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Applications of Beta Negative Binomial Distribution and Laguerre Polynomials on Ozaki Bi-Close-to-Convex Functions

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Abstract: In the present paper, due to beta negative binomial distribution series and Laguerre polynomials, we investigate a new family $\mathcal{F}_{\Sigma}(\delta,\eta,\lambda,\theta;h)$ of normalized holomorphic and bi-univalent functions associated with Ozaki close-to-convex functions. We provide estimates on the initial Taylor–Maclaurin coefficients and discuss Fekete–Szegő type inequality for functions in this family.

Keywords: bi-univalent function; Laguerre polynomial; coefficient bound; Fekete–Szegő problem; beta negative binomial distribution; subordination

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1. Introduction

Consider the set $\mathcal{A}$ of functions f which are holomorphic in the unit disk $\mathbb{D} = \{|z| < 1\}$ in the complex plane $\mathbb{C}$, of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ be the subset of $\mathcal{A}$ which contains univalent functions in $\mathbb{D}$ having the form (1). As we can see in [1], due to the Koebe one-quarter theorem, every function $f \in \mathcal{S}$ has an inverse f^{-1} such that $f^{-1}(f(z)) = z$, $(z \in \mathbb{D})$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$. With f on the form (1), we have

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots, \quad |w| < r_0(f). \tag{2}$$

We called a function $f \in \mathcal{A}$ as bi-univalent in $\mathbb{D}$, if both f and f^{-1} are univalent in $\mathbb{D}$. The set of bi-univalent functions in $\mathbb{D}$ is denoted by Σ.

In recent years, Srivastava et al. [2] reconsidered the study of holomorphic and bi-univalent functions. In this sense, we pursued a kind of surveys represented by those of Ali et al. [3], Bulut et al. [4], Srivastava et al. [5] and others (see, for example, [6–18]).

The polynomial solution $\phi(\tau)$ of the differential equation (see [19])

$$\tau\phi'' + (1 + \gamma - \tau)\phi' + n\phi = 0,$$

consists on the generalized Laguerre polynomial $L_n^{\gamma}(\tau)$, where $\gamma > -1$ and n is non-negative integers.

We defined by

$$H_{\gamma}(\tau, z) = \sum_{n=0}^{\infty} L_n^{\gamma}(\tau) z^n = \frac{e^{-\frac{\tau z}{1-z}}}{(1-z)^{\gamma+1}}, \tag{3}$$

the generating function of generalized Laguerre polynomial $L_n^{\gamma}(\tau)$, where $\tau \in \mathbb{R}$ and $z \in \mathbb{D}$. Similarly, the generalized Laguerre polynomials is given by the following recurrence relations:

$$L_{n+1}^{\gamma}(\tau) = \frac{2n+1+\gamma-\tau}{n+1} L_n^{\gamma}(\tau) - \frac{n+\gamma}{n+1} L_{n-1}^{\gamma}(\tau) \quad (n \geq 1),$$

with the initial conditions

$$L_0^{\gamma}(\tau) = 1, \quad L_1^{\gamma}(\tau) = 1+\gamma-\tau \quad \text{and} \quad L_2^{\gamma}(\tau) = \frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}. \tag{4}$$

Obviously, if $\gamma = 0$ the generalized Laguerre polynomial implies the simple Laguerre polynomial, i.e., $L_n^0(\tau) = L_n(\tau)$.

Consider two functions f and g holomorphic in $\mathbb{D}$. We say that the function f is subordinate to g, if there exists a function w, holomorphic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, $(z \in \mathbb{D})$ such that $f(z) = g(w(z))$. We denote this relation by $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{D})$. In addition, if the function g is univalent in $\mathbb{D}$, then we get the following equivalence (see [20]), $f(z) \prec g(z) \iff f(0) = g(0)$ *and* $f(\mathbb{D}) \subset g(\mathbb{D})$.

From a theoretical standpoint, the Poisson, Pascal, logarithmic, binomial and Borel distributions have all been examined in some depth in geometric function theory (see for example [21–26]).

For a discrete random variable x, we say that it has a beta negative binomial distribution if it takes the values $0, 1, 2, 3, \cdots$ with the probabilities

$$\frac{B(\eta+\theta, \lambda)}{B(\eta, \lambda)}, \quad \theta \frac{B(\eta+\theta, \lambda+1)}{B(\eta, \lambda)}, \quad \frac{1}{2}\theta(\theta+1)\frac{B(\eta+\theta, \lambda+2)}{B(\eta, \lambda)}, \cdots,$$

respectively, where η, θ and λ are the parameters.

$$\begin{aligned} \mathrm{Prob}(x = \tau) &= \binom{\theta+\tau-1}{\tau} \frac{B(\eta+\theta, \lambda+\tau)}{B(\eta, \lambda)} \\ &= \frac{\Gamma(\theta+\tau)}{\tau!\Gamma(\theta)} \frac{\Gamma(\eta+\theta)\Gamma(\lambda+\tau)\Gamma(\eta+\lambda)}{\Gamma(\eta+\theta+\lambda+\tau)\Gamma(\eta)\Gamma(\lambda)} \\ &= \frac{(\eta)_\theta(\theta)_\tau(\lambda)_\tau}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_\tau \tau!}, \end{aligned}$$

where $(\alpha)_n$ is the Pochhammer symbol defined by

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1 & (n = 0), \\ \alpha(\alpha+1)\ldots(\alpha+n-1) & (n \in \mathbb{N}). \end{cases}$$

Wanas and Al-Ziadi [27] developed the following power series whose coefficients are beta negative binomial distribution probabilities:

$$\mathfrak{X}_{\eta,\lambda}^{\theta}(z) = z + \sum_{n=2}^{\infty} \frac{(\eta)_\theta(\theta)_{n-1}(\lambda)_{n-1}}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_{n-1}(n-1)!} z^n \qquad (z \in \mathbb{D};\ \eta, \lambda, \theta > 0).$$

By the well-known ratio test, we deduce that the radius of convergence of the above power series is infinity.

We recall the linear operator $\mathfrak{B}_{\eta,\lambda}^{\theta} : \mathcal{A} \longrightarrow \mathcal{A}$, as can be found in (see [27])

$$\mathfrak{B}_{\eta,\lambda}^{\theta} f(z) = \mathfrak{X}_{\eta,\lambda}^{\theta}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(\eta)_\theta(\theta)_{n-1}(\lambda)_{n-1}}{(\eta+\lambda)_\theta(\theta+\eta+\lambda)_{n-1}(n-1)!} a_n z^n \qquad z \in \mathbb{D},$$

where $(*)$ represents the Hadamard product (or convolution) of two series.

2. Main Results

We open the main section by introducing the family $\mathcal{F}_{\Sigma}(\delta,\eta,\lambda,\theta;h)$ as follows:

Definition 1. *Suppose that $\frac{1}{2}\leq\delta\leq 1$, $\eta,\lambda,\theta>0$ and h is analytic in $\mathbb{D}$, $h(0)=1$. We say that the function $f\in\Sigma$ is in the family $\mathcal{F}_{\Sigma}(\delta,\eta,\lambda,\theta;h)$ if the following subordinations hold:*

$$\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(1+\frac{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)'}\right)\prec h(z)$$

and

$$\frac{2\delta-1}{2\delta+1}+\frac{2}{2\delta+1}\left(1+\frac{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f^{-1}(w)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f^{-1}(w)\right)'}\right)\prec h(w),$$

where f^{-1} is given by (2).

For $\delta=\frac{1}{2}$ in Definition 1, the family $\mathcal{F}_{\Sigma}(\delta,\eta,\lambda,\theta;h)$ reduces to the family $\mathcal{S}_{\Sigma}(\eta,\lambda,\theta;h)$ of bi-starlike functions such that the following subordinations hold:

$$1+\frac{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f(z)\right)'}\prec h(z)$$

and

$$1+\frac{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f^{-1}(w)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda}f^{-1}(w)\right)'}\prec h(w).$$

Theorem 1. *Suppose that $\frac{1}{2}\leq\delta\leq 1$ and $\eta,\lambda,\theta>0$. If $f\in\Sigma$ of the form (1) is in the family $\mathcal{F}_{\Sigma}(\delta,\eta,\lambda,\theta;h)$, with $h(z)=1+e_1z+e_2z^2+\cdots$, then*

$$|a_2|\leq\frac{(2\delta+1)\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)|e_1|}{4\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}=\frac{|e_1|}{\Upsilon}\tag{5}$$

and

$$|a_3|\leq\min\left\{\max\left\{\left|\frac{e_1}{\Phi}\right|,\left|\frac{e_2}{\Phi}+\frac{\Psi e_1^2}{\Upsilon^2\Phi}\right|\right\},\max\left\{\left|\frac{e_1}{\Phi}\right|,\left|\frac{e_2}{\Phi}-\frac{(2\Phi-\Psi)e_1^2}{\Upsilon^2\Phi}\right|\right\}\right\},\tag{6}$$

where

$$\begin{aligned}\Upsilon&=\frac{4\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{(2\delta+1)\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)},\\ \Phi&=\frac{6\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{(2\delta+1)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)},\\ \Psi&=\frac{8\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{(2\delta+1)\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)}.\end{aligned}\tag{7}$$

Proof. Assume that $f\in\mathcal{F}_{\Sigma}(\delta,\eta,\lambda,\theta;h)$. Then, there exist two holomorphic functions $\phi,\psi:\mathbb{D}\longrightarrow\mathbb{D}$ given by

$$\phi(z)=r_1z+r_2z^2+r_3z^3+\cdots\quad(z\in\mathbb{D})\tag{8}$$

and

$$\psi(w)=s_1w+s_2w^2+s_3w^3+\cdots\quad(w\in\mathbb{D}),\tag{9}$$

with $\phi(0) = \psi(0) = 0$, $|\phi(z)| < 1$, $|\psi(w)| < 1$, $z, w \in \mathbb{D}$ such that

$$1 + \frac{2}{2\delta+1} \frac{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f(z)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f(z)\right)'} = 1 + e_1\phi(z) + e_2\phi^2(z) + \cdots \tag{10}$$

and

$$1 + \frac{2}{2\delta+1} \frac{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f^{-1}(w)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f^{-1}(w)\right)'} = 1 + e_1\psi(w) + e_2\psi^2(w) + \cdots . \tag{11}$$

Using (8)–(11), one obtains

$$1 + \frac{2}{2\delta+1} \frac{z\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f(z)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f(z)\right)'} = 1 + e_1 r_1 z + \left[e_1 r_2 + e_2 r_1^2\right] z^2 + \cdots \tag{12}$$

and

$$1 + \frac{2}{2\delta+1} \frac{w\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f^{-1}(w)\right)''}{\left(\mathfrak{B}^{\theta}_{\eta,\lambda} f^{-1}(w)\right)'} = 1 + e_1 s_1 w + \left[e_1 s_2 + e_2 s_1^2\right] w^2 + \cdots . \tag{13}$$

Since $|\phi(z)| < 1$ and $|\psi(w)| < 1$, $z, w \in \mathbb{D}$, we deduce

$$|r_j| \leq 1 \quad \text{and} \quad |s_j| \leq 1 \ (j \in \mathbb{N}). \tag{14}$$

In view of (12) and (13), after simplifying, we obtain

$$\frac{4\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{(2\delta+1)\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)} a_2 = e_1 r_1, \tag{15}$$

$$\begin{aligned} &\frac{6\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{(2\delta+1)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)} a_3 - \frac{8\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{(2\delta+1)\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} a_2^2 \\ &= e_1 r_2 + e_2 r_1^2, \end{aligned} \tag{16}$$

$$-\frac{4\theta\Gamma(\eta+\theta)\Gamma(\lambda+1)\Gamma(\eta+\lambda)}{(2\delta+1)\Gamma(\eta+\theta+\lambda+1)\Gamma(\eta)\Gamma(\lambda)} a_2 = e_1 s_1 \tag{17}$$

and

$$\begin{aligned} &\frac{6\theta(\theta+1)\Gamma(\eta+\theta)\Gamma(\lambda+2)\Gamma(\eta+\lambda)}{(2\delta+1)\Gamma(\eta+\theta+\lambda+2)\Gamma(\eta)\Gamma(\lambda)} \left(2a_2^2 - a_3\right) - \frac{8\theta^2\Gamma^2(\eta+\theta)\Gamma^2(\lambda+1)\Gamma^2(\eta+\lambda)}{(2\delta+1)\Gamma^2(\eta+\theta+\lambda+1)\Gamma^2(\eta)\Gamma^2(\lambda)} a_2^2 \\ &= e_1 s_2 + e_2 s_1^2. \end{aligned} \tag{18}$$

From (15) and (17), we derive inequality (5). Applying (7), then (15) and (16) become

$$\Upsilon a_2 = e_1 r_1, \quad \Phi a_3 - \Psi a_2^2 = e_1 r_2 + e_2 r_1^2 \tag{19}$$

which yields

$$\frac{\Phi}{e_1} a_3 = r_2 + \left(\frac{e_2}{e_1} + \frac{\Psi e_1}{\Upsilon^2}\right) r_1^2, \tag{20}$$

and on using the known sharp result ([28], p. 10):

$$|r_2 - \mu r_1^2| \leq \max\{1, |\mu|\} \tag{21}$$

for all $\mu \in \mathbb{C}$, we obtain

$$\left|\frac{\Phi}{e_1}\right||a_3| \leq \max\left\{1, \left|\frac{e_2}{e_1} + \frac{\Psi e_1}{\Upsilon^2}\right|\right\}. \tag{22}$$

Similarly, (17) and (18) become

$$-\Upsilon a_2 = e_1 s_1, \quad \Phi(2a_2^2 - a_3) - \Psi a_2^2 = e_1 s_2 + e_2 s_1^2. \tag{23}$$

These equalities provide

$$-\frac{\Phi}{e_1} a_3 = s_2 + \left(\frac{e_2}{e_1} - \frac{(2\Phi - \Psi)e_1}{\Upsilon^2}\right) s_1^2. \tag{24}$$

Applying (21), we deduce

$$\left|\frac{\Phi}{e_1}\right||a_3| \leq \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Phi - \Psi)e_1}{\Upsilon^2}\right|\right\}. \tag{25}$$

Inequality (6) follows from (22) and (25). □

Furthermore, we use the generating function (3) of the generalized Laguerre polynomials $L_n^{\gamma}(\tau)$ as $h(z)$. As a consequence, from (4), we obtain $e_1 = 1 + \gamma - \tau$ and $e_2 = \frac{\tau^2}{2} - (\gamma + 2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}$, and then, Theorem 1 is reduced to the following corollary.

Corollary 1. *If $f \in \Sigma$ of the form (1) is in the class $\mathcal{F}_{\Sigma}(\delta, \eta, \lambda, \theta; H_{\gamma}(\tau, z))$, then*

$$|a_2| \leq \frac{(2\delta + 1)\Gamma(\eta + \theta + \lambda + 1)\Gamma(\eta)\Gamma(\lambda)|1 + \gamma - \tau|}{4\theta\Gamma(\eta + \theta)\Gamma(\lambda + 1)\Gamma(\eta + \lambda)} = \frac{|1 + \gamma - \tau|}{\Upsilon}$$

and

$$\begin{aligned} |a_3| \quad \leq \quad & \min\left\{\max\left\{\left|\frac{1 + \gamma - \tau}{\Phi}\right|, \left|\frac{\frac{\tau^2}{2} - (\gamma + 2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{\Phi} + \frac{\Psi(1 + \gamma - \tau)^2}{\Upsilon^2\Phi}\right|\right\},\right. \\ & \left.\max\left\{\left|\frac{1 + \gamma - \tau}{\Phi}\right|, \left|\frac{\frac{\tau^2}{2} - (\gamma + 2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{\Phi} - \frac{(2\Phi - \Psi)(1 + \gamma - \tau)^2}{\Upsilon^2\Phi}\right|\right\}\right\}, \end{aligned}$$

for all $\delta, \eta, \lambda, \theta$ such that $\frac{1}{2} \leq \delta \leq 1$ and $\eta, \lambda, \theta > 0$, where Υ, Φ, Ψ are defined by (7) and $H_{\gamma}(\tau, z)$ is given by (3).

In the following theorem, we develop "the Fekete–Szegő Problem" for the family $\mathcal{F}_{\Sigma}(\delta, \eta, \lambda, \theta; h)$.

Theorem 2. *If $f \in \Sigma$ of the form (1) is in the class $\mathcal{F}_{\Sigma}(\delta, \eta, \lambda, \theta; h)$, then*

$$\left|a_3 - \eta a_2^2\right| \leq \frac{|e_1|}{\Phi} \min\left\{\max\left\{1, \left|\frac{e_2}{e_1} + \frac{(\Psi + \eta\Phi)e_1}{\Upsilon^2}\right|\right\}, \max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Phi - \Psi - \eta\Phi)e_1}{\Upsilon^2}\right|\right\}\right\}, \tag{26}$$

for all $\delta, \eta, \lambda, \theta$ such that $\frac{1}{2} \leq \delta \leq 1$ and $\eta, \lambda, \theta > 0$, where Υ, Φ, Ψ are defined by (7).

Proof. According to the notations from the proof of Theorem 1 and from (19) and (20), we obtain

$$a_3 - \eta a_2^2 = \frac{e_1}{\Phi}\left(r_2 + \left(\frac{e_2}{e_1} + \frac{(\Psi + \eta\Phi)e_1}{\Upsilon^2}\right) r_1^2\right). \tag{27}$$

Applying the well-known sharp result $|r_2 - \mu r_1^2| \leq \max\{1, |\mu|\}$, one obtains

$$|a_3 - \eta a_2^2| \leq \frac{|e_1|}{\Phi} \max\left\{1, \left|\frac{e_2}{e_1} + \frac{(\Psi + \eta\Phi)e_1}{\Upsilon^2}\right|\right\}. \tag{28}$$

Similarly, from (23) and (24), we derive

$$a_3 - \eta a_2^2 = -\frac{e_1}{\Phi}\left(s_2 + \left(\frac{e_2}{e_1} - \frac{(2\Phi - \Psi - \eta\Phi)e_1}{\Upsilon^2}\right)s_1^2\right) \tag{29}$$

and in view of $|s_2 - \mu s_1^2| \leq \max\{1, |\mu|\}$, we get

$$|a_3 - \eta a_2^2| \leq \frac{|e_1|}{\Phi}\max\left\{1, \left|\frac{e_2}{e_1} - \frac{(2\Phi - \Psi - \eta\Phi)e_1}{\Upsilon^2}\right|\right\}. \tag{30}$$

Inequality (26) follows from (28) and (30). □

Corollary 2. *If $f \in \Sigma$ of the form (1) is in the class $\mathcal{F}_\Sigma(\delta, \eta, \lambda, \theta; H_\gamma(\tau, z))$, then*

$$\begin{aligned} &\left|a_3 - \eta a_2^2\right| \\ \leq\ &\frac{|1+\gamma-\tau|}{\Phi}\min\left\{\max\left\{1, \left|\frac{\frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{1+\gamma-\tau} + \frac{(\Psi+\eta\Phi)(1+\gamma-\tau)}{\Upsilon^2}\right|\right\},\right. \\ &\left.\max\left\{1, \left|\frac{\frac{\tau^2}{2} - (\gamma+2)\tau + \frac{(\gamma+1)(\gamma+2)}{2}}{1+\gamma-\tau} - \frac{(2\Phi - \Psi - \eta\Phi)(1+\gamma-\tau)}{\Upsilon^2}\right|\right\}\right\}, \end{aligned}$$

for all $\delta, \eta, \lambda, \theta$ such that $\frac{1}{2} \leq \delta \leq 1$ and $\eta, \lambda, \theta > 0$, where Υ, Φ, Ψ are given by (7) and $H_\gamma(\tau, z)$ is given by (3).

3. Conclusions

In the present survey, we considered a certain class of bi-univalent functions, denoted by $\mathcal{F}_\Sigma(\delta, \eta, \lambda, \theta; h)$, representable in the form of a Hadamard product of two power series. The coefficients of the first one, developed by Wanas and Al-Ziadi in [27], are beta negative binomial distribution probabilities. Furthermore, the Fekete–Szegő Problem was developed, by making use of the newly introduced family. Consequently, inequalities of Fekete–Szegő type were obtained in the special case of generalized Laguerre polynomials.

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