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Stochastic Modelling of Red Palm Weevil Using Chemical Injection and Pheromone Traps

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Abstract: This paper deals with the mathematical modelling of the red palm weevil (RPW), *Rhynchophorus ferrugineus* (Olivier) (Coleoptera: Curculionidae), in date palms using chemical control by utilizing injection and sex pheromone traps. A deterministic and stochastic model for RPW is proposed and analyzed. The existence of a positive global solution for the stochastic RPW model is investigated, and the conditions for the extinction of RPWs from the stochastic system are obtained. The adequate criteria for the presence of a unique ergodic stationary distribution for the RPW system are established by creating suitable Lyapunov functions. The impact of chemical injection and pheromone traps on RPW is demonstrated. The importance of environmental noise on RPW is highlighted and simulated using the Milstein method.

Keywords: stochastic models; stability; sex pheromone trap; *Rhynchophorus ferrugineus*; chemical injection

MSC: 37N25; 92D30; 93E03

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1. Introduction

The red palm weevil (RPW) is considered one of the most dangerous insects to date palms. Recently, RPW expanded its distribution within palm varieties and outbreaks to be a major invasive agricultural pest in date palm cultivars, which helped in becoming a key date-palm pest within a short time [1]. RPW as an invasive species is defined by its ability to invade, colonize and adapt to new agricultural areas worldwide [2]. The global movement of commercial goods helped spread the species of RPW, which was previously confined to areas of its initial discovery in India [3,4]. The insect is voracious in feeding, especially in the larval stage, which requires at least 60–90 days to complete its development and transfer to a pupal stage. RPW is a soft tissue insect that feeds on the inner tissue of palm trunk [5–8]. Consequently, larvae feeding leads to severe damage, resulting in destroying the inner palm trunk and finally complete palm death. Reducing RPW feeding, damage and distribution is not possible to by following one control method, but an integrated control program must be developed to reduce RPW activity and limit its damage. The integrated management program for RPW should include different control techniques such as mechanical, chemical, biological control and other possible methods that can be relied upon to control the numbers of the RPW in the affected areas. When RPW damage is detected on the trunk at one or more points, the chemical injection method can be followed, and it is considered one of the most successful methods of treatment and is a more effective method than spraying pesticides [9–11]. According to [12], early and intermediate infestation can be treated by injecting insecticides into the trunk. When the trunk is locally injected, the chemical pesticide spreads within the stem under the influence of diffusion and gravity, which leads to the killing of both the larval and pupal stages. Sex aggregated pheromone traps are used to control the level of RPW in farms by attracting both males and females, which affects the pest populations and kills them because the traps contain insecticides that kill what is being caught, thus reducing the number of RPW [3,13]. The researchers showed that using pheromone traps for the RPW are one of the most effective ways to monitor and reduce the numbers of this harmful insect in the Arab Gulf countries [14]. Mathematical models can help understand and explain the spread of this pest and the methods and factors that control it. This paper aims to develop and analyze a mathematical model of RPW with sex pheromone traps and chemical injection. The paper is organized as follows: The RPW mathematical model is described in Section 2, and the conditions for stability of the RPW model are obtained. In Section 3, the stochastic RPW model is performed, and the existence of a positive global solution for the stochastic RPW model is investigated, as well as the sufficient conditions for population extinction from the stochastic system. Sufficient criteria for the existence of a unique ergodic stationary distribution for the RPW system are established. The numerical simulations described in Section 4 are used to verify the theoretical results. The discussion and conclusion are found in Section 5.

2. Mathematical Model

- In this model, the total date palm tree population is divided into two classes: susceptible date palm tree denoted by $P_1(t)$ and infected date palm denoted by $P_2(t)$. In the absence of RPW, the date palm tree grows logistically with an intrinsic growth rate r and carrying capacity k. The natural death rate of the susceptible and infected date palm tree is μ_1.
- Assume that the date palm tree is susceptible to infection according to simple mass kinematics with β as the RPW transmission coefficient. Experimental studies on palm pests indicated that the functional response pattern of the predator is consistent with the Holling II functional response [15,16]. As a result, we assume that RPW larvae $L(t)$ harvest palm trees with Holing type-II functional response. The predation rate of RPW larvae on a date palm tree is c, and constant a is the half-saturation constant. The larvae population decreases by α rate due to the transformation from the larva stage to the adult stage. The transition rate of RPW larvae to adults females $F(t)$ is given by ν, whereas a complementing fraction $(1-\nu)$ will emerge as males $M(t)$. We assume that larvae $L(t)$ decreases at the θ rate due to the injection of chemical compounds. The natural death of larvae is assumed to be μ_2, while the natural death for adult RPW is μ.
- To indicate the trap's effect, one can consider the approach proposed by Barclay [17]. We assume the pheromone trap attracted additional η females. As a result, the RPW males attracted to pheromone traps according to $\frac{\gamma\eta M}{F+\eta}$, where γ represents the effective rate of pheromone traps on mortality of RPW males. Recently, this approach was used by [18–21] to investigate the dynamics of the mirid population under mating disruption and trapping.

The following system describes the model of RPW with sex pheromone traps and chemical injection.

$$
\begin{aligned}
\frac{dP_1}{dt} &= rP_1(1-\frac{P_1}{k}) - \beta P_1 P_2 - \mu_1 P_1 \\
\frac{dP_2}{dt} &= \beta P_1 P_2 - \frac{cP_2 L}{a+P_2} - \mu_1 P_2 \\
\frac{dL}{dt} &= \frac{cP_2 L}{a+P_2} - (\mu_2+\alpha+\theta)L \\
\frac{dF}{dt} &= \nu\alpha L - \mu F \\
\frac{dM}{dt} &= (1-\nu)\alpha L - \mu\, M - \frac{\gamma\eta M}{F+\eta}.
\end{aligned} \tag{1}
$$

The RPW model (1) has four equilibrium points. The trivial equilibrium point $E_0 = (0,0,0,0,0)$ is stable if $r < \mu_1$. The free RPW equilibrium point $E_1 = (\frac{k}{r}(r-\mu_1), 0, 0, 0, 0)$ exists if $r > \mu_1$ and E_1 stable if $R_0 < 1$, where $R_0 = \frac{kr\beta}{\mu_1(k\beta+r)}$. The equilibrium point $E_2 = (P_{12}, P_{22}, 0, 0, 0)$, where $P_{12} = \frac{\mu_1}{\beta}$ and $P_{22} = \frac{kr\beta - k\beta\mu_1 - r\mu_1}{k\beta^2}$. E_2 exists if $R_0 > 1$. The first three eigenvalues of $J(E_2)$ are $\lambda_1 = -\mu$, $\lambda_2 = -\mu-\gamma$ and $\lambda_3 = \frac{cP_{22}}{a+P_{22}} - \Psi$, where $\Psi = (\mu_2+\alpha+\theta)$. The other two eigenvalues are given by $\lambda^2 + \frac{r\mu_1}{k\beta}\lambda + \mu_1 r\left(1-\frac{1}{R_0}\right) = 0$, and the roots have negative real parts. As a result, E_2 is stable if $1 < R_0 < 1 + \frac{ak\beta^2\Psi}{\mu_1(k\beta+r)}$. The coexistence equilibrium point $E_3 = (P_{13}, P_{23}, L_3, F_3, M_3)$, where

$$
P_{13} = \frac{k\rho}{r}\left[1 - \frac{a\beta\Psi}{\rho(c-\Psi)}\right],\quad P_{23} = \frac{a\Psi}{c-\Psi},\quad L_3 = \frac{a(\beta P_{13}-\mu_1)}{c-\Psi},\ F_3 = \frac{\nu\alpha a(\beta P_{13}-\mu_1)}{\mu(c-\Psi)},
$$

$$
M_3 = \frac{(1-\nu)\alpha L_3(F_3+\eta)}{\mu(F_3+\eta)+\nu\eta}.
$$

E_3 exists if $\Psi < \frac{\rho c}{a\beta+\rho}$ and $\beta P_{13} > \frac{\mu_1}{a}$, where $\rho = r - \mu_1$. The stability of the RPW system around E_3 is now investigated. The Jacobian matrix of the RPW model (1) around E_3 is given as follows.

$J(E_3) =$

$$
\begin{pmatrix}
-\frac{rP_{13}}{k} & -\beta P_{13} & 0 & 0 & 0 \\
\beta P_{23} & \frac{cL_3P_{23}}{(a+P_{23})^2} & -\frac{cP_{23}}{a+y} & 0 & 0 \\
0 & \frac{acL_3}{(a+P_{23})^2} & 0 & 0 & 0 \\
0 & 0 & \alpha\epsilon & -\mu & 0 \\
0 & 0 & \alpha(1-\epsilon) & \frac{\eta\gamma M_3}{(\eta+F_3)^2} & -\mu - \frac{\eta\gamma}{\eta+F_3}
\end{pmatrix}.
$$

The first two eigenvalues of $J(E_3)$ are $\lambda_1 = -\mu$, and $\lambda_2 = -\mu - \frac{\eta\gamma}{\eta+F_3}$. The other three eigenvalues are determined by $\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$, where $c_1 = \frac{rP_{13}}{k} - \frac{cL_3P_{23}}{(a+P_{23})^2}$, $c_2 = P_{23}\left(\frac{ac^2L_3}{(a+P_{23})^3} + P_{13}\left(\beta^2 - \frac{cL_3 r}{k(a+P_{23})^2}\right)\right)$ and $c_3 = \frac{arc^2L_3P_{13}P_{23}}{k(a+P_{23})^3}$. The coexistence equilibrium point $E_3 = (P_{13}, P_{23}, L_3, F_3, M_3)$ is stable if $c_1 > 0$, $c_2 > 0$ and $c_1c_2 > c_3$.

3. Dynamics of the Stochastic Model

Stochastic effects can be significant in the case of RPW because the environmental conditions of its transmission are subject to randomness. The deterministic RPW (1) ignores the possible importance of a stochastic environment. In [22], a deterministic and stochastic prey–predator model for three predators and a single prey was proposed and analyzed. In this paper, we study a stochastic eco-epidemiological model for one of the agricultural pests. The RPW model (1) will be extended to include the stochastic effects as follows:

$$
\begin{aligned}
dP_1 &= \left(rP_1(1-\frac{P_1}{k}) - \beta P_1P_2 - \mu_1 P_1\right)dt + \sigma_1 P_1\, dW_1, \\
dP_2 &= \left(\beta P_1P_2 - \frac{cP_2L}{a+P_2} - \mu_1 P_2\right)dt + \sigma_2 P_2\, dW_2, \\
dL &= \left(\frac{cP_2L}{a+P_2} - (\mu_2+\alpha+\theta)L\right)dt + \sigma_3 L\, dW_3, \\
dF &= (\nu\alpha L - \mu F)dt + \sigma_4 F\, dW_4, \\
dM &= \left((1-\nu)\alpha L - \mu\, M - \frac{\gamma\eta M}{F+\eta}\right)dt + \sigma_5 M\, dW_5.
\end{aligned} \tag{2}
$$

where $W = \{W_1, W_2, W_3, W_4, W_5, t \geq 0\}$ represents the five-dimensional standard Brownian motions with $W_i(0) = 0$, and $\sigma_i^2 (i = 1,2,3,4,5)$ is the intensities of the white noise defined in a complete probability space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$ with a filtration $\mathcal{F}_{t\geq 0}$ satisfying the usual conditions. In the next theorem, we will prove the existence and uniqueness of a global positive solution of the system (2). This approach has recently been used in many papers for the analysis of stochastic predator–prey systems [23–27], stochastic epidemic models [28–34] and stochastic eco-epidemiological models [35].

Theorem 1. *For any given initial value $(P_1(0), P_2(0), L(0), F(0), M(0)) \in \mathbb{R}_+^5$, there exists a unique solution $(P_1(t), P_2(t), L(t), F(t), M(t))$ of system (2) for $t \geq 0$ and the global positive solution remains in $\mathbb{R}_+^5$ with probability one.*

Proof. Firstly, one can consider the local solution $(P_1(t), P_2(t), L(t), F(t), M(t))$ of system (2) for $t \in [0, \tau_e)$, where τ_e is the explosion time [36], by conducting the transformation of variables.

$$
X_1(t) = \ln P_1(t),\; X_2(t) = \ln P_2(t),\; X_3(t) = \ln L(t),\; X_4(t) = \ln F(t),\; X_5 = \ln M(t).
$$

Using the Itô formula, one can change system (2) as follows.

$$
\begin{aligned}
d\,X_1(t) &= \left[r(1-\frac{e^{Z_1}}{k}) - \beta e^{Z_2} - \mu_1 - \frac{\sigma_1^2}{2}\right]dt + \sigma_1\, dW_1, \\
d\,X_2(t) &= \left[\beta e^{Z_1} - \frac{c\,e^{Z_3}}{a+e^{Z_2}} - \mu_1 - \frac{\sigma_2^2}{2}\right]dt + \sigma_2\, dW_2, \\
d\,X_3(t) &= \left[\frac{c\,e^{Z_2}}{a+e^{Z_2}} - (\mu_2+\alpha+\theta) - \frac{\sigma_3^2}{2}\right]dt + \sigma_3\, dW_3, \\
d\,X_4(t) &= \left[\frac{\nu\alpha e^{Z_3}}{e^{Z_4}} - \mu - \frac{\sigma_4^2}{2}\right]dt + \sigma_4\, dW_4, \\
d\,X_5(t) &= \left[(1-\nu)\alpha\frac{e^{Z_3}}{e^{Z_5}} - \mu - \frac{\gamma\eta}{e^{Z_4}+\eta} - \frac{\sigma_5^2}{2}\right]dt + \sigma_5\, dW_5.
\end{aligned} \tag{3}
$$

The coefficients of system (3) satisfy the local Lipschitz conditions; consequently, there exists a unique local solution

$$
(P_1(t), P_2(t), L(t), F(t), M(t)) = (e^{X_1(t)}, e^{X_2(t)}, e^{X_3(t)}, e^{X_4(t)}, e^{X_5(t)})
$$

on $[0, \tau_e)$. To ensure that this solution is global, one needs to prove that $\tau_e = \infty$ a.s. Let $s_0 > 0$ be sufficiently large for every coordinate in the interval $[\frac{1}{s_0}, s_0]$. For each integer $s > s_0$, we define the stopping time.

$$
\tau_s = \inf\left\{t \in [0, \tau_e) : \min\{P_1, P_2, L, F, M\} \notin (\frac{1}{s}, s) \text{ or } \max\{P_1, P_2, L, F, M\} \notin (\frac{1}{s}, s)\right\}.
$$

One can note that τ_s is increasing as $s \to \infty$. Assume $\tau_\infty = \lim_{s\to\infty} \tau_s$, then $\tau_\infty \le \tau_e$. In the next step, one needs to verify that $\tau_\infty = \infty$. If this is not true, then there exists a constant $T > 0$ and $\epsilon \in (0,1)$ such that $\mathbb{P}(\tau_\infty \le T) \ge \epsilon$. As a result, there exists an integer $s_1 \ge s_0$ such that $\mathbb{P}(\tau_s \le T) \ge \epsilon,\ \ s \ge s_1$. Define the following C^2 positive definite function $V_1(P_1, P_2, L, F, M)$ as

$$V_1 = (P_1 + 1 - \ln P_1) + (P_2 + 1 - \ln P_2) + (L + 1 - \ln L) + (F + 1 - \ln F) + (M + 1 - \ln M).$$

Using Itô's formula, one obtains

$$\begin{aligned}
dV_1 =& \Bigg[(P_1-1)\Big(r(1-\frac{P_1}{k}) - \beta P_2 - \mu_1\Big) + (P_2-1)\Big(\beta P_1 - \frac{cL}{a+P_2} - \mu_1\Big) + (L-1)\Big(\frac{cP_2}{a+P_2} - \Psi\Big) \\
&+(1-\frac{1}{F})(\nu\alpha L - \mu F) + (1-\frac{1}{M})\Big((1-\nu)\alpha L - \mu M - \frac{\gamma\eta M}{F+\eta}\Big) + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2\Bigg]dt + \sigma_1(P_1-1)dW_1 \\
&+\sigma_2(P_2-1)dW_2 + \sigma_3(L-1)dW_3 + \sigma_4(F-1)dW_4 + \sigma_5(M-1)dW_5 \\
&\le \Bigg[(r+\frac{1}{k})P_1 + \beta P_2 + cL + (\Psi + \mu_1 + 2\mu + \nu) + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2\Bigg]dt + \sigma_1(P_1-1)dW_1 \\
&+\sigma_2(P_2-1)dW_2 + \sigma_3(L-1)dW_3 + \sigma_4(F-1)dW_4 + \sigma_5(M-1)dW_5.
\end{aligned}$$

Using inequality $x \le 2(x+1-\ln x)$, for any $x > 0$, one obtains

$$\begin{aligned}
dV_1 \le& \Bigg[(\Psi + \mu_1 + 2\mu + \nu) + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2 + 2(r+\frac{1}{k})(P_1 + 1 - \ln P_1) + 2\beta(P_2 + 1 - \ln P_2) + 2c(L + 1 - \ln L)\Bigg]dt \\
&+\sigma_1(P_1-1)dW_1 + \sigma_2(P_2-1)dW_2 + \sigma_3(L-1)dW_3 + \sigma_4(F-1)dW_4 + \sigma_5(M-1)dW_5,
\end{aligned}$$

which means that

$$dV_1 \le K(1+V_1)dt + \sigma_1(P_1-1)dW_1 + \sigma_2(P_2-1)dW_2 + \sigma_3(L-1)dW_3 + \sigma_4(F-1)dW_4 + \sigma_5(M-1)dW_5,$$

where $K_1 = (\Psi + \mu_1 + 2\mu + \nu) + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2$, $K_2 = \max\Big\{2(r+\frac{1}{k}), 2\beta, 2c\Big\}$ and $K = \max\{K_1, K_2\}$.
Taking the expectation of the above inequality, one obtain the following.

$$\begin{aligned}
&EV_1(P_1(t_1\wedge\tau_s), P_2(t_1\wedge\tau_s), L(t_1\wedge\tau_s), F(t_1\wedge\tau_s), M(t_1\wedge\tau_s)) \\
&\quad\le V_1(P_1(0), P_2(0), L(0), F(0), M(0)) + KE\int_0^{t_1\wedge\tau_s}(1+V_1)\ dt \\
&\le V_1(P_1(0), P_2(0), L(0), F(0), M(0)) + KT + K\int_0^{t_1\wedge\tau_s} EV_1\ dt.
\end{aligned}$$

Following [35,37], applying Grownwall's inequality, one obtains

$$\begin{aligned}
&EV_1(P_1(t_1\wedge\tau_s), P_2(t_1\wedge\tau_s), L(t_1\wedge\tau_s), F(t_1\wedge\tau_s), M(t_1\wedge\tau_s)) \\
&\le [V_1((P_1(0), P_2(0), L(0), F(0), M(0))) + KT]e^{KT} = K_3.
\end{aligned}$$

The remainder of the proof is similar to [37,38] and is therefore omitted. The proof is now complete. □

The above theorem shows that the stochastic RPW system (2) has a positive global solution remaining in $\mathbb{R}^5_+$ with a probability of one. In the following, we will establish the boundedness property of the RPW model (2).

Lemma 1. *Let $W(t) = P_1(t) + P_2(t) + L(t) + F(t) + M(t)$, then for any positive initial value, the following inequality holds:*

$$\limsup_{t\to\infty} \ E[W(t)] \leq \frac{rk}{4\xi} \ a.s,$$

where $\xi = min\{\mu_1, \mu_2, \mu\}$.

Proof. According to the stochastic RPW system (2), we have the following.

$$\begin{aligned} dW(t) &\leq \left[rP_1(1-\frac{P_1}{k}) - \xi \ W(t) \right] dt + \sigma_1 P_1 dW_1 + \sigma_2 P_2 dW_2 + \sigma_3 L dW_3 + \sigma_4 F dW_4 + \sigma_5 M dW_5 \\ &\leq \left[\frac{rk}{4} - \xi \ W(t) \right] dt + \sigma_1 P_1 dW_1 + \sigma_2 P_2 dW_2 + \sigma_3 L dW_3 + \sigma_4 F dW_4 + \sigma_5 M dW_5. \end{aligned}$$

Integrating from 0 to t yields

$$W(t) \leq W(0) + \int_0^t \left(\frac{rk}{4} - \xi \ W(s) \right) ds + \int_0^t [\sigma_1 P_1 dW_1 + \sigma_2 P_2 dW_2 + \sigma_3 L dW_3 + \sigma_4 F dW_4 + \sigma_5 M dW_5] ds.$$

According to strong law of large numbers, one obtains

$$E[W(t)] \leq W(0) + \int_0^t E\left(\frac{rk}{4} - \xi \ W(s) \right) ds.$$

Consequently,

$$\frac{dE[W(t)]}{dt} + \xi E[W(t)] \leq \frac{rk}{4}$$

Thus, one obtains

$$\limsup_{t\to\infty} \ E[W(t)] \leq \frac{rk}{4\xi}.$$

□

The above theorem tells us the solution of RPW system (2) is uniformly bounded in mean, and as a result, the deterministic RPW system (1) is uniformly bounded. The conditions for RPW extinction will be established using the following theorem.

Theorem 2. *If $r < \frac{\sigma_1^2}{2} + \mu_1$, then the populations will be extinct with a probability of one for any positive initial conditions.*

Proof. Applying Itô's formula to the first equation of stochastic RPW system (2), one obtains

$$d(\ln P_1) = \left[r(1-\frac{P_1}{k}) - \beta P_2 - \mu_1 - \frac{\sigma_1^2}{2} \right] dt + \sigma_1 \, dW_1,$$

and integrating both sides of the above equation from 0 to t leads to

$$\ln P_1(t) \leq \ln P_1(0) + (r - \mu_1 - \frac{\sigma_1^2}{2})\, t - \frac{r}{k} \int_0^t P_1(s) ds - \beta \int_0^t P_2(s) ds + \sigma_1 \, W_1.$$

It follows that

$$\limsup_{t\to\infty} \frac{\ln P_1(t)}{t} \leq r - \mu_1 - \frac{\sigma_1^2}{2} < 0 \quad a.s.$$

which implies that

$$\lim_{t\to\infty} P_1(t) = 0.$$

Applying Itô's formula to the second equation of stochastic RPW system (2), one obtains

$$d(\ln P_2(t)) = \left[\beta P_1 - \frac{cL}{a+P_2} - \mu_1 - \frac{\sigma_2^2}{2}\right]dt + \sigma_2\, dW_2.$$

Consequently,

$$\ln P_2(t) \le \ln P_2(0) + \beta\int_0^t P_1(s)ds - \int_0^t \frac{cL}{a+P_2}ds - (\mu_1 + \frac{\sigma_1^2}{2})\, t + \sigma_1\, W_1,$$

Taking the superior limit, one obtains

$$\lim_{t\to\infty}\sup\frac{\ln P_2(t)}{t} \le -(\mu_1 + \frac{\sigma_2^2}{2}) < 0 \quad a.s.$$

Thus, $\lim_{t\to\infty} P_2(t) = 0$. The other classes of the RPW system (2) also proceed to extinction a.s. Thus,

$$\lim_{t\to\infty} L(t) = 0, \quad \lim_{t\to\infty} F(t) = 0, \quad \lim_{t\to\infty} M(t) = 0.$$

□

In the following theorem, we will establish the asymptotic stability of the RPW system (2).

Theorem 3. *If* $K_1 = \frac{\sigma_1^2}{2} + r - \mu_1 < 0$, $K_2 = \frac{\sigma_2^2}{2} - \mu_1 < 0$, $K_3 = \frac{\sigma_3^2}{2} - \psi < 0$, $K_4 = \frac{\sigma_4^2}{2} - \mu < 0$, $K_5 = \frac{\sigma_5^2}{2} - (\mu+\nu) < 0$ *and* $\nu\alpha^3(1-\nu)^2 + 8K_3K_4K_5 < 2\nu^2\alpha^2$*, then the trivial solution of the RPW system (2) is stochastically asymptotically stable in probability for any positive initial conditions.*

Proof. Firstly, one can consider the following linearized RPW system.

$$\begin{aligned} dP_1 &= (r-\mu_1)P_1\, dt + \sigma_1 P_1\, dW_1, \\ dP_2 &= -\mu_1 P_2 dt + \sigma_2 P_2\, dW_2, \\ dL &= -\psi L dt + \sigma_3 L\, dW_3, \\ dF &= (\nu\alpha L - \mu F)dt + \sigma_4 F\, dW_4, \\ dM &= [(1-\nu)\alpha L - (\mu+\gamma)\, M]dt + \sigma_5 M\, dW_5. \end{aligned} \tag{4}$$

Consider the following Lyapunov function.

$$V_2 = \frac{1}{2}\left[P_1^2(t) + P_2^2(t) + L^2(t) + F^2(t) + M^2(t)\right].$$

Applying Itô's formula to the linearized stochastic RPW system (4), one computes

$$LV_2 = \left[\frac{\sigma_1^2}{2} + r - \mu_1\right]P_1^2 + \left[\frac{\sigma_2^2}{2} - \mu_1\right]P_2^2 + \left[\frac{\sigma_3^2}{2} - \Psi\right]L^2 + \left[\frac{\sigma_4^2}{2} - \mu\right]F^2 + \left[\frac{\sigma_5^2}{2} - (\mu+\gamma)\right]M^2 + \nu\alpha LF + (1-\nu)\alpha LM.$$

LV_2 can be written in the form $LV_2 = \frac{1}{2}x^T Qx$, where $x = (P_1, P_2, L, F, M)$ and

$$Q = \begin{pmatrix} 2K_1 & 0 & 0 & 0 & 0 \\ 0 & K_2 & 0 & 0 & 0 \\ 0 & 0 & K_3 & \nu\alpha & (1-\nu)\alpha \\ 0 & 0 & \nu\alpha & K_4 & 0 \\ 0 & 0 & (1-\nu)\alpha & 0 & K_5 \end{pmatrix}.$$

Matrix Q will be negatively definite if $K_1 = \frac{\sigma_1^2}{2} + r - \mu_1 < 0$, $K_2 = \frac{\sigma_2^2}{2} - \mu_1 < 0$, $K_3 = \frac{\sigma_3^2}{2} + -\psi < 0$, $K_4 = \frac{\sigma_4^2}{2} - \mu < 0$, $K_5 = \frac{\sigma_5^2}{2} - (\mu + \gamma) < 0$ and $\nu\alpha^3(1-\nu)^2 + 8K_3K_4K_5 < 2\nu^2\alpha^2$. According to Theorem 2.4 [36], if there exists a positive-definite decreasing unbounded function V_2 such that LV_2 is negative-definite, then the trivial solution of the linearized stochastic RPW system (4) is stochastically stable in the large. As indicated by Arnold [39] (Theorem 11.6.1), if the trivial solution of the linear stochastic RPW system (4) is stochastically asymptotically stable, then the trivial solution of the non-linear stochastic RPW system (2) is stochastically asymptotically stable. □

In the following, based on the method of Khasminskii [40], we establish the conditions for the existence of an ergodic stationary distribution of the positive solutions to the RPW model (2). The positive equilibrium E_3 for system (1) is locally asymptotically stable, but there is non positive equilibrium point for RPW system (2). According to [41,42], one can investigate the stationary distribution for the RPW system (2) instead of asymptotically stable equilibria. Before providing the main theorem, we first state the following Lemma

Lemma 2 ([40])**.** *The Markov process $X(t)$ has a unique ergodic stationary distribution $\pi(.)$ if there exists a bounded closed domain $U \subset \mathbb{R}^d$ with regular boundary Γ possessing the following properties:*

$C_1:$ *There is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(x)\eta_i\eta_j \geq M|\eta^2|, x \in U, \eta \in \mathbb{R}^d$;*
$C_2:$ *There exists a non-negative C^2 function V such that LV is negative on $\mathbb{R}^d \backslash U$.*

Theorem 4. *Assume $\beta > \mu_1$, then for any positive initial value, system* (2) *has a unique ergodic stationary distribution $\pi(.)$.*

Proof. In order to prove Theorem 4, one needs only to validate conditions C_1 and C_2 of Lemma 2. The first step is to validate conditions C_1 of Lemma 2. The diffusion matrix A_1 of the system (2) is given by

$$A_1 = \begin{pmatrix} \sigma_1^2 P_1^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2^2 P_2^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3^2 L^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_4^2 F^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5^2 M^2 \end{pmatrix}.$$

Following [37,43,44], choose $M_1 = \min\{\sigma_1^2 P_1^2, \sigma_2^2 P_2^2, \sigma_3^2 L^2, \sigma_4^2 F^2, \sigma_5^2 M^2\}$; then, there is a positive number M_1 such that

$$\sum_{i,j=1}^{5} a_{ij}(P_1, P_2, L, F, M)\eta_i\eta_j = \sigma_1^2 P_1^2 \eta_1^2 + \sigma_2^2 P_2^2 \eta_2^2 + \sigma_3^2 L^2 \eta_3^2 + \sigma_4^2 F^2 \eta_4^2 + \sigma_5^2 M^2 \eta_5^2 \geq M_1 |\eta^2|,$$

for all $(P_1, P_2, L, F, M) \in U$, $\eta = (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \in \mathbb{R}^5$. This implies condition that C_1 in Lemma 2 is satisfied. The second step is to prove that there exists a non-negative C^2 function V_3 such that $LV_3 < 0$ as follows. Define the following function.

$$V_3 = (P_1 - 1 + \ln P_1) + (P_2 - 1 + \ln P_2) + (L - 1 + \ln L) + (F - 1 + \ln F) + (M - 1 + \ln M).$$

Applying Itô formula leads to the following:

$$
\begin{aligned}
LV_3 = & \frac{(P_1-1)}{P_1}\left(rP_1(1-\frac{P_1}{k}) - \beta P_1P_2 - \mu_1 P_1\right) + \frac{(P_2-1)}{P_2}\left(\beta P_1 P_2 - \frac{cP_2L}{a+P_2} - \mu_1 P_2\right) \\
& + \frac{(L-1)}{L}\left(\frac{cP_2L}{a+P_2} - \Psi L\right) + \frac{(F-1)}{F}(\nu\alpha L - \mu F) + \frac{(M-1)}{M}\left((1-\nu)\alpha L - \mu M - \frac{\gamma\eta M}{I+\eta}\right) + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2 \\
\leq & -\frac{rP_1^2}{k} + \left[r + \frac{r}{k} - \mu_1 - \beta\right]P_1 + (\beta-\mu_1)P_2 + \theta + \mu_2 + 2\mu_1 + 2\mu + \nu + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2 - r \\
\leq & (\beta-\mu_1)P_2 - \left[\frac{r}{2k}\left(r + \frac{r}{k} - \mu_1 - \beta\right)^2 + r - (\theta + \mu_2 + 2\mu_1 + 2\mu + \nu + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2)\right] \\
\leq & (\beta-\mu_1)P_2 - h_1,
\end{aligned}
$$

where $h_1 = \left[\frac{r}{2k}\left(r + \frac{r}{k} - \mu_1 - \beta\right)^2 + r - (\theta + \mu_2 + 2\mu_1 + 2\mu + \nu + \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2)\right]$.

Define $U_1 = \{(P_1, P_2, L, F, M) \in \mathbb{R}_+^5 : P_2 > h_1\}$; then, LV_3 is negative on $\mathbb{R}^d \backslash U_1$, which implies that condition C_2 in Lemma 2 is satisfied. As a result, the RPW system (2) is ergodic and has a stationary distribution. This completes the proof. □

4. Numerical Simulations

The RPW model is simulated in this section to demonstrate some of the previously obtained analytical results. The following parameters will be used to simulate the interactions between palm trees and different stages of the RPW $r = 5; k = 3; \beta = 0.6; \mu_1 = 0.1; \mu_2 = 0.002; \alpha = 0.5; a = 3; \theta = 0.03; \mu = 0.001; \gamma = 0.2; \eta = 0.2; \nu = 0.6; c = 1.4$. To understand the effect of the chemical injection coefficient θ on the dynamic behavior of the RPW (1), one can increase the θ value and keep the rest of the parameters as above. Figure 1 indicates the occurrence of transcritical bifurcation at $\theta^* = 0.5$ and supercritical Hoph bifurcation at $\theta^{**} = 0.1285$, as shown in Figure 2. When $\theta > \theta^*$, the density of larvae will become extinct; consequently, the other classes of RPW will be extinct. Figure 3 indicates that the population density of RPW males decreases with an increase in sex pheromone trap parameter η. One can conclude that pheromone trap parameters η can limit the spread of RPWs. The effect of intrinsic date palm growth rate r can be shown by drawing the bifurcation diagram regarding r as a bifurcation parameter. From Figures 4 and 5, it can be seen that two transcritical bifurcation values localized at $r^* = 0.1$ and $r^{**} = 1.275$. When $r < 0.1$, the trivial equilibrium point is locally asymptotically stable. For $0.1 < r < 1.275$, the equilibrium point $E_2 = (0.166667, 1.57407r - 0.166667, 0, 0, 0)$ is locally stable. It can be seen that supercritical Hopf bifurcation value localized at $r = 4.19817$ as shown in Figure 5. When $r > 4.19817$, the RPW model (2) proceeds through limit cycle oscillation and for $r < 4.19817$, E_3 is locally stable as indicated in Figure 4 and coincides with Figure 5.

To provide some numerical findings to the stochastic RPW system (2), we use the Milstein method mentioned in [45]. The stochastic RPW system (2) reduces to the following discrete system:

$$
\begin{aligned}
P_{1(j+1)} &= P_{1j} + h\left(rP_{1j}(1-\frac{P_{1j}}{k}) - \beta P_{1j}P_{2j} - \mu P_{1j}\right) + \sigma_1 P_{1j}\sqrt{h}\epsilon_{1j} + \frac{\sigma_1^2}{2}P_{1j}\left[\epsilon_{1j}^2 - 1\right]h \\
P_{2(j+1)} &= P_{2j} + h\left(\beta P_{1j}P_{2j} - \frac{cP_{2j}L_j}{a+P_{2j}} - \mu_1 P_{2j}\right) + \sigma_2 P_{2j}\sqrt{h}\epsilon_{2j} + \frac{\sigma_2^2}{2}P_{2j}\left[\epsilon_{2j}^2 - 1\right]h \\
L_{(j+1)} &= L_j + h\left(\frac{cP_{2j}L_j}{a+P_{2j}} - \Psi L_j\right) + \sigma_3 L_j\sqrt{h}\epsilon_{3j} + \frac{\sigma_3^2}{2}L_j\left[\epsilon_{3j}^2 - 1\right]h \\
F_{(j+1)} &= F_j + h(\nu\alpha L_j - \mu F_j) + \sigma_4 F_j\sqrt{h}\epsilon_{4j} + \frac{\sigma_4^2}{2}F_j\left[\epsilon_{4j}^2 - 1\right]h \\
M_{(j+1)} &= M_j + h\left((1-\nu)\alpha L_j - \mu M_j - \frac{\gamma\eta M_j}{F_j+\eta}\right) + \sigma_5 M_j\sqrt{h}\epsilon_{5j} + \frac{\sigma_5^2}{2}M_j\left[\epsilon_{5j}^2 - 1\right]h,
\end{aligned}
\tag{5}
$$

where ϵ_{ij}, $(i, j = 1, 2, 3, 4, 5)$ are independent random Gaussian variables $N(0, 1)$, and h is a positive time increment. For the given parameters, one can note that the conditions of Theorems 2 and 3 are verified and the populations will be extinct with probability one if $r < \frac{\sigma_1^2}{2} + \mu_1$ as indicated in Figure 6, when $r = 0.8$. The time series for the stochastic system (2) and its histograms of probability density function are shown in Figure 7. The conditions of Theorem 3 hold and system (2) has a unique stationary distribution and it has ergodic properties. If one gradually increases the intensities of fluctuation σ_i and keeps the remaining parameters unchanged, the RPW female oscillates around coexistence point E_3, as shown in Figure 8.

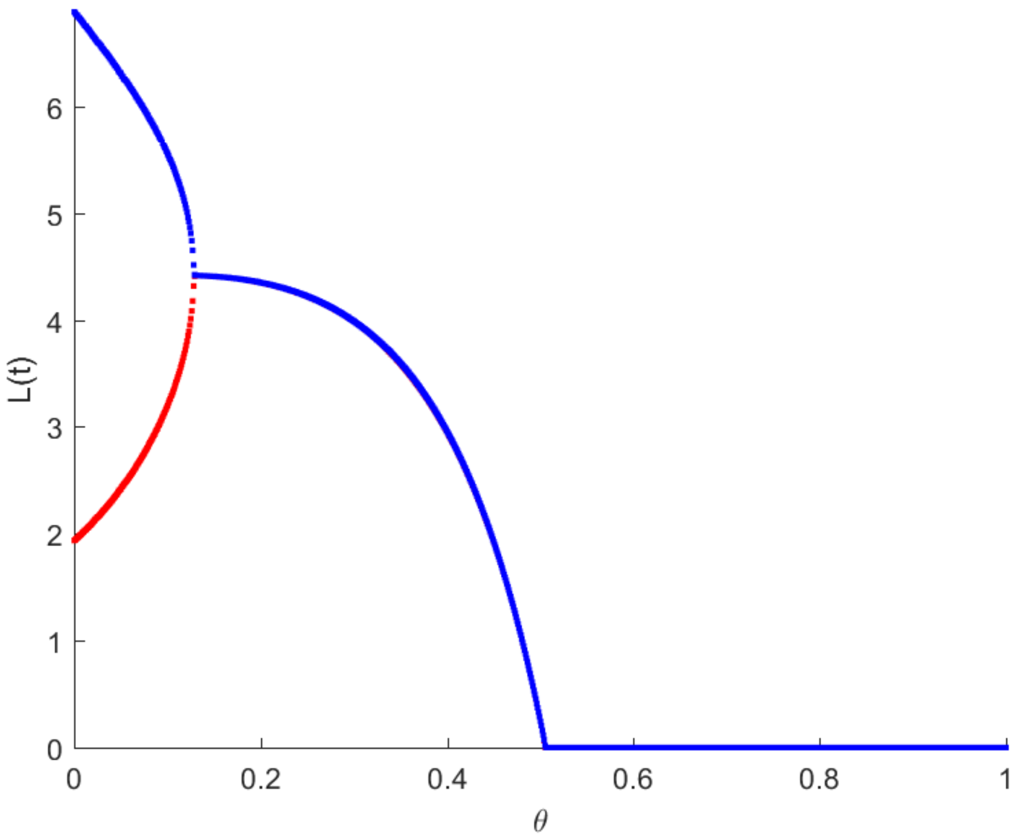

Figure 1. Bifurcation diagram of RPW system (1) with respect to θ.

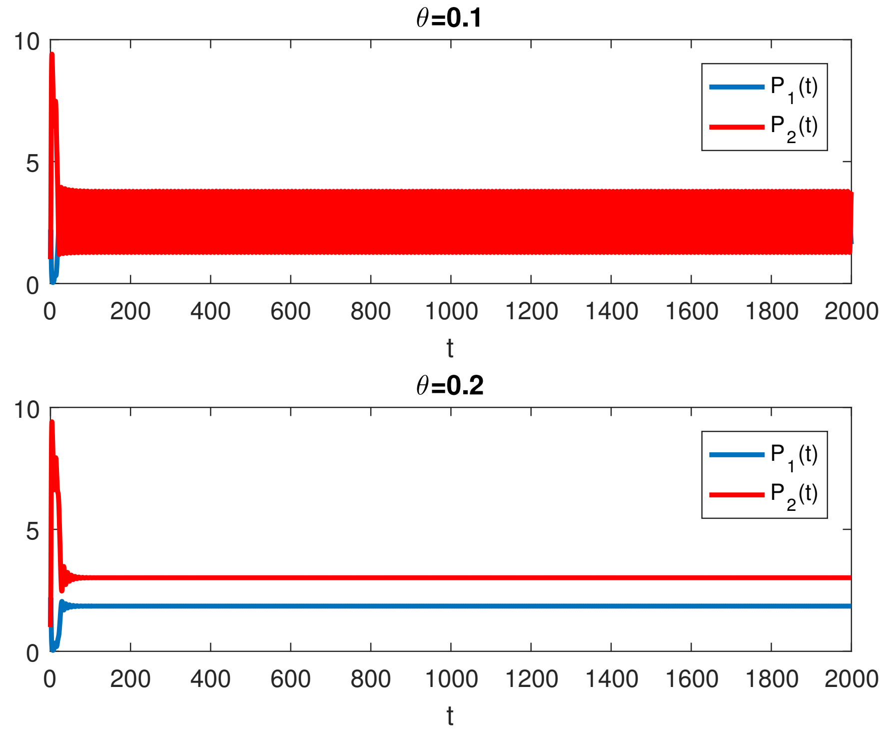

Figure 2. The RPW system (1) with $\theta = 0.1$ and $\theta = 0.2$.

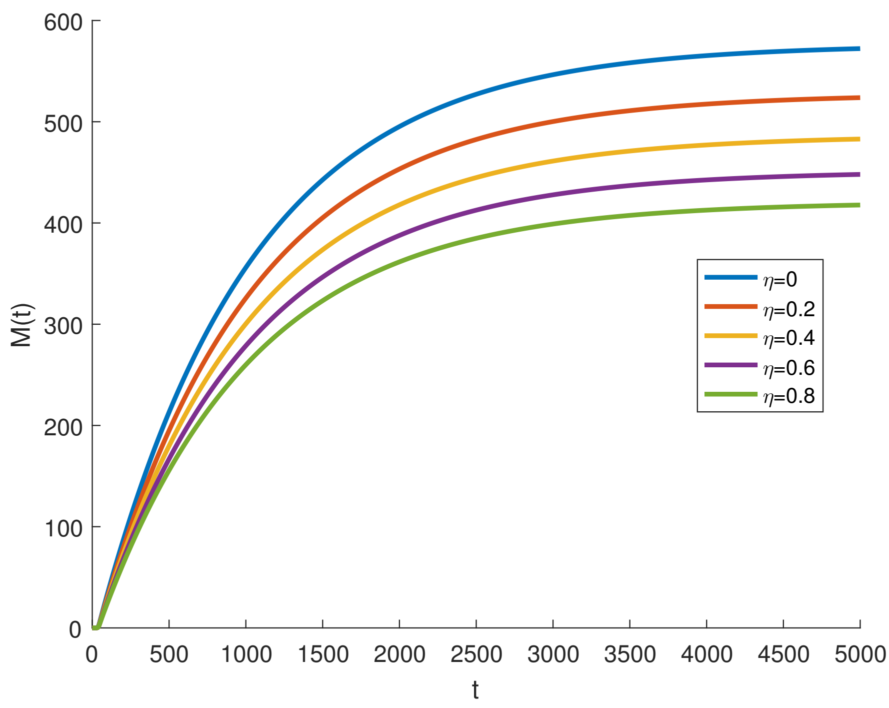

Figure 3. The RPW system (1) with different values of η.

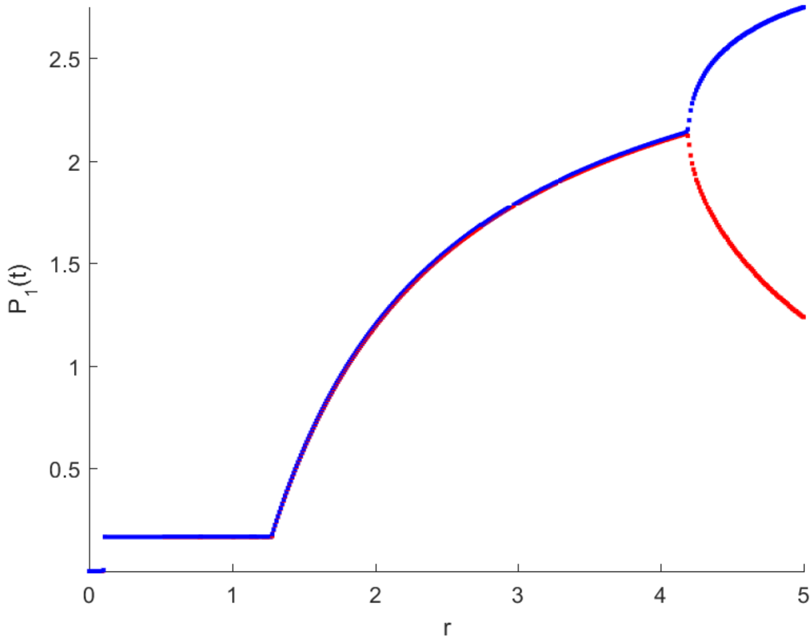

Figure 4. Bifurcation diagram of RPW system (1) with respect to r.

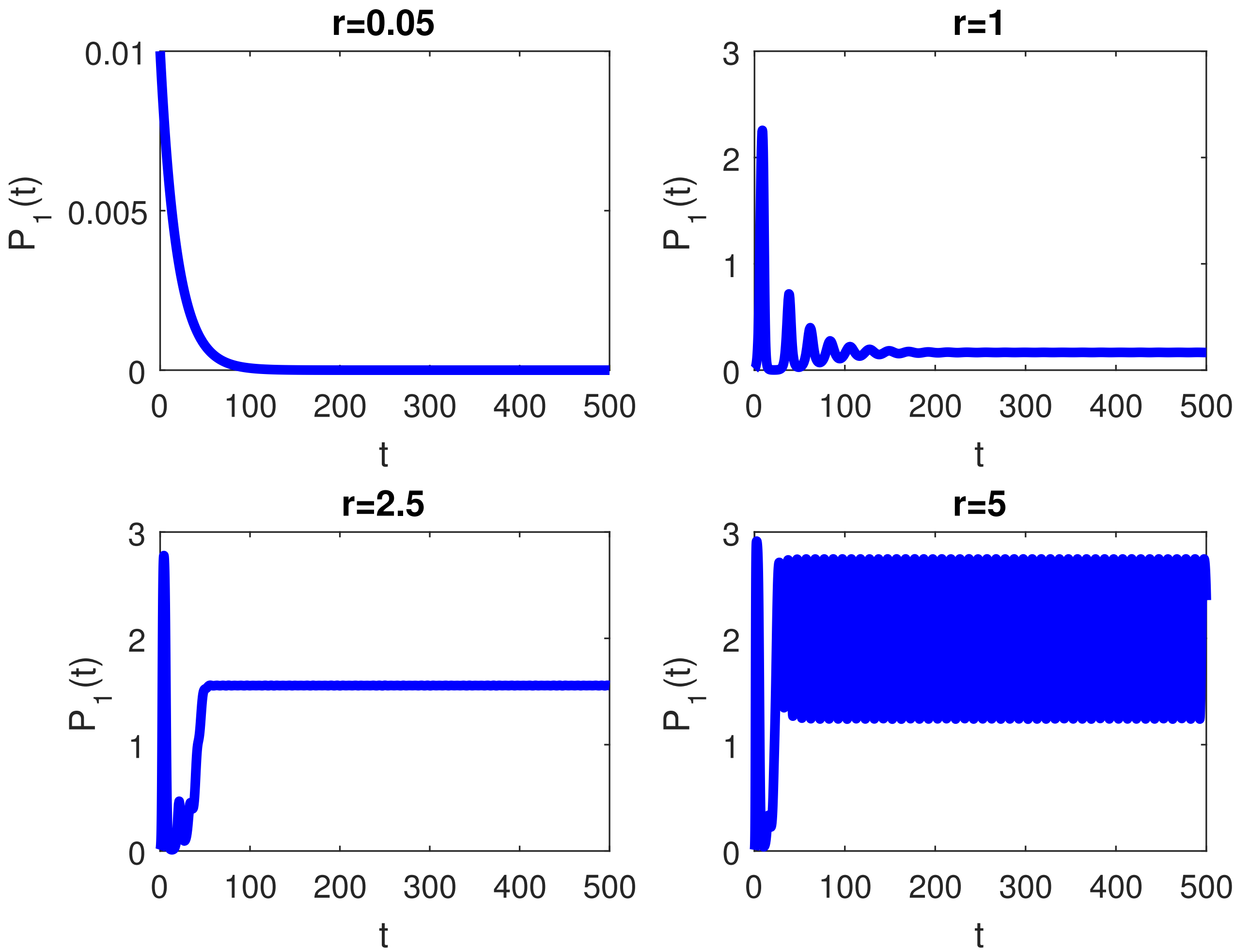

Figure 5. The RPW system (1) with $r = 0.05,\ 1,\ 2.5,\ 5$.

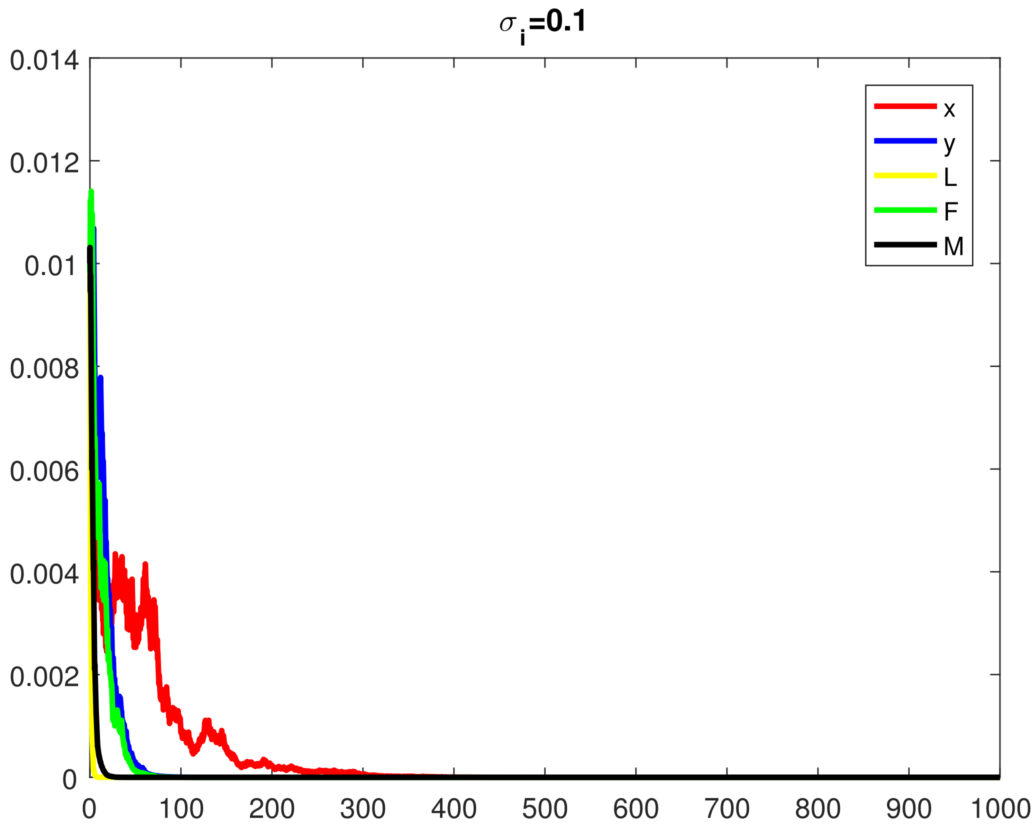

Figure 6. The stochastic RPW system (2) with $\sigma_i = 0.1$.

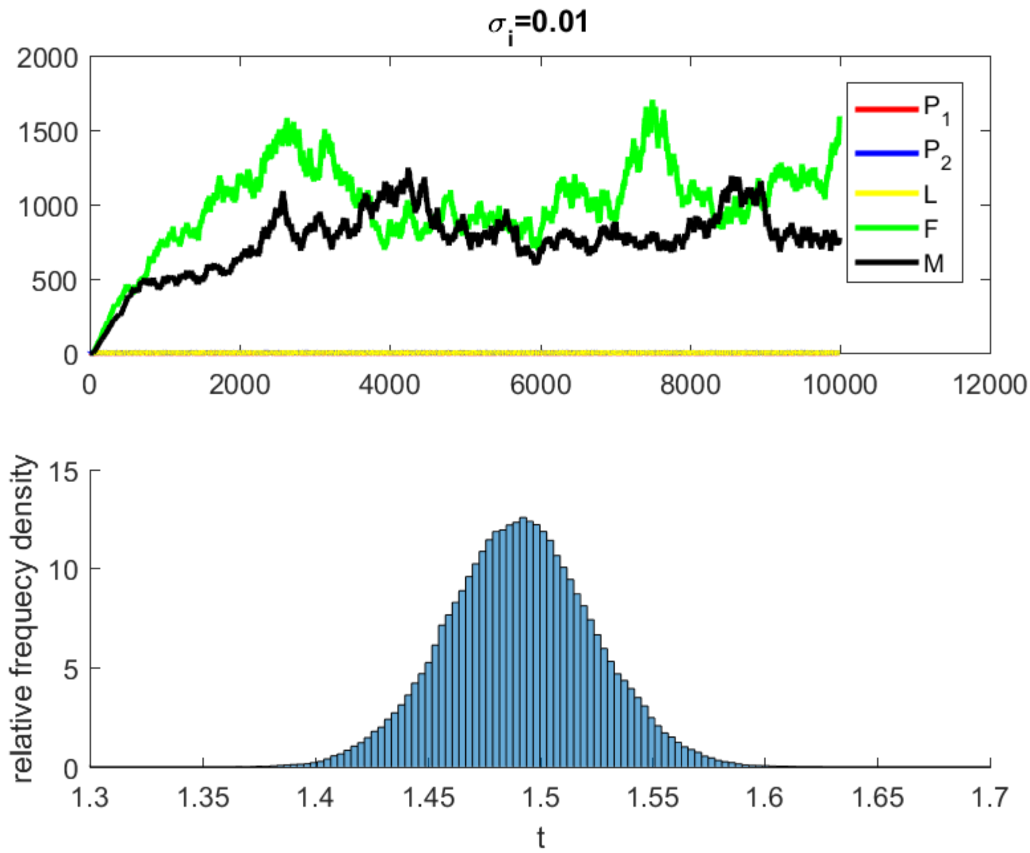

Figure 7. The stochastic system (2) and its histograms of probability density function.

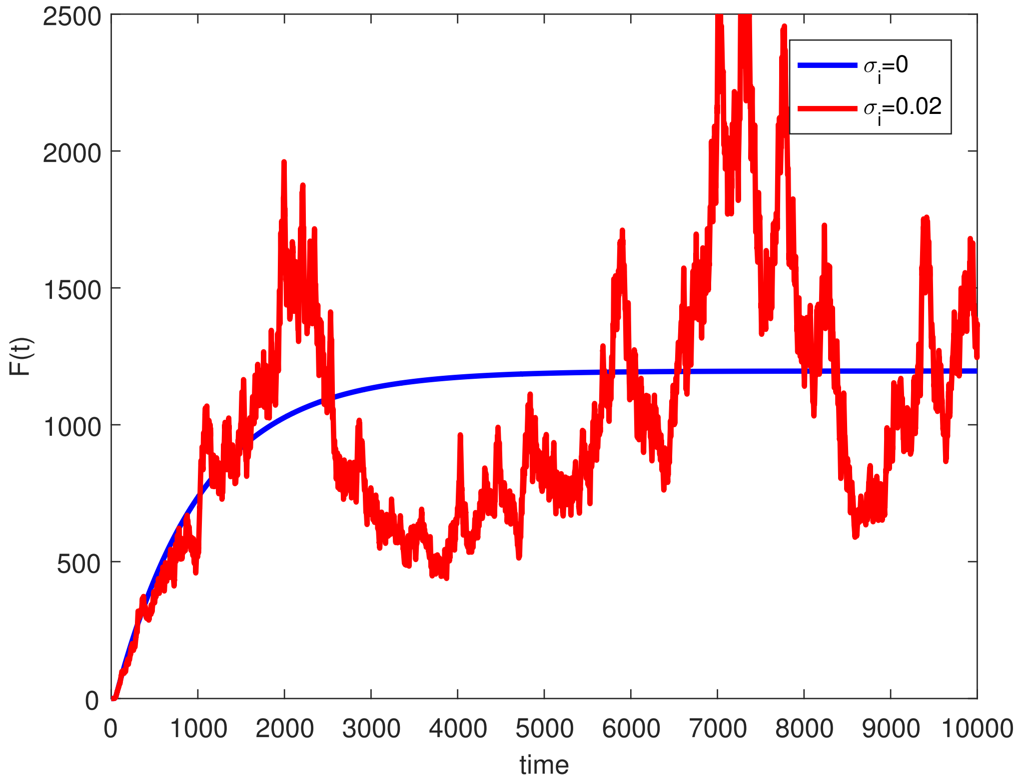

Figure 8. The stochastic RPW system (2) with $\sigma_i = 0$ and $\sigma_i = 0.02$.

5. Discussion and Conclusions

In this paper, a deterministic and stochastic model for RPW has been proposed and analyzed. For the deterministic model, the stability of the solution has been studied. The chemical injection parameter θ plays an essential role in controlling the RPW insect, because by increasing the rate of injection, the insect is killed in the larval stage, and as a result, the adult stages of the insect do not appear. The θ^* parameter is biologically important, as by knowing the parameters of the insect infestation of palm farms, it is possible to determine the critical injection rate that leads to the disappearance of the palm weevil from the farms. Moreover, from the numerical results, one can find that the population density of RPW males decreases with increasing sex pheromone trap parameters η. The numerical simulation for RPW indicates that white noise has a significant impact on the dynamical behavior of the RPW system. The conditions for the extinction of RPW insects from the stochastic model have been obtained. The adequate criteria for the presence of a unique ergodic stationary distribution for the RPW system have been established by creating suitable Lyapunov functions. The importance of environmental noise in RPW has been simulated using the Milstein method. If one inserts the intensities of fluctuation $\sigma_i = 0$ and the chemical injection parameter $\theta = 0$, the results of the stochastic model in this paper coincide with the results of the deterministic model considered by [18]. Moreover, it is interesting to study controlling RPW in date palms using sterile insect technique and the effects of other factors, such as time delays and impulsive perturbations. We leave these cases for future work.

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