligned} \frac{\partial l}{\partial \rho_{11}} &= \frac{a_k}{(\alpha_1 - 1)(\mu_1 - 1)} \left(\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1 - e^{-\theta x}} \right) \left(\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1 - e^{-\lambda y}} \right) \\ &\times \left\{ \frac{\rho_{11} \exp\left[\frac{-\rho_{11}}{2(1-\rho_{11}^2)}\right] \{\rho_{11}(z_1^2 + z_2^2) - 2z_1 z_2\}}{\sqrt{(1-\rho_{11}^2)^3}} + \sqrt{1 - \rho_{11}^2} \left(\exp\left[\frac{-\rho_{11}}{2(1-\rho_{11}^2)}\right] \{\rho_{11}(z_1^2 + z_2^2) - 2z_1 z_2\} \right) \right. \\ &\quad \left. + (1 - 3\rho_{11}^2) \{\rho_{11}(z_1^2 + z_2^2) - 2z_1 z_2\} + \{(\rho_{11} - \rho_{11}^3)(z_1^2 + z_2^2)\} \right\}, \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{\partial l}{\partial \rho_{12}} &= \frac{b_k}{(\alpha_1 - 1)(\mu_2 - 1)} \left(\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1 - e^{-\theta x}} \right) \left(\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1 - e^{-\lambda y}} \right) \\ &\times \left\{ \frac{\rho_{12} \exp\left[\frac{-\rho_{12}}{2(1-\rho_{12}^2)}\right] \{\rho_{12}(z_1^2 + z_2^2) - 2z_1 z_2\}}{\sqrt{(1-\rho_{12}^2)^3}} + \sqrt{1 - \rho_{12}^2} \left(\exp\left[\frac{-\rho_{12}}{2(1-\rho_{12}^2)}\right] \{\rho_{12}(z_1^2 + z_2^2) - 2z_1 z_2\} \right) \right. \\ &\quad \left. + (1 - 3\rho_{12}^2) \{\rho_{12}(z_1^2 + z_2^2) - 2z_1 z_2\} + \{(\rho_{12} - \rho_{12}^3)(z_1^2 + z_2^2)\} \right\}, \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{\partial l}{\partial \rho_{21}} &= \frac{c_k}{(\alpha_2 - 1)(\mu_1 - 1)} \left(\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1 - e^{-\theta x}} \right) \left(\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1 - e^{-\lambda y}} \right) \\ &\times \left\{ \frac{\rho_{21} \exp\left[\frac{-\rho_{21}}{2(1-\rho_{21}^2)}\right] \{\rho_{21}(z_1^2 + z_2^2) - 2z_1 z_2\}}{\sqrt{(1-\rho_{21}^2)^3}} + \sqrt{1 - \rho_{21}^2} \left(\exp\left[\frac{-\rho_{21}}{2(1-\rho_{21}^2)}\right] \{\rho_{21}(z_1^2 + z_2^2) - 2z_1 z_2\} \right) \right. \\ &\quad \left. + (1 - 3\rho_{21}^2) \{\rho_{21}(z_1^2 + z_2^2) - 2z_1 z_2\} + \{(\rho_{21} - \rho_{21}^3)(z_1^2 + z_2^2)\} \right\}, \end{aligned} \quad (37)$$

and

$$\begin{aligned} \frac{\partial l}{\partial \rho_{22}} &= \frac{d_k}{(\alpha_2 - 1)(\mu_2 - 1)} \left(\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1 - e^{-\theta x}} \right) \left(\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1 - e^{-\lambda y}} \right) \\ &\times \left\{ \frac{\rho_{22} \exp\left[\frac{-\rho_{22}}{2(1-\rho_{22}^2)}\right] \{\rho_{22}(z_1^2 + z_2^2) - 2z_1 z_2\}}{\sqrt{(1-\rho_{22}^2)^3}} + \sqrt{1 - \rho_{22}^2} \left(\exp\left[\frac{-\rho_{22}}{2(1-\rho_{22}^2)}\right] \{\rho_{22}(z_1^2 + z_2^2) - 2z_1 z_2\} \right) \right. \\ &\quad \left. + (1 - 3\rho_{22}^2) \{\rho_{22}(z_1^2 + z_2^2) - 2z_1 z_2\} + \{(\rho_{22} - \rho_{22}^3)(z_1^2 + z_2^2)\} \right\}. \end{aligned} \quad (38)$$

The method uses a two-step strategy to estimate the marginal of X and Y as well as the copula function separately, yielding the MLEs of $(\alpha_1, \mu_1, \alpha_2, \mu_2, \theta, \lambda, \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22})$. The MLEs of $\alpha_1, \mu_1, \alpha_2, \mu_2, \theta, \lambda, \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}$ are obtained by solving the nonlinear equation in (28)–(37). The density of copulas is then calculated as follows:

$$\log L_{ij}(\rho) = \sum_{k=1}^n \log C(\hat{F}_i(x_k), \hat{F}_j(y_k)), \quad (39)$$

where $\hat{F}_i(x_k)$ and $\hat{F}_j(y_k)$ are the maximum likelihood estimates of the PDF from the first step. The MLEs of $\rho_{11}, \rho_{12}, \rho_{21}$, and ρ_{22} are obtained by solving the nonlinear (39). The M-step is completed by setting $\hat{a}_d = n^{-1} \sum_{k=1}^n a_{dk}$. To solve these equations numerically, we use the copula R package. Next, we replace these estimates in (a_{dk}, b_{dk}, c_{dk}) after obtaining MLEs for $\alpha_1, \mu_1, \alpha_2, \mu_2, \theta, \lambda, \rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}$. We finish the M-step by setting $\hat{a}_d = n^{-1} \sum_{k=1}^n a_{dk}$, and so on.

The method of matching moments, which is generated from the marginal univariate APE and the Gaussian copula parameter separately, are used to obtain the initial values of the parameters for the mixing proportions. The BAPE parameter estimators obtained are used as starting values for the EM method. Then, assuming that the dependency between two variables X and Y is zero, we combine the moment estimators of the marginal mixing parameters to generate initial values for the bivariate mixing parameters. Ref. [31] for more information. Following that, we present the process for estimating the unknown parameters for the density in (28). We use two approaches in copula-based estimation: parametric and semiparametric.

5.3. Maximum Likelihood Estimation

We look at how to estimate the unknown parameters of BMAPE (type-II) distributions using the maximum likelihood technique and two-step estimation. This entails a two-step process in which the marginal and copula functions are individually estimated; see [32]. The log-likelihood function is defined as follows:

$$\log L = \sum_{i=1}^n [\log f_1(x_i) + \log f_2(y_i) + \log c(F_1(x_i), F_2(y_i))]. \tag{40}$$

The log-likelihood function in (40) may be rewritten as

$$\log L = \sum_{i=1}^n \log f_1(x_i) + \sum_{i=1}^n \log f_2(x_{2i}) + \sum_{i=1}^n \log c(F_1(x_{1i}), F_2(x_{2i})). \tag{41}$$

The MLEs are separately used to estimate the parameters of the marginal distribution F_1 and F_2 , as follows:

$$\log L_1 = \sum_{i=1}^n \log \log f_1(x_i). \log L_2 = \sum_{i=1}^n \log f_2(y_i). \tag{42}$$

Then, by maximizing the copula density, the copula parameters may be estimated.

$$\log L = \sum_{i=1}^n \log c(F_1(x_i), F_2(y_i)). \tag{43}$$

The maximum likelihood estimation will estimate the parameters of each marginal distribution by evaluating the first step with APE distributions. If (x_1, \dots, x_n) is a random sample from $APE(\alpha, \theta)$ and (y_1, \dots, y_n) is a random sample from $APE(\mu, \lambda)$, the log-likelihood functions are provided by

$$\log L_1(x, \alpha, \theta) = n \log(\theta) + n \log(\log(\alpha)) - \theta \sum_{i=1}^n x_i + \log(\alpha) \sum_{i=1}^n (1 - e^{-\theta x_i}) \tag{44}$$

and

$$\log L_2(y, \mu, \lambda) = n \log(\lambda) + n \log(\log(\mu)) - \lambda \sum_{i=1}^n y_i + \log(\mu) \sum_{i=1}^n (1 - e^{-\lambda y_i}). \tag{45}$$

As a result, the maximum likelihood equations are as follows:

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^n y_i + \log(\mu) \sum_{i=1}^n y_i e^{-\lambda y_i} = 0 \tag{46}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i + \log(\alpha) \sum_{i=1}^n x_i e^{-\theta x_i} = 0, \tag{47}$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha \log(\alpha)} + \frac{1}{\alpha} \sum_{i=1}^n (1 - e^{-\theta x_i}) = 0 \tag{48}$$

and

$$\frac{\partial l}{\partial \mu} = \frac{n}{\mu \log(\mu)} + \frac{1}{\mu} \sum_{i=1}^n (1 - e^{-\lambda y_i}) = 0, \tag{49}$$

The MLEs of α, μ, θ and λ are obtained by solving the nonlinear (46)–(49) system. The density of copulas is then calculated as follows:

$$\log L(\gamma) = \sum_{i=1}^n \log c(\hat{F}_1(x_i), \hat{F}_2(y_i)), \tag{50}$$

where $\hat{F}_1(x)$ and $\hat{F}_2(y)$ are the maximum likelihood estimates of the first-step parameters. Therefore, the maximum likelihood estimate of γ can be obtained by solving the nonlinear (50).

5.4. Estimation via Semiparametric Methods

The inversion Kendall’s and inversion of Spearman’s semiparametric methods for estimating the copula parameter in copula models are compared to the two methods of moments approaches, inversion Kendall’s τ and inversion of Spearman’s ρ , respectively.

5.5. Moments Method

We present a brief description of moment’s method of inversion of Kendall’s τ and the inversion of Spearman’s ρ as indicated in Ref. [33]. Let c be a bivariate random sample from a CDF, $C_\gamma [F_1(x), F_2(y)]$, where F_1 and F_2 are continuous CDF and C_γ is an absolutely continuous copula such that $\gamma \in \mathcal{O}$. Moreover, unless otherwise specified, let R_1, \dots, R_n be the vectors of ranks associated with (x_1, \dots, x_n) . All vectors in the following are row vectors. The inversion of a consistent estimator of a moment of the copula C_γ is used in moments methods. The two most well-known moments, Spearman’s ρ and Kendall’s τ , are given by Spearman and Kendall, respectively, as

$$\rho(\gamma) = 12 \int \int_{[0,1]^2} u v dC_\gamma(u, v) - 3, \tag{51}$$

and

$$\tau(\gamma) = 4 \int \int_{[0,1]^2} C_\gamma(u, v) dC_\gamma(u, v) - 1. \tag{52}$$

These two moments’ consistent estimators can be stated as

$$\rho_n = \frac{12}{n(n+1)(n-1)} \sum_{i=1}^n R_{i,1}R_{i,2} - 3 \frac{n+1}{n-1}, \tag{53}$$

and

$$\tau_n = \frac{4}{n(n-1)} \sum_{i=1}^n 1[x_{i,1} \leq x_{j,1}]1[x_{i,2} \leq x_{j,2}] - 1. \tag{54}$$

Consistent estimators of ρ and τ will be if and are one-to-one $\gamma_{n,\rho} = \rho^{-1}(\rho_n)$, $\gamma_{n,\tau} = \tau^{-1}(\tau_n)$, respectively.

Inversion of Kendall’s τ and inversion of Spearman’s ρ are two terms for the same thing. Ref. [33] and the references referenced therein for more information. As previously stated, the moment’s method of τ and ρ estimation for copula can be classified as semiparametric approach estimation.

5.6. Copula Fit Tests

We want to compare the empirical copula with the parametric estimator generated under the null hypothesis; for more information, see [34,35]. According to the theory, a test to see if C is well-represented by a given copula C_γ can be represented as

$$H_0 : C = C_\gamma, \quad \text{against} \quad H_1 : C \neq C_\gamma.$$

There are several well-known ways in the literature, such as Ref. [36] or the rapid multiplier approach, as described by Refs. [37,38]. The empirically based goodness of fit test is written as

$$\mathbb{C}_n(u, v) = \sqrt{n} \{C_n(u, v) - C_{\gamma_n}(u, v)\},$$

where $C_n(u, v)$ is the empirical copula of the X and Y data.

$C_n(u, v) = \frac{1}{n} \sum_{i=1}^n 1(U_{i,n} \leq u, V_{i,n} \leq v)$, $u, v \in [0, 1]$, the pseudo-observations $U_{i,n}$ and $V_{i,n}$ from C , are calculated as:

$$U_{i,n} = \frac{R_{1i}}{n+1}, \quad V_{i,n} = \frac{R_{2i}}{n+1}; \quad R_{1i} \text{ and } R_{2i} \text{ are, respectively, the ranks of } X_i, Y_i.$$

$C_n(u, v)$ is an estimator obtained using the pseudo observations and $C_n(u, v)$ is a consistent estimator. The relevant test statistic, is the Cramer-von Miss, which is defined as

$$S_n = \sum_{i=1}^n \{C_n(U_{i,n}, V_{i,n}) - C_{\gamma_n}(U_{i,n}, V_{i,n})\}^2; \text{ see [39] as well as the references within.}$$

6. Bayesian Estimation

The Bayes estimates of the joint posterior density (54) model parameters are computed in this part under the assumption that the random variable $\Phi = (\alpha_1, \alpha_2, \mu_1, \mu_2, \theta, \lambda)$ have independent gamma prior distributions with hyperparameter w_k and m_k and $k = 1, 2, 3, 4, 5, 6$, provided by

$$f(\Phi; w, m) = \frac{m_k^{w_k}}{\Gamma w_k} \Phi^{w_k-1} e^{-m_k \Phi}, \Phi > 0, \quad (55)$$

and $\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}$ theses have a non-informative prior. The joint posterior density for the vector Φ given the data is obtained by multiplying (9) or (28) as

$$\pi(\Phi|x) \propto L(x|\Phi)f(\Phi; w_m, s_m). \quad (56)$$

Integrating out the (nuisance) hyperparameters yields marginal distributions of Φ . As a result, under the square error loss function, the Bayesian estimators of the parameters can be derived as follows:

$$\hat{\Phi} \propto \int_0^\infty \Phi \pi(\Phi|x) d\Phi. \quad (57)$$

6.1. Bayes MCMC Estimates

As the integrals in (56) are not in a closed form, the Markov-chain Monte-Carlo (MCMC) method is used. The posterior distribution and intractable integrals are derived using simulated samples from the posterior distribution in MCMC algorithms. Gibbs sampling and the Metropolis-Hastings (M-H) algorithm are also employed as MCMC techniques, see [40,41].

The M-H algorithm assumes that a candidate value can be generated from a proposed distribution for each iteration of the process. As a result, the applicant value is allowed if the approval chance is high enough. This method ensures that the Markov chain for the target density will converge. Finally, we can see that the advantage of the MCMC technique over the MLE method is that by creating probability intervals based on empirical posterior distribution, we can always obtain an acceptable interval estimate of the parameters. In MLE, this is frequently unavailable.

6.2. Bayes Credible Intervals

The following expression can be used to obtain a symmetric $100(1 - \epsilon)\%$ percent two-sided Bayes probability interval estimate of Φ , denoted by $[L_\Phi, U_\Phi]$.

$$p[L(t) < \Phi < U(t)] = \int_{L(t)}^{U(t)} \pi(\theta, \beta, \lambda|t) d\Phi = 1 - \epsilon, \quad (58)$$

We need appropriate numerical approaches to solve this non-linear equation because finding the interval L_Φ and U_Φ analytically is difficult.

7. Monte-Carlo Simulation

To assess the adaptability and flexibility of the proposed MBAPE model for different combinations of the true parameter value and sample size, an extensive Monte-Carlo simulation is conducted based on both independent and dependent cases. The point (MLE and MCMC) estimates and interval (ACI and BCI) estimates are evaluated based on 2,000 random samples of sizes $n (= 50, 100, 150, 200)$. Extensive computations were performed using R statistical software via two useful statistical packages, namely, (i) the ‘‘coda’’ package proposed by Ref. [42], and (ii) the ‘‘maxLik’’ package, which uses the Newton–Raphson method of maximization in the likelihood computations, proposed by Ref. [43].

Comparison between point estimators is made in terms of their mean absolute bias (MAB) and root mean squared-error (RMSE) values. In addition, the behavior of 95% asymptotic/credible intervals are compared by their average confidence lengths (ACLs). In

Table 1, different actual values of the model parameters and the associated hyperparameters are reported. Clearly, the values of hyperparameters are specified in such a way that the prior mean becomes the expected value of the target parameter. In this study, for both sets III and IV, we take $(\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22})$ as $(0.2, 0.15, 0.05, 0.1)$ and $(0.5, 0.4, 0.2, 0.3)$ as well as also assuming they have improper gamma priors. The MCMC estimates under squared-error loss are computed by running the M-H algorithm 12,000 times and ignoring the first 2,000 MCMC iterations. The calculated MLE of each unknown parameter is taken as an initial guess to obtain the corresponding Bayes estimate; for more information in this topic see [44–46]. The simulation outcomes are provided in Tables 2–6. In these tables, the Bayes MCMC results under Prior 1 (as an example) are denoted as “MCMC.1”; similarly, the BCI estimates under Prior 1 are also denoted as “BCI.1”, for brevity. From Tables 2–6, it can be seen that the proposed estimates of the unknown MBAPE parameters are sufficiently efficient with respect to their minimum simulated values of RMSEs, MABs and ACLs. For all true parametric combinations, when n increases under independent and dependent assumptions, the RMSEs, MABs and ACLs of the classical and Bayesian estimates decrease, as expected. Therefore, one can maximize the efficiency associated with all estimates by increasing the overall sample size. As $\alpha, \theta, \mu, \lambda$ and ρ increase, it is observed that their RMSEs, MABs and ACLs increase. A similar pattern is observed in the case of $\alpha_1, \alpha_2, \mu_1, \mu_2$ and ρ_{ij} for $i, j = 1, 2$ increases, a dependent case. Since the Bayesian estimates include the flexible gamma information, the Bayesian estimates of all unknown parameters have overall lower RMSE and MAB values, so they performed better compared to the classical estimates in both independent and dependent cases. Consequently, the credible intervals estimates are also better than the asymptotic confidence intervals in terms of shortest ACLs. In particular, as the calculated variance in Prior 2 is less than Prior 1, it is also seen that the Bayes (point and interval) estimates have performed better based on Prior 2 than those obtained based on Prior 1. All accuracy criteria developed in this work support this statement.

Table 1. Different parameter values under independent/dependent cases.

Parameter	True Value	Prior 1		Prior 2		True Value	Prior 1		Prior 2	
		w	m	w	m		w	m	w	m
Set →		I					II			
α	0.4	0.8	2	2.0	5	0.8	1.6	2	4.0	5
θ	0.5	1.0	2	2.5	5	1.5	3.0	2	7.5	5
μ	0.2	0.4	2	1.0	5	0.4	2.0	2	2.0	5
λ	0.8	1.6	2	4.0	5	1.2	2.4	2	6.0	5
ρ	0.3	0.6	2	1.5	5	0.5	1.0	2	5.0	5
Set →		III					IV			
α_1	0.1	0.2	2	0.5	5	0.3	0.6	2	1.5	5
α_2	0.3	0.6	2	1.5	5	0.5	1.0	2	2.5	5
θ	0.5	1.0	2	2.5	5	1.5	3.0	2	7.5	5
μ_1	0.1	0.2	2	0.5	5	0.2	0.4	2	1.0	5
μ_2	0.1	0.2	2	0.5	5	0.2	0.4	2	1.0	5
λ	0.8	1.6	2	4.0	5	1.2	2.4	2	5.0	5

Table 2. Simulated point and interval estimates of the BMAPE parameters for Sets I and II.

Set	n	Par.	MLE		MCMC.1		MCMC.2		ACI	BCI.1	BCI.2
			MAB	RMSE	MAB	RMSE	MAB	RMSE			
I	50	α	0.618	1.106	0.143	0.199	0.037	0.064	3.056	0.348	0.092
		θ	0.171	0.217	0.108	0.154	0.039	0.059	0.727	0.300	0.099
		μ	0.401	0.764	0.179	0.205	0.102	0.177	2.040	0.296	0.250
		λ	0.314	0.409	0.262	0.395	0.203	0.343	1.402	0.658	0.492
		ρ	0.304	0.363	0.261	0.335	0.166	0.290	0.636	0.525	0.392
	100	α	0.351	0.527	0.127	0.187	0.036	0.058	1.771	0.346	0.089
		θ	0.118	0.155	0.096	0.125	0.037	0.056	0.552	0.250	0.093
		μ	0.246	0.388	0.153	0.198	0.097	0.165	1.226	0.274	0.242
		λ	0.242	0.375	0.240	0.350	0.200	0.301	1.110	0.648	0.474
		ρ	0.302	0.317	0.231	0.291	0.164	0.279	0.575	0.405	0.390
	150	α	0.263	0.377	0.122	0.181	0.034	0.055	1.338	0.286	0.087
		θ	0.097	0.151	0.079	0.101	0.036	0.051	0.458	0.249	0.086
		μ	0.186	0.277	0.103	0.163	0.096	0.141	0.918	0.266	0.236
		λ	0.202	0.340	0.198	0.258	0.152	0.226	0.975	0.480	0.352
		ρ	0.301	0.311	0.225	0.285	0.162	0.277	0.555	0.399	0.319
	200	α	0.224	0.314	0.116	0.155	0.032	0.054	1.121	0.282	0.084
		θ	0.084	0.109	0.038	0.061	0.032	0.045	0.398	0.113	0.076
		μ	0.177	0.221	0.099	0.153	0.095	0.140	0.751	0.259	0.230
		λ	0.192	0.335	0.179	0.226	0.150	0.220	0.847	0.464	0.347
		ρ	0.300	0.306	0.211	0.280	0.160	0.267	0.503	0.392	0.276
II	50	α	0.907	1.583	0.272	0.443	0.107	0.183	4.988	0.676	0.265
		θ	0.373	0.482	0.276	0.437	0.223	0.396	1.815	0.726	0.550
		μ	0.591	1.046	0.368	0.628	0.360	0.621	3.161	1.062	0.867
		λ	0.395	0.562	0.311	0.461	0.266	0.453	1.794	0.883	0.644
		ρ	0.503	0.551	0.470	0.505	0.236	0.412	1.057	0.573	0.551
	100	α	0.546	0.819	0.268	0.404	0.105	0.177	2.866	0.640	0.258
		θ	0.293	0.452	0.262	0.387	0.221	0.392	1.295	0.698	0.544
		μ	0.423	0.693	0.355	0.621	0.354	0.568	1.810	1.033	0.857
		λ	0.346	0.509	0.288	0.451	0.255	0.372	1.328	0.799	0.630
		ρ	0.502	0.525	0.348	0.476	0.233	0.400	0.836	0.571	0.392
	150	α	0.427	0.610	0.254	0.398	0.101	0.173	2.179	0.609	0.248
		θ	0.268	0.437	0.226	0.378	0.216	0.338	1.069	0.695	0.542
		μ	0.407	0.672	0.351	0.618	0.270	0.392	1.326	1.030	0.851
		λ	0.315	0.487	0.258	0.444	0.234	0.299	1.103	0.764	0.626
		ρ	0.501	0.512	0.339	0.401	0.230	0.399	0.789	0.558	0.319
	200	α	0.365	0.514	0.218	0.320	0.098	0.169	1.827	0.521	0.243
		θ	0.244	0.392	0.223	0.277	0.184	0.233	0.914	0.614	0.539
		μ	0.400	0.648	0.326	0.558	0.225	0.313	1.124	0.862	0.832
		λ	0.307	0.453	0.255	0.442	0.200	0.257	0.959	0.624	0.619
		ρ	0.500	0.509	0.317	0.399	0.229	0.381	0.764	0.552	0.277

Table 3. Simulated point and interval estimates of the BMAPE parameters using Set III when $(\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) = (0.2, 0.15, 0.05, 0.1)$.

<i>n</i>	Par.	MLE		MCMC.1		MCMC.2		ACI	BCI.P1	BCI.P2
		MAB	RMSE	MAB	RMSE	MAB	RMSE			
50	α_1	0.085	0.113	0.058	0.061	0.039	0.044	0.606	0.085	0.074
	α_2	0.274	0.295	0.073	0.081	0.029	0.031	0.630	0.143	0.077
	θ	0.273	0.263	0.228	0.062	0.208	0.060	0.779	0.099	0.056
	μ_1	0.087	0.123	0.061	0.063	0.049	0.053	0.455	0.133	0.055
	μ_2	0.099	0.106	0.087	0.104	0.045	0.046	0.528	0.116	0.068
	λ	0.323	0.359	0.053	0.063	0.018	0.019	1.814	0.129	0.041
	ρ_{11}	0.388	0.411	0.224	0.234	0.172	0.178	0.514	0.247	0.195
	ρ_{12}	0.428	0.447	0.184	0.194	0.141	0.144	0.554	0.228	0.162
	ρ_{21}	0.409	0.442	0.166	0.176	0.112	0.115	0.602	0.180	0.156
	ρ_{22}	0.427	0.451	0.166	0.180	0.163	0.171	0.678	0.254	0.154
100	α_1	0.084	0.101	0.048	0.051	0.023	0.025	0.206	0.083	0.070
	α_2	0.268	0.287	0.041	0.053	0.024	0.024	0.549	0.105	0.067
	θ	0.239	0.260	0.220	0.035	0.176	0.015	0.557	0.079	0.038
	μ_1	0.086	0.108	0.050	0.058	0.045	0.051	0.388	0.104	0.053
	μ_2	0.085	0.101	0.035	0.044	0.025	0.030	0.321	0.101	0.061
	λ	0.322	0.358	0.047	0.058	0.015	0.016	1.045	0.117	0.040
	ρ_{11}	0.364	0.391	0.173	0.181	0.145	0.158	0.394	0.192	0.139
	ρ_{12}	0.420	0.445	0.157	0.172	0.109	0.119	0.381	0.201	0.117
	ρ_{21}	0.407	0.439	0.127	0.136	0.050	0.059	0.451	0.161	0.097
	ρ_{22}	0.422	0.450	0.113	0.119	0.053	0.061	0.449	0.130	0.128
150	α_1	0.083	0.096	0.025	0.028	0.022	0.022	0.201	0.069	0.066
	α_2	0.267	0.273	0.040	0.045	0.018	0.022	0.539	0.092	0.040
	θ	0.235	0.257	0.212	0.034	0.170	0.013	0.519	0.076	0.037
	μ_1	0.086	0.104	0.043	0.046	0.012	0.015	0.321	0.099	0.052
	μ_2	0.083	0.097	0.026	0.031	0.019	0.024	0.244	0.078	0.037
	λ	0.319	0.356	0.038	0.041	0.014	0.015	0.821	0.069	0.037
	ρ_{11}	0.365	0.388	0.087	0.089	0.061	0.062	0.348	0.074	0.040
	ρ_{12}	0.419	0.435	0.129	0.130	0.068	0.078	0.350	0.139	0.060
	ρ_{21}	0.401	0.429	0.059	0.067	0.041	0.044	0.364	0.094	0.076
	ρ_{22}	0.409	0.440	0.075	0.080	0.017	0.018	0.410	0.121	0.026
200	α_1	0.079	0.086	0.021	0.025	0.019	0.021	0.163	0.050	0.016
	α_2	0.258	0.269	0.035	0.040	0.014	0.018	0.347	0.080	0.019
	θ	0.232	0.247	0.171	0.024	0.167	0.010	0.413	0.070	0.013
	μ_1	0.084	0.099	0.039	0.045	0.005	0.006	0.265	0.068	0.015
	μ_2	0.081	0.091	0.022	0.023	0.011	0.013	0.201	0.049	0.020
	λ	0.299	0.343	0.022	0.028	0.009	0.011	0.801	0.057	0.013
	ρ_{11}	0.358	0.382	0.030	0.031	0.018	0.023	0.298	0.063	0.011
	ρ_{12}	0.415	0.434	0.034	0.038	0.010	0.011	0.304	0.054	0.019
	ρ_{21}	0.392	0.425	0.050	0.050	0.040	0.042	0.361	0.057	0.008
	ρ_{22}	0.408	0.438	0.043	0.059	0.014	0.015	0.375	0.095	0.008

Table 4. Simulated point and interval estimates of the BMAPE parameters using Set III when $(\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) = (0.5, 0.4, 0.2, 0.3)$.

<i>n</i>	Par.	MLE		MCMC.1		MCMC.2		ACI	BCI.P1	BCI.P2
		MAB	RMSE	MAB	RMSE	MAB	RMSE			
50	α_1	0.095	0.144	0.065	0.067	0.045	0.056	0.807	0.129	0.083
	α_2	0.253	0.394	0.070	0.073	0.033	0.035	1.565	0.143	0.090
	θ	0.212	0.182	0.174	0.087	0.143	0.022	1.232	0.154	0.042
	μ_1	0.091	0.124	0.053	0.054	0.041	0.049	0.971	0.122	0.058
	μ_2	0.084	0.125	0.056	0.064	0.032	0.037	0.786	0.094	0.068
	λ	0.232	0.277	0.112	0.115	0.046	0.048	2.056	0.126	0.067
	ρ_{11}	0.153	0.189	0.098	0.107	0.053	0.056	0.516	0.182	0.097
	ρ_{12}	0.230	0.252	0.198	0.214	0.063	0.072	0.534	0.260	0.124
	ρ_{21}	0.335	0.348	0.112	0.120	0.032	0.036	0.620	0.138	0.093
	ρ_{22}	0.268	0.286	0.217	0.232	0.035	0.040	0.645	0.235	0.122
100	α_1	0.078	0.133	0.060	0.063	0.043	0.054	0.696	0.115	0.077
	α_2	0.220	0.238	0.059	0.066	0.030	0.032	0.787	0.120	0.079
	θ	0.210	0.177	0.170	0.049	0.137	0.011	1.030	0.117	0.039
	μ_1	0.077	0.091	0.047	0.052	0.034	0.038	0.758	0.101	0.054
	μ_2	0.079	0.101	0.039	0.042	0.019	0.023	0.695	0.092	0.061
	λ	0.216	0.255	0.068	0.072	0.027	0.032	1.752	0.091	0.052
	ρ_{11}	0.142	0.171	0.082	0.102	0.042	0.047	0.482	0.146	0.066
	ρ_{12}	0.227	0.244	0.102	0.111	0.038	0.040	0.524	0.170	0.070
	ρ_{21}	0.333	0.346	0.099	0.105	0.031	0.033	0.576	0.133	0.073
	ρ_{22}	0.266	0.283	0.106	0.119	0.029	0.034	0.602	0.182	0.087
150	α_1	0.067	0.087	0.057	0.062	0.036	0.041	0.562	0.113	0.068
	α_2	0.219	0.236	0.037	0.047	0.027	0.030	0.716	0.111	0.060
	θ	0.207	0.165	0.152	0.046	0.131	0.010	0.831	0.080	0.024
	μ_1	0.075	0.089	0.045	0.050	0.013	0.014	0.708	0.100	0.034
	μ_2	0.078	0.098	0.033	0.038	0.012	0.014	0.549	0.079	0.054
	λ	0.205	0.249	0.059	0.070	0.020	0.021	1.692	0.090	0.040
	ρ_{11}	0.135	0.167	0.045	0.053	0.036	0.044	0.452	0.100	0.065
	ρ_{12}	0.215	0.242	0.086	0.094	0.017	0.022	0.418	0.154	0.068
	ρ_{21}	0.332	0.345	0.060	0.069	0.023	0.024	0.530	0.124	0.039
	ρ_{22}	0.257	0.281	0.100	0.111	0.021	0.032	0.527	0.178	0.083
200	α_1	0.066	0.074	0.032	0.039	0.031	0.037	0.494	0.085	0.052
	α_2	0.217	0.233	0.025	0.031	0.023	0.028	0.547	0.090	0.043
	θ	0.180	0.158	0.147	0.025	0.130	0.008	0.804	0.068	0.022
	μ_1	0.068	0.081	0.035	0.041	0.009	0.012	0.691	0.092	0.032
	μ_2	0.067	0.076	0.029	0.032	0.011	0.011	0.544	0.076	0.052
	λ	0.202	0.243	0.046	0.048	0.019	0.020	1.510	0.052	0.038
	ρ_{11}	0.133	0.163	0.042	0.047	0.022	0.027	0.390	0.084	0.062
	ρ_{12}	0.214	0.233	0.065	0.076	0.015	0.020	0.388	0.117	0.043
	ρ_{21}	0.330	0.344	0.036	0.043	0.020	0.022	0.528	0.099	0.035
	ρ_{22}	0.249	0.260	0.045	0.056	0.019	0.030	0.510	0.150	0.060

Table 5. Simulated point and interval estimates of the BMAPE parameters using Set IV when $(\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) = (0.2, 0.15, 0.05, 0.1)$.

<i>n</i>	Par.	MLE		MCMC.1		MCMC.2		ACI	BCI.P1	BCI.P2
		MAB	RMSE	MAB	RMSE	MAB	RMSE			
50	α_1	0.235	0.307	0.065	0.070	0.038	0.044	2.610	0.109	0.084
	α_2	0.349	0.432	0.070	0.078	0.056	0.020	2.926	0.141	0.079
	θ	0.974	0.456	0.383	0.062	0.070	0.027	3.167	0.104	0.060
	μ_1	0.160	0.213	0.076	0.078	0.061	0.068	1.758	0.191	0.073
	μ_2	0.179	0.250	0.097	0.103	0.049	0.053	2.360	0.113	0.110
	λ	0.338	0.394	0.075	0.083	0.023	0.029	2.843	0.151	0.048
	ρ_{11}	0.427	0.448	0.229	0.240	0.171	0.180	0.507	0.253	0.199
	ρ_{12}	0.475	0.492	0.186	0.197	0.147	0.150	0.541	0.228	0.163
	ρ_{21}	0.524	0.538	0.167	0.177	0.112	0.115	0.696	0.182	0.159
	ρ_{22}	0.495	0.508	0.164	0.181	0.162	0.166	0.677	0.252	0.147
100	α_1	0.198	0.229	0.049	0.053	0.032	0.039	1.388	0.099	0.081
	α_2	0.338	0.379	0.042	0.055	0.029	0.029	2.324	0.102	0.078
	θ	0.944	0.445	0.348	0.057	0.021	0.019	2.478	0.100	0.044
	μ_1	0.144	0.175	0.062	0.065	0.057	0.063	1.410	0.104	0.066
	μ_2	0.152	0.196	0.044	0.049	0.029	0.034	1.261	0.099	0.065
	λ	0.305	0.373	0.065	0.079	0.014	0.017	1.982	0.122	0.036
	ρ_{11}	0.417	0.427	0.180	0.185	0.146	0.160	0.471	0.192	0.147
	ρ_{12}	0.473	0.483	0.154	0.169	0.111	0.122	0.435	0.218	0.140
	ρ_{21}	0.508	0.520	0.128	0.138	0.046	0.058	0.576	0.163	0.100
	ρ_{22}	0.481	0.493	0.113	0.120	0.054	0.062	0.535	0.131	0.130
150	α_1	0.183	0.244	0.039	0.045	0.022	0.027	1.280	0.076	0.069
	α_2	0.320	0.353	0.040	0.045	0.027	0.031	1.816	0.087	0.044
	θ	0.873	0.428	0.266	0.041	0.007	0.013	2.199	0.082	0.038
	μ_1	0.138	0.173	0.039	0.046	0.013	0.015	1.004	0.100	0.053
	μ_2	0.147	0.186	0.037	0.048	0.017	0.020	1.105	0.061	0.056
	λ	0.301	0.362	0.047	0.057	0.013	0.015	1.860	0.112	0.034
	ρ_{11}	0.407	0.420	0.090	0.092	0.064	0.065	0.441	0.078	0.067
	ρ_{12}	0.467	0.481	0.141	0.143	0.070	0.080	0.427	0.119	0.098
	ρ_{21}	0.507	0.517	0.057	0.065	0.040	0.042	0.543	0.093	0.089
	ρ_{22}	0.479	0.492	0.075	0.081	0.046	0.052	0.525	0.125	0.096
200	α_1	0.182	0.236	0.028	0.028	0.019	0.021	1.016	0.042	0.019
	α_2	0.318	0.352	0.031	0.036	0.016	0.056	1.757	0.084	0.034
	θ	0.382	0.399	0.038	0.026	0.003	0.011	1.866	0.077	0.025
	μ_1	0.128	0.150	0.028	0.033	0.010	0.011	0.857	0.061	0.026
	μ_2	0.132	0.153	0.013	0.018	0.012	0.013	0.920	0.050	0.025
	λ	0.285	0.354	0.028	0.030	0.012	0.014	1.625	0.070	0.032
	ρ_{11}	0.402	0.412	0.061	0.063	0.019	0.026	0.439	0.044	0.038
	ρ_{12}	0.464	0.474	0.036	0.040	0.005	0.006	0.386	0.055	0.021
	ρ_{21}	0.506	0.516	0.049	0.049	0.038	0.041	0.511	0.059	0.010
	ρ_{22}	0.466	0.476	0.071	0.073	0.022	0.025	0.510	0.056	0.051

Table 6. Simulated point and interval estimates of the BMAPE parameters using Set IV when $(\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22}) = (0.5, 0.4, 0.2, 0.3)$.

<i>n</i>	Par.	MLE		MCMC.1		MCMC.2		ACI	BCI.P1	BCI.P2
		MAB	RMSE	MAB	RMSE	MAB	RMSE			
50	α_1	0.250	0.412	0.072	0.077	0.056	0.071	2.545	0.157	0.087
	α_2	0.374	0.553	0.085	0.093	0.078	0.081	3.738	0.143	0.077
	θ	0.996	0.352	0.277	0.059	0.085	0.051	2.793	0.104	0.060
	μ_1	0.207	0.312	0.078	0.083	0.074	0.080	2.369	0.205	0.071
	μ_2	0.219	0.347	0.093	0.099	0.077	0.090	2.621	0.139	0.108
	λ	0.281	0.354	0.060	0.072	0.022	0.027	2.786	0.132	0.068
	ρ_{11}	0.175	0.211	0.160	0.168	0.060	0.072	0.648	0.236	0.113
	ρ_{12}	0.218	0.247	0.099	0.101	0.049	0.061	0.541	0.137	0.120
	ρ_{21}	0.345	0.363	0.169	0.176	0.081	0.093	0.758	0.191	0.130
	ρ_{22}	0.261	0.280	0.234	0.247	0.032	0.043	0.679	0.241	0.107
100	α_1	0.247	0.347	0.060	0.064	0.040	0.045	1.836	0.104	0.085
	α_2	0.326	0.407	0.050	0.056	0.034	0.037	2.958	0.117	0.074
	θ	0.985	0.327	0.260	0.050	0.050	0.021	2.195	0.098	0.055
	μ_1	0.201	0.284	0.065	0.076	0.043	0.050	1.818	0.154	0.063
	μ_2	0.211	0.330	0.062	0.069	0.031	0.037	2.247	0.110	0.066
	λ	0.246	0.314	0.050	0.060	0.018	0.025	2.147	0.125	0.043
	ρ_{11}	0.159	0.190	0.144	0.159	0.031	0.041	0.460	0.163	0.096
	ρ_{12}	0.214	0.230	0.061	0.070	0.020	0.024	0.459	0.130	0.064
	ρ_{21}	0.344	0.361	0.137	0.148	0.038	0.046	0.527	0.180	0.125
	ρ_{22}	0.258	0.276	0.136	0.146	0.031	0.037	0.575	0.208	0.097
150	α_1	0.218	0.329	0.046	0.047	0.025	0.029	1.701	0.101	0.083
	α_2	0.311	0.414	0.036	0.045	0.019	0.023	2.623	0.089	0.070
	θ	0.974	0.304	0.241	0.041	0.013	0.012	1.872	0.094	0.039
	μ_1	0.172	0.227	0.063	0.067	0.013	0.016	1.167	0.098	0.057
	μ_2	0.203	0.295	0.040	0.045	0.016	0.019	1.628	0.094	0.058
	λ	0.236	0.301	0.037	0.041	0.013	0.014	1.600	0.096	0.041
	ρ_{11}	0.157	0.187	0.086	0.089	0.029	0.033	0.430	0.100	0.082
	ρ_{12}	0.209	0.225	0.034	0.042	0.012	0.016	0.423	0.066	0.054
	ρ_{21}	0.337	0.351	0.110	0.117	0.024	0.028	0.487	0.140	0.058
	ρ_{22}	0.254	0.268	0.103	0.116	0.030	0.032	0.490	0.179	0.081
200	α_1	0.159	0.201	0.038	0.045	0.010	0.012	1.233	0.048	0.039
	α_2	0.268	0.314	0.031	0.037	0.014	0.019	2.081	0.084	0.042
	θ	0.922	0.271	0.184	0.029	0.010	0.010	1.570	0.080	0.036
	μ_1	0.148	0.202	0.029	0.034	0.011	0.012	1.089	0.065	0.050
	μ_2	0.156	0.209	0.022	0.025	0.011	0.014	1.269	0.049	0.041
	λ	0.226	0.281	0.029	0.033	0.011	0.012	1.571	0.071	0.037
	ρ_{11}	0.149	0.175	0.048	0.055	0.023	0.029	0.363	0.092	0.052
	ρ_{12}	0.200	0.219	0.029	0.035	0.011	0.015	0.371	0.061	0.044
	ρ_{21}	0.333	0.343	0.084	0.094	0.018	0.020	0.429	0.132	0.034
	ρ_{22}	0.252	0.265	0.074	0.082	0.018	0.024	0.449	0.176	0.042

8. Motor Data Analysis

To demonstrate the practical utility of the proposed methodologies in an engineering phenomenon, one real data set reported by Ref. [47] is analyzed. This data represents the failure times (in days) of a parallel system constituted by two identical motors, namely, (X) and (Y), see Table 7. We first check the suitability of the APE model to the complete motor data sets. For this purpose, for each X and Y data point, the Kolmogorov–Smirnov (KS) statistics and its *p*-value, as well as the MLEs with their standard errors (SEs), are obtained and also provided in Table 7.

Table 7. Failure times of two motors.

Motor	Failure Times	MLE(SE)		KS	
		α	θ	Statistic	p -Value
X	84, 88, 102, 139, 148, 156, 207, 212, 213, 220, 220, 235, 243, 245, 250, 257, 263, 300	39.044 (29.274)	0.0099 (0.0016)	0.2829	0.112
Y	65, 121, 123, 148, 150, 156, 172, 192, 202, 212, 214, 220, 248, 265, 275, 300, 330, 350	47.906 (38.493)	0.0096 (0.0015)	0.2289	0.260

It shows that the APE distribution fits the motor data sets quite well. Moreover, some graphical methods for goodness of fit for the motor data sets are developed. Plots of fitted density, empirical cumulative distribution, probability-probability (PP) and quantile-quantile (QQ) for X and Y data sets are displayed in Figures 3 and 4, respectively. It is clear that the two variables X and Y are fitted for the marginal APE distribution.

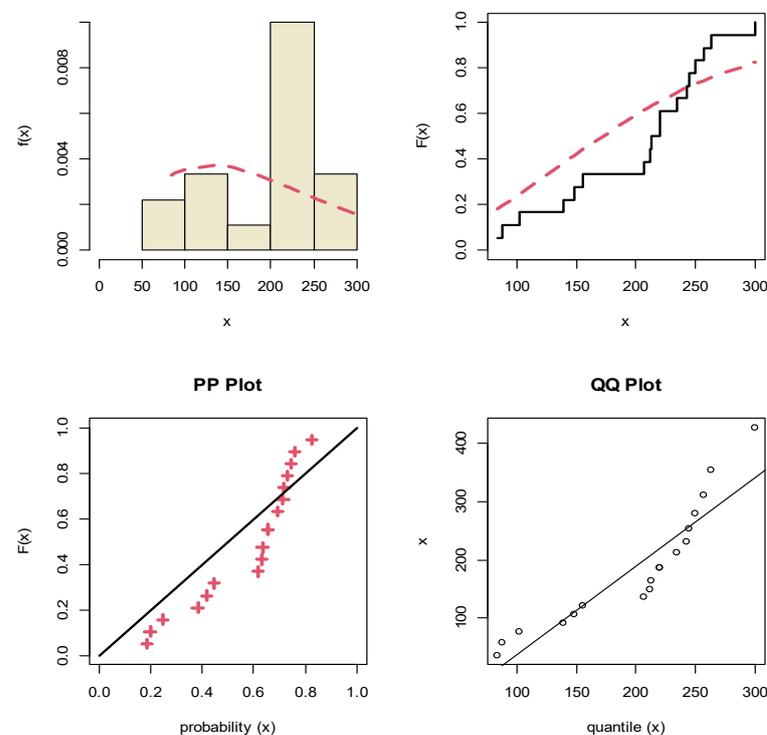


Figure 3. The fitted density, empirical CDF, PP and QQ plots under X data.

Using the complete X and Y data sets, the MLEs and 95% ACIs of $\alpha, \theta, \mu, \lambda$ and ρ are calculated. Furthermore, since no prior information is available about the MBAPE parameters, the Bayesian (point/credible) estimates of $\alpha, \theta, \mu, \lambda$ and ρ are developed using the non-informative priors. Taking the maximum likelihood estimates of the same unknown quantities as initial values, the MCMC sampler is replicated 30,000 times and then the first 5,000 iterations removed as a burn-in. The point (with their SEs) estimates and interval (with their lengths) estimates of the unknown parameters based on X and Y data sets are computed and listed in Tables 8 and 9, respectively. This indicates that the estimates of the unknown parameters obtained by maximum likelihood and Bayesian methods are very similar. Moreover, trace plots (in Figure 5) for MCMC draws of the posterior generated samples of $\alpha, \theta, \mu, \lambda$ and ρ showed that the MCMC simulated variates converge very well. It represents the sample mean and two bounds of 95% BCIs with solid (—) and dashed (---) lines, respectively. Moreover, the MCMC frequency estimates of $\alpha, \theta,$

μ , λ and ρ using the Gaussian kernel are represented in Figure 6. It represents the sample mean with a vertical dash-dotted (:) line. It is evident that the generated posteriors of all unknown parameters are quite symmetrical. To sum up, the numerical findings of the suggested estimation methods based on motor data provide a good demonstration of the proposed model.

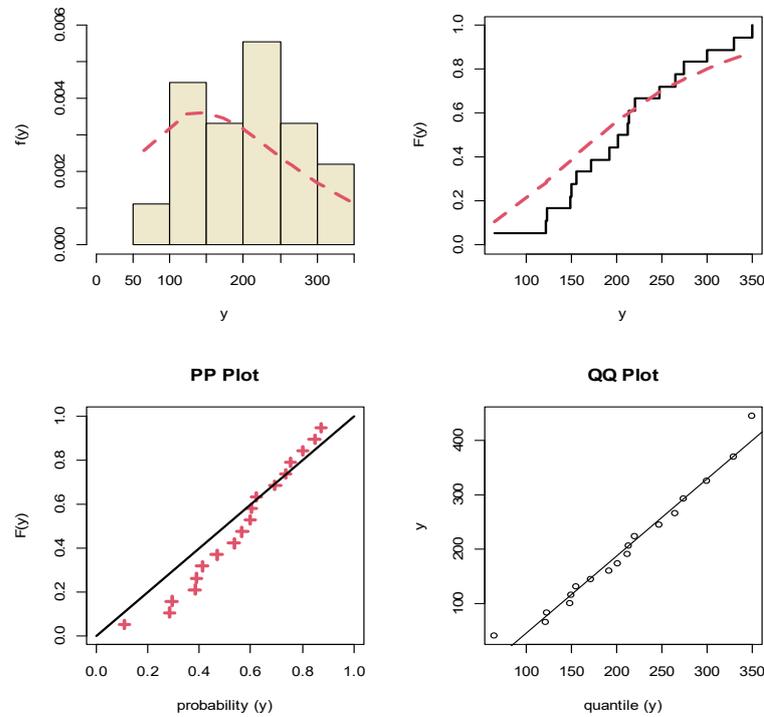


Figure 4. The fitted density, empirical CDF, PP and QQ plots under Y data.

Table 8. The point estimates of α , θ , μ , λ and ρ under motor data.

Method	α	θ	μ	λ	ρ
MLE	3256.43	0.01312	3112.19	0.01262	−0.30736
(SE)	(0.0056)	(0.0014)	(8.3910)	(0.0014)	(0.2181)
Bayes	3256.41	0.01263	3112.17	0.01212	−0.30733
(SE)	(0.0032)	(0.0001)	(0.0031)	(0.0001)	(0.0001)

Table 9. The 95% interval estimates of α , θ , μ , λ and ρ under motor data.

Method	α	θ	μ	λ	ρ
ACI	(3256.4,3256.4)	(0.0103,0.0160)	(3095.8,3128.6)	(0.0099,0.0154)	(−0.7347,0.1200)
(length)	(0.0218)	(0.0057)	(32.894)	(0.0055)	(0.8547)
BCI	(3255.4,3257.4)	(0.0118,0.0136)	(3111.2,3113.2)	(0.0112,0.0131)	(−0.3083,−0.3064)
(length)	(1.9753)	(0.0018)	(1.9131)	(0.0019)	(0.0019)

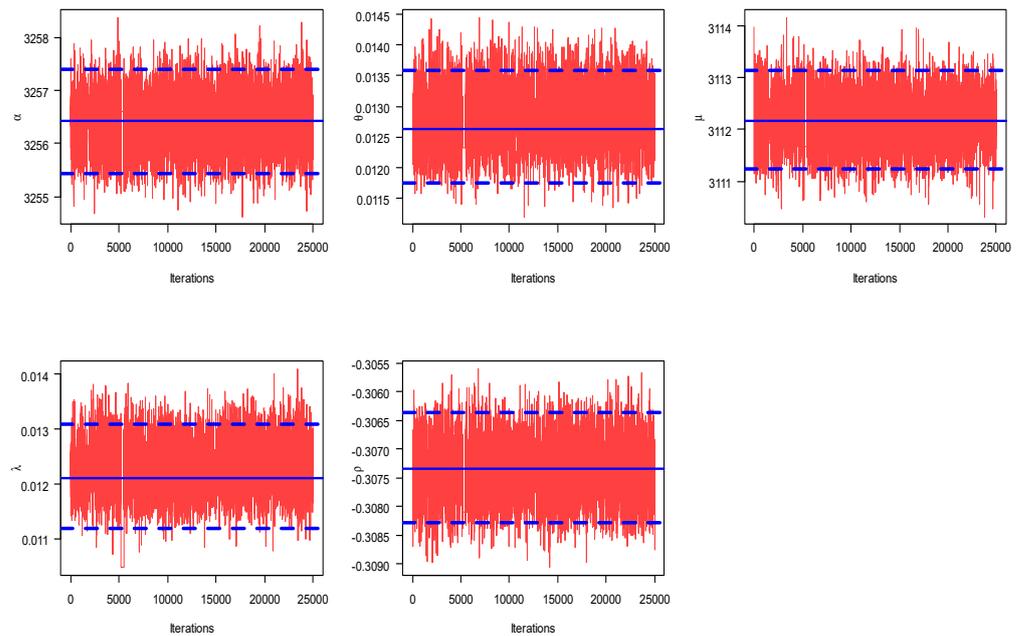


Figure 5. MCMC trace plots of α , θ , μ , λ and ρ under motor data set.

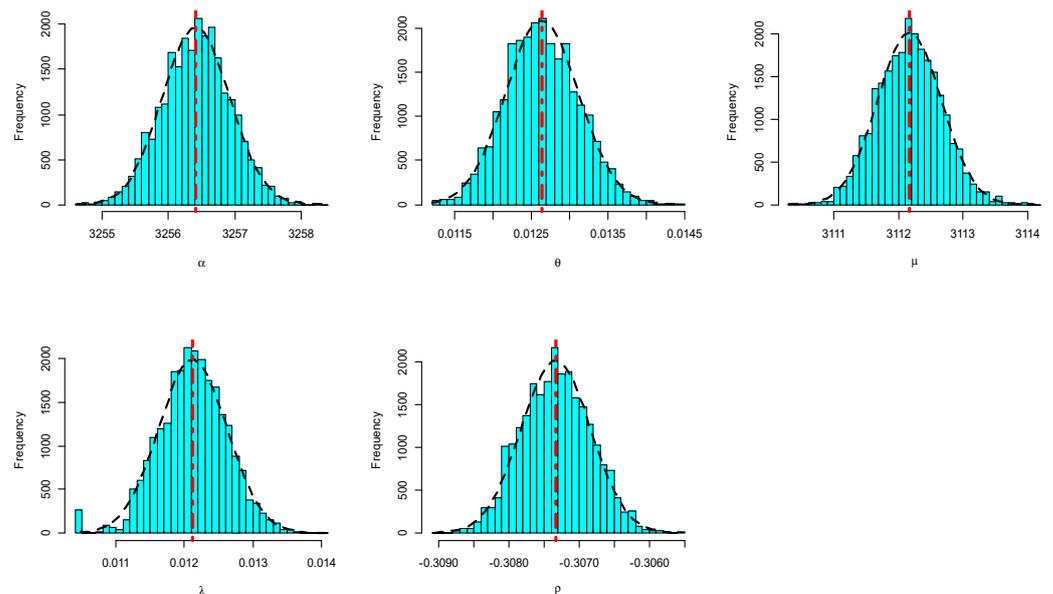


Figure 6. MCMC frequency estimates plots of α , θ , μ , λ and ρ under motor data set.

9. Conclusions

In this paper, we propose and study a new class of bivariate alpha power distributions via finite mixtures involving APE distributions as marginals, and also consider the copula-based construction of bivariate alpha power distributions. The developed class of distribution is made up of two different types of mixture, namely, type I, which starts with two independent APE distributions, and type II, which starts with a bivariate Gaussian copula. The model parameters are estimated using both frequentist (for example, method of moments and the method of maximum likelihood) and under the Bayesian (using independent gamma priors) paradigm. This distribution can be employed in practice for non-negative and positively correlated random variables because the joint distribution function and joint density function are in closed forms; in addition, it can be applied to model censored data as well. Under the Bayesian paradigm, we adopted the EM approach, which works quite well and may be efficiently employed to compute the MLEs, because

the MLEs of the unknown parameters cannot be computed in closed form. To compare the behavior/efficiency of the proposed methods and to see the applicability of the different estimators, a small simulation study is considered, and one real data set is investigated to examine the applicability of the proposed model. From the simulation study, it appears that the point and interval estimates of all unknown parameters using the MCMC algorithm are performing better. Therefore, the Bayes M-H method used to estimate the unknown MBAPE parameters is recommended. However, whether this is a general trend or not is subject to a separate study with a wide range of possible values of the model parameters.

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Appendix A

Let us have a look at the BMAPE (type-I) distribution’s shape. From the joint density in (5), it is clear that there may be more than one critical point for the BMAPE (type-I) distribution. A numerical examination must be adopted for varying model parameter choices to obtain an idea of the critical points for this distribution

$$\begin{aligned} \frac{\partial h_1(x,y)}{\partial x} &= a \left(\left(\frac{1}{(\alpha_1-1)(\mu_1-1)} \right) \left[\theta^2 \log(\alpha_1) e^{-2\theta x} \alpha_1^{-e^{-\theta x}} (1 - e^{-\theta x}) + \theta e^{-\theta x} \alpha_1^{1-e^{-\theta x}} \right] \left[\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}} \right] \right) \\ &+ b \left(\left(\frac{1}{(\alpha_1-1)(\mu_2-1)} \right) \left[\theta^2 \log(\alpha_1) e^{-2\theta x} \alpha_1^{-e^{-\theta x}} (1 - e^{-\theta x}) + \theta e^{-\theta x} \alpha_1^{1-e^{-\theta x}} \right] \left[\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}} \right] \right) \\ &+ c \left(\left(\frac{1}{(\alpha_2-1)(\mu_1-1)} \right) \left[\theta^2 \log(\alpha_2) e^{-2\theta x} \alpha_2^{-e^{-\theta x}} (1 - e^{-\theta x}) + \theta e^{-\theta x} \alpha_2^{1-e^{-\theta x}} \right] \left[\lambda \log(\mu_1) e^{-\lambda y} \mu_1^{1-e^{-\lambda y}} \right] \right) \\ &+ d \left(\left(\frac{1}{(\alpha_2-1)(\mu_2-1)} \right) \left[\theta^2 \log(\alpha_2) e^{-2\theta x} \alpha_2^{-e^{-\theta x}} (1 - e^{-\theta x}) + \theta e^{-\theta x} \alpha_2^{1-e^{-\theta x}} \right] \left[\lambda \log(\mu_2) e^{-\lambda y} \mu_2^{1-e^{-\lambda y}} \right] \right) = 0, \end{aligned} \tag{A1}$$

Also,

$$\begin{aligned} \frac{\partial h_1(x,y)}{\partial y} &= a \left(\left(\frac{1}{(\alpha_1-1)(\mu_1-1)} \right) \left[\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta x}} + \theta e^{-\theta x} \alpha_1^{1-e^{-\theta x}} \right] \left[\lambda^2 \log(\mu_1) e^{-2\lambda y} (1 - e^{-\lambda y}) \mu_1^{-e^{-\lambda y}} + \lambda e^{-\lambda y} \mu_1^{1-e^{-\lambda y}} \right] \right) \\ &+ b \left(\left(\frac{1}{(\alpha_1-1)(\mu_2-1)} \right) \left[\theta \log(\alpha_1) e^{-\theta x} \alpha_1^{1-e^{-\theta x}} + \theta e^{-\theta x} \alpha_1^{1-e^{-\theta x}} \right] \left[\lambda^2 \log(\mu_2) e^{-2\lambda y} (1 - e^{-\lambda y}) \mu_2^{-e^{-\lambda y}} + \lambda e^{-\lambda y} \mu_2^{1-e^{-\lambda y}} \right] \right) \\ &+ c \left(\left(\frac{1}{(\alpha_2-1)(\mu_1-1)} \right) \left[\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}} + \theta e^{-\theta x} \alpha_2^{1-e^{-\theta x}} \right] \left[\lambda^2 \log(\mu_1) e^{-2\lambda y} (1 - e^{-\lambda y}) \mu_1^{-e^{-\lambda y}} + \lambda e^{-\lambda y} \mu_1^{1-e^{-\lambda y}} \right] \right) \\ &+ d \left(\left(\frac{1}{(\alpha_2-1)(\mu_2-1)} \right) \left[\theta \log(\alpha_2) e^{-\theta x} \alpha_2^{1-e^{-\theta x}} + \theta e^{-\theta x} \alpha_2^{1-e^{-\theta x}} \right] \left[\lambda^2 \log(\mu_2) e^{-2\lambda y} (1 - e^{-\lambda y}) \mu_2^{-e^{-\lambda y}} + \lambda e^{-\lambda y} \mu_2^{1-e^{-\lambda y}} \right] \right) = 0. \end{aligned} \tag{A2}$$

From (A1) and (A2), it is evident that there will be more than one point for the BMAPE (type-I) distribution.

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