axioms

MDPI

Editorial

A Brief Overview and Survey of the Scientific Work by Feng Qi

Ravi Prakash Agarwal [1,*], Erdal Karapinar [2,3,*], Marko Kostić [4], Jian Cao [5] and Wei-Shih Du [6]

[1] Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363-8202, USA
[2] Department of Mathematics, Çankaya University, Etimesgut, Ankara 06790, Turkey
[3] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[4] Faculty of Technical Sciences, University of Novi Sad, 21125 Novi Sad, Serbia
[5] School of Mathematics, Hangzhou Normal University, Hangzhou 311121, China
[6] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
* Correspondence: ravi.agarwal@tamuk.edu (R.P.A.); erdalkarapinar@yahoo.com (E.K.)

Abstract: In the paper, the authors present a brief overview and survey of the scientific work by Chinese mathematician Feng Qi and his coauthors.

Keywords: overview; survey; inequality; series expansion; partial Bell polynomial; convex function; special function; mathematical mean; Bernoulli number; matrix; completely monotonic degree; logarithmically completely monotonic function; gamma function; polygamma function; Bell number; Wallis ratio; additivity; complete elliptic integral; Pólya inequality; statistics

MSC: 00-02; 01-02; 05-02; 11-02; 12-02; 15-02; 26-02; 33-02; 40-02; 41-02; 44-02; 53-02

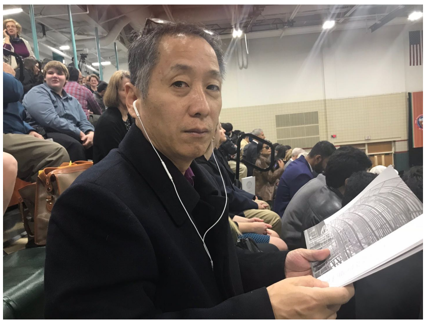
check for updates

Citation: Agarwal, R.P.; Karapinar, E.; Kostić, M.; Cao, J.; Du, W.-S. A Brief Overview and Survey of the Scientific Work by Feng Qi. *Axioms* **2022**, *11*, 385. https://doi.org/10.3390/axioms11080385

Received: 11 July 2022
Accepted: 30 July 2022
Published: 5 August 2022

1. Introduction

Professor Feng Qi, whose ORCID profile is at https://orcid.org/0000-0001-6239-2968, received his PhD degree from the University of Science and Technology of China in 1999 and is currently a full Professor at Tiangong University and Henan Polytechnic University, China. On 17 May 2022, he moved to Dallas as an independent researcher in mathematics.



December 2017 in Dallas

Among other institutions and universities, he has visited Victoria University in Australia and the University of Hong Kong twice, the University of Copenhagen, Antalya IC Hotel for attending a conference, several universities in South Korea, Sun Yat-sen University, Kaohsiung Normal University, and so on. He is, or was, the editor-in-chief, an associate editor, or a member of the editorial board of over 40 reputable international journals. In 1993, Qi published his first academic paper in China. In 1996, Qi published his first academic paper abroad. To date, he has published over 670 papers in 220 journals, collections, or proceedings. Currently, his academic interests and research fields mainly include the theory of special functions, classical analysis, mathematical inequalities and applications, mathematical means and applications, analytic combinatorics, analytic number theory, the convex theory of functions, and so on.

Now, let us start out by briefly presenting an overview and survey of some research results obtained by Dr. Professor Feng Qi and his coauthors.

2. Concrete Contributions

2.1. Bell Numbers and Inequalities

From 2013 on, Dr. Qi began to consider some problems related to combinatorial number theory and applied the logarithmically complete monotonicity to combinatorial number theory.

December 2017 in Dallas

In the research article [1], Qi presented derivatives of the generating functions for the Bell numbers by induction and by the well-known Faà di Bruno formula. Using this approach, he recovered an explicit formula in terms of the Stirling numbers of the second kind, found the logarithmically absolute and complete monotonicity of the generating functions, and deduced some inequalities for the Bell numbers. The logarithmic convexity of the sequence of the Bell numbers is shown after that.

As is well known, the Bell number B_n is defined as the number of all equivalence relations on the set $\mathbb{N}_n = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$. These numbers have been known already in medieval Japan, but they are named after Eric Temple Bell, who systematically analyzed them in the 1930s.

Let us recall that

$$B_1 = 1, \quad B_2 = 2, \quad B_3 = 5, \quad B_4 = 15, \quad B_5 = 52.$$

Since

$$\mathrm{e}^{\mathrm{e}^x} = \mathrm{e} \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \quad \text{and} \quad \mathrm{e}^{\mathrm{e}^{-x}} = \mathrm{e} \sum_{k=0}^{\infty} (-1)^k B_k \frac{x^k}{k!},$$

the functions $\mathrm{e}^{\mathrm{e}^{\pm x}}$ are called the generating functions for the Bell numbers B_k. The Bell numbers are also called exponential numbers.

It is known that, for every positive integer $n \in \mathbb{N}$, we have

$$\frac{\mathrm{d}^n \mathrm{e}^{\mathrm{e}^x}}{\mathrm{d} x^n} = \mathrm{e}^{\mathrm{e}^x} \sum_{k=1}^{n} S(n,k) \mathrm{e}^{kx} \quad \text{and} \quad \frac{\mathrm{d}^n \mathrm{e}^{\mathrm{e}^{-x}}}{\mathrm{d} x^n} = (-1)^n \mathrm{e}^{\mathrm{e}^{-x}} \sum_{k=1}^{n} S(n,k) \mathrm{e}^{-kx},$$

where $S(n,k)$ is the Stirling number of the second kind, which can be computed by

$$S(n,k) = \frac{1}{k!} \sum_{\ell=1}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n.$$

The Stirling numbers of the second kind satisfy the recurrence relation

$$S(n+1, k+1) = S(n,k) + (k+1) S(n, k+1), \quad 1 \le k \le n-1.$$

From the above, we have

$$B_n = \frac{1}{\mathrm{e}} \lim_{x \to 0} \frac{\mathrm{d}^n \mathrm{e}^{\mathrm{e}^x}}{\mathrm{d} x^n}$$

and therefore
$$B_n = \sum_{k=1}^{n} S(n,k).$$
Several inequalities for the Bell numbers B_n have been proven, including the following ones:

1. Let $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ be two non-increasing tuples of non-negative integers such that $\sum_{i=1}^{k} a_i \ge \sum_{i=1}^{k} b_i$ for $1 \le k \le n-1$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. Then
$$B_{a_1} B_{a_2} \cdots B_{a_n} \ge B_{b_1} B_{b_2} \cdots B_{b_n}.$$
2. If $\ell \ge 0$ and $n \ge k \ge 0$, then we have
$$B_{n+\ell}^{k} B_{\ell}^{n-k} \ge B_{k+\ell}^{n}.$$
3. If $\ell \ge 0$, $n \ge k \ge m$, $2k \ge n$, and $2m \ge n$, then we have
$$B_{k+\ell} B_{n-k+\ell} \ge B_{m+\ell} B_{n-m+\ell}.$$
4. If $k \ge 0$ and $n \in \mathbb{N}$, then we have
$$\left(\prod_{\ell=0}^{n} B_{k+2\ell} \right)^{1/(n+1)} \ge \left(\prod_{\ell=0}^{n-1} B_{k+2\ell+1} \right)^{1/n}.$$

These results have been extended and generalized in [2–5] by Qi and his coauthors.

2.2. Partial Bell Polynomials

Partial Bell polynomials are also called the Bell polynomials of the second kind. They are usually denoted by $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$. They are closely connected with the famous Faà di Bruno formula in combinatorics. In recent years, Qi and his coauthors creatively considered some special values of $B_{n,k}$ for special sequences $x_1, x_2, \dots, x_{n-k+1}$ and successfully applied to some mathematical problems.

The survey article [6] is worth to be mentioned. We now just introduce the newest results obtained by Qi and his coauthors.

1. In the papers [7,8], the following conclusions were proved.
   (a) For $m \in \mathbb{N}$ and $|t| < 1$, the function $\left(\frac{\arcsin t}{t}\right)^m$, whose value at $t = 0$ is defined to be 1, has Maclaurin's series expansion
$$\left(\frac{\arcsin t}{t} \right)^m = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{Q(m, 2k; 2)}{\binom{m+2k}{m}} \frac{(2t)^{2k}}{(2k)!}, \tag{1}$$
   where
$$Q(m,k;\alpha) = \sum_{\ell=0}^{k} \binom{m+\ell-1}{m-1} s(m+k-1, m+\ell-1) \left(\frac{m+k-\alpha}{2} \right)^{\ell} \tag{2}$$
   for $m, k \in \mathbb{N}$, the constant $\alpha \in \mathbb{R}$ such that $m + k \ne \alpha$, and the Stirling numbers of the first kind $s(m+k-1, m+\ell-1)$ are analytically generalized by
$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}, \quad |x| < 1.$$

(b) For $k,n \geq 0$ and $x_m \in \mathbb{C}$ with $m \in \mathbb{N}$, we have

$$B_{2n+1,k}\left(0,x_2,0,x_4,\dots,\frac{1+(-1)^k}{2}x_{2n-k+2}\right)=0. \tag{3}$$

For $k,n \in \mathbb{N}$ such that $2n \geq k \in \mathbb{N}$, we have

$$\begin{aligned}B_{2n,k}&\left(0,\frac{1}{3},0,\frac{9}{5},0,\frac{225}{7},\dots,\frac{1+(-1)^{k+1}}{2}\frac{[(2n-k)!!]^2}{2n-k+2}\right)\\&=(-1)^{n+k}\frac{(4n)!!}{(2n+k)!}\sum_{q=1}^{k}(-1)^q\binom{2n+k}{k-q}Q(q,2n;2),\end{aligned} \tag{4}$$

where $Q(q,2n;2)$ is given by (2).

Maclaurin's series expansion (1) was recovered in (Section 6 [9]) and was generalized in (Section 4 [10]) as

$$\left(\frac{\arcsin t}{t}\right)^{\alpha}=1+\sum_{n=1}^{\infty}(-1)^n\left[\sum_{k=1}^{2n}\frac{(-\alpha)_k}{(2n+k)!}\sum_{q=1}^{k}(-1)^q\binom{2n+k}{k-q}Q(q,2n;2)\right](2t)^{2n} \tag{5}$$

for $\alpha \in \mathbb{R}$ and $|t|<1$ by rediscovering a special case of (3) and the closed-form Formula (4), where $Q(q,2n;2)$ is given by (2) and the rising factorial of a complex number $\alpha \in \mathbb{C}$ is defined by

$$(\alpha)_m=\prod_{k=0}^{m-1}(\alpha+k)=\begin{cases}\alpha(\alpha+1)\cdots(\alpha+m-1), & m\in\mathbb{N};\\ 1, & m=0.\end{cases} \tag{6}$$

2. In [9], among other things, by establishing the Taylor series expansion

$$\left[\frac{(\arccos x)^2}{2(1-x)}\right]^k=1+(2k)!\sum_{n=1}^{\infty}\frac{Q(2k,2n;2)}{(2k+2n)!}[2(x-1)]^n \tag{7}$$

for $k \in \mathbb{N}$ and $|x|<1$, Qi derived the specific value

$$\begin{aligned}B_{m,k}&\left(-\frac{1}{12},\frac{2}{45},-\frac{3}{70},\frac{32}{525},-\frac{80}{693},\dots,\frac{(2m-2k+2)!!}{(2m-2k+4)!}Q(2,2m-2k+2;2)\right)\\&=(-1)^k[2(m-k)]!!\binom{m}{k}\sum_{j=1}^{k}(-1)^j(2j)!\binom{k}{j}\frac{Q(2j,2m;2)}{(2j+2m)!}\end{aligned}$$

for $m \geq k \in \mathbb{N}$ and then generalized the series expansion (7) to

$$\left[\frac{(\arccos x)^2}{2(1-x)}\right]^{\alpha}=1+\sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}\frac{(-\alpha)_j}{j!}\sum_{\ell=1}^{j}(-1)^{\ell}(2\ell)!\binom{j}{\ell}\frac{Q(2\ell,2n;2)}{(2\ell+2n)!}\right][2(x-1)]^n$$

for $\alpha \in \mathbb{R}$, where $Q(2j,2m;2)$ is defined by (2).

3. In [10], among other things, by establishing the specific values

$$B_{2r+k,k}\left(1,0,1,0,9,0,225,0,\dots,[(2r-3)!!]^2,0,[(2r-1)!!]^2\right)=(-1)^r2^{2r}Q(k,2r;2)$$

and

$$B_{2r+k-1,k}\left(1,0,1,0,9,0,225,0,\dots,[(2r-3)!!]^2,0\right)=0$$

for $r,k \in \mathbb{N}$, Qi concluded

$$
\begin{aligned}
\left(\frac{2\arccos t}{\pi}\right)^{\alpha} &= 1+\sum_{r=1}^{\infty}(-1)^{r}\left[\sum_{\ell=1}^{r}(-1)^{\ell}\frac{(-\alpha)_{2\ell-1}}{\pi^{2\ell-1}}Q(2\ell-1,2r-2\ell;2)\right]\frac{(2t)^{2r-1}}{(2r-1)!}\\
&\quad+\frac{(-\alpha)_2}{\pi^2}\frac{(2t)^2}{2!}+\sum_{r=2}^{\infty}(-1)^{r}\left[\sum_{\ell=1}^{r}(-1)^{\ell}\frac{(-\alpha)_{2\ell}}{\pi^{2\ell}}Q(2\ell,2r-2\ell;2)\right]\frac{(2t)^{2r}}{(2r)!}
\end{aligned}
$$

for $\alpha\in\mathbb{R}$ and $|t|<1$, where $(\alpha)_r$ for $\alpha\in\mathbb{R}$ and $r\in\mathbb{N}$ is defined by (6) and $Q(k,2r;2)$ is given by (2).

4. In [11], among other things, by establishing a special case of (3) and the explicit formula

$$
\begin{aligned}
&B_{2m,k}\left(0,-\frac{1}{3},0,\frac{1}{5},\dots,\frac{(-1)^m}{2m-k+2}\sin\frac{k\pi}{2}\right)\\
&\qquad=(-1)^{m+k}\frac{2^{2m}}{k!}\sum_{j=1}^{k}(-1)^{j}\binom{k}{j}\frac{T(2m+j,j)}{\binom{2m+j}{j}},\quad 2m\ge k\ge 1,
\end{aligned}
$$

Qi showed that,

(a) when $\alpha\ge 0$, the series expansions

$$
\operatorname{sinc}^{\alpha} z=1+\sum_{q=1}^{\infty}(-1)^{q}\left[\sum_{k=1}^{2q}\frac{(-\alpha)_k}{k!}\sum_{j=1}^{k}(-1)^{j}\binom{k}{j}\frac{T(2q+j,j)}{\binom{2q+j}{j}}\right]\frac{(2z)^{2q}}{(2q)!}\tag{8}
$$

is convergent in $z\in\mathbb{C}$;

(b) when $\alpha<0$, the series expansion (8) is convergent in $|z|<\pi$;

where

$$
\operatorname{sinc} z=\begin{cases}\dfrac{\sin z}{z}, & z\ne 0\\ 1, & z=0\end{cases}
$$

is called the sinc function,

$$
T(n,\ell)=\begin{cases}1, & (n,\ell)=(0,0)\\ 0, & n\in\mathbb{N},\ell=0\\ \dfrac{1}{\ell!}\displaystyle\sum_{j=0}^{\ell}(-1)^{j}\binom{\ell}{j}\left(\frac{\ell}{2}-j\right)^{n}, & n,\ell\in\mathbb{N}\end{cases}
$$

for $n\ge\ell\in\mathbb{N}_0=\{0,1,2,\dots\}$ is called the central factorial numbers of the second kind [12,13], and the rising factorial $(\alpha)_k$ is defined by (6).

On new results and applications of special values of partial Bell polynomials $B_{n,k}$ in recent years by Qi and his coauthors, please refer to [14–25] and closely related references therein.

2.3. Wallis Ratio

Starting from 1999, Qi began to be interested in special functions and applications. Through these work, he posed mathematical notions such as logarithmically completely monotonic function and completely monotonic degree.

The new approximation formula and the inequalities for the Wallis ratio

$$
W_n=\frac{(2n-1)!!}{(2n)!!},\quad n\in\mathbb{N}
$$

have been examined in a joint research article [26] with C. Mortici. In (Theorems 4.1 and 4.2 [26]), the authors have proved the asymptotic formula

$$W_n \sim \sqrt{\frac{\mathrm{e}}{n}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{1}{24n^2}+\frac{1}{48n^3}+\frac{1}{160n^4}+\frac{1}{960n^5}+\cdots\right), \quad n\to\infty$$

and the inequality

$$W_n > \sqrt{\frac{\mathrm{e}}{n}}\left(1-\frac{1}{2n}\right)^n \frac{1}{\sqrt{n}} \exp\left(\frac{1}{24n^2}+\frac{1}{48n^3}+\frac{1}{160n^4}+\frac{1}{960n^5}\right), \quad n\ge 1$$

respectively. In (Theorem 5.2 [26]), the double inequality

$$\sqrt{\frac{\mathrm{e}}{\pi}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3}\frac{1}{\sqrt{n}} < W_n < \sqrt{\frac{\mathrm{e}}{\pi}}\left[1-\frac{1}{2(n+1/3)}\right]^{n+1/3}\frac{\mathrm{e}^{1/144n^3}}{\sqrt{n}}$$

has been proved for each integer $n\ge 1$.

In the branch of the Wallis ratio and inequalities, Qi and his coauthors published also the papers [27–33] and applied some results from [31] to the derivation of the series expansion (8).

2.4. Additivity of Polygamma Functions

The classical Euler gamma function $\Gamma(x)$ is defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty \mathrm{e}^{-t}\,t^{x-1}\,\mathrm{d}\,t.$$

The function $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is usually called the psi or digamma function, while the function $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ is called the polygamma function.

August 2008 in Sydney

The properties of the gamma function, the digamma function, and the polygamma functions have been investigated in many research papers by now. In a joint work [34] with B.-N. Guo and Q.-M. Luo, F. Qi proved that for each positive integer $i\in\mathbb{N}$ the function $|\psi^{(i)}(\mathrm{e}^x)|$ is subadditive on $(\ln\theta_i,\infty)$ and superadditive on $(-\infty,\ln\theta_i)$, where $\theta_i\in(0,1)$ is the unique root of the equation $2|\psi^{(i)}(\theta)|=|\psi^{(i)}(\theta^2)|$.

An earlier paper similar to [34] is [35] in which the convexity and concavity of the functions $\psi^{(k)}(\mathrm{e}^x)$ and $\psi^{(k)}(x^c)$ for $x\in\mathbb{R}$ and $c\ne 0$ were considered by Qi and his two coauthors.

2.5. Bounds for Mathematical Means in Terms of Mathematical Means

In [36], a joint work with X.-T. Shi, F.-F. Liu, and Z.-H. Yang, Qi examined a double inequality for an integral mean in terms of the exponential and logarithmic means. Among many other results, it has been proved that, for every two distinct positive real numbers $a > 0$ and $b > 0$, we have

$$L(a,b) < \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta}\,\mathrm{d}\,\theta < I(a,b),$$

where

$$L(a,b) = \frac{b-a}{\ln b - \ln a} \quad \text{and} \quad I(a,b) = \frac{1}{\mathrm{e}}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}$$

are called [37] the logarithmic and exponential means, respectively.

The paper [36] is a starting point of [38,39] and many other papers such as [40–48] by other mathematicians.

In a joint research article [49] with W.-D. Jiang, F. Qi proved a double inequality for the combination of the Toader mean and the arithmetic mean in terms of the contraharmonic mean. Qi and his coauthors also published many other papers such as [50–56] in which some special means are bounded in terms of elementary and simple mathematical means.

2.6. Complete Elliptic Integrals

There is no need to say that the theory of complete elliptic integrals has attracted F. Qi and his coauthors, who provided many significant contributions in this field. Some new bounds for the complete elliptic integrals of the first and second kind and their generalizations were given in [57–62], for example.

2.7. Matrices

In [63], Qi and his two coauthors analytically discovered the inverse of the interesting matrix

$$A_n = (a_{i,j})_{n\times n} = \begin{pmatrix}
\binom{1}{0} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
\binom{1}{1} & \binom{2}{0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & \binom{2}{1} & \binom{3}{0} & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & \binom{2}{2} & \binom{3}{1} & \binom{4}{0} & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & \binom{3}{2} & \binom{4}{1} & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & \binom{3}{3} & \binom{4}{2} & \cdots & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \binom{4}{3} & \cdots & 0 & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & 0 & \cdots & \binom{n-3}{0} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & \binom{n-3}{1} & \binom{n-2}{0} & 0 & 0\\
0 & 0 & 0 & 0 & \cdots & \binom{n-3}{2} & \binom{n-2}{1} & \binom{n-1}{0} & 0\\
0 & 0 & 0 & 0 & \cdots & \binom{n-3}{3} & \binom{n-2}{2} & \binom{n-1}{1} & \binom{n}{0}
\end{pmatrix}_{n\times n} \tag{9}$$

for $n \in \mathbb{N}$, where

$$a_{i,j} = \begin{cases} 0, & i < j\\ \dbinom{j}{i-j}, & j \le i \le 2j\\ 0, & i > 2j \end{cases}$$

for $1 \le i, j \le n$. Basing on this result, they presented an inversion theorem which states that

$$\frac{s_n}{n!} = \sum_{k=1}^{n}(-1)^k\binom{k}{n-k}S_k \quad \text{if and only if} \quad nS_n = \sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}s_k,$$

where s_k and S_k are two sequences independent of n such that $n \geq k \geq 1$. Moreover, they deduced several identities, including

$$\sum_{\ell=0}^{\lfloor (j-1)/2 \rfloor} (-1)^{\ell} \binom{j-\ell-1}{\ell} C_{i-\ell-1} = \frac{j}{i} \binom{2i-j-1}{i-1}, \quad i \geq j \geq 1$$

and

$$\frac{\sum_{\ell=0}^{m-1} (-1)^{\ell} \binom{2m-\ell-1}{\ell} \frac{n+2\ell+1}{n-\ell+1} C_{n-\ell-1}}{\sum_{\ell=0}^{m-1} (-1)^{\ell} \binom{2m-\ell-2}{\ell} \frac{1}{2m-2\ell-1} C_{n-\ell-1}} = m(2m-1), \quad n \geq 2m \geq 2, \tag{10}$$

relating to the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, where $\lfloor x \rfloor$ denotes the floor function whose value is the largest integer less than or equal to x.

We remark that the inverse of the matrix A_n defined in (9) was also combinatorially studied and connected in ([64] p. 8), while the identity in (10) was also combinatorially discussed and compared at the end on ([65] p. 3162). We emphasize that the approaches and methods used in [64,65] are quite different from those in [63]. This means that the approaches and methods used by Qi and his coauthors in [63] are novel and innovative.

October 2007 at Weinan Normal University, China

By the way, as for the Catalan numbers C_n, we recommend the new papers [66–69] by Qi and his coauthors. In these papers, the Catalan numbers C_n were generalized, some new properties of C_n were discovered by considering logarithmically complete monotonicity of their generating functions, integral representations of C_n were surveyed in [70] and applied in [63].

In [71], Hong and Qi clarified several new inequalities for generalized eigenvalues of perturbation problems on Hermitian matrices. If $A \in \mathbb{C}^{n\times n}$ is a Hermitian complex matrix of format $n \times n$, then A has the pure real spectrum. Let us denote its eigenvalues by $\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)$ and assume that

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A).$$

By $\|\cdot\|_2$ we denote the spectral norm of a matrix. If $E \in \mathbb{C}^{n\times n}$ is also a Hermitian complex matrix of format $n \times n$, then the famous Weyl theorem states that

$$\max_{1\leq i\leq n} |\lambda_i(A) - \lambda_i(A+E)| \leq \|E\|_2.$$

Besides this result, we know that the following inequalities hold: If $A, B \in \mathbb{C}^{n\times n}$ are Hermitian complex matrices of format $n \times n$ and $i, j, k, \ell, m \in \mathbb{N}$ satisfy $j + k - 1 \leq i \leq \ell + m - n - 1$, then we have

$$\lambda_\ell(A) + \lambda_m(B) \leq \lambda_i(A+B) \leq \lambda_j(A) + \lambda_k(B).$$

In particular,

$$\lambda_i(A) + \lambda_n(B) \leq \lambda_i(A+B) \leq \lambda_j(A) + \lambda_1(B).$$

Accurately, Hong and Qi proved in [71] the following results:

1. Suppose that $A, B, H, E \in \mathbb{C}^{n\times n}$ are Hermitian complex matrices of format $n \times n$, that B is positive definite, that $\nu = \|E\|_2/\lambda_n(B) < 1$, and that the positive integers $i, j, k, \ell, m \in \mathbb{N}$ satisfy $j + k - 1 \le i \le \ell + m - n - 1$.
   (a) If $\lambda_i(A+H) \ge 0$, then
   $$\frac{\lambda_\ell(AB^{-1}) + \lambda_m(HB^{-1})}{1+\nu} \le \lambda_i\big((A+H)(B+H)^{-1}\big) \le \frac{\lambda_j(AB^{-1}) + \lambda_k(HB^{-1})}{1-\nu}.$$
   (b) If $\lambda_i(A+H) \le 0$, then
   $$\frac{\lambda_j(AB^{-1}) + \lambda_k(HB^{-1})}{1-\nu} \le \lambda_i\big((A+H)(B+H)^{-1}\big) \le \frac{\lambda_\ell(AB^{-1}) + \lambda_m(HB^{-1})}{1+\nu}.$$
2. Suppose that $A, B, H, E \in \mathbb{C}^{n\times n}$ are Hermitian complex matrices of format $n \times n$, that B is positive definite, and that $\nu = \|E\|_2/\lambda_n(B) < 1$. Then we have
   $$\begin{aligned}\beta_i(A)\lambda_i(AB^{-1}) + \beta_n(H)\lambda_n(HB^{-1}) &\le \lambda_i\big((A+H)(B+H)^{-1}\big)\\ &\le \alpha_i(A)\lambda_i(AB^{-1}) + \alpha_1(H)\lambda_1(HB^{-1}).\end{aligned}$$

For more information on this topic, see also the joint papers [72,73] with Y. Hong, in which the authors considered determinantal inequalities of the Hua–Marcus–Zhang type for quaternion matrices and refined two determinantal inequalities for positive semidefinite matrices.

2.8. Bounds for Ratio of Bernoulli Numbers

One of the most influential scientific results of F. Qi was presented in [74], in which Qi considered a double inequality for the ratio of two non-zero neighboring Bernoulli numbers. This result has been quoted almost one hundred times in recent years.

It is well known that the Bernoulli numbers B_n can be generated by

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi.$$

Since the function $\frac{x}{e^x-1} - 1 + \frac{x}{2}$ is even on $\mathbb{R}$, all of the Bernoulli numbers B_{2n+1} for $n \in \mathbb{N}$ are equal to 0. Due to (Theorem 1.1 [74]), we have

$$\frac{2^{2k-1}-1}{2^{2k+1}-1}\frac{(2k+1)(2k+2)}{\pi^2} < \frac{|B_{2k+2}|}{|B_{2k}|} < \frac{2^{2k}-1}{2^{2k+2}-1}\frac{(2k+1)(2k+2)}{\pi^2}, \quad k \in \mathbb{N}. \tag{11}$$

This double inequality immediately implies

$$\lim_{k\to\infty} \frac{|B_{2k+2}|}{k^2|B_{2k}|} = \frac{1}{\pi^2}.$$

In order to achieve his aims, Qi used the well-known identity

$$B_{2k} = 2\frac{(-1)^{k+1}(2k)!}{(2\pi)^{2k}}\zeta(2k), \quad k \in \mathbb{N},$$

where $\zeta(\cdot)$ is the Riemann zeta function.

The double inequality (11) and related results in [75,76] have been extended, refined, generalized, improved, non-self-cited, and applied in over 50 preprints and papers such as [77–92] by many mathematicians, combinatorists, and physicists around the world.

2.9. Special Polynomials

The Boole polynomials $Bl_n(x;\alpha)$ are defined by

$$\frac{(1+t)^x}{1+(1+t)^\alpha}=\sum_{n=0}^{\infty}Bl_n(x;\alpha)\frac{t^n}{n!}.$$

The Peters polynomials (or higher-order Boole polynomials) $s_n(x;\alpha,\nu)$, defined by

$$\frac{(1+t)^x}{[1+(1+t)^\alpha]^\nu}=\sum_{n=0}^{\infty}s_n(x;\alpha,\nu)\frac{t^n}{n!},$$

clearly generalize the Boole polynomials. It is also known that the Peters polynomials can be further generalized. For example, the degenerate Peters polynomials $s_n(x;\alpha,\nu;\lambda)$, which are defined by

$$\frac{\mathrm{e}^{x[(1+t)^\lambda 1]/\lambda}}{\left(1+\mathrm{e}^{\alpha[(1+t)^\lambda 1]/\lambda}\right)^\nu}=\sum_{n=0}^{\infty}s_n(x;\alpha,\nu;\lambda)\frac{t^n}{n!},$$

generalize the Peters polynomials.

In a joint research article [93] with Y.-W. Li and M. C. Dağlı, F. Qi showed that

$$\begin{aligned}s_n(x;\alpha,\nu;\lambda)=(n-1)!\sum_{k=1}^{n}&\left[\frac{(-1)^k}{\lambda^{k-1}k!}\sum_{\ell=1}^{k}(-1)^\ell\ell\binom{k}{\ell}\binom{\lambda\ell-1}{n-1}\right]\\&\times\left[\sum_{\ell=1}^{k}\frac{\langle-\nu\rangle_\ell}{2^{\nu+\ell}}\sum_{r+s=\ell}\sum_{i+j=k}\binom{k}{i}\left(-\frac{x}{\nu}\right)^i\left(\alpha-\frac{x}{\nu}\right)^jS(i,r)S(j,s)\right],\end{aligned}$$

where the falling factorial $\langle z\rangle_n$ is defined for $z\in\mathbb{C}$ by

$$\langle z\rangle_n=\prod_{k=0}^{n-1}(z-k)=\begin{cases}z(z-1)\cdots(z-n+1), & n\ge1;\\1, & n=0.\end{cases}$$

Setting $x=0$ in this formula, we obtain the special result stated in (Theorem 4.1 [93]).

In addition to the paper [93], Dr. Feng Qi and his coauthors conducted more work in the papers [94–114], for example, in this branch. Many of these papers are related to partial Bell polynomials $B_{n,k}$ mentioned above.

2.10. Complete Monotonicity Properties Related to Polygamma Functions

In [115], Qi employed the convolution theorem for the Laplace transform, Bernstein's theorem for completely monotonic functions, and some other analytic techniques to reveal some necessary and sufficient conditions for two functions defined by two derivatives of a function involving trigamma function to be completely monotonic or monotonic. See also a joint paper [116] with R. P. Agarwal, where the authors analyzed the complete monotonicity for several classes of functions related to ratios of gamma functions, and a joint paper [117] with D. Lim, where the authors investigated a ratio of finite many gamma functions and its monotonicity properties. We notice that the papers [115,117] are companions of the papers [118–126]. This series of articles originate from the paper [127] and its preprints.

2008 at Victoria University, Footscray, Melbourne, Australia

2.11. Convex Functions and Inequalities

From 2012 on, F. Qi collaborated with Professor Bo-Yan Xi and his academic group at Inner Mongolia University for Nationalities and paid much attention on generalizations of convex functions and on establishment of integral inequalities of the Hermite–Hadamard type.

The theory of convex functions is extremely significant in many areas of pure and applied sciences. The Jensen inequality and the Hermite–Hadamard type inequalities are still very attractive fields of research within the theory of convex functions. Concerning the scientific work of Professor Feng Qi in this area, we would like to mention the research articles [128–140] and references cited therein.

In this issue, we will briefly describe the results obtained in collaboration of Professor Feng Qi with Y. Wang and M.-M. Zheng in [133] only. Suppose that $\alpha \in (0,1]$ and $m \in (0,1]$. Let us recall that a function $f : [0,b] \to \mathbb{R}$, where $0 < b < \infty$, is said to be (α, m)-convex if and only if

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for $x, y \in [0,b]$ and $t \in [0,1]$. If $\alpha = 1$, then an (α, m)-convex function $f : [0,b] \to \mathbb{R}$ is also said to be m-convex. Further on, a non-empty set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\nu : S \times S \to \mathbb{R}^n$ if and only if $x + t\nu(x,y) \in S$ for all $t \in [0,1]$ and $x, y \in S$. If this is the case, a function $f : S \to \mathbb{R}$ is said to be preinvex with respect to ν if and only if

$$f(y + t\nu(x,y)) \le tf(x) + (1-t)f(y), \quad x, y \in S, \quad t \in [0,1].$$

We know the following conclusions:

1. If $-\infty < c < a < b < d < \infty$, the function $f : [c,d] \to \mathbb{R}$ is differentiable, and the derivative $|f'|$ is convex on $[a,b]$, then we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{8}\left(|f'(a)| + |f'(b)|\right).$$

2. For $0 \le a < b < \infty$, if the function $f : [0,b] \to \mathbb{R}$ is m-convex for $m \in (0,1]$ and the Lebesgue integrable, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \min\left\{\frac{f(a)+mf(b/m)}{2}, \frac{f(b)+mf(a/m)}{2}\right\}.$$

3. For $0 \le a < b < \infty$ and $\alpha, m \in (0,1]$, if the function $f : [0,b] \to \mathbb{R}$ is (α, m)-convex and differentiable and its first derivative is the Lebesgue integrable, then we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{2}\frac{1}{2^{1-1/q}}$$
$$\times \min\left\{\left[v_1|f'(a)|^q + v_2 m|f'(b)|^q\right]^{1/q}, \left[v_1|f'(b)|^q + v_2 m|f'(a)|^q\right]^{1/q}\right\},$$

provided that the function $|f'|^q$ is (α, m)-convex for some real number $q \ge 1$, where

$$v_1 = \frac{\alpha + 1/2^\alpha}{(\alpha+1)(\alpha+2)} \quad \text{and} \quad v_2 = \frac{1}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha+2}{2} - \frac{1}{2^\alpha}\right).$$

August 2014 in China

In (Definition 7 [133]), the authors introduced the following notion: Suppose that a non-empty set $S \subseteq \mathbb{R}^n$ is invex with respect to ν for $\alpha \in (0,1]$. We say that a function $f : S \to \mathbb{R}$ is α-preinvex with respect to ν if and only if

$$f(y + t\nu(x,y)) \le t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for $x, y \in S$ and $t \in [0,1]$. The main results are the Hermite–Hadamard type inequalities in (Theorems 5 to 9 [133]), where the authors mainly use the assumption that the function $|f'|^q$ is α-preinvex for some real number $\alpha \in (0,1]$ and $q \ge 1$. Until now, Qi and Xi's academic group have jointly published over 120 papers in reputable peer-review journals. Due to their better work in generalizing convex functions and in establishing the Hermite–Hadamard type inequalities, Qi and Xi's group acquired financial support from the National Natural Science Foundation of China with Grant No. 11361038 between 2014 and 2017.

2.12. Fractional Derivatives and Integrals

Let us note that Professor F. Qi analyzed, in three joint work [141–143] with W.-S. Du, A. Ghaffar, C.-J. Huang, S. M. Hussain, K. S. Nisar, and G. Rahman, the Čebyšev and Grüss type inequalities for conformable k-fractional integral operators, where the authors investigated the Hermite–Hadamard type inequalities for k-fractional conformable integrals.

Concerning the integral inequalities, it is also worth noting that F. Qi and his coauthors have generalized, in [144–147], the Young integral inequality using the Taylor theorems in terms of higher order derivatives and their norms; the authors have applied their results for the estimation of several concrete definite integrals.

2.13. Differential Geometry

From September 1982 to July 1986, F. Qi majored in mathematical education as a bachelor student at Department of Mathematics, Henan University, China. From September 1986 to June 1989, he majored in differential geometry as his master's research supervised by Professor Yi-Pei Chen at the Department of Mathematics, Xiamen University, China.

From March 1996 to January 1999, he majored in analysis and topology as his doctoral supervised by Professor Sen-Lin Xu at the Department of Mathematics, University of Science and Technology of China. In this period, he jointly published over 10 papers, including [148–152], in differential geometry.

2.14. Pólya Type Integral Inequalities

Starting from 1993, Qi's research was extended to mathematical inequalities and applications, including generalizations of the Pólya integral inequality [153]. As for the Pólya type integral inequalities, his first paper is [154], his last paper is [155]. On this topic, he also published the papers [156–163]. Then, he surveyed the Pólya type integral inequalities from the origin to date in [164]. Some of these results have been applied to refine the famous Young's integral inequality in the papers [145–147].

October 2015 in Huizhou, Guangdong, China

2.15. Properties of Special Mathematical Means

Starting from 1997, Qi's research was further extended to mathematical means and applications. He started out by publishing [165,166]. His newest and creative papers in this area are [38,167–178], for example. In these papers, he discovered logarithmic convexity and Schur convexity of the extended mean values (or say, Stolarsky's means), considered the logarithmically complete monotonicity of special mathematical means, established integral and Lévy–Khintchine representations of some special mathematical means and their reciprocals. Concretely speaking, for example, Qi and his coauthors obtained the following results:

1. Let $n \in \mathbb{N}$ be not less than 2 and $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ be a positive sequence, that is, $a_k > 0$ for $1 \le k \le n$. The arithmetic and geometric means $A_n(\boldsymbol{a})$ and $G_n(\boldsymbol{a})$ of the positive sequence $\boldsymbol{a}$ are defined, respectively, as

$$A_n(\boldsymbol{a}) = \frac{1}{n}\sum_{k=1}^{n} a_k \quad \text{and} \quad G_n(\boldsymbol{a}) = \left(\prod_{k=1}^{n} a_k\right)^{1/n}.$$

For $z \in \mathbb{C} \setminus (-\infty, -\min\{a_k, 1 \le k \le n\}]$ and $n \ge 2$, let $\boldsymbol{e} = (\overbrace{1, 1, \dots, 1}^{n})$ and

$$G_n(\boldsymbol{a} + z\boldsymbol{e}) = \left[\prod_{k=1}^{n}(a_k + z)\right]^{1/n}.$$

In (Theorem 1.1 [176]), by virtue of the Cauchy integral formula in the theory of complex functions, the following integral representation was established.

Let σ be a permutation of the sequence $\{1, 2, \ldots, n\}$ such that the sequence $\sigma(\boldsymbol{a}) = \left(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}\right)$ is a rearrangement of $\boldsymbol{a}$ in an ascending order $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)}$. Then the principal branch of the geometric mean $G_n(\boldsymbol{a} + z\boldsymbol{e})$ has the integral representation

$$G_n(\boldsymbol{a} + z\boldsymbol{e}) = A_n(\boldsymbol{a}) + z - \frac{1}{\pi}\sum_{\ell=1}^{n-1} \sin\frac{\ell\pi}{n} \int_{a_{\sigma(\ell)}}^{a_{\sigma(\ell+1)}} \left|\prod_{k=1}^{n}(a_k - t)\right|^{1/n} \frac{\mathrm{d}\,t}{t+z} \quad (12)$$

for $z \in \mathbb{C} \setminus (-\infty, -\min\{a_k, 1 \leq k \leq n\}]$.

Taking $z = 0$ in the integral representation (12) yields the fundamental inequality

$$G_n(\boldsymbol{a}) = A_n(\boldsymbol{a}) - \frac{1}{\pi}\sum_{\ell=1}^{n-1} \sin\frac{\ell\pi}{n} \int_{a_{\sigma(\ell)}}^{a_{\sigma(\ell+1)}} \left[\prod_{k=1}^{n}|a_k - t|\right]^{1/n} \frac{\mathrm{d}\,t}{t} \leq A_n(\boldsymbol{a}). \quad (13)$$

For $0 < a_1 \leq a_2 \leq a_3$, taking $n = 2, 3$ in (13) gives

$$\frac{a_1 + a_2}{2} - \sqrt{a_1 a_2} = \frac{1}{\pi}\int_{a_1}^{a_2} \sqrt{\left(1 - \frac{a_1}{t}\right)\left(\frac{a_2}{t} - 1\right)}\,\mathrm{d}\,t \geq 0$$

and

$$\frac{a_1 + a_2 + a_3}{3} - \sqrt[3]{a_1 a_2 a_3} = \frac{\sqrt{3}}{2\pi}\int_{a_1}^{a_3} \sqrt[3]{\left|\left(1 - \frac{a_1}{t}\right)\left(1 - \frac{a_2}{t}\right)\left(1 - \frac{a_3}{t}\right)\right|}\,\mathrm{d}\,t \geq 0.$$

These texts are excerpted from the site https://math.stackexchange.com/a/4256320/945479 on 10 July 2022.

2. The weighted version of the integral representation (12) can be found in the paper (Theorem 3.1 [175]). We recite the weighted version as follows.

For $n \geq 2$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, and $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ with $a_k, w_k > 0$ and $\sum_{k=1}^{n} w_k = 1$, the weighted arithmetic and geometric means $A_{\boldsymbol{w},n}(\boldsymbol{a})$ and $G_{\boldsymbol{w},n}(\boldsymbol{a})$ of $\boldsymbol{a}$ with the positive weight $\boldsymbol{w}$ are defined, respectively, as

$$A_{\boldsymbol{w},n}(\boldsymbol{a}) = \sum_{k=1}^{n} w_k a_k \quad \text{and} \quad G_{\boldsymbol{w},n}(\boldsymbol{a}) = \prod_{k=1}^{n} a_k^{w_k}.$$

Let us denote $\alpha = \min\{a_k, 1 \leq k \leq n\}$. For a complex variable $z \in \mathbb{C} \setminus (-\infty, -\alpha]$, we introduce the complex function

$$G_{\boldsymbol{w},n}(\boldsymbol{a} + z) = \prod_{k=1}^{n}(a_k + z)^{w_k}.$$

With the aid of the Cauchy integral formula in the theory of complex functions, the following integral representation was established in (Theorem 3.1 [175]).

Let $0 < a_k \leq a_{k+1}$ for $1 \leq k \leq n-1$ and $z \in \mathbb{C} \setminus (-\infty, -a_1]$. Then the principal branch of the weighted geometric mean $G_{\boldsymbol{w},n}(\boldsymbol{a} + z)$ with a positive weight $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ has the integral representation

$$\begin{aligned} &G_{\boldsymbol{w},n}(\boldsymbol{a} + z) - A_{\boldsymbol{w},n}(\boldsymbol{a}) \\ &\quad = z - \frac{1}{\pi}\sum_{\ell=1}^{n-1} \sin\left[\left(\sum_{k=1}^{\ell} w_k\right)\pi\right] \int_{a_\ell}^{a_{\ell+1}} \prod_{k=1}^{n}|a_k - t|^{w_k} \frac{\mathrm{d}\,t}{t+z}. \end{aligned} \quad (14)$$

Letting $z = 0$ in the integral representation (14) gives the fundamental inequality

$$G_{\boldsymbol{w},n}(\boldsymbol{a}) = A_{\boldsymbol{w},n}(\boldsymbol{a}) - \frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left[\left(\sum_{k=1}^{\ell} w_k\right)\pi\right]\int_{a_\ell}^{a_{\ell+1}}\prod_{k=1}^{n}|a_k - t|^{w_k}\frac{\mathrm{d}\,t}{t} \le A_{\boldsymbol{w},n}(\boldsymbol{a}). \tag{15}$$

Setting $n = 2$ in (15) leads to

$$a_1^{w_1}a_2^{w_2} = w_1a_1 + w_2a_2 - \frac{\sin(w_1\pi)}{\pi}\int_{a_1}^{a_2}\left(1 - \frac{a_1}{t}\right)^{w_1}\left(\frac{a_2}{t} - 1\right)^{w_2}\mathrm{d}\,t \le w_1a_1 + w_2a_2 \tag{16}$$

for $w_1, w_2 > 0$ such that $w_1 + w_2 = 1$. These texts are excerpted from the site https://math.stackexchange.com/a/4256320/945479 on 10 July 2022.

3. For $a_k < a_{k+1}$ and $w_k > 0$ with $\sum_{k=1}^{n} w_k = 1$ and $n \ge 2$, the principal branch of the reciprocal $H_{\boldsymbol{a},\boldsymbol{w},n}(z)$ of the weighted geometric mean $G_{\boldsymbol{w},n}(\boldsymbol{a} + z)$ can be represented by

$$\begin{aligned} H_{\boldsymbol{a},\boldsymbol{w},n}(z) &= \frac{1}{\prod_{k=1}^{n}(z + a_k)^{w_k}} \\ &= \frac{1}{\pi}\sum_{\ell=1}^{n-1}\sin\left(\pi\sum_{k=1}^{\ell} w_k\right)\int_{a_\ell}^{a_{\ell+1}}\frac{1}{\prod_{k=1}^{n}|t - a_k|^{w_k}}\frac{\mathrm{d}\,t}{t+z}, \end{aligned} \tag{17}$$

where $z \in \mathbb{C} \setminus [-a_n, -a_1]$. Consequently, the reciprocal $H_{\boldsymbol{a},\boldsymbol{w},n}(t - a_1)$ of the weighted geometric mean $G_{\boldsymbol{w},n}(\boldsymbol{a} + t - a_1)$ is a Stieltjes function and a logarithmically completely monotonic function. See (Theorem 2.1 [172]).

2.16. Invited Visits and Promotions

Due to his better work in mathematical inequalities and applications, F. Qi and his academic groups obtained support from the National Natural Science Foundation of China with Grant No. 10001016 between 2001 and 2003. Due to this, Qi obtained an invitation and support from Dr. Professor Sever S. Dragomir to visit Victoria University (Melbourne, Australia) for collaboration between November 2001 and January 2002. This is his first visit abroad. Supported by the China Scholarship Council, he visited Victoria University again to collaborate with Dr. Professor Pietro Cerone and Sever S. Dragomir between March 2008 and February 2009.

May 2017 in Jiaozuo, China

Due to inventing the notion of logarithmically completely monotonic functions and his better work in special functions, Qi obtained an invitation and support from Dr. Professor Christian Berg at Copenhagen University to attend the Workshop on Integral Transforms, Positivity and Applications between 1 and 3 September 2010.

Dr. Feng Qi was also invited and supported by Dr. Professor Ahmet Ocak Akdemir, Wing-Sum Cheung, Yeol Je Cho, Junesang Choi, Wei-Shih Du, Taekyun Kim, and Jen-Chih

Yao, to visit the University of Hong Kong twice in 2004, to visit Dongguk University at Gyeongju, Gyeongsang National University, Kwangwoon University, Kyungpook National University, and several other universities in South Korea from 2012 to 2015, to visit Antalya in Turkey in 2016, and to visit Sun Yat-sen University and Kaohsiung Normal University in Taiwan in 2018, for academic collaborations and international conferences, including taking part in the International Congress of Mathematicians 2014.

Due to his excellent works in university mathematics education, administration, and academic research, Qi was promptly and quicker promoted from a lecturer to an associate professor, to a full professor, and to a Specially-Appointed-Professor for Universities of Henan Province at Henan Polytechnic University in November 1995, October 1999, and November 2005.

2.17. Editorial and Refereeing Appointments

Currently, Dr. Qi is editors-in-chief, associate editor, editor, member of editorial board for over 25 internationally-reputed and peer-reviewed journals such as the Journal of Inequalities and Applications which is being indexed by the Science Citation Index-Expanded and Scopus.

The first academic journal specializing in mathematical inequalities, the Journal of Inequalities and Applications, was found by Dr. Professor Ravi Prakash Agarwal in 1997. This history was cultivated in Qi's survey article [164]. In addition, the following seven journals have also specialized in mathematical inequalities:

1. *Advances in Inequalities and Applications* (since 2012);
2. *Advances in Nonlinear Variational Inequalities* (since 1998);
3. *Journal of Inequalities and Special Functions* (since 2010);
4. *Journal of Inequalities in Pure and Applied Mathematics* (since 2000 to 2009);
5. *Journal of Mathematical Inequalities* (since 2007);
6. *Mathematical Inequalities and Applications* (since 1998);
7. *Turkish Journal of Inequalities* (since 2017).

It is also worth to mentioning the Monographs in Inequalities: Series in Inequalities at the site http://books.ele-math.com/ accessed on 10 July 2022.

Professor Qi was a recipient of the Top Peer Reviewer powered by Publons in the years 2016 and 2019. See Certificates in Figure 1.

Figure 1. Qi's Certificates for Top Peer Reviewer powered by Publons in 2016 and 2019.

3. Statistics of Qi's Contributions

Since 1993, Qi has published over 670 peer-reviewed articles, including over 42 papers published in Chinese, in over 220 journals, book chapters, collections, and conference proceedings in mathematics, see Table 1.

Table 1. The year distribution of Qi's papers formally published since 1993.

Year	Papers	Year	Papers	Year	Papers	Year	Papers
1993	9	1994	5	1995	5	1996	6
1997	7	1998	9	1999	14	2000	7
2001	7	2002	7	2003	37	2004	21
2005	23	2006	31	2007	21	2008	22
2009	14	2010	14	2011	7	2012	26
2013	38	2014	51	2015	52	2016	43
2017	35	2018	45	2019	39	2020	23
2021	26	2022	23	2023	3	2024	1

In Feng Qi's Google Scholar profile dated on 2 August 2022, his over 850 papers, preprints, and other works were indexed and they were totally cited 16858 times. See the screenshot in Figure 2.

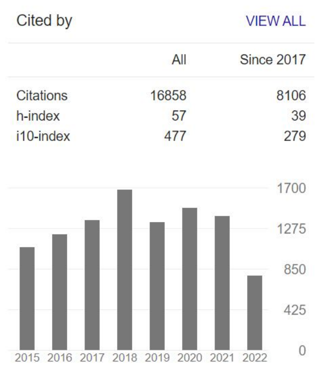


Figure 2. Statistics from Qi's Google Scholar profile dated on 2 August 2022.

In Feng Qi's Scopus profile dated on 2 August 2022, his 417 articles were indexed and they were cited 6590 times by 2007 documents. See the screenshot in Figure 3.

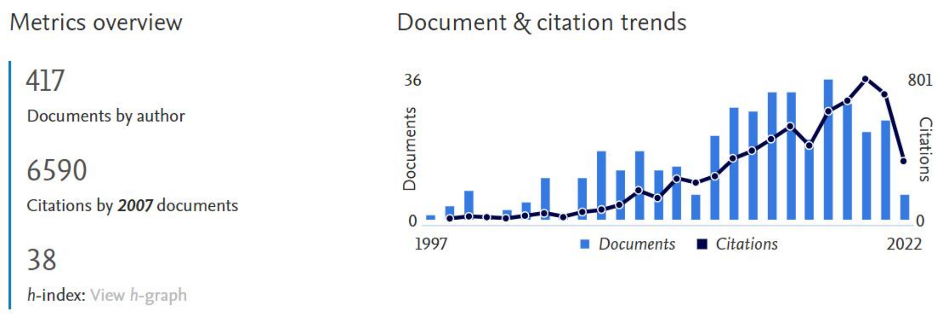


Figure 3. Statistics from Qi's Scopus profile dated on 2 August 2022.

In Qi's Publons profile dated on 2 August 2022, his 412 papers were indexed by the Web of Science Core Collection and they were cited 5915 times. See the screenshots in Figure 4.

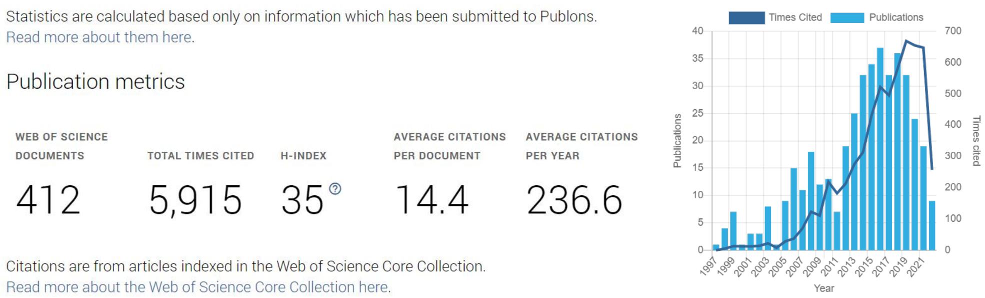


Figure 4. Statistics from Qi's Publons profile dated on 2 August 2022.

From 2014 to 2021, Qi consecutively ranked as the Most Cited Chinese Researchers in Mathematics. These rankings were carried out jointly by Elsevier and ShanghaiRanking Consultancy. See Figure 5.

Figure 5. Certificate of the 2021 Most Cited Chinese Researchers.

In the Stanford University's 2021 list of World's Top 2% Scientists, Qi ranked 61510/190064 in the Single Year Impact Data (2020) and ranked 96040/186178 in the Career-long Data (1960–2020). For more data, please click the link https://doi.org/10.17632/btchxktzyw.3 accessed on 10 July 2022.

In the 2022 edition of the World's Top Mathematics Scientists by Research.com dated on 2 August 2022, Qi ranked 330 worldwide and his 580 papers were indexed and were cited 11291 times. See Figure 6 or click the link https://research.com/u/feng-qi accessed on 8 July 2022.

H-Index & Metrics

Discipline name	H-index	Citations	Publications	World Ranking	National Ranking
Mathematics	57	11,291	580	330	12

Figure 6. Statistics from Qi's Research.com profile dated on 2 August 2022.

Since the year 1992, Qi took charge of and participated in two national research projects supported by the National Natural Science Foundation of China, several provincial scientific projects supported by Henan Province, and several university scientific projects supported by Henan Polytechnic University and Tianjin Polytechnic University. Totally he acquired about one and a half millions CNY of funding support.

Since 2002, his names "Feng Qi", "F. Qi", and "Qi" have appeared in titles of over 89 papers or preprints which were published or announced by hundreds of mathematicians in the globe. See, for example, the papers [40–48,88,179–185].

Currently, Qi's 49 papers or preprints were cited at the Wikipedia site https://en.wikipedia.org/wiki/Euler_numbers accessed on 10 July 2022 and in eight monographs or handbooks [37,186–192].

After the notion "logarithmically completely monotonic function" was explicitly defined in the preprints [193,194] and the papers [195,196], an important paper on logarithmically completely monotonic functions is [197], and the notion has been seemingly and gradually becoming a standard terminology in mathematical community. Currently, except over 60 preprints and papers by Qi and his coauthors, there have been over 40 papers and preprints whose titles contain the phrases "logarithmically completely monotonic function", "logarithmically complete monotonicity", and "logarithmically completely monotone" by other mathematicians. See, for example, the monographs [191,192] and the papers [198–201]. Qi pointed out several times that the terminology of the logarithmically completely monotonic function was first used without explicit definition in the paper [202].

By the Web of Science Core Collection, Feng Qi's papers have been cited at least in the following 50 research areas: mathematics, science technology other topics, computer science, plant sciences, mathematical computational biology, engineering, business economics, physics, mechanics, communication, telecommunications, biochemistry molecular biology, agriculture, operations research management science, food science technology, genetics heredity, life sciences biomedicine other topics, materials science, nutrition dietetics, anatomy morphology, chemistry, pharmacology pharmacy, biodiversity conservation, cell biology, environmental sciences ecology, instruments instrumentation, mathematical methods in social sciences, physiology, psychology, thermodynamics, transportation, acoustics, astronomy astrophysics, behavioral sciences, biophysics, biotechnology applied microbiology, cardiovascular system cardiology, developmental biology, energy fuels, government law, health care sciences services, infectious diseases, pathology, polymer science, psychiatry, public administration, public environmental occupational health, social issues, sociology, toxicology, and the like.

4. Conclusions

Recommended by Dr. Professor Ravi Prakash Agarwal, Dr. Professor Feng Qi is currently an editor of the *Journal of Inequalities and Applications*, the first academic journal specializing in mathematical inequalities in the world and in the history, founded by Dr. Professor Ravi Prakash Agarwal in 1997, as mentioned in Section 2.17. As one of the first two master students supervised by Dr. Professor Feng Qi between September 2004 and June 2007, Dr. Professor Jian Cao published the papers [27,35,66,203–218] jointly with Qi. As one of academic friends, Dr. Professor Wei-Shih Du published the papers [95,101,110,142] jointly with Qi. Currently Dr. Professor Feng Qi is an editor of the journal *Results in Nonlinear Analysis* founded by Dr. Professor Erdal Karapinar. As one of international colleagues, Dr. Professor Marko Kostić and his coauthors published the papers [77–79,82,84] in which Qi's results mentioned in Section 2.8 were cited and applied many times.

There have been more mathematical studies by Dr. Professor Feng Qi and his coauthors than those summarized in this paper. From the review articles [116,164,219–224], for example, we can also see more systematic contributions by F. Qi and his coauthors in mathematics. We think that we just summarized a small part of works and ideas created by Dr. Professor Feng Qi. This manuscript is the survey of the scientific work by Feng Qi and his coauthors, but not a total survey of all the topics Feng Qi and his coauthors have worked on. If this manuscript were an overall survey of or an almost complete overview of Qi's work, then it would be a book of more than 500 pages.

Author Contributions: Writing—original draft, R.P.A., J.C., W.-S.D., E.K., and M.K. All authors contributed equally to the manuscript and read and approved the final manuscript.

Funding: Marco Kostić is partially supported by Grant No. 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia. Jian Cao is partially supported by Grant No. LY21A010019 of the Zhejiang Provincial Natural Science Foundation of China. Wei-Shih Du is partially supported by Grant No. MOST 111-2115-M-017-002 of the Ministry of Science and Technology of the Republic of China.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The study did not report any data.

Acknowledgments: The authors thank anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Qi, F. Some inequalities for the Bell numbers. *Proc. Indian Acad. Sci. Math. Sci.* **2017**, *127*, 551–564. [CrossRef]
2. Qi, F. Integral representations for multivariate logarithmic polynomials. *J. Comput. Appl. Math.* **2018**, *336*, 54–62. [CrossRef]
3. Qi, F. On multivariate logarithmic polynomials and their properties. *Indag. Math.* **2018**, *29*, 1179–1192. [CrossRef]
4. Qi, F. Some inequalities and an application of exponential polynomials. *Math. Inequal. Appl.* **2020**, *23*, 123–135. [CrossRef]
5. Qi, F.; Niu, D.-W.; Lim, D.; Guo, B.-N. Some properties and an application of multivariate exponential polynomials. *Math. Methods Appl. Sci.* **2020**, *43*, 2967–2983. [CrossRef]
6. Qi, F.; Niu, D.-W.; Lim, D.; Yao, Y.-H. Special values of the Bell polynomials of the second kind for some sequences and functions. *J. Math. Anal. Appl.* **2020**, *491*, 124382. [CrossRef]
7. Guo, B.-N.; Lim, D.; Qi, F. Maclaurin's series expansions for positive integer powers of inverse (hyperbolic) sine and tangent functions, closed-form formula of specific partial Bell polynomials, and series representation of generalized logsine function. *Appl. Anal. Discrete Math.* **2023**, *17*, *in press*. [CrossRef]
8. Guo, B.-N.; Lim, D.; Qi, F. Series expansions of powers of arcsine, closed forms for special values of Bell polynomials, and series representations of generalized logsine functions. *AIMS Math.* **2021**, **6**, 7494–7517. [CrossRef]
9. Qi, F. Taylor's series expansions for real powers of two functions containing squares of inverse cosine function, closed-form formula for specific partial Bell polynomials, and series representations for real powers of Pi. *Demonstr. Math.* **2022**, *55*, *in press*.
10. Qi, F. Explicit formulas for partial Bell polynomials, Maclaurin's series expansions of real powers of inverse (hyperbolic) cosine and sine, and series representations of powers of Pi. *Res. Sq.* **2021**. [CrossRef]
11. Qi, F.; Taylor, P. Several series expansions for real powers and several formulas for partial Bell polynomials of sinc and sinhc functions in terms of central factorial and Stirling numbers of second kind. *arXiv* **2022**, https://arxiv.org/abs/2204.05612v4.
12. Qi, F.; Guo, B.-N. Relations among Bell polynomials, central factorial numbers, and central Bell polynomials. *Math. Sci. Appl. E-Notes* **2019**, *7*, 191–194. [CrossRef]
13. Qi, F.; Wu, G.-S.; Guo, B.-N. An alternative proof of a closed formula for central factorial numbers of the second kind. *Turkish J. Anal. Number Theory* **2019**, *7*, 56–58. [CrossRef]
14. Guo, B.-N.; Qi, F. Viewing some ordinary differential equations from the angle of derivative polynomials. *Iran. J. Math. Sci. Inform.* **2021**, *16*, 77–95. [CrossRef]
15. Jin, S.; Guo, B.-N.; Qi, F. Partial Bell polynomials, falling and rising factorials, Stirling numbers, and combinatorial identities. *CMES Comput. Model. Eng. Sci.* **2022**, *132*, 781–799. [CrossRef]
16. Qi, F. Simplifying coefficients in a family of ordinary differential equations related to the generating function of the Laguerre polynomials. *Appl. Appl. Math.* **2018**, *13*, 750–755.
17. Qi, F.; Guo, B.-N. Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials. *Mediterr. J. Math.* **2017**, *14*, 140. [CrossRef]
18. Qi, F.; Lim, D. Closed formulas for special Bell polynomials by Stirling numbers and associate Stirling numbers. *Publ. Inst. Math. (Beograd) (N.S.)* **2020**, *108*, 131–136. [CrossRef]
19. Qi, F.; Lim, D.; Guo, B.-N. Explicit formulas and identities for the Bell polynomials and a sequence of polynomials applied to differential equations. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2019**, *113*, 1–9. [CrossRef]
20. Qi, F.; Lim, D.; Yao, Y.-H. Notes on two kinds of special values for the Bell polynomials of the second kind. *Miskolc Math. Notes* **2019**, *20*, 465–474. [CrossRef]
21. Qi, F.; Natalini, P.; Ricci, P.E. Recurrences of Stirling and Lah numbers via second kind Bell polynomials. *Discrete Math. Lett.* **2020**, *3*, 31–36.
22. Qi, F.; Niu, D.-W.; Lim, D.; Guo, B.-N. Closed formulas and identities for the Bell polynomials and falling factorials. *Contrib. Discrete Math.* **2020**, *15*, 163–174. [CrossRef]
23. Qi, F.; Shi, X.-T.; Liu, F.-F.; Kruchinin, D.V. Several formulas for special values of the Bell polynomials of the second kind and applications,. *J. Appl. Anal. Comput.* **2017**, *7*, 857–871. [CrossRef]
24. Qi, F.; Wan, A. A closed-form expression of a remarkable sequence of polynomials originating from a family of entire functions connecting the Bessel and Lambert functions. *São Paulo J. Math. Sci.* **2021**, *15*, *in press*. [CrossRef]
25. Qi, F.; Zheng, M.-M. Explicit expressions for a family of the Bell polynomials and applications. *Appl. Math. Comput.* **2015**, *258*, 597–607. [CrossRef]
26. Qi, F.; Mortici, C. Some best approximation formulas and inequalities for the Wallis ratio. *Appl. Math. Comput.* **2015**, *253*, 363–368. [CrossRef]
27. Cao, J.; Niu, D.-W.; Qi, F. A Wallis type inequality and a double inequality for probability integral. *Aust. J. Math. Anal. Appl.* **2007**, *4*, 3. Available online: http://ajmaa.org/cgi-bin/paper.pl?string=v4n1/V4I1P3.tex (accessed on 10 July 2022).
28. Chen, C.-P.; Qi, F. Best upper and lower bounds in Wallis' inequality. *J. Indones. Math. Soc. (MIHMI)* **2005**, *11*, 137–141.
29. Chen, C.-P.; Qi, F. Completely monotonic function associated with the gamma function and proof of Wallis' inequality. *Tamkang J. Math.* **2005**, *36*,303–307. [CrossRef]

30. Chen, C.-P.; Qi, F. The best bounds in Wallis' inequality. *Proc. Am. Math. Soc.* **2005**, *133*, 397–401. [CrossRef]
31. Guo, B.-N.; Qi, F. On the Wallis formula. *Internat. J. Anal. Appl.* **2015**, *8*, 30–38.
32. Guo, S.; Xu, J.-G.; Qi, F. Some exact constants for the approximation of the quantity in the Wallis' formula. *J. Inequal. Appl.* **2013**, *2013*, 7. [CrossRef]
33. Qi, F. An improper integral, the beta function, the Wallis ratio, and the Catalan numbers. *Probl. Anal. Issues Anal.* **2018**, **7**, 104–115. [CrossRef]
34. Guo, B.-N.; Qi, F.; Luo, Q.-M. The additivity of polygamma functions. *Filomat* **2015**, *29*, 1063–1066. FIL1505063G. [CrossRef]
35. Cao, J.; Niu, D.-W.; Qi, F. Convexities of some functions involving the polygamma functions. *Appl. Math. E-Notes* **2008**, *8*, 53–57.
36. Qi, F.; Shi, X.-T.; Liu, F.-F.; Yang, Z.-H. A double inequality for an integral mean in terms of the exponential and logarithmic means. *Period. Math. Hungar.* **2017**, *75*, 180–189. [CrossRef]
37. Bullen, P.S. *Handbook of Means and Their Inequalities*; Mathematics and Its Application; Kluwer Academic Publishers Group: Dordrecht, The Netherlands, 2003; Volume 560. [CrossRef]
38. Qi, F.; Guo, B.-N. Lévy–Khintchine representation of Toader–Qi mean. *Math. Inequal. Appl.* **2018**, *21*, 421–431. [CrossRef]
39. Qi, F.; Yao, S.-W.; Guo, B.-N. Arithmetic means for a class of functions and the modified Bessel functions of the first kind. *Mathematics* **2019**, *7*, 60. [CrossRef]
40. Qian, W.-M.; Zhang, W.; Chu, Y.-M. Optimal bounds for Toader–Qi mean with applications. *J. Comput. Anal. Appl.* **2020**, *28*, 526–536.
41. Qian, W.-M.; Zhang, X.-H.; Chu, Y.-M. Sharp bounds for the Toader–Qi mean in terms of harmonic and geometric means. *J. Math. Inequal.* **2017**, *11*, 121–127. [CrossRef]
42. Xu, H.Z.; Qian, W.M. Some sharp bounds for Toader–Qi mean and other bivariate means. *J. Zhejiang Univ. Sci. Ed. (Zhejiang Daxue Xuebao, Lixue Ban)* **2017**, *44*, 526–530. (In Chinese) [CrossRef]
43. Yang, Z.-H.; Chu, Y.-M. A sharp lower bound for Toader–Qi mean with applications. *J. Funct. Spaces* **2016**, *2016*, 4165601. [CrossRef]
44. Yang, Z.-H.; Chu, Y.-M. On approximating the modified Bessel function of the first kind and Toader–Qi mean. *J. Inequal. Appl.* **2016**, *2016*, 21. [CrossRef]
45. Yang, Z.-H.; Chu, Y.-M.; Song, Y.-Q. Sharp bounds for Toader–Qi mean in terms of logarithmic and identric means. *Math. Inequal. Appl.* **2016**, *19*, 721–730. [CrossRef]
46. Yang, Z.-H.; Tian, J.-F. A new chain of inequalities involving the Toader–Qi, logarithmic and exponential means. *Appl. Anal. Discrete Math.* **2021**, *15*, 467–485. [CrossRef]
47. Yang, Z.-H.; Tian, J.-F.; Zhu, Y.-R. New sharp bounds for the modified Bessel function of the first kind and Toader–Qi mean. *Mathematics* **2020**, *8*, 901. [CrossRef]
48. Zhu, L. New bounds for the modified Bessel function of the first kind and Toader–Qi mean. *Mathematics* **2021**, *9*, 2867. [CrossRef]
49. Jiang, W.-D.; Qi, F. A double inequality for the combination of Toader mean and the arithmetic mean in terms of the contraharmonic mean. *Publ. Inst. Math. (Beograd) (N.S.)* **2016**, *99*, 237–242. [CrossRef]
50. Hua, Y.; Qi, F. A double inequality for bounding Toader mean by the centroidal mean. *Proc. Indian Acad. Sci. Math. Sci.* **2014**, *124*, 527–531. [CrossRef]
51. Hua, Y.; Qi, F. The best bounds for Toader mean in terms of the centroidal and arithmetic means. *Filomat* **2014**, *28*, 775–780. [CrossRef]
52. Jiang, W.-D.; Qi, F. Sharp bounds for the Neuman—Sándor mean in terms of the power and contraharmonic means. *Cogent Math.* **2015**, *2*, 7. [CrossRef]
53. Jiang, W.-D.; Qi, F. Sharp bounds for Neuman—Sándor's mean in terms of the root-mean-square. *Period. Math. Hungar.* **2014**, *69*, 134–138. [CrossRef]
54. Li, W.-H.; Miao, P.; Guo, B.-N. Bounds for the Neuman—Sándor mean in terms of the arithmetic and contra-harmonic means. *Axioms* **2022**, *11*, 236. [CrossRef]
55. Li, W.-H.; Shen, Q.-X.; Guo, B.-N. Several double inequalities for integer powers of the sinc and sinhc functions with applications to the Neuman–Sándor mean and the first Seiffert mean. *Axioms* **2022**, *11*, 304. [CrossRef]
56. Qi, F.; Li, W.-H. A unified proof of inequalities and some new inequalities involving Neuman–Sándor mean. *Miskolc Math. Notes* **2014**, *15*, 665–675. [CrossRef]
57. Guo, B.-N.; Qi, F. Some bounds for the complete elliptic integrals of the first and second kind. *Math. Inequal. Appl.* **2011**, *14*, 323–334. [CrossRef]
58. Qi, F.; Huang, Z. Inequalities of the complete elliptic integrals. *Tamkang J. Math.* **1998**, *29*, 165–169. [CrossRef]
59. Wang, F.; Guo, B.-N.; Qi, F. Monotonicity and inequalities related to complete elliptic integrals of the second kind. *AIMS Math.* **2020**, *5*, 2732–2742. [CrossRef]
60. Wang, F.; Qi, F. Monotonicity and sharp inequalities related to complete (p,q)-elliptic integrals of the first kind. *C. R. Math. Acad. Sci. Paris* **2020**, *358*, 961–970. [CrossRef]
61. Yin, L.; Lin, X.-L.; Qi, F. Monotonicity, convexity and inequalities related to complete (p,q,r)-elliptic integrals and generalized trigonometric functions. *Publ. Math. Debrecen* **2020**, *97*, 181–199. [CrossRef]
62. Yin, L.; Qi, F. Some inequalities for complete elliptic integrals. *Appl. Math. E-Notes* **2014**, *14*, 192–199.

63. Qi, F.; Zou, Q.; Guo, B.-N. The inverse of a triangular matrix and several identities of the Catalan numbers. *Appl. Anal. Discrete Math.* **2019**, *13*, 518–541. [CrossRef]
64. Beck, G.; Dilcher, K. A matrix related to Stern polynomials and the Prouhet–Thue–Morse sequence. *Integers* **2022**, *22*, 30.
65. Chu, W. Further identities on Catalan numbers. *Discrete Math.* **2018**, *341*, 3159–3164. [CrossRef]
66. Li, W.-H.; Cao, J.; Niu, D.-W.; Zhao, J.-L.; Qi, F. An analytic generalization of the Catalan numbers and its integral representation. *arXiv* **2005**. https://arxiv.org/abs/2005.13515v2.
67. Qi, F.; Cerone, P. Some properties of the Fuss–Catalan numbers. *Mathematics* **2018**, *6*, 277. [CrossRef]
68. Qi, F.; Shi, X.-T.; Cerone, P. A unified generalization of the Catalan, Fuss, and Fuss–Catalan numbers. *Math. Comput. Appl.* **2019**, *24*, 49. [CrossRef]
69. Qi, F.; Shi, X.-T.; Liu, F.-F. An integral representation, complete monotonicity, and inequalities of the Catalan numbers. *Filomat* **2018**, *32*, 575–587. [CrossRef]
70. Qi, F.; Guo, B.-N. Integral representations of the Catalan numbers and their applications. *Mathematics* **2017**, *5*, 40. [CrossRef]
71. Hong, Y.; Lim, D.; Qi, F. Some inequalities for generalized eigenvalues of perturbation problems on Hermitian matrices. *J. Inequal. Appl.* **2018**, *2018*, 6. [CrossRef]
72. Hong, Y.; Qi, F. Determinantal inequalities of Hua-Marcus-Zhang type for quaternion matrices. *Open Math.* **2021**, *19*, 562–568. [CrossRef]
73. Hong, Y.; Qi, F. Refinements of two determinantal inequalities for positive semidefinite matrices. *Math. Inequal. Appl.* **2022**, *25*, 673–678. [CrossRef]
74. Qi, F. A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers. *J. Comput. Appl. Math.* **2019**, *351*, 1–5. [CrossRef]
75. Qi, F. Notes on a double inequality for ratios of any two neighbouring non-zero Bernoulli numbers. *Turkish J. Anal. Number Theory* **2018**, *6*, 129–131. [CrossRef]
76. Shuang, Y.; Guo, B.-N.; Qi, F. Logarithmic convexity and increasing property of the Bernoulli numbers and their ratios. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2021**, *115*, 12. [CrossRef]
77. Bagul, Y.J.; Banjac, B.; Chesneau, C.; Kostić, M.; Malešević, B. New refinements of Cusa–Huygens inequality. *Results Math.* **2021**, *76*, 16. [CrossRef]
78. Bagul, Y.J.; Chesneau, C.; Kostić, M. The Cusa-Huygens inequality revisited. *Novi Sad J. Math.* **2020**, *50*, 149–159. [CrossRef]
79. Bagul, Y.J.; Kostić, M.; Chesneau, C.; Dhaigude, R.M. On the generalized Becker-Stark type inequalities. *Acta Univ. Sapientiae Math.* **2021**, *13*, 88–104. [CrossRef]
80. Bouali, M. Double inequalities for complete monotonicity degrees of remainders of asymptotic expansions of the gamma and digamma functions. *arXiv* **2022**, https://arxiv.org/abs/2202.01801v1.
81. Chen, X.-D.; Wang, H.; Yu, J.; Cheng, Z.; Zhu, P. New bounds of Sinc function by using a family of exponential functions. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2022**, *116*, 17. [CrossRef]
82. Chouchi, B.; Fedorov, V.; Kostić, M. Monotonicity of certain classes of functions related with Cusa–Huygens inequality. *Chelyab. Fiz.-Mat. Zh. (Chelyabinsk Phys. Math. J.)* **2021**, *6*, 331–337.
83. Englert, C.; Giudice, G.F.; Greljo, A.; McCullough, M. The $\hat{H}$-parameter: An oblique Higgs view. *J. High Energy Phys.* **2019**, 28. [CrossRef]
84. Kostić, M.; Pilipović, S.; Velinov, D.; Fedorov, V.E. *c*-almost periodic type distributions. *Chelyab. Fiz.-Mat. Zh. (Chelyabinsk Phys. Math. J.)* **2021**, *6*, 190–207. [CrossRef]
85. Liu, J.-C. On two supercongruences for sums of Apéry-like numbers. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2021**, *115*, 7. [CrossRef]
86. Tan, S.-Y.; Huang, T.-R.; Chu, Y.-M. Functional inequalities for Gaussian hypergeometric function and generalized elliptic integral of the first kind. *Math. Slovaca* **2021**, *71*, 667–682. [CrossRef]
87. Volkov, Y.S. Efficient computation of Favard constants and their connection to Euler polynomials and numbers. *Sib. Èlektron. Mat. Izv. (Sib. Electron. Math. Rep.)* **2020**, *17*, 1921–1942. [CrossRef]
88. Xu, A.-M.; Cen, Z.-D. Qi's conjectures on completely monotonic degrees of remainders of asymptotic formulas of di- and tri-gamma functions. *J. Inequal. Appl.* **2020**, 10. [CrossRef]
89. Yang, Z.-H. Approximations for certain hyperbolic functions by partial sums of their Taylor series and completely monotonic functions related to gamma function. *J. Math. Anal. Appl.* **2016**, *441*, 549–564. [CrossRef]
90. Yang, Z.-H.; Tian, J.-F. Sharp bounds for the ratio of two zeta functions. *J. Comput. Appl. Math.* **2020**, *364*, 112359. [CrossRef]
91. Zhao, T.-H.; Qian, W.-M.; Chu, Y.-M. Sharp power mean bounds for the tangent and hyperbolic sine means. *J. Math. Inequal.* **2021**, *15*, 1459–1472. [CrossRef]
92. Zhu, L. New bounds for arithmetic mean by the Seiffert-like means. *Mathematics* **2022**, *10*, 14. [CrossRef]
93. Li, Y.-W.; Dağlı, M.C.; Qi, F. Two explicit formulas for degenerate Peters numbers and polynomials. *Discrete Math. Lett.* **2022**, *8*, 1–5. [CrossRef]
94. Guo, B.-N.; Polatlı, E.; Qi, F. *Determinantal Formulas and Recurrent Relations for Bi-Periodic Fibonacci and Lucas Polynomials*; Chapter 18 in the Springer Proceedings of the International Conference on Advances in Mathematics and Computing (ICAMC-2020) organized by Veer Surendra Sai University of Technology, Odisha, India, during 7–8 February 2020; Paikray, S.K., Dutta, H., Mordeson, J.N., Eds.; New Trends in Applied Analysis and Computational Mathematics; Springer Book Series; Advances in Intelligent Systems and Computing; Springer: Singapore, 2021; Volume 1356. [CrossRef]

95. Kızılateş, C.; Du, W.-S.; Qi, F. Several determinantal expressions of generalized Tribonacci polynomials and sequences. *Tamkang J. Math.* **2022**, *53*, *in press*. [CrossRef]
96. Qi, F. Determinantal expressions and recurrence relations for Fubini and Eulerian polynomials. *J. Interdiscip. Math.* **2019**, *22*, 317–335. [CrossRef]
97. Qi, F.; Determinantal expressions and recursive relations of Delannoy polynomials and generalized Fibonacci polynomials. *J. Nonlinear Convex Anal.* **2021**, *22*, 1225–1239.
98. Qi, F. Simplifying coefficients in differential equations related to generating functions of reverse Bessel and partially degenerate Bell polynomials. *Bol. Soc. Paran. Mat.* **2021**, *39*, 73–82. [CrossRef]
99. Qi, F.; Čerňanová, V.; Semenov, Y.S. Some tridiagonal determinants related to central Delannoy numbers, the Chebyshev polynomials, and the Fibonacci polynomials. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **2019**, *81*, 123–136.
100. Qi, F.; Chapman, R.J. Two closed forms for the Bernoulli polynomials. *J. Number Theory* **2016**, *159*, 89–100. [CrossRef]
101. Qi, F.; Dağlı, M.C.; Du, W.-S. Determinantal forms and recursive relations of the Delannoy two-functional sequence. *Adv. Theory Nonlinear Anal. Appl.* **2020**, *4*, 184–193. [CrossRef]
102. Qi, F.; Dağlı, M.C.; Lim, D. Several explicit formulas for (degenerate) Narumi and Cauchy polynomials and numbers. *Open Math.* **2021**, *19*, 833–849. [CrossRef]
103. Qi, F.; Guo, B.-N. A closed form for the Stirling polynomials in terms of the Stirling numbers. *Tbilisi Math. J.* **2017**, **10**, 153–158. [CrossRef]
104. Qi, F.; Guo, B.-N. A determinantal expression and a recurrence relation for the Euler polynomials. *Adv. Appl. Math. Sci.* **2017**, *16*, 297–309.
105. Qi, F.; Guo, B.-N. Explicit formulas and recurrence relations for higher order Eulerian polynomials. *Indag. Math.* **2017**, *28*, 884–891. [CrossRef]
106. Qi, F.; Guo, B.-N. Expressing the generalized Fibonacci polynomials in terms of a tridiagonal determinant. *Matematiche (Catania)* **2017**, *72*, 167–175. [CrossRef]
107. Qi, F.; Guo, B.-N. Some determinantal expressions and recurrence relations of the Bernoulli polynomials. *Mathematics* **2016**, *4*, 65. [CrossRef]
108. Qi, F.; Guo, B.-N. Some properties of the Hermite polynomials. *Georgian Math. J.* **2022**, *28*, 925–935. [CrossRef]
109. Qi, F.; Guo, B.-N. Two nice determinantal expressions and a recurrence relation for the Apostol–Bernoulli polynomials. *J. Indones. Math. Soc. (MIHMI)* **2017**, *23*, 81–87. [CrossRef]
110. Qi, F.; Kızılateş, C.; Du, W.-S. A closed formula for the Horadam polynomials in terms of a tridiagonal determinant. *Symmetry* **2019**, *11*, 8. [CrossRef]
111. Qi, F.; Lim, D.; Guo, B.-N. Some identities related to Eulerian polynomials and involving the Stirling numbers. *Appl. Anal. Discrete Math.* **2018**, *12*, 467–480. [CrossRef]
112. Wang, Y.; Dağlı, M.C.; Liu, X.-M.; Qi, F. Explicit, determinantal, and recurrent formulas of generalized Eulerian polynomials. *Axioms* **2021**, *10*, 37. [CrossRef]
113. Wu, L.; Chen, X.-Y.; Dağlı, M.C.; Qi, F. On degenerate array type polynomials. *CMES Comput. Model. Eng. Sci.* **2022**, *131*, 295–305. [CrossRef]
114. Zhao, J.-L.; Wang, J.-L.; Qi, F. Derivative polynomials of a function related to the Apostol–Euler and Frobenius–Euler numbers. *J. Nonlinear Sci. Appl.* **2017**, *10*, 1345–1349. [CrossRef]
115. Qi, F.; Necessary and sufficient conditions for complete monotonicity and monotonicity of two functions defined by two derivatives of a function involving trigamma function. *Appl. Anal. Discrete Math.* **2021**, *15*, 378–392. [CrossRef]
116. Qi, F.; Agarwal, R.P. On complete monotonicity for several classes of functions related to ratios of gamma functions. *J. Inequal. Appl.* **2019**, Paper No. 36, 42 pages. [CrossRef]
117. Qi, F.; Lim, D. Monotonicity properties for a ratio of finite many gamma functions. *Adv. Differ. Equ.* **2020**, 193. [CrossRef] [PubMed]
118. Qi, F. Alternative proofs for monotonicity of some functions related to sectional curvature of Fisher–Rao manifold of beta distributions. In *Applied Nonlinear Analysis and Soft Computing: Proceedings of ANASC 2020, Guwahati, India, 22–23 December 2020*; Advance in Intelligent Systems and Computing; Springer: Singapore, 2022; *in press*.
119. Qi, F. Decreasing properties of two ratios defined by three and four polygamma functions. *C. R. Math. Acad. Sci. Paris* **2022**, *360*, 89–101. [CrossRef]
120. Qi, F. Decreasing property and complete monotonicity of two functions constituted via three derivatives of a function involving trigamma function. *Math. Slovaca* **2022**, *72*, *in press*. [CrossRef]
121. Qi, F. Lower bound of sectional curvature of Fisher–Rao manifold of beta distributions and complete monotonicity of functions involving polygamma functions. *Results Math.* **2021**, *76*, 217. [CrossRef]
122. Qi, F. Necessary and sufficient conditions for a difference constituted by four derivatives of a function involving trigamma function to be completely monotonic. *Math. Inequal. Appl.* **2021**, *24*, 845–855. [CrossRef]
123. Qi, F. Necessary and sufficient conditions for a difference defined by four derivatives of a function containing trigamma function to be completely monotonic. *Appl. Comput. Math.* **2022**, *21*, 61–70. [CrossRef]
124. Qi, F. Necessary and sufficient conditions for a ratio involving trigamma and tetragamma functions to be monotonic. *Turkish J. Inequal.* **2021**, *5*, 50–59.

125. Qi, F. Some properties of several functions involving polygamma functions and originating from the sectional curvature of the beta manifold. *São Paulo J. Math. Sci.* **2020**, *14*, 614–630. [CrossRef]
126. Qi, F. Two monotonic functions defined by two derivatives of a function involving trigamma function. *TWMS J. Pure Appl. Math.* **2022**, *13*, 91–104.
127. Brigant, A.L.; Preston, S.C.; Puechmorel, S. Fisher–Rao geometry of Dirichlet distributions. *Differ. Geom. Appl.* **2021**, *74*, 16. [CrossRef]
128. Bai, Y.-M.; Qi, F. Some integral inequalities of the Hermite–Hadamard type for log-convex functions on co-ordinates. *J. Nonlinear Sci. Appl.* **2016**, *9*, 5900–5908. [CrossRef]
129. Bai, S.-P.; Wang, S.-H.; Qi, F. On HT-convexity and Hadamard-type inequalities. *J. Inequal. Appl.* **2020**, 1–12.[CrossRef]
130. Hua, J.; Xi, B.-Y.; Qi, F. Some new inequalities of Simpson type for strongly s-convex functions. *Afr. Mat.* **2015**, *26*, 741–752. [CrossRef]
131. Qi, F.; Zhang, T.-Y.; Xi, B.-Y. Hermite–Hadamard-type integral inequalities for functions whose first derivatives are convex. *Ukrainian Math. J.* **2015**, *67*, 625–640. [CrossRef]
132. Shuang, Y.; Qi, F. Integral inequalities of the Hermite–Hadamard type for (α, m)-GA-convex functions. *J. Nonlinear Sci. Appl.* **2017**, *10*, 1854–1860. [CrossRef]
133. Wang, Y.; Zheng, M.-M.; Qi, F. Integral inequalities of Hermite–Hadamard type for functions whose derivatives are (α, m)-preinvex. *J. Inequal. Appl.* **2014**, *2014*, 10. [CrossRef]
134. Wu, Y.; Qi, F. Discussions on two integral inequalities of Hermite–Hadamard type for convex functions. *J. Comput. Appl. Math.* **2022**, *406*, 6. [CrossRef]
135. Wu, Y.; Qi, F.; Pei, Z.-L.; Bai, S.-P. Hermite–Hadamard type integral inequalities via (s, m)-P-convexity on co-ordinates. *J. Nonlinear Sci. Appl.* **2016**, **9**, 876–884. [CrossRef]
136. Xi, B.-Y.; Bai, S.-P.; Qi, F. On integral inequalities of the Hermite–Hadamard type for co-ordinated (α, m_1)-(s, m_2)-convex functions. *J. Interdiscip. Math.* **2018**, *21*, 1505–1518. [CrossRef]
137. Xi, B.-Y.; Gao, D.-D.; Qi, F. Integral inequalities of Hermite–Hadamard type for (α, s)-convex and (α, s, m)-convex functions. *Ital. J. Pure Appl. Math.* **2020**, *44*, 499–510.
138. Xi, B.-Y.; Qi, F. Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means. *J. Nonlinear Convex Anal.* **2015**, *16*, 873–890.
139. Xi, B.-Y.; Qi, F. Properties and inequalities for the (h_1, h_2)- and (h_1, h_2, m)-GA-convex functions. *Cogent Math.* **2016**, *3*, 1176620. [CrossRef]
140. Xi, B.-Y.; Qi, F. Some inequalities of Hermite–Hadamard type for geometrically P-convex functions. *Adv. Stud. Contemp. Math. (Kyungshang)* **2016**, *26*, 211–220.
141. Huang, C.-J.; Rahman, G.; Nisar, K.S.; Ghaffar, A.; Qi, F. Some inequalities of the Hermite–Hadamard type for k-fractional conformable integrals. *Austral. J. Math. Anal. Appl.* **2019**, *16*, 7. Available online: http://ajmaa.org/cgi-bin/paper.pl?string=v16n1/V16I1P7.tex (accessed on 10 July 2022).
142. Qi, F.; Rahman, G.; Hussain, S.M.; Du, W.-S.; Nisar, K.S. Some inequalities of Čebyšev type for conformable k-fractional integral operators. *Symmetry* **2018**, *10*, 614. [CrossRef]
143. Rahman, G.; Nisar, K.S.; Ghaffar, A.; Qi, F. Some inequalities of the Grüss type for conformable k-fractional integral operators. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **2020**, *114*, 9. [CrossRef]
144. Hoorfar, A.; Qi, F. A new refinement of Young's inequality. *Math. Inequal. Appl.* **2008**, *11*, 689–692. [CrossRef]
145. Qi, F.; Li, W.-H.; Wu, G.-S.; Guo, B.-N. Refinements of Young's integral inequality via fundamental inequalities and mean value theorems for derivatives. In *Topics in Contemporary Mathematical Analysis and Applications*; Dutt, H., Ed.; CRC Press: Boca Raton, FL, USA, 2021; Chapter 8, pp. 193–227. [CrossRef]
146. Qi, F.; Wan, A. Geometric interpretations and reversed versions of Young's integral inequality. *Adv. Theory Nonlinear Anal. Appl.* **2021**, *5*, 1–6. [CrossRef]
147. Wang, J.-Q.; Guo, B.-N.; Qi, F. Generalizations and applications of Young's integral inequality by higher order derivatives. *J. Inequal. Appl.* **2019**, 18. [CrossRef]
148. Mei, J.-Q.; Xu, S.-L.; Qi, F. Curvature pinching for minimal submanifolds in unit spheres. *Math. Appl. (Wuhan)* **1999**, *12*, 5–10.
149. Qi, F.; Guo, B.-N. Lower bound of the first eigenvalue for the Laplace operator on compact Riemannian manifold. *Chinese Quart. J. Math.* **1993**, *8*, 40–49.
150. Qi, F.; Yu, L.-Q.; Luo, Q.-M. Estimates for the upper bounds of the first eigenvalue on submanifolds. *Chin. Quart. Math.* **1994**, *9*, 40–43.
151. Qi, F.; Zheng, M.-M. Absolute monotonicity of functions related to estimates of first eigenvalue of Laplace operator on Riemannian manifolds. *Int. J. Anal. Appl.* **2014**, *6*, 123–131.
152. Xu, S.-L.; Huang, Z.; Qi, F. A rigidity theorem for manifold with a nice submanifold. *Math. Appl. (Wuhan)* **1999**, *12*, 72–75.
153. Pólya, G. Ein mittelwertsatz für Funktionen mehrerer Veränderlichen. *Tôhoku Math. J.* **1921**, *19*, 1–3.
154. Qi, F. Inequalities for an integral. *Math. Gaz.* **1996**, *80*, 376–377. [CrossRef]
155. Sun, Y.; Yang, H.-T.; Qi, F. Some inequalities for multiple integrals on the n-dimensional ellipsoid, spherical shell, and ball. *Abstr. Appl. Anal.* **2013**, *2013*, 904721. [CrossRef]
156. Guo, B.-N.; Qi, F. *Estimates for An Integral in L^p Norm of the $(n+1)$-th Derivative of Its Integrand, Inequality Theory and Applications*; Cho, Y.J., Kim, J.K., Dragomir, S.S., Eds.; Nova Science Publishers: Hauppauge, NY, USA, 2003; Volume 3, pp. 127–131..

157. Guo, B.-N.; Qi, F. Proofs of an integral inequality. *Math. Informatics Q.* **1997**, *7*, 182–184.
158. Guo, B.-N.; Qi, F. Some estimates of an integral in terms of the L^p-norm of the $(n+1)$st derivative of its integrand. *Anal. Math.* **2003**, *29*, 1–6. http://dx.doi.org/10.1023/A:1022894413541. [CrossRef]
159. Qi, F. Further generalizations of inequalities for an integral. *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* **1997**, *8*, 79–83. Available online: https://www.jstor.org/stable/i40147167 (accessed on 10 July 2022).
160. Qi, F. Inequalities for a multiple integral. *Acta Math. Hungar.* **1999**, *84*, 19–26.:1006642601341. [CrossRef]
161. Qi, F. Inequalities for a weighted multiple integral. *J. Math. Anal. Appl.* **2001**, *253*, 381–388. [CrossRef]
162. Qi, F.; Cerone, P.; Dragomir, S.S. Some new Iyengar type inequalities. *Rocky Mountain J. Math.* **2005**, *35*, 997–1015. [CrossRef]
163. Qi, F.; Zhang, Y.-J. Inequalities for a weighted integral. *Adv. Stud. Contemp. Math. (Kyungshang)* **2002**, *4*, 93–101.
164. Qi, F. Pólya type integral inequalities: Origin, variants, proofs, refinements, generalizations, equivalences, and applications. *Math. Inequal. Appl.* **2015**, *18*, 1–38. [CrossRef]
165. Qi, F.; Luo, Q.-M. Refinements and extensions of an inequality. *Math. Inform. Q.* **1999**, *9* , 23–25.
166. Qi, F.; Xu, S.-L. Refinements and extensions of an inequality, II. *J. Math. Anal. Appl.* **1997**, *211*, 616–620. [CrossRef]
167. Guo, B.-N.; Qi, F. On the degree of the weighted geometric mean as a complete Bernstein function. *Afr. Mat.* **2015**, *26*, 1253–1262. [CrossRef]
168. Qi, F. Bounding the difference and ratio between the weighted arithmetic and geometric means. *Int. J. Anal. Appl.* **2017**, *13*, 132–135.
169. Qi, F.; Chen, S.-X. Complete monotonicity of the logarithmic mean. *Math. Inequal. Appl.* **2007**, *10*, 799–804. [CrossRef]
170. Qi, F.; Guo, B.-N. The reciprocal of the geometric mean of many positive numbers is a Stieltjes transform. *J. Comput. Appl. Math.* **2017**, *311*, 165–170. [CrossRef]
171. Qi, F.; Guo, B.-N. The reciprocal of the weighted geometric mean is a Stieltjes function. *Bol. Soc. Mat. Mex. (3)* **2018**, *24*, 181–202. [CrossRef]
172. Qi, F.; Guo, B.-N. The reciprocal of the weighted geometric mean of many positive numbers is a Stieltjes function. *Quaest. Math.* **2018**, *41*, 653–664. [CrossRef]
173. Qi, F.; Lim, D. Integral representations of bivariate complex geometric mean and their applications. *J. Comput. Appl. Math.* **2018**, *330*, 41–58. [CrossRef]
174. Qi, F.; Zhang, X.-J.; Li, W.-H. An elementary proof of the weighted geometric mean being a Bernstein function. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **2015**, *77*, 35–38.
175. Qi, F.; Zhang, X.-J.; Li, W.-H. An integral representation for the weighted geometric mean and its applications. *Acta Math. Sin. (Engl. Ser.)* **2014**, *30*, 61–68. [CrossRef]
176. Qi, F.; Zhang, X.-J.; Li, W.-H. Lévy–Khintchine representation of the geometric mean of many positive numbers and applications. *Math. Inequal. Appl.* **2014**, *17*, 719–729. [CrossRef]
177. Qi, F.; Zhang, X.-J.; Li, W.-H. Lévy–Khintchine representations of the weighted geometric mean and the logarithmic mean. *Mediterr. J. Math.* **2014**, *11*, 315–327. [CrossRef]
178. Qi, F.; Zhang, X.-J.; Li, W.-H. The harmonic and geometric means are Bernstein functions. *Bol. Soc. Mat. Mex. (3)* **2017**, *23*, 713–736. [CrossRef]
179. Aljinović, A.A.; Pečarić, J. Note on an integral inequality similar to Qi's inequality. *Math. Maced.* **2009**, *7*, 1–7.
180. Chammam, W. Catalan–Qi numbers, series involving the Catalan–Qi numbers and a Hankel determinant evaluation. *J. Math.* **2020**, 8101725. [CrossRef]
181. Corcino, R.B.; Vega, M.A.R.P.; Dibagulun, A.M. A (p,q)-analogue of Qi-type formula for r-Dowling numbers. *J. Math. Computer Sci.* **2022**, *24*, 273–286. [CrossRef]
182. Mao, Z.-X.; Zhu, Y.-R.; Guo, B.-H.; Wang, F.-H.; Yang, Y.-H.; Zhao, H.-Q. Qi type diamond-alpha integral inequalities. *Mathematics* **2021**, *9*, 24. [CrossRef]
183. Nemes, G. A solution to an open problem on Mathieu series posed by Hoorfar and Qi. *Acta Math. Vietnam.* **2012**, *37*, 301–310.
184. Pečarić, J.; T Pejković, T. Note on Feng Qi's integral inequality. *J. Inequal. Pure Appl. Math.* **2004**, *5*, 51. Available online: http://www.emis.de/journals/JIPAM/article418.html (accessed on 10 July 2022).
185. Pogány, T.K. On an open problem of F. Qi. *J. Inequal. Pure Appl. Math.* **2002**, *3*, 54. Available online: http://www.emis.de/journals/JIPAM/article206.html (accessed on 10 July 2022).
186. Bernstein, D.S. *Scalar, Vector, and Matrix Mathematics: Theory, Facts, and Formulas-Revised and Expanded Edition*; Expanded, R.A., Ed.; Princeton University Press: Princeton, NJ, USA, 2018. Available online: https://press.princeton.edu/books/hardcover/9780691151205/scalar-vector-and-matrix-mathematics (accessed on 10 July 2022).
187. Bernstein, D.S. *Matrix Mathematics: Theory, Facts, and Formulas*, 2nd ed.; Princeton University Press: Princeton, NJ, USA, 2009. [CrossRef]
188. Bullen, P.S. *Dictionary of Inequalities*, 2nd ed.; Monographs and Research Notes in Mathematics; CRC Press: Boca Raton, FL, USA, 2015. [CrossRef]
189. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Digital Library of Mathematical Functions*; U.S. Department of Commerce, National Institute of Standards and Technology: Washington, DC, USA; Cambridge University Press: Cambridge, UK, 2010–2020. Available online: https://dlmf.nist.gov/ (accessed on 10 July 2022).

190. Olver, F.W.J.; Lozier, D.W.; Boisvert, R.F.; Clark, C.W. (Eds.) *NIST Handbook of Mathematical Functions*; U.S. Department of Commerce, National Institute of Standards and Technology: Washington, DC, USA; Cambridge University Press, Cambridge, UK, 2010. Available online: http://www.cambridge.org/catalogue/catalogue.asp?isbn=9780521192255 (accessed on 10 July 2022).
191. Schilling, R.L.; Song, R.; Vondraček, Z. *Bernstein Functions. Theory and Applications*, de Gruyter Studies in Mathematics; Walter de Gruyter & Co.: Berlin, Germany, 2010; Volume 37.
192. Schilling, R.L.; Song, R.; Vondraček, Z. *Bernstein Functions: Theory and Applications*, 2nd ed.; De Gruyter Studies in Mathematics; Walter de Gruyter & Co.: Berlin, Germany, 2012; Volume 37. [CrossRef]
193. Qi, F.; Guo, B.-N. Complete monotonicities of functions involving the gamma and digamma functions. *RGMIA Res. Rep. Coll.* **2004**, *7*, 63–72. Available online: http://rgmia.org/v7n1.php (accessed on 10 July 2022).
194. Qi, F.; Guo, B.-N.; Chen, C.-P. Some completely monotonic functions involving the gamma and polygamma functions. *RGMIA Res. Rep. Coll.* **2004**, *7*, 31–36. Available online: http://rgmia.org/v7n1.php (accessed on 10 July 2022). [CrossRef]
195. Qi, F.; Chen, C.-P. A complete monotonicity property of the gamma function. *J. Math. Anal. Appl.* **2004**, *296*, 603–607. [CrossRef]
196. Qi, F.; Guo, B.-N.; Chen, C.-P. Some completely monotonic functions involving the gamma and polygamma functions. *J. Aust. Math. Soc.* **2006**, *80*, 81–88. [CrossRef]
197. Berg, C. Integral representation of some functions related to the gamma function. *Mediterr. J. Math.* **2004**, *1*, 433–439. [CrossRef]
198. Guo, S.; Srivastava, H.M.; Cheung, W.-S. Some properties of functions related to certain classes of completely monotonic functions and logarithmically completely monotonic functions. *Filomat* **2014**, *28*, 821–828. [CrossRef]
199. Li, A.-J.; Zhao, W.-Z.; Chen, C.-P. Logarithmically complete monotonicity properties for the ratio of gamma function. *Adv. Stud. Contemp. Math. (Kyungshang)* **2006**, *13*, 183–191.
200. Mehrez, K.; Das, S. Logarithmically completely monotonic functions related to the q-gamma function and its applications. *Anal. Math. Phys.* **2022**, *12*, 20. [CrossRef]
201. Tian, J.-F.; Yang, Z.-H. Logarithmically complete monotonicity of ratios of q-gamma functions. *J. Math. Anal. Appl.* **2022**, *508*, 13. [CrossRef]
202. Atanassov, R.D.; Tsoukrovski, U.V. Some properties of a class of logarithmically completely monotonic functions. *C. R. Acad. Bulgare Sci.* **1988**, *41*, 21–23.
203. Cao, J.; Niu, D.-W.; Qi, F. A refinement of Carleman's inequality. *Adv. Stud. Contemp. Math. (Kyungshang)* **2006**, *13*, 57–62.
204. Cao, J.; Niu, D.-W.; Qi, F. An extension and a refinement of van der Corput's inequality. *Internat. J. Math. Math. Sci.* **2006**, 70786. [CrossRef]
205. Huo, Z.-H.; Niu, D.-W.; Cao, J.; Qi, F. A generalization of Jordan's inequality and an application. *Hacet. J. Math. Stat.* **2011**, *40*, 53–61.
206. Jiang, W.-D.; Cao, J.; Qi, F. Sharp inequalities for bounding Seiffert mean in terms of the arithmetic, centroidal, and contra-harmonic means. *Math. Slovaca* **2016**, *66*, 1115–1118. [CrossRef]
207. Niu, D.-W.; Cao, J.; Qi, F. A class of logarithmically completely monotonic functions related to $(1+1/x)^x$ and an application. *Gen. Math.* **2006**, *14*, 97–112.
208. Niu, D.-W.; Cao, J.; Qi, F. A refinement of van der Corput's inequality. *J. Inequal. Pure Appl. Math.* **2006**,*7*, 127. Available online: http://www.emis.de/journals/JIPAM/article744.html (accessed on 10 July 2022).
209. Niu, D.-W.; Cao, J.; Qi, F. Generalizations of Jordan's inequality and concerned relations. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* **2010**, *72*, 85–98.
210. Niu, D.-W.; Huo, Z.-H.; Cao, J.; Qi, F. A general refinement of Jordan's inequality and a refinement of L. Yang's inequality. *Integral Transforms Spec. Funct.* **2008**, *19*, 157–164. [CrossRef]
211. Qi, F.; Cao, J.; Niu, D.-W. A generalization of van der Corput's inequality. *Appl. Math. Comput.* **2008**, *203*, 770–777. [CrossRef]
212. Qi, F.; Cao, J.; Niu, D.-W. More notes on a functional equation. *Internat. J. Math. Ed. Sci. Tech.* **2006**, *37*, 865–868. [CrossRef]
213. Qi, F.; Cao, J.; Niu, D.-W.; Ujević, N. An upper bound of a function with two independent variables. *Appl. Math. E-Notes* **2006**, *6*, 17.
214. Qi, F.; Jiang, W.-D.; Cao, J. Two double inequalities for the Seiffert mean in terms of the arithmetic, centroidal, and contra-harmonic means. *Adv. Stud. Contemp. Math. (Kyungshang)* **2015**, *25*, 547–552.
215. Qi, F.; Li, A.-J.; Zhao, W.-Z.; Niu, D.-W.; Cao, J. Extensions of several integral inequalities. *J. Inequal. Pure Appl. Math.* **2006**, *7*, 107. Available online: http://www.emis.de/journals/JIPAM/article706.html (accessed on 10 July 2022).
216. Qi, F.; Niu, D.-W.; Cao, J. An infimum and an upper bound of a function with two independent variables. *Octogon Math. Mag.* **2006**, *14*, 248–250.
217. Qi, F.; Niu, D.-W.; Cao, J. Logarithmically completely monotonic functions involving gamma and polygamma functions. *J. Math. Anal. Approx. Theory* **2006**, *1*, 66–74.
218. Qi, F.; Niu, D.-W.; Cao, J.; Chen, S.-X. Four logarithmically completely monotonic functions involving gamma function. *J. Korean Math. Soc.* **2008**, *45*, 559–573. [CrossRef]
219. Mahmoud, M.; Qi, F. Bounds for completely monotonic degrees of remainders in asymptotic expansions of the digamma function. *Math. Inequal. Appl.* **2022**, *25*, 291–306. [CrossRef]
220. Ouimet, F.; Qi, F. Logarithmically complete monotonicity of a matrix-parametrized analogue of the multinomial distribution. *Math. Inequal. Appl.* **2022**, *25*, 703–714. [CrossRef]
221. Qi, F. Bounds for the ratio of two gamma functions. *J. Inequal. Appl.* **2010**, *2010*, 493058. [CrossRef]

222. Qi, F. Bounds for the ratio of two gamma functions: from Gautschi's and Kershaw's inequalities to complete monotonicity. *Turkish J. Anal. Number Theory* **2014**, *2*, 152–164. [CrossRef]
223. Qi, F. Complete monotonicity for a new ratio of finitely many gamma functions. *Acta Math. Sci. Ser. B (Engl. Ed.)* **2022**, *42B*, 511–520. [CrossRef]
224. Qi, F.; Niu, D.-W.; Guo, B.-N. Refinements, generalizations, and applications of Jordan's inequality and related problems. *J. Inequal. Appl.* **2009**, *2009*, 271923. [CrossRef]