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On Implicative Derivations of MTL-Algebras

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Abstract: This paper introduces the implicative derivations and gives some of their characterizations on MTL-algebras. Furthermore, we provide some representation of MTL-algebras by implicative derivations and obtain some representation of Boolean algebra via the algebra of all implicative derivations. Finally, we explore the relationship between implicative derivation and other operators on MTL-algebras and show that there exists a bijection between the sets of multiplier and implicative derivations on IMTL-algebras. The results of this paper can provide the common properties of implicative derivations in the t-norm-based fuzzy logical algebras.

Keywords: MTL-algebra; implicative derivation; k-modal operator; Galois connection

MSC: 06D35

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1. Introduction

In order to capture the logic of t-norm-based fuzzy logics and their residual [1], Esteva et al. introduced the t-norm-based fuzzy logic **MTL** [2], and the resulting class of algebras called MTL-algebras. They have interesting algebraic properties and cover all the mathematical structures that appear in a t-norm-based fuzzy logic framework, such as, MV-algebras, BL-algebras, Gödel algebras, IMTL-algebras and R_0-algebras. Therefore, MTL-algebras are important algebraic structures in which the community of fuzzy logicians have become interested [1–5].

The notion of derivations is instrumental in studying properties and structure in fuzzy logical algebraic structures. In 1957, Posner [6] studied kinds of derivations in a prime ring and some of their basic algebraic properties. Afterward, Jun, Borzooei and Zhan et al. [7–9] produced some characterizations of p-semisimple BCI-algebras via derivations with respect to BCI-algebras with derivation. In 2008, Xin, Çeven et al. [10–12] characterized modular lattices and distributive lattices by isotone derivations with respect to lattices with derivations. Furthermore, Alshehri, Ghorbain, Yazarli, et al. [13–15] derived the derivations on MV-algebras and gave some conditions under which an additive derivation is isotone, in fact, for a linearly ordered MV-algebra. In 2013, Lee et al. [16,17] introduced and studied derivations and f-derivations on lattice implication algebras and discussed the relations between derivations and filters. In 2016, He et al. [18] investigated the kinds of derivations in residuated lattices, and characterized Heyting algebras with respect to the above derivations. In 2017, Hua [19] studied derivations in R_0-algebras, which are equivalent to NM-algebras, and discussed the relation between filters and the fixed point set of these derivations. The paper is motivated by the following considerations:

(1) It is well-known that derivations have been studied on MV-algebras, BL-algebras IMTL-algebras and residuated lattices and so on. Although they are essentially different logical algebras they are all particular types of MTL-algebras. Thus, it is

meaningful for us to establish the derivation theory of MTL-algebras for studying the common properties of derivations in t-norm-based fuzzy logical algebras.

(2) The previous research regarding derivations on logical algebras is multiplicative derivation, which is a map that satisfies

$$d(x \boxdot y) = (d(x) \boxdot y) \Cup (x \boxdot d(y)).$$

There are few studies, however, regarding derivations defined by $\rightarrow$ and any other operations on residuated structures so far. Therefore, it is interesting to study these derivations on logical algebras.

(3) It has always been known that Galois connections play a central role in studying logical algebras, and the relation between derivations and Galois connections is an important research topic to study. However, there are few research works regarding the relation between derivations and Galois connections on logical algebras so far. Thus, it is necessary for us to study the relation between derivations and Galois connections on logical algebras. Given these considerations, we propose a new type of derivation on MTL-algebras. Indeed,

(1) The notion of implicative derivations, which are defined by the operations $\hookrightarrow$ and $\Cup$, is introduced on MTL-algebras, and some characterizations of them are given. (see Definition 3, Theorem 1).
(2) Every implicative derivation is principle on IMTL-algebras (see Theorem 2, Remark 2).
(3) Every Boolean algebra represents the sets of all implicative derivations on Boolean algebras (see Theorem 6).
(4) There is an isotone Galois connection between the sets of multipliers and implicative derivations on IMTL-algebras (see Theorem 7).
(5) There is a bijection between the sets of multiplier and implicative derivations on IMTL-algebras (see Theorem 8).

In Section 2, we review some basic nations and definitions of MTL-algebras. In Section 3, we introduce implicative derivations on MTL-algebras and provide some of their characterizations. In Section 4, we give some representations of MTL-algebras by implicative derivations. In Section 5, we discuss the relationships between implicative derivations with other operators on MTL-algebras.

2. Preliminaries

First, some basic notions of MTL-algebras and their related algebraic results are presented.

Definition 1 ([5]). *An algebra* $(\mathcal{M}, \boxdot, \hookrightarrow, \Cap, \Cup, 0, 1)$ *is said to be a* residuated lattice *if*

(1) $(\mathcal{M}, \Cap, \Cup, 0, 1)$ *is a bounded lattice,*
(2) $(\mathcal{M}, \boxdot, 1)$ *is a commutative monoid,*
(3) $u \boxdot v \leq w$ *iff* $u \leq v \hookrightarrow w$*, for any* $u, v, w \in L$.

By $\mathcal{M}$ we mean that the universe of a residuated lattice $(\mathcal{M}, \boxdot, \hookrightarrow, \Cap, \Cup, 0, 1)$. On $\mathcal{M}$, we define

$$u \leq v \text{iff} u \hookrightarrow v = 1.$$

Then $\leq$ is a binary partial order on $\mathcal{M}$ and for $u \in \mathcal{M}, 0 \leq u \leq 1$.

A residuated lattice $\mathcal{M}$ is an *MTL-algebra* if it satisfies the prelinearlity equation:

$$\text{(PRE)} \quad (u \hookrightarrow v) \Cup (v \hookrightarrow u) = 1.$$

An MTL-algebra $\mathcal{M}$ is a *Gödel algebra* if it satisfies

$$\text{(IDE)} \quad u \boxdot u = u.$$

We denote the set $\{u | u \boxdot u = u\}$ of $\mathcal{M}$ by $\mathcal{I}(\mathcal{M})$.

An MTL-algebra $\mathcal{M}$ is an *IMTL-algebra* if it satisfies the double negation property:

$$\text{(DNP)} \quad \neg\neg u = u.$$

In every IMTL-algebra, we define further operations as follows:

$$u \boxplus v = \neg(\neg u \boxdot \neg v),$$

and also check

$$u \boxplus v = \neg(\neg u \boxdot \neg v), u \hookrightarrow v = \neg u \boxplus v.$$

An IMTL-algebra $\mathcal{M}$ is called an R_0*-algebra* if it satisfies:

$$\text{(WNM)}(u \boxdot v \hookrightarrow 0) \Cup (u \Cap v \hookrightarrow u \boxdot v) = 1.$$

Proposition 1 ([2]). *The following hold in any MTL-algebra $\mathcal{M}$, for all $u, v, w \in \mathcal{M}$,*
(1) *if $u \leq v$, then $v \hookrightarrow w \leq u \hookrightarrow w$, $w \hookrightarrow u \leq w \hookrightarrow v$ and $u \boxdot w \leq v \boxdot w$,*
(2) $u \hookrightarrow (u \Cap v) = u \hookrightarrow v$,
(3) $(u \hookrightarrow v) \boxdot w \leq (u \boxdot w) \hookrightarrow (v \boxdot w)$,
(4) $u \leq v \hookrightarrow u$,
(5) $u \hookrightarrow (v \Cap w) \leq (u \hookrightarrow v) \Cap (u \hookrightarrow w)$,
(6) $u \boxdot (u \Cap w) = (u \boxdot v) \Cap (u \boxdot w)$,
(7) $u \hookrightarrow v \leq (u \Cup w) \hookrightarrow (v \Cup w)$,
(8) $u \hookrightarrow v = (u \Cup v) \hookrightarrow v$,
(9) $u \boxdot (v \Cup w) = (u \boxdot v) \Cup (u \boxdot w)$,
(10) $u \hookrightarrow (v \Cap w) = (u \hookrightarrow v) \Cap (u \hookrightarrow w)$,
(11) $u \Cup v = ((u \hookrightarrow v) \hookrightarrow v) \Cap (v \hookrightarrow u) \hookrightarrow u)$,
(12) $u \hookrightarrow (v \Cup w) = (u \hookrightarrow v) \Cup (u \hookrightarrow w)$,
(13) $(\mathcal{M}, \Cap, \Cup)$ *is a distributive lattice,*
(14) *if $u \in \mathcal{I}(\mathcal{M})$ and $v, w \in \mathcal{M}$, then*
(i) $u \boxdot v = u \Cap v$,
(ii) $u \hookrightarrow (v \hookrightarrow w) = (u \hookrightarrow v) \hookrightarrow (u \hookrightarrow w)$.

Definition 2 ([20]). *Given sets $\mathcal{E}, \mathcal{F}$ and two order-preserving maps $f : \mathcal{E} \longrightarrow \mathcal{F}$ and $g : \mathcal{F} \longrightarrow \mathcal{E}$, the pair (f, g) establishes a* Galois connection *between $\mathcal{E}$ and $\mathcal{F}$ if $fg \geq id_{\mathcal{F}}$ and $gf \leq id_{\mathcal{E}}$.*

3. Implicative Derivations of MTL-Algebras

Then, we introduce derivations in MTL-algebras and give some of their characterizations.

Definition 3. *Let $\mathcal{M}$ be an MTL-algebra. A mapping $g : L \longrightarrow L$ is called an* implicative derivation *on L if*

$$g(u \hookrightarrow v) = (g(u) \hookrightarrow v) \Cup (u \hookrightarrow g(v)),$$

for any $u, v \in \mathcal{M}$.

Denoting $\mathcal{G}(\mathcal{M})$ to be the set of implicative derivations of $\mathcal{M}$.
Some examples of implicative derivations on MTL-algebras are presented.

Example 1. *Let $\mathcal{M}$ be an MTL-algebra. Define a mapping $1_g : \mathcal{M} \to \mathcal{M}$ by*

$$1_g(u) = 1$$

for all $u \in \mathcal{M}$. Then $1_g \in \mathcal{G}(\mathcal{M})$. Moreover, defining $g_1 : \mathcal{M} \to \mathcal{M}$ by

$$g_1(u) = u$$

for all $u \in \mathcal{M}$. Then $g_1 \in \mathcal{G}(\mathcal{M})$.

Example 2. *Let $\mathcal{M} = \{0, u, v, w, 1\}$ be a chain. Defining operations $\boxdot$ and $\hookrightarrow$ are*

$\boxdot$	0	u	v	w	1
0	0	0	0	0	0
u	0	u	0	u	u
v	0	0	v	v	v
w	0	u	v	w	w
1	0	u	v	w	1

$\hookrightarrow$	0	u	v	w	1
0	1	1	1	1	1
u	v	1	1	1	1
v	u	u	1	1	1
w	0	u	v	1	1
1	0	u	v	w	1

Then $(\mathcal{M}, \boxdot, \hookrightarrow, \Cap, \Cup, 0, 1)$ is an MTL-algebra. Now, we define $g : \mathcal{M} \to \mathcal{M}$ as follows:

$$g(x) = \begin{cases} 0, & x = 0, \\ u, & x = u, \\ v, & x = v, \\ 1, & x = w, 1. \end{cases}$$

Then $g \in \mathcal{G}(\mathcal{M})$.

Example 3. *Let $\mathcal{M}_n$ be a standard n-valued R_0-algebra for some $n \geq 2$.*

$$g(x) = \begin{cases} \frac{1}{n-1}, & u = 0 \\ \frac{n-2}{n-1} \hookrightarrow x, & u \neq 0' \end{cases}$$

Then $g \in \mathcal{G}(\mathcal{M})$.

Example 4. *Let $\mathcal{M} = [0, 1]$ and we define $\otimes, \Rightarrow$ on $\mathcal{M}$ are*

$$u \boxdot v = \min\{u, v\}, \; u \hookrightarrow v = \begin{cases} 1, & u \leq v, \\ v, & otherwise \end{cases}.$$

Then $(\mathcal{M}, \min, \max, \boxdot, \hookrightarrow, 0, 1)$ is an MTL-algebra. Now, we define $g : \mathcal{M} \to \mathcal{M}$ as follows:

$$g(u) = \begin{cases} u, & u \leq 0.5, \\ 1, & u \geq 0.5 \end{cases},$$

then $g \in \mathcal{G}(\mathcal{M})$.

Example 5. *Let $\mathcal{M}$ be an MTL-algebra and $a \in \mathcal{M}$. Then $g_a(u) = a \hookrightarrow u$ for any $u \in \mathcal{M}$ is an implication derivation on $\mathcal{M}$.*

Proposition 2. *Let $g \in \mathcal{G}(\mathcal{M})$. Then, for any $u, v \in \mathcal{M}$,*
(1) $g(1) = 1$,
(2) $u \leq g(u)$,
(3) $g(u) \Cap g(v) \leq g(u \hookrightarrow v)$,
(4) $g(u) \hookrightarrow v \leq u \hookrightarrow g(v)$,
(5) $g(u \hookrightarrow v) = u \hookrightarrow g(v)$,
(6) $g(u \hookrightarrow v) \geq g(u) \hookrightarrow g(v)$.

Proof. (1)–(3) are easily verified, we only show (4)–(6).
(4) It follows from (2) and Proposition 1(1).
(5) From (4), we have

$$\begin{aligned} g(u \hookrightarrow v) &= (g(u) \hookrightarrow v) \cup (u \hookrightarrow g(v)) \\ &= u \hookrightarrow g(v), \end{aligned}$$

for any $u, v \in \mathcal{M}$.
(6) It can be observed directly from (2) and Proposition 1(1). □

Theorem 1. *Let $g : \mathcal{M} \to \mathcal{M}$ be a map on an MTL-algebra $\mathcal{M}$. Then the following are equivalent:*
(1) $g \in \mathcal{D}(\mathcal{M})$,
(2) $g(u \hookrightarrow v) = u \hookrightarrow g(v)$ *for any* $u, v \in \mathcal{M}$.

Proof. (1) $\Rightarrow$ (2) Obviously from Proposition 2(5).
(2) $\Rightarrow$ (1) From (2), we have

$$g(1) = g(0 \hookrightarrow u) = 0 \hookrightarrow g(u) = 1,$$

and hence

$$1 = g(u \hookrightarrow u) = u \hookrightarrow g(u),$$

which implies $u \leq g(u)$. Then by Proposition 1(1),

$$\begin{aligned} g(u \hookrightarrow v) &= u \hookrightarrow g(v) \\ &= (g(u) \hookrightarrow v) \Cup (u \hookrightarrow g(v)), \end{aligned}$$

for any $u, v \in \mathcal{M}$. □

Remark 1. *The map $g_p : \mathcal{M} \to \mathcal{M}$, as defined by*

$$g_p(u) = p \hookrightarrow u$$

for any $u \in \mathcal{M}$, $g_p \in \mathcal{G}(\mathcal{M})$, which is said to be the principle implicative derivation. Indeed,

$$\begin{aligned} g_p(u \hookrightarrow v) &= p \hookrightarrow (u \hookrightarrow v) \\ &= u \hookrightarrow (p \hookrightarrow v) \\ &= u \hookrightarrow g_p(v), \end{aligned}$$

for any $u, v \in \mathcal{M}$.

By Remark 1, **whether any implicative derivation g can be represented as the form of g_p.**

Indeed, this assertion is not true for MTL-algebra.

Example 6. *Let $\mathcal{M} = \{0, u, v, 1\}$ be a chain. Defining operations $\boxdot$ and $\hookrightarrow$ are*

$\otimes$	0	u	v	1
0	0	0	0	0
u	0	u	u	u
v	0	u	v	v
1	0	u	v	1

$\Rightarrow$	0	u	v	1
0	1	1	1	1
u	0	1	1	1
v	0	u	1	1
1	0	u	v	1

Then $(\mathcal{M}, \min, \max, \boxdot, \hookrightarrow, 0, 1)$ is an MTL-algebra. Defining $g : \mathcal{M} \to \mathcal{M}$ as follows:

$$g(x) = \begin{cases} u, & x = 0 \\ v, & x = v \\ 1, & x = u, 1 \end{cases},$$

and $g \in \mathcal{G}(\mathcal{M})$. But

$$g_0(0) = 1 \neq u, g_u(0) = g_v(0) = g_1(0) = 0 \neq u.$$

So g is not representative of g_p, for any $p \in \mathcal{M}$.

However, some positive answers are given under certain conditions.

Theorem 2. *Let g be an implicative derivation on an IMTL-algebra $\mathcal{M}$. Then the following are equivalent:,*
(1) $g \in \mathcal{D}(\mathcal{M})$,
(2) $g(u) = \neg g(0) \hookrightarrow u$.

Proof. (1) For any $u \in \mathcal{M}$, by Theorem 1, we have

$$\begin{aligned} g(u) &= g(\neg\neg u) \\ &= g(\neg u \hookrightarrow 0) \\ &= \neg u \hookrightarrow g(0) \\ &= u \boxplus g(0) \\ &= g(0) \boxplus u \\ &= \neg g(0) \hookrightarrow u, \end{aligned}$$

for any $u \in \mathcal{M}$.
(2) Taking $p = \neg g(0)$ in Remark 1. □

Remark 2. (1) *Theorem 2 shows that g is determined by the element $\neg g(0)$ on IMTL-algebras. If we take $p = \neg g(0)$, then every implicative derivation on IMTL-algebras is principle.*
(2) *Every implicative derivation on IMTL-algebra is isotone. If $u \leq v$, then*

$$g(u) = \neg g(0) \hookrightarrow u \leq \neg g(0) \hookrightarrow v = g(v).$$

4. Characterizations of MTL-Algebras Based on Implicative Derivations

Here we study the algebraic structure of the set of implicative derivations and give some representations of MTL-algebras via them.

Theorem 3. *If $\mathcal{M}$ is an MTL-algebra, then $(\mathcal{G}(\mathcal{M}), \cap, \cup, g_1, 1_g)$ is a bounded distributive lattice, where*

$$(g_i \cap g_j)(u) = g_i(x) \Cap g_j(x),$$
$$(g_i \cup g_j)(u) = g_i(x) \Cup g_j(x).$$

for all $g_i, g_j \in \mathcal{G}(\mathcal{M})$, and $u \in \mathcal{M}$.

Proof. For any $g_i, g_j \in \mathcal{G}(\mathcal{M})$, and $u \in \mathcal{M}$, by Proposition 1(10) and (12), we have

$$\begin{aligned} (g_i \cap g_j)(u \hookrightarrow v) &= g_i(u \hookrightarrow v) \Cap g_j(u \hookrightarrow v) \\ &= (u \hookrightarrow g_i(v)) \Cap (u \hookrightarrow g_j(v)) \\ &= u \hookrightarrow (g_i(v) \Cap g_j(v)) \\ &= u \hookrightarrow (g_i \cap g_j)(v), \end{aligned}$$

and

$$\begin{aligned} (g_i \cup g_j)(u \hookrightarrow v) &= g_i(u \hookrightarrow v) \Cup g_j(u \hookrightarrow v) \\ &= (u \hookrightarrow g_i(v)) \Cup (u \hookrightarrow g_j(v)) \\ &= u \hookrightarrow (g_i(v) \Cup g_j(v)) \\ &= u \hookrightarrow (g_i \cup g_j)(v), \end{aligned}$$

which implies $g_i \cap g_j, g_i \cup g_j \in \mathcal{G}(\mathcal{M})$.
Furthermore, for any $g_i \in \mathcal{G}(\mathcal{M})$ and $x \in \mathcal{M}$, we have

$$(g_i \cap 1_g)(u) = g_i(u) \Cap 1_g(u) = g_i(u) \Cap 1 = g_i(u),$$
$$(g_i \cup 1_g)(u) = g_i(u) \Cup 1_g(u) = g_i(u) \Cup 1 = g_1(u)$$

which implies $g_i \cap 1_g = 1_g, g_i \cup 1_g = 1_g$.

Moreover, $(\mathcal{G}(\mathcal{M}), \cap, \cup, g_1, 1_g)$ is a bounded distributive lattice. □

Theorem 4. *If* $(\mathcal{M}, \Cap, \Cup, \hookrightarrow, 0, 1)$ *is a Gödel algebra (or an idempotent MTL-algebra), then* $(\mathcal{G}(\mathcal{M}), \cap, \cup, \Rightarrow, g_1, 1_g)$ *is also a Gödel algebra, where*

$$\begin{aligned}(g_i \cap g_j)(u) &= g_i(u) \Cap g_j(u),\\ (g_i \cup g_j)(u) &= g_i(u) \Cup g_j(u),\\ (g_i \Rightarrow g_j)(u) &= g_i(u) \hookrightarrow g_j(u).\end{aligned}$$

for all $g_i, g_j \in \mathcal{G}(\mathcal{M})$*, and* $u \in \mathcal{M}$.

Proof. By Theorem 3, $(\mathcal{G}(\mathcal{M}), \cap, \cup, g_1, 1_g)$ is a bounded distributive lattice if $\mathcal{M}$ is an MTL-algebra. Now, we prove that $(\mathcal{G}(\mathcal{M}), \cap, \cup, \Rightarrow, g_1, 1_g)$ is a Gödel algebra if $\mathcal{M}$ is a Gödel algebra. For any $g_i, g_j \in \mathcal{G}(\mathcal{M})$ and $u \in \mathcal{M}$, by Proposition 1(14)(ii), we have

$$\begin{aligned}(g_i \Rightarrow g_j)(u \Rightarrow v) &= g_i(u \hookrightarrow v) \hookrightarrow g_j(u \hookrightarrow v)\\ &= (u \hookrightarrow g_i(v)) \hookrightarrow (u \hookrightarrow g_j(v))\\ &= u \hookrightarrow (g_i(v) \hookrightarrow g_j(v))\\ &= u \hookrightarrow (g_i \Rightarrow g_j)(v),\end{aligned}$$

which implies $g_i \Rightarrow g_j \in \mathcal{G}(\mathcal{M})$.

By Theorems 3 and 4, the operations $\cap, \cup, \Rightarrow$ are well defined if $\mathcal{M}$ is a Gödel algebra. □

As a result of Theorems 3 and 4, some important findings are obtained.

Theorem 5. *If* $(\mathcal{M}, \Cap, \Cup, 0, 1)$ *is a Boolean algebra (or an idempotent IMTL-algebra), then* $(\mathcal{G}(\mathcal{M}), \cap, \cup, \star, g_1, 1_g)$ *is also a Boolean algebra, where*

$$\begin{aligned}(g_i \cap g_j)(u) &= g_i(u) \Cap g_j(u),\\ (g_i \cup g_j)(u) &= g_i(u) \Cup g_j(u),\\ (g_i)^\star(u) &= g_i(u) \hookrightarrow g_1(u).\end{aligned}$$

for any $g_i, g_j \in \mathcal{G}(\mathcal{M})$*, and* $u \in \mathcal{M}$.

Proof. By Theorem 3 $(\mathcal{G}(\mathcal{M}), \cap, \cup, g_1, 1_g)$ is a bounded distributive lattice if $\mathcal{M}$ is an MTL-algebra. Moreover, for any $g_i \in \mathcal{G}(\mathcal{M})$, and $u \in \mathcal{M}$, we have

$$\begin{aligned}(g_i \cap (g_i)^\star)(u) &= g_i(u) \Cap (g_i(u))^\star\\ &= g_i(u) \Cap (g_i(u) \hookrightarrow g_1(u))\\ &= g_i(u) \Cap g_1(u)\\ &= g_1(u),\\ (g_i \cup (g_i)^\star)(u) &= g_i(u) \Cup (g_i(u) \hookrightarrow g_1(u))\\ &= g_i(u) \Cup g_i(u))^\star \Cup g_1(u)\\ &= 1 \Cup g_1(u)\\ &= 1\\ &= 1_g(u),\end{aligned}$$

that is, $g_i \cap (g_i)^\star = g_1$ and $g_i \cup (g_i)^\star = 1_g$. □

Theorem 6. *Every Boolean algebra* $\mathcal{M}$ *is isomorphic to* $(\mathcal{G}(\mathcal{M}), \cap, \cup, \star, g_1, 1_g)$.

Proof. Define $\chi : \mathcal{M} \to \mathcal{G}(\mathcal{M})$ by

$$\chi(x)(u) = x \Cup u,$$

for any $a, u \in \mathcal{M}$. By Theorem 1, χ is well defined.

(1) If $\chi(x) = \chi(y)$, then $\chi(x)(u) = \chi(y)(u)$, and hence $x \Cup u = y \Cup u$ for all $u \in \mathcal{M}$. Now, if $u = x$, then $x = x \Cup x = x \Cup y$, that is, $y \leq x$. If $u = y$, then $x \Cup y = y \Cup y = y$, and hence $x \Cup y = y$, that is, $x \leq y$. So $x = y$, which shows that χ is an injective function.

(2) For any $g \in \mathcal{G}(\mathcal{M})$, there exists a $g(0) \in \mathcal{M}$ such that $g = \chi((g(0)))$, which implies that χ is a surjection function. Indeed, by Theorem 1(2), we have $g(u) = \neg u \hookrightarrow g(0) = u \boxplus g(0) = u \Cup g(0) = \chi((g(0)))(u)$, for any $u \in \mathcal{M}$.

(3) For any $x, y \in \mathcal{M}$, we have

$$\begin{aligned}
\chi(x \Cap y)(u) &= (x \Cap y) \Cup u \\
&= (x \Cup u) \Cap (y \cup u) \\
&= (\chi(x) \cap \chi(y))(u), \\
\chi(x \Cup y)(u) &= (x \Cup y) \Cap u \\
&= (x \Cap u) \Cup (y \Cap u) \\
&= (\chi(x) \cup \chi(y))(u), \\
\chi(\neg x)(u) &= \neg x \Cup u \\
&= x \hookrightarrow u \\
&= (x \hookrightarrow u) \Cap (u \hookrightarrow u) \\
&= (x \Cup u) \hookrightarrow u \\
&= (\chi(x)(u))^{\star}.
\end{aligned}$$

which implies that χ is a homomorphism.

Therefore $(\mathcal{M}, \Cap, \Cup, 0, 1)$ is isomorphic to $(\mathcal{G}(\mathcal{M}), \cap, \cup, \star, g_1, 1_g)$. □

5. Relations between Implicative Derivations and Other Operators on MTL-Algebras

Recall in [21] that a self map f is called a multiplier of a distributive lattice L if

$$f(u \Cap v) = u \Cap f(v),$$

for any $u, v \in \mathcal{M}$. Applying this notion to MTL-algebras as a self f satisfies

$$f(u \boxdot v) = u \boxdot f(v).$$

Denoting $\mathcal{M}(\mathcal{M})$ by the set of all multipliers of $\mathcal{M}$.

Proposition 3. *Let f be a multiplier on an MTL-algebra $\mathcal{M}$. Then, for any $u, v \in \mathcal{M}$,*
(1) $f(0) = 0$,
(2) $f(u) = u \boxdot f(1)$,
(3) $f(u) \leq u$,
(4) *if $u \leq v$, then $f(u) \leq f(v)$.*

Proof. The proof is easy, and hence omitted. □

Now, we discuss the relations between $\mathcal{M}(\mathcal{M})$ and $\mathcal{G}(\mathcal{M})$.

Let $\varphi : \mathcal{M}(\mathcal{M}) \to \mathcal{G}(\mathcal{M})$ be the map

$$\varphi(f)(u) = \neg(f(\neg u))$$

for any $f \in \mathcal{M}(\mathcal{M})$ and $x \in \mathcal{M}$, and $\psi : \mathcal{G}(\mathcal{M}) \to \mathcal{M}(\mathcal{M})$ be the map such that

$$\psi(g)(u) = \neg(g(\neg u))$$

for any $g \in \mathcal{G}(\mathcal{M})$, and $u \in \mathcal{M}$.

Theorem 7. *Let $\mathcal{M}$ be an IMTL-algebra. There exists here an isotone Galois connection between $\mathcal{M}(\mathcal{M})$ and $\mathcal{G}(\mathcal{M})$. Namely,*

$$f \leq \psi(g) \text{iff} g \leq \varphi(f)$$

for any $f \in \mathcal{M}(\mathcal{M})$ and $g \in \mathcal{G}(\mathcal{M})$.

Proof. (1) By Propositions 3(4) and Remark 2(2) that φ and ψ are isotone.

(2) If $f \leq \psi(g)$, that is $f(u) \leq \psi(g)(u) = \neg(g(\neg u))$, then $\neg(f(u)) \geq g(\neg u)$ for any $u \in \mathcal{M}$. So $\varphi(f)(u) = \neg(f(\neg u)) \geq g(u)$, which implies $g \leq \varphi(f)$.

Conversely, if $g \leq \varphi(f)$, that is $g(u) \leq \varphi(f)(u) = \neg(f(\neg u))$ for any $u \in \mathcal{M}$, then $f(\neg u) \leq \neg(g(u))$. So $f(u) \leq \neg(g(\neg u)) = \psi(g)(u)$ for any $u \in \mathcal{M}$, which implies $f \leq \psi(g)$. □

Theorem 8. *Let $\mathcal{M}$ be an IMTL-algebra. Then there exists a bijection between $\mathcal{G}(\mathcal{M})$ and $\mathcal{M}(\mathcal{M})$. Namely,*

(1) *if $f \in \mathcal{M}(\mathcal{M})$, then $\varphi(f) \in \mathcal{G}(\mathcal{M})$,*
(2) *if $g \in \mathcal{G}(\mathcal{M})$, then $\psi(g) \in \mathcal{M}(\mathcal{M})$,*
(3) *$\psi\varphi(f) = f$ and $\varphi\psi(g) = g$.*

Proof. If f is a multiplier on $\mathcal{M}$, then

$$\begin{aligned} \varphi(f)(u \hookrightarrow v) &= \neg(f\neg(u \hookrightarrow v)) \\ &= \neg(f(u \boxdot \neg v)) \\ &= \neg(u \boxdot f(\neg v)) \\ &= u \hookrightarrow \varphi(f)(v), \end{aligned}$$

for any $u, v \in \mathcal{M}$, by Theorem 1, $\varphi(f) \in \mathcal{G}(\mathcal{M})$.

Conversely, if $g \in \mathcal{G}(\mathcal{M})$, then

$$\begin{aligned} \psi(g)(u \boxdot v) &= \neg(g(\neg(u \boxdot v))) \\ &= \neg(g(u \boxdot \neg v)) \\ &= \neg(u \hookrightarrow g(\neg v)) \\ &= u \boxdot \neg(g(\neg v)) \\ &= u \boxdot \psi(g)(v), \end{aligned}$$

for any $u, v \in \mathcal{M}$, which implies that $\psi(g)$ is a multiplier on $\mathcal{M}$.

Moreover, by Proposition 3(2), we have

$$\begin{aligned} \psi\varphi(f)(u) &= f(\neg(f(\neg u))) \\ &= f(1) \otimes ((f(1)) \hookrightarrow u) \\ &= f(1) \boxdot u \\ &= f(u) \end{aligned}$$

for any $u \in \mathcal{M}$, and so $\psi\varphi(f) = f$. Similarly, $\varphi\psi(g) = g$. □

Borumand Saeid et al. introduced in [22] that a k-modal operator in BL-algebra, which is a map satisfies the following conditions:

(M1) $\Box u \boxdot \Box v \leq \Box(u \boxdot v)$,

(M2) if $u \leq v$, then $\Box u \leq \Box v$,
(M3) $1 \leq \Box 1$.

Proposition 4. *If L is an MTL-algebra and $a \in \mathcal{G}(\mathcal{M})$, then g_a is a k-modal operator on $\mathcal{M}$.*

Proof. (M2) and (M3) are easily verified. Then, we will show that (M1) also holds. Indeed, by Proposition 1(14)(ii), we have

$$\begin{aligned} g_a(u \hookrightarrow v) &= a \hookrightarrow (u \hookrightarrow v) \\ &= (a \hookrightarrow u) \hookrightarrow (a \hookrightarrow v) \\ &= g_a(u) \hookrightarrow g_a(v), \end{aligned}$$

for any $u, v \in \mathcal{M}$. Then, by Definition 1(3), we get $g_a(u) \boxdot g_a(v) \leq g_a(u \boxdot v)$ for any $u, v \in \mathcal{M}$. □

The condition $a \in G(L)$ is necessary.

Example 7. *Let $L = \{0, u, v, w, x, y, 1\}$ with lattice order $0 \leq u \leq w \leq 1$, $0 \leq v \leq x \leq 1$ and $v \leq w$. Defining operations $\boxdot$ and $\hookrightarrow$ as follows:*

$\boxdot$	0	u	v	w	x	y	1
0	0	0	0	0	0	0	0
u	0	u	u	u	u	u	u
v	0	u	v	u	v	u	v
w	0	u	u	u	u	w	w
x	0	u	v	u	v	w	x
y	0	u	u	w	w	x	x
1	0	u	v	w	x	y	1

$\hookrightarrow$	0	u	v	w	x	y	1
0	1	1	1	1	1	1	1
u	0	1	1	1	1	1	1
v	0	y	1	e	1	e	1
w	0	x	x	1	1	1	1
x	0	w	x	y	1	y	1
y	0	v	v	x	x	1	1
1	0	u	v	w	x	y	1

Then $(\mathcal{M}, \boxdot, \hookrightarrow, \Cap, \Cup, 0, 1)$ is an MTL-algebra. Defining $g_d : \mathcal{M} \to \mathcal{M}$ as follows:

$$g(x) = \begin{cases} 0, & x = 0 \\ w, & x = u \\ x, & x = v \\ y, & x = w, y \\ 1, & x = x, 1 \end{cases},$$

Then $g_d \in \mathcal{G}(\mathcal{M})$. However, it is not a k-modal operator on $\mathcal{M}$ since

$$g_x(w) \boxdot g_x(v) = w \nleqslant u = g_x(w \boxdot v).$$

The k-modal operator is not the implicative derivation on MTL-algebra.

Example 8. *Let $\mathcal{M}$ be the MTL-algebra in Example 7. Now, we define $\Box : \mathcal{M} \to \mathcal{M}$ as follows:*

$$\Box(x) = \begin{cases} 0, & x = 0, u, v, w, y \\ x, & x = x \\ 1, & x = 1 \end{cases},$$

Then $\Box$ is a k-modal operator on $\mathcal{M}$. However, it is not an implicative derivation on $\mathcal{M}$, since

$$\Box(y \hookrightarrow y) = 1 \neq 0 = y \hookrightarrow \Box y.$$

It is interesting to consider under which conditions, is every implicative derivation a k-modal operator on an MTL-algebra.

Proposition 5. *Let $\mathcal{M}$ be an MTL-algebra and $g \in \mathcal{G}(\mathcal{M})$ satisfies*

$$(*) \quad g(u \hookrightarrow v) = g(u) \hookrightarrow g(u \Cap v).$$

Then the following statements are equivalent:
(1) $g(u \hookrightarrow v) = g(u) \hookrightarrow g(v)$,
(2) *if* $u \leq v$, *then* $g(u) \leq g(v)$.

Proof. (1) $\Rightarrow$ (2) If $u \leq v$, then it follows from (1) that

$$1 = g(1) = g(u \hookrightarrow v) = g(u) \hookrightarrow g(v),$$

which implies $g(u) \leq g(v)$.
(2) $\Rightarrow$ (1) If g satisfies (2), then by $(*)$, we have

$$\begin{aligned} g(u) \boxdot g(u \hookrightarrow v) &= g(u) \boxdot (g(u) \hookrightarrow g(u \Cap v)) \\ &\leq g(u \Cap v) \\ &\leq g(v), \end{aligned}$$

which implies $g(u \hookrightarrow v) \leq g(u) \hookrightarrow g(v)$. Then by Proposition 2(6), we obtain $g(u \hookrightarrow v) = g(u) \hookrightarrow g(v)$ for any $u, v \in \mathcal{M}$. □

Corollary 1. *Let $\mathcal{M}$ be an IMTL-algebra and $g \in \mathcal{G}(\mathcal{M})$ satisfies $(*)$. Then g is a k-modal operator on $\mathcal{M}$.*

Proof. By Remark 2(2), and Propositions 2(1) and 5. □

6. Conclusions

The notion of implicative derivations is beneficial for discussing structures and properties in fuzzy logic algebraic. In order to provide the common properties of implicative derivations in the t-norm-based logical algebras, we introduce the implicative derivations on MTL-algebras and obtain some of their characterizations. We also obtain some characterizations of Boolean algebras via implicative derivations and show the relations between implicative derivations and other operators, for example, multiplier and k-modal operators, on IMTL-algebras. In the future, we will study some of their algebraic properties of derivations on algebraic hyperstructures [23,24].

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