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New Diamond-α Steffensen-Type Inequalities for Convex Functions over General Time Scale Measure Spaces

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Abstract: In this paper, we extend some Steffensen-type inequalities to time scales by using the diamond-α-dynamic integral. Further, we prove some new Steffensen-type inequalities for convex functions utilizing positive σ-finite measures in time scale calculus. Moreover, as a special case, we obtain these inequalities for the delta and the nabla integral. By using the relation between calculus on time scales $\mathbb{T}$ and differential calculus on $\mathbb{R}$, we obtain already-known Steffensen-type inequalities.

Keywords: time scales; diamond-α integral; measure spaces; Steffensen-type inequality; convex function

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1. Introduction

The theory of time scale calculus appeared in 1988. Hilger introduced it to connect discrete and continuous analysis (see [1,2]). A combined $\Diamond_\alpha$ (diamond-α) dynamic derivative appeared in [3] as a linear combination of the well-known Δ (delta) and ∇ (nabla) dynamic derivatives on time scales. As a special case of the diamond-α dynamic derivative, we obtain the Δ derivative for $\alpha = 1$ and the ∇ derivative for $\alpha = 0$.

Since time scale calculus unifies a discrete and a continuous case, it can be used not only to obtain generalizations of the integral or discrete Steffensen inequality on time scales, but also to connect the integral and the discrete Steffensen inequalities. Some of the papers dealing with this topic are [4–6]. A comprehensive review of Steffensen's inequality not only in time scale calculus, but also in fractional calculus, general measure spaces, and calculus on $\mathbb{R}$ can be found in the monographs [7,8].

The integral Steffensen inequality [9] states:

Theorem 1. *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)dt$. Then, we have*

$$\int_{b-\lambda}^{b} f(t)dt \leq \int_a^b f(t)g(t)dt \leq \int_a^{a+\lambda} f(t)dt. \tag{1}$$

The inequalities are reversed for f nondecreasing.

In [10], Anderson proved the Steffensen inequality for the nabla integral.

Theorem 2 ([10])**.** *Let $a, b \in \mathbb{T}_\kappa^\kappa$ and $f, g : [a,b] \to \mathbb{R}$ be nabla integrable functions with f decreasing and $0 \leq g \leq 1$ on $[a,b]$. Assume $\lambda = \int_a^b g(t)\nabla t$ such that $b-\lambda, a+\lambda \in \mathbb{T}$. Then,*

$$\int_{b-\lambda}^{b} f(t)\nabla t \leq \int_a^b f(t)g(t)\nabla t \leq \int_a^{a+\lambda} f(t)\nabla t. \tag{2}$$

For the reader's convenience, we will recall the definition of $\mathbb{T}_\kappa^\kappa$, mentioned in Theorem 2, in Section 2.

By replacing the ∇ integral with Δ integral in the theorem above, we can obtain an analogous result for the Δ integral. Further, a generalization of Steffensen's inequality for the diamond-α integral was proven by Ozkan and Yildrim in [11].

Now, let us recall the Steffensen-type inequality proven by Masjed-Jamei et al. in [12].

Theorem 3. *If f and g are integrable functions such that f is nonincreasing and*

$$-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \leq g(x) \leq 1-\frac{\sigma}{b-a}\left(1-\frac{1}{q}\right) \tag{3}$$

on (a,b), where $q \neq 0$ and

$$\sigma = q\int_a^b g(x)dx,$$

then

$$\begin{aligned}\int_{b-\sigma}^b f(x)dx - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx &\leq \int_a^b f(x)g(x)dx \\ &\leq \int_a^{a+\sigma} f(x)dx - \frac{\sigma}{b-a}\left(1-\frac{1}{q}\right)\int_a^b f(x)dx.\end{aligned} \tag{4}$$

The inequalities (4) are reversed for f nondecreasing.

The purpose of this paper is to extend the Steffensen-type inequality obtained by Masjed-Jamei et al. to a general time scale calculus using the diamond-α-integral. Furthermore, we will prove diamond-α-dynamic Steffensen-type inequalities for the class of convex functions. As a special case of the obtained results for the diamond-α-integral, we obtain results for the Δ-integral and for the ∇-integral. Moreover, by taking $\mathbb{T} = \mathbb{R}$, we recapture already-known Steffensen-type inequalities.

2. Preliminaries

Let us begin by recalling some basic facts about time scale calculus. By $\mathbb{T}$, we denote a time scale, which is an arbitrary nonempty closed subset of $\mathbb{R}$. The topology of a time scale $\mathbb{T}$ is inherited from the standard topology in $\mathbb{R}$.

The jump operators $\rho, \sigma : \mathbb{T} \to \mathbb{T}$ are defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

We say that the point $t \in \mathbb{T}$ is left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a right-scattered minimum m, we define $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. We define $\mathbb{T}_\kappa^\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

We continue with the definition of the delta and the nabla dynamic derivative.

Definition 1 ([13])**.** *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function, and let $t \in \mathbb{T}^\kappa$. Then, we define $f^\Delta(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s| \quad \textit{for all } s \in U.$$

We call $f^\Delta(t)$ the delta derivative of f at t.

We say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided that $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Definition 2 ([13])**.** *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function, and let $t \in \mathbb{T}_\kappa$. Then, we define $f^\nabla(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that*

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s| \quad \textit{for all } s \in U.$$

We call $f^\nabla(t)$ the nabla derivative of f at t.

We say that f is *nabla differentiable* on $\mathbb{T}_\kappa$ provided that $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

Let us also note, if $\mathbb{T} = \mathbb{R}$, then $f^\Delta = f^\nabla = f'$.

By $[a, b]_\mathbb{T} = \{t \in \mathbb{T} : a \leq t \leq b\}$, we denote the time scale interval.

Now, we define the diamond-α dynamic derivative and the diamond-α dynamic integral. For more details, see [3].

Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t \in \mathbb{T}_\kappa^\kappa$, we define the diamond-$\alpha$ dynamic derivative $f^{\diamondsuit_\alpha}(t)$ by

$$f^{\diamondsuit_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t), \quad 0 \leq \alpha \leq 1.$$

Hence, we say that f is diamond-α differentiable if and only if f is Δ and ∇ differentiable. If $\alpha = 1$, we obtain the Δ derivative, and if, $\alpha = 0$, we obtain the ∇ derivative. Further, if $\alpha \in (0, 1)$, we obtain a "weighted dynamic derivative".

Let $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$. Then, the diamond-$\alpha$ integral from a to b of the function f is defined by

$$\int_a^b f(t)\diamondsuit_\alpha(t) = \alpha \int_a^b f(t)\Delta t + (1 - \alpha) \int_a^b f(t)\nabla t$$

provided that there exist delta and nabla integrals of the function f on $\mathbb{T}$.

In the following theorem, we recall relations between calculus on time scales $\mathbb{T}$ and differential calculus on $\mathbb{R}$.

Theorem 4 ([14])**.** *Let $a, b \in \mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then a bounded function f on $[a, b]$ is Δ-integrable from a to b if and only if f is Riemann integrable on $[a, b]$ in the classical sense, and in this case,*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the ordinary Riemann integral.

Remark 1 ([14])**.** *In the case $\mathbb{T} = \mathbb{R}$, the Riemann ∇-integral, as in the case of the Δ-integral, coincides with the usual Riemann integral.*

Corollary 1 ([15])**.** *Let $a, b \in \mathbb{T}$ and $a < b$. If $\mathbb{T} = \mathbb{R}$, then a bounded function f on $[a, b]$ is $\diamondsuit_\alpha$-integrable from a to b if and only if is Riemann integrable on $[a, b]$ in the classical sense, and in this case,*

$$\int_a^b f(t)\diamondsuit_\alpha t = \int_a^b f(t)dt.$$

A comprehensive survey of time scale calculus is given in monographs [13,14]. In particular, for a survey on classical inequalities on time scales, see [16], and for Riemann and Lebesgue integration on time scales, see [17]. Some other recent approaches in time scale calculus can be found in papers [18–21]. Further, for an overview of recent developments of multivariable time scale calculus, we refer the reader to [22].

3. Extension of Steffensen-Type Inequalities to Time Scale Calculus

Through the article by $\mathcal{B}([a, b]_\mathbb{T})$, we denote the Borel σ-algebra on $[a, b]_\mathbb{T}$.

In order to extend Theorem 3 to a general time scale calculus for the diamond-α integral, we first prove the following extension of Theorem 2 to a general time scale calculus for the diamond-α integral, i.e., the Steffensen inequality for the diamond-α integral.

Theorem 5. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\diamondsuit_\alpha \mu$-integrable functions such that f is nonincreasing and $0 \le g \le 1$ on $[a,b]_{\mathbb{T}}$:*

(i) Let $\lambda \in \mathbb{T}$ be a positive constant such that

$$\mu([a, a+\lambda]_{\mathbb{T}}) = \int_{[a,b]_{\mathbb{T}}} g(t) \diamondsuit_\alpha \mu(t), \tag{5}$$

then

$$\int_{[a,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t) \le \int_{[a,a+\lambda]_{\mathbb{T}}} f(t) \diamondsuit_\alpha \mu(t). \tag{6}$$

(ii) Let $\lambda \in \mathbb{T}$ be a positive constant such that

$$\mu((b-\lambda, b]_{\mathbb{T}}) = \int_{[a,b]_{\mathbb{T}}} g(t) \diamondsuit_\alpha \mu(t), \tag{7}$$

then

$$\int_{(b-\lambda,b]_{\mathbb{T}}} f(t) \diamondsuit_\alpha \mu(t) \le \int_{[a,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t). \tag{8}$$

Proof. (i) Subtracting the left-hand side of Inequality (6) from the right-hand side of Inequality (6) and by using (5) and the properties of functions f and g, we arrive at

$$\begin{aligned}
&\int_{[a,a+\lambda]_{\mathbb{T}}} f(t) \diamondsuit_\alpha t - \int_{[a,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t) \\
&\quad = \int_{[a,a+\lambda]_{\mathbb{T}}} f(t)(1 - g(t)) \diamondsuit_\alpha \mu(t) - \int_{(a+\lambda,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t) \\
&\quad \ge f(a+\lambda) \int_{[a,a+\lambda]_{\mathbb{T}}} (1 - g(t)) \diamondsuit_\alpha \mu(t) - \int_{(a+\lambda,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t) \\
&\quad = f(a+\lambda) \left(\mu([a, a+\lambda]_{\mathbb{T}}) - \int_{[a,a+\lambda]_{\mathbb{T}}} g(t) \diamondsuit_\alpha \mu(t) \right) - \int_{(a+\lambda,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t) \\
&\quad = f(a+\lambda) \int_{(a+\lambda,b]_{\mathbb{T}}} g(t) \diamondsuit_\alpha \mu(t) - \int_{(a+\lambda,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t) \\
&\quad = \int_{(a+\lambda,b]_{\mathbb{T}}} (f(a+\lambda) - f(t)) g(t) \diamondsuit_\alpha \mu(t) \ge 0,
\end{aligned}$$

which proves our assertion.

(ii) Similar to Part (i), subtracting the left-hand side of Inequality (8) from the right-hand side of Inequality (8) and by using (7) and the properties of functions f and g, we arrive at

$$\begin{aligned}
&\int_{[a,b]_{\mathbb{T}}} f(t) g(t) \diamondsuit_\alpha \mu(t) - \int_{(b-\lambda,b]_{\mathbb{T}}} f(t) \diamondsuit_\alpha \mu(t) \\
&\qquad \ge \int_{[a,b-\lambda)_{\mathbb{T}}} (f(t) - f(b-\lambda)) g(t) \diamondsuit_\alpha \mu(t) \ge 0,
\end{aligned}$$

which proves our assertion.

□

Remark 2. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure $\diamondsuit_\alpha x$ in Theorem 5, the inequality (6) reduces to the right-hand side Steffensen's inequality (1) and the inequality (8) reduces to the left-hand side Steffensen's inequality (1).*

Corollary 2. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Delta\mu$-integrable functions such that f is nonincreasing and $0 \leq g \leq 1$ on $[a,b]_{\mathbb{T}}$:*

(i) Let $\lambda \in \mathbb{T}$ be a positive constant such that

$$\mu([a, a+\lambda]_{\mathbb{T}}) = \int_{[a,b]_{\mathbb{T}}} g(t)\Delta\mu(t),$$

then

$$\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\mu(t) \leq \int_{[a,a+\lambda]_{\mathbb{T}}} f(t)\Delta\mu(t). \tag{9}$$

(ii) Let $\lambda \in \mathbb{T}$ be a positive constant such that

$$\mu((b-\lambda, b]_{\mathbb{T}}) = \int_{[a,b]_{\mathbb{T}}} g(t)\Delta\mu(t),$$

then

$$\int_{(b-\lambda,b]_{\mathbb{T}}} f(t)\Delta\mu(t) \leq \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\mu(t). \tag{10}$$

Proof. Apply Theorem 5 for $\alpha = 1$. □

Remark 3. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure Δx in Corollary 2, the inequality (9) reduces to the right-hand side Steffensen's inequality and the inequality (10) reduces to the left-hand side Steffensen's inequality.*

Corollary 3. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\nabla\mu$-integrable functions such that f is nonincreasing and $0 \leq g \leq 1$ on $[a,b]_{\mathbb{T}}$:*

(i) Let $\lambda \in \mathbb{T}$ be a positive constant such that

$$\mu([a, a+\lambda]_{\mathbb{T}}) = \int_{[a,b]_{\mathbb{T}}} g(t)\nabla\mu(t),$$

then

$$\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\nabla\mu(t) \leq \int_{[a,a+\lambda]_{\mathbb{T}}} f(t)\nabla\mu(t). \tag{11}$$

(ii) Let $\lambda \in \mathbb{T}$ be a positive constant such that

$$\mu((b-\lambda, b]_{\mathbb{T}}) = \int_{[a,b]_{\mathbb{T}}} g(t)\nabla\mu(t),$$

then

$$\int_{(b-\lambda,b]_{\mathbb{T}}} f(t)\nabla\mu(t) \leq \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\nabla\mu(t). \tag{12}$$

Proof. Apply Theorem 5 for $\alpha = 0$. □

Remark 4. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure ∇x in Corollary 3, the inequality (11) reduces to the right-hand side Steffensen's inequality and the inequality (12) reduces to the left-hand side Steffensen's inequality.*

Now, we prove an extension of the right-hand side Steffensen-type inequality from Theorem 3. We use positive σ-finite measures and the diamond-α-integral to obtain this extension on a general time scale calculus.

Theorem 6. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Diamond_\alpha\mu$-integrable functions such that f is nonincreasing and*

$$-\frac{\mu([a,a+\sigma]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \le g(x) \le 1 - \frac{\mu([a,a+\sigma]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \tag{13}$$

on $(a,b)_{\mathbb{T}}$, where $q \neq 0$. Let $\sigma \in \mathbb{T}$ be a positive constant such that

$$\mu([a,a+\sigma]_{\mathbb{T}}) = q\int_{[a,b]_{\mathbb{T}}} g(x)\Diamond_\alpha\mu(x), \tag{14}$$

then

$$\int_{[a,b]_{\mathbb{T}}} f(x)g(x)\Diamond_\alpha\mu(x) \le \int_{[a,a+\sigma]_{\mathbb{T}}} f(x)\Diamond_\alpha\mu(x) - \frac{\mu([a,a+\sigma]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,b]_{\mathbb{T}}} f(x)\Diamond_\alpha\mu(x). \tag{15}$$

The inequality (15) is reversed for f nondecreasing.

Proof. Assume $p, q \in \mathbb{R}$, and define the functions

$$F(x) = f(x) + p\int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) \tag{16}$$

and

$$G(x) = g(x) + \frac{q-1}{\mu([a,b]_{\mathbb{T}})}\int_{[a,b]_{\mathbb{T}}} g(t)\Diamond_\alpha\mu(t). \tag{17}$$

Applying the inequality (6), we arrive at

$$\int_{[a,b]_{\mathbb{T}}} F(t)G(t)\Diamond_\alpha\mu(t) \le \int_{[a,a+\sigma]_{\mathbb{T}}} F(t)\Diamond_\alpha\mu(t). \tag{18}$$

By simple calculation using (14), we have

$$\begin{aligned}
\int_{[a,b]_{\mathbb{T}}} F(t)G(t)\Diamond_\alpha\mu(t) &= \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Diamond_\alpha\mu(t) \\
&\quad + \left[pq + \frac{q-1}{\mu([a,b]_{\mathbb{T}})}\right]\int_{[a,b]_{\mathbb{T}}} g(t)\Diamond_\alpha\mu(t)\int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) \\
&= \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Diamond_\alpha\mu(t) \\
&\quad + \left[p\cdot\mu([a,a,+\sigma]_{\mathbb{T}}) + \frac{\mu([a,a+\sigma]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\right]\int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t)
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
\int_{[a,a+\sigma]_{\mathbb{T}}} F(t)\Diamond_\alpha\mu(t) &= \int_{[a,a+\sigma]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) + p\int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t)\int_{[a,a+\sigma]_{\mathbb{T}}}\Diamond_\alpha\mu(t) \\
&= \int_{[a,a+\sigma]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) + p\cdot\mu([a,a+\sigma]_{\mathbb{T}})\int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t).
\end{aligned} \tag{20}$$

Hence, from (18), we have that the inequality (15) holds. □

Remark 5. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure $\Diamond_\alpha x$ in Theorem 6, the inequality (15) reduces to the right-hand-side inequality in (4).*

Corollary 4. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Delta\mu$-integrable functions such that f is* *nonincreasing and* (13) *holds on* $(a,b)_{\mathbb{T}}$, *where* $q \neq 0$. *Let* $\sigma \in \mathbb{T}$ *be a positive constant such that* $\mu([a,a+\sigma]_{\mathbb{T}}) = q\int_{[a,b]_{\mathbb{T}}} g(x)\Delta\mu(x)$, *then*

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} f(x)g(x)\Delta\mu(x) \leq &\int_{[a,a+\sigma]_{\mathbb{T}}} f(x)\Delta\mu(x) \\ &- \frac{\mu([a,a+\sigma]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta\mu(x).\end{aligned} \tag{21}$$

The inequality (21) *is reversed for* f *nondecreasing.*

Proof. Apply Theorem 6 for $\alpha = 1$. □

Remark 6. *If* $\mathbb{T} = \mathbb{R}$, *taking the Lebesgue scale measure* Δx *in Corollary* 4, *the inequality* (21) *reduces to the right-hand-side inequality in* (4).

Corollary 5. *Let* $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ *be the time scale measure space with the positive* σ*-finite measure on* $\mathcal{B}([a,b]_{\mathbb{T}})$, *and let* $f,g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *be* $\nabla\mu$*-integrable functions such that* f *is nonincreasing and* (13) *holds on* $(a,b)_{\mathbb{T}}$, *where* $q \neq 0$. *Let* $\sigma \in \mathbb{T}$ *be a positive constant such that* $\mu([a,a+\sigma]_{\mathbb{T}}) = q\int_{[a,b]_{\mathbb{T}}} g(x)\nabla\mu(x)$, *then*

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} f(x)g(x)\nabla\mu(x) \leq &\int_{[a,a+\sigma]_{\mathbb{T}}} f(x)\nabla\mu(x) \\ &- \frac{\mu([a,a+\sigma]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,b]_{\mathbb{T}}} f(x)\nabla\mu(x).\end{aligned} \tag{22}$$

The inequality (22) *is reversed for* f *nondecreasing.*

Proof. Apply Theorem 6 for $\alpha = 0$. □

Remark 7. *If* $\mathbb{T} = \mathbb{R}$, *taking the Lebesgue scale measure* ∇x *in Corollary* 5, *the inequality* (22) *reduces to the right-hand-side inequality in* (4).

Let us also state and prove an extension of the left-hand side Steffensen-type inequality from Theorem 3 to a general time scale calculus with positive σ-finite measures and the diamond-α-integral.

Theorem 7. *Let* $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ *be the time scale measure space with the positive* σ*-finite measure on* $\mathcal{B}([a,b]_{\mathbb{T}})$, *and let* $f,g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ *be* $\Diamond_\alpha\mu$*-integrable functions such that* f *is nonincreasing and*

$$-\frac{\mu((b-\sigma,b]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \leq g(x) \leq 1 - \frac{\mu((b-\sigma,b]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \tag{23}$$

on $(a,b)_{\mathbb{T}}$, *where* $q \neq 0$. *Let* $\sigma \in \mathbb{T}$ *be a positive constant such that*

$$\mu((b-\sigma,b]_{\mathbb{T}}) = q\int_{[a,b]_{\mathbb{T}}} g(x)\Diamond_\alpha\mu(x), \tag{24}$$

then

$$\begin{aligned}\int_{(b-\sigma,b]_{\mathbb{T}}} f(x)\Diamond_\alpha\mu(x) - \frac{\mu((b-\sigma,b]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,b]_{\mathbb{T}}} f(x)\Diamond_\alpha\mu(x) \\ \leq \int_{[a,b]_{\mathbb{T}}} f(x)g(x)\Diamond_\alpha\mu(x).\end{aligned} \tag{25}$$

The inequality (25) *is reversed for* f *nondecreasing.*

Proof. Assume $p, q \in \mathbb{R}$. Let us define the functions F and G by (16) and (17), respectively. Applying the inequality (8), we arrive at

$$\int_{(b-\sigma,b]_{\mathbb{T}}} F(t)\Diamond_\alpha\mu(t) \leq \int_{[a,b]_{\mathbb{T}}} F(t)G(t)\Diamond_\alpha\mu(t). \tag{26}$$

By simple calculation using (24), we arrive at

$$\begin{aligned}
\int_{[a,b]_{\mathbb{T}}} F(t)G(t)\Diamond_\alpha\mu(t) &= \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Diamond_\alpha\mu(t) \\
&+ \left[pq + \frac{q-1}{\mu([a,b]_{\mathbb{T}})}\right] \int_{[a,b]_{\mathbb{T}}} g(t)\Diamond_\alpha\mu(t) \int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) \\
&= \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Diamond_\alpha\mu(t) \\
&+ \left[p \cdot \mu((b-\sigma,b]_{\mathbb{T}}) + \frac{\mu((b-\sigma,b]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\right] \int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t)
\end{aligned}$$

and

$$\begin{aligned}
\int_{(b-\sigma,b]_{\mathbb{T}}} F(t)\Diamond_\alpha\mu(t) &= \int_{(b-\sigma,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) + p\int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) \int_{(b-\sigma,b]_{\mathbb{T}}} \Diamond_\alpha\mu(t) \\
&= \int_{(b-\sigma,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) + p \cdot \mu((b-\sigma,b]_{\mathbb{T}}) \int_{[a,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t).
\end{aligned}$$

Hence, from (26), we have that the inequality (25) holds. □

Remark 8. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure $\Diamond_\alpha x$ in Theorem 7, the inequality (25) reduces to the left-hand-side inequality in (4).*

Corollary 6. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\Delta\mu$-integrable functions such that f is nonincreasing and (23) holds on $(a,b)_{\mathbb{T}}$, where $q \neq 0$. Let $\sigma \in \mathbb{T}$ be a positive constant such that $\mu((b-\sigma,b]_{\mathbb{T}}) = q\int_{[a,b]_{\mathbb{T}}} g(x)\Delta\mu(x)$, then*

$$\int_{(b-\sigma,b]_{\mathbb{T}}} f(x)\Delta\mu(x) - \frac{\mu((b-\sigma,b]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,b]_{\mathbb{T}}} f(x)\Delta\mu(x) \leq \int_{[a,b]_{\mathbb{T}}} f(x)g(x)\Delta\mu(x). \tag{27}$$

The inequality (27) is reversed for f nondecreasing.

Proof. Apply Theorem 7 for $\alpha = 1$. □

Remark 9. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure Δx in Corollary 6, the inequality (27) reduces to the left-hand-side inequality in (4).*

Corollary 7. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\nabla\mu$-integrable functions such that f is nonincreasing and (23) holds on $(a,b)_{\mathbb{T}}$, where $q \neq 0$. Let $\sigma \in \mathbb{T}$ be a positive constant such that $\mu((b-\sigma,b]_{\mathbb{T}}) = q\int_{[a,b]_{\mathbb{T}}} g(x)\nabla\mu(x)$, then*

$$\int_{(b-\sigma,b]_{\mathbb{T}}} f(x)\nabla\mu(x) - \frac{\mu((b-\sigma,b]_{\mathbb{T}})}{\mu([a,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,b]_{\mathbb{T}}} f(x)\nabla\mu(x) \leq \int_{[a,b]_{\mathbb{T}}} f(x)g(x)\nabla\mu(x). \tag{28}$$

The inequality (28) is reversed for f nondecreasing.

Proof. Apply Theorem 7 for $\alpha = 0$. □

Remark 10. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure ∇x in Corollary 7, the inequality (28) reduces to the left-hand-side inequality in (4).*

4. New Diamond-α Steffensen-Type Inequalities for Convex Functions

In [23,24], Pečarić and the author introduced the following class of convex functions in a point, denoted $\mathcal{M}_1^c[a,b]$.

Definition 3 ([24])**.** *Let $f : [a,b] \to \mathbb{R}$ be a function and $c \in (a,b)$. We say that f belongs to class $\mathcal{M}_1^c[a,b]$ ($\mathcal{M}_2^c[a,b]$) if there exists a constant A such that the function $F(x) = f(x) - Ax$ is nonincreasing (nondecreasing) on $[a,c]$ and nondecreasing (nonincreasing) on $[c,b]$.*

Now, let us prove the diamond-α-dynamic Steffensen-type inequalities for the class $\mathcal{M}_1^c[a,b]$ on a general time scale.

Theorem 8. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\diamondsuit_\alpha\mu$-integrable function. For given $c \in (a,b)_{\mathbb{T}}$ and $q \neq 0$, denote $\mu([a,a+\sigma_1]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\diamondsuit_\alpha\mu(t)$ and $\mu((b-\sigma_2,b]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\diamondsuit_\alpha\mu(t)$. Assume*

$$-\frac{\mu([a,a+\sigma_1]_{\mathbb{T}}}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \le g(x) \le 1 - \frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right), \text{ for } x \in (a,c)_{\mathbb{T}}, \tag{29}$$

$$-\frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \le g(x) \le 1 - \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right), \text{ for } x \in (c,b)_{\mathbb{T}}, \tag{30}$$

and

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} tg(t)\diamondsuit_\alpha\mu(t) &= \int_{[a,a+\sigma_1]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) + \int_{(b-\sigma_2,b]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) \\ &- \left(1-\frac{1}{q}\right)\left[\frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\int_{[a,c]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) + \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\int_{[c,b]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t)\right].\end{aligned} \tag{31}$$

If $f \in \mathcal{M}_1^c[a,b]_{\mathbb{T}}$, then

$$\begin{aligned}&\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\diamondsuit_\alpha\mu(t) \\ &\le \int_{[a,a+\sigma_1]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) - \frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu[a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) \\ &+ \int_{(b-\sigma_2,b]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) - \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t).\end{aligned} \tag{32}$$

If $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, the inequality in (32) is reversed.

Proof. Let the function f be from the class $\mathcal{M}_1^c[a,b]_{\mathbb{T}}$. Define the function F by $F(x) = f(x) - Ax$, for a constant A defined as in Definition 3.

Since F is nonincreasing on $[a,c]_{\mathbb{T}}$, applying the inequality (15), we arrive at

$$\begin{aligned}0 \le &\int_{[a,a+\sigma_1]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) - \frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) - \int_{[a,c]_{\mathbb{T}}} f(t)g(t)\diamondsuit_\alpha\mu(t) \\ &- A\left(\int_{[a,a+\sigma_1]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) - \frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) - \int_{[a,c]_{\mathbb{T}}} tg(t)\diamondsuit_\alpha\mu(t)\right).\end{aligned} \tag{33}$$

Since F is nondecreasing on $[c,b]_{\mathbb{T}}$, applying the reverse inequality (25), we arrive at

$$
\begin{aligned}
0 \leq & \int_{(b-\sigma_2,b]_{\mathbb{T}}} f(t)\Diamond_\alpha \mu(t) - \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\Diamond_\alpha \mu(t) - \int_{[c,b]_{\mathbb{T}}} f(t)g(t)\Diamond_\alpha \mu(t) \\
& - A\left(\int_{(b-\sigma_2,b]_{\mathbb{T}}} t\Diamond_\alpha \mu(t) - \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} t\Diamond_\alpha \mu(t) - \int_{[c,b]_{\mathbb{T}}} tg(t)\Diamond_\alpha \mu(t)\right).
\end{aligned} \tag{34}
$$

Clearly, combining Relations (33) and (34), we have

$$
\begin{aligned}
& \int_{[a,a+\sigma_1]_{\mathbb{T}}} f(t)\Diamond_\alpha \mu(t) - \frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\Diamond_\alpha \mu(t) + \int_{(b-\sigma_2,b]_{\mathbb{T}}} f(t)\Diamond_\alpha \mu(t) \\
& - \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\Diamond_\alpha \mu(t) - \int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Diamond_\alpha \mu(t) \\
& \geq A\left(\int_{[a,a+\sigma_1]_{\mathbb{T}}} t\Diamond_\alpha \mu(t) + \int_{(b-\sigma_2,b]_{\mathbb{T}}} t\Diamond_\alpha \mu(t) - \left(1-\frac{1}{q}\right)\frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\int_{[a,c]_{\mathbb{T}}} t\Diamond_\alpha \mu(t)\right. \\
& \left. -\left(1-\frac{1}{q}\right)\frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\int_{[c,b]_{\mathbb{T}}} t\Diamond_\alpha \mu(t) - \int_{[a,b]_{\mathbb{T}}} tg(t)\Diamond_\alpha \mu(t)\right).
\end{aligned}
$$

Therefore, if the condition (31) is satisfied, the inequality (32) holds.
Similarly, for $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, we obtain the reversed inequality. □

Remark 11. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure $\Diamond_\alpha x$ in Theorem 8, the inequality (32) reduces to the results obtained in [25].*

Corollary 8. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\Delta\mu$-integrable function. For given $c \in (a,b)_{\mathbb{T}}$ and $q \neq 0$, denote $\mu([a,a+\sigma_1]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\Delta\mu(t)$ and $\mu((b-\sigma_2,b]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\Delta\mu(t)$. Assume (29) and (30) hold and*

$$
\begin{aligned}
\int_{[a,b]_{\mathbb{T}}} tg(t)\Delta\mu(t) = & \int_{[a,a+\sigma_1]_{\mathbb{T}}} t\Delta\mu(t) + \int_{(b-\sigma_2,b]_{\mathbb{T}}} t\Delta\mu(t) \\
& - \left(1-\frac{1}{q}\right)\left[\frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\int_{[a,c]_{\mathbb{T}}} t\Delta\mu(t) + \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\int_{[c,b]_{\mathbb{T}}} t\Delta\mu(t)\right].
\end{aligned} \tag{35}
$$

If $f \in \mathcal{M}_1^c[a,b]_{\mathbb{T}}$, then

$$
\begin{aligned}
\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\mu(t) \leq & \int_{[a,a+\sigma_1]_{\mathbb{T}}} f(t)\Delta\mu(t) - \frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu[a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\Delta\mu(t) \\
& + \int_{(b-\sigma_2,b]_{\mathbb{T}}} f(t)\Delta\mu(t) - \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\Delta\mu(t).
\end{aligned} \tag{36}
$$

If $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, the inequality in (36) is reversed.

Proof. Apply Theorem 8 for $\alpha = 1$. □

Remark 12. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure Δx in Corollary 8, the inequality (36) reduces to the results obtained in [25].*

Corollary 9. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\nabla\mu$-integrable function. For given*

$c \in (a,b)_{\mathbb{T}}$ and $q \neq 0$, denote $\mu([a,a+\sigma_1]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\nabla\mu(t)$ and $\mu((b-\sigma_2,b]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\nabla\mu(t)$. Assume (29) and (30) hold and

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} tg(t)\nabla\mu(t) &= \int_{[a,a+\sigma_1]_{\mathbb{T}}} t\nabla\mu(t) + \int_{(b-\sigma_2,b]_{\mathbb{T}}} t\nabla\mu(t) \\ &\quad - \left(1-\frac{1}{q}\right)\left[\frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\int_{[a,c]_{\mathbb{T}}} t\nabla\mu(t) + \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\int_{[c,b]_{\mathbb{T}}} t\nabla\mu(t)\right].\end{aligned} \tag{37}$$

If $f \in \mathcal{M}_1^c[a,b]_{\mathbb{T}}$ then

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\nabla\mu(t) \le& \int_{[a,a+\sigma_1]_{\mathbb{T}}} f(t)\nabla\mu(t) - \frac{\mu([a,a+\sigma_1]_{\mathbb{T}})}{\mu[a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\nabla\mu(t) \\ &+ \int_{(b-\sigma_2,b]_{\mathbb{T}}} f(t)\nabla\mu(t) - \frac{\mu((b-\sigma_2,b]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\nabla\mu(t).\end{aligned} \tag{38}$$

If $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, the inequality in (38) is reversed.

Proof. Apply Theorem 8 for $\alpha = 0$. □

Remark 13. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure ∇x in Corollary 9, the inequality (38) reduces to the results obtained in [25].*

Now, let us prove another diamond-α-dynamic Steffensen-type inequality for functions that are convex in a point on a general time scale.

Theorem 9. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\Diamond_\alpha\mu$-integrable function. For given $c \in (a,b)_{\mathbb{T}}$ and $q \neq 0$, denote $\mu((c-\sigma_1,c]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\Diamond_\alpha\mu(t)$ and $\mu([c,c+\sigma_2]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\Diamond_\alpha\mu(t)$. Assume*

$$-\frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \le g(x) \le 1 - \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right), \text{ for } x \in (a,c)_{\mathbb{T}}, \tag{39}$$

$$-\frac{\mu([c,c+\sigma_2]_{\mathbb{T}}}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right) \le g(x) \le 1 - \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right), \text{ for } x \in (c,b)_{\mathbb{T}}, \tag{40}$$

and

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} tg(t)\Diamond_\alpha\mu(t) =& \int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} t\Diamond_\alpha\mu(t) - \frac{\mu(c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} t\Diamond_\alpha\mu(t) \\ &- \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} t\Diamond_\alpha\mu(t).\end{aligned} \tag{41}$$

If $f \in \mathcal{M}_1^c[a,b]_{\mathbb{T}}$, then

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Diamond_\alpha\mu(t) \ge& \int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) \\ &- \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t) \\ &- \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\Diamond_\alpha\mu(t).\end{aligned} \tag{42}$$

If $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, the inequality in (42) is reversed.

Proof. Let the function f be from the class $\mathcal{M}_1^c[a,b]_{\mathbb{T}}$. Define the function F by $F(x) = f(x) - Ax$, for a constant A defined as in Definition 3.

Since F is nonincreasing on $[a,c]_{\mathbb{T}}$, applying the inequality (25), we arrive at

$$\begin{aligned}0 \le &\int_{[a,c]_{\mathbb{T}}} f(t)g(t)\diamondsuit_\alpha\mu(t) - \int_{(c-\sigma_1,c]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) + \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) \\ &- A\left(\int_{[a,c]_{\mathbb{T}}} tg(t)\diamondsuit_\alpha\mu(t) - \int_{(c-\sigma_1,c]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) + \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t)\right).\end{aligned} \tag{43}$$

Since F is nondecreasing on $[c,b]_{\mathbb{T}}$, applying the reverse inequality (15), we arrive at

$$\begin{aligned}0 \le &\int_{[c,b]_{\mathbb{T}}} f(t)g(t)\diamondsuit_\alpha\mu(t) - \int_{[c,c+\sigma_2]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) + \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) \\ &- A\left(\int_{[c,b]_{\mathbb{T}}} tg(t)\diamondsuit_\alpha\mu(t) - \int_{[c,c+\sigma_2]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) + \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t)\right).\end{aligned} \tag{44}$$

Clearly, combining Relations (43) and (44), we have

$$\begin{aligned}&\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\diamondsuit_\alpha\mu(t) - \int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) + \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) \\ &+ \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\diamondsuit_\alpha\mu(t) \ge A\left(\int_{[a,b]_{\mathbb{T}}} tg(t)\diamondsuit_\alpha\mu(t) - \int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t)\right. \\ &\left.+ \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t) + \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} t\diamondsuit_\alpha\mu(t)\right).\end{aligned}$$

Therefore, if the condition (41) holds, the inequality (42) holds.
Similarly, for $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, we obtain the reversed inequality. □

Remark 14. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure $\diamondsuit_\alpha x$ in Theorem 9, the inequality (42) reduces to the results obtained in [25].*

Corollary 10. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\Delta\mu$-integrable function. For given $c \in (a,b)_{\mathbb{T}}$ and $q \ne 0$, denote $\mu((c-\sigma_1,c]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\Delta\mu(t)$ and $\mu([c,c+\sigma_2]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\Delta\mu(t)$. Assume (39) and (40) hold and*

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} tg(t)\Delta\mu(t) = &\int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} t\Delta\mu(t) - \frac{\mu(c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} t\Delta\mu(t) \\ &- \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} t\Delta\mu(t).\end{aligned} \tag{45}$$

If $f \in \mathcal{M}_1^c[a,b]_{\mathbb{T}}$, then

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\Delta\mu(t) \ge &\int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} f(t)\Delta\mu(t) \\ &- \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\Delta\mu(t) \\ &- \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\Delta\mu(t).\end{aligned} \tag{46}$$

If $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, the inequality in (46) is reversed.

Proof. Apply Theorem 9 for $\alpha = 1$. □

Remark 15. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure Δx in Corollary 10, the inequality (46) reduces to the results obtained in [25].*

Corollary 11. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\nabla\mu$-integrable function. For given $c \in (a,b)_{\mathbb{T}}$ and $q \neq 0$, denote $\mu((c-\sigma_1, c]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\nabla\mu(t)$ and $\mu([c, c+\sigma_2]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\nabla\mu(t)$. Assume (39) and (40) hold and*

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} tg(t)\nabla\mu(t) = \int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} t\nabla\mu(t) - \frac{\mu(c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} t\nabla\mu(t) \\ - \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} t\nabla\mu(t).\end{aligned} \tag{47}$$

If $f \in \mathcal{M}_1^c[a,b]_{\mathbb{T}}$, then

$$\begin{aligned}\int_{[a,b]_{\mathbb{T}}} f(t)g(t)\nabla\mu(t) \geq &\int_{(c-\sigma_1,c+\sigma_2]_{\mathbb{T}}} f(t)\nabla\mu(t) \\ &- \frac{\mu((c-\sigma_1,c]_{\mathbb{T}})}{\mu([a,c]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[a,c]_{\mathbb{T}}} f(t)\nabla\mu(t) \\ &- \frac{\mu([c,c+\sigma_2]_{\mathbb{T}})}{\mu([c,b]_{\mathbb{T}})}\left(1-\frac{1}{q}\right)\int_{[c,b]_{\mathbb{T}}} f(t)\nabla\mu(t).\end{aligned} \tag{48}$$

If $f \in \mathcal{M}_2^c[a,b]_{\mathbb{T}}$, the inequality in (48) is reversed.

Proof. Apply Theorem 9 for $\alpha = 10$. □

Remark 16. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure ∇x in Corollary 11, the inequality (48) reduces to the results obtained in [25].*

The following relationship between the class of convex functions and the class of functions convex in a point was proven in [24].

Theorem 10. *The function f is convex on $[a,b]$ if and only if it is convex in every $c \in (a,b)$.*

Let us prove new diamond-α-dynamic Steffensen-type inequalities on a general time scale by utilizing convex functions.

Corollary 12. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\Diamond_\alpha\mu$-integrable function. For given $c \in (a,b)_{\mathbb{T}}$ and $q \neq 0$, denote $\mu([a, a+\sigma_1]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\Diamond_\alpha\mu(t)$ and $\mu((b-\sigma_2, b]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\Diamond_\alpha\mu(t)$. Assume (29), (30), and (31) hold. If $f : [a,b] \to \mathbb{R}$ is a convex function, then (32) holds.*
If $f : [a,b] \to \mathbb{R}$ is a concave function, the inequality in (32) is reversed.

Proof. From Theorem 10, we have that the convex function f is from the class $\mathcal{M}_1^c[a,b]$, for every $c \in (a,b)$. Now, applying Theorem 8, the statement of this corollary follows. □

Remark 17. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure $\Diamond_\alpha x$ in Corollary 12, the inequality (32) reduces to the results obtained in [25].*

Corollary 13. *Let $([a,b]_{\mathbb{T}}, \mathcal{B}([a,b]_{\mathbb{T}}), \mu)$ be the time scale measure space with the positive σ-finite measure on $\mathcal{B}([a,b]_{\mathbb{T}})$, and let $g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be a $\Delta\mu$-integrable function. For given $c \in (a,b)_{\mathbb{T}}$ and $q \neq 0$, denote $\mu([a, a+\sigma_1]_{\mathbb{T}}) = q\int_{[a,c]_{\mathbb{T}}} g(t)\Delta\mu(t)$ and $\mu((b-\sigma_2, b]_{\mathbb{T}}) = q\int_{[c,b]_{\mathbb{T}}} g(t)\Delta\mu(t)$.*

Assume (29), (30) *and* (35) *hold. If* $f : [a, b] \to \mathbb{R}$ *is a convex function, then* (36) *holds. If* $f : [a, b] \to \mathbb{R}$ *is a concave function, the inequality in* (36) *is reversed.*

Proof. Taking $\alpha = 1$ in Corollary 12, we obtain the statement of this corollary. □

Remark 18. *If* $\mathbb{T} = \mathbb{R}$*, taking the Lebesgue scale measure* Δx *in Corollary 13, the inequality* (36) *reduces to the results obtained in [25].*

Corollary 14. *Let* $([a, b]_{\mathbb{T}}, \mathcal{B}([a, b]_{\mathbb{T}}), \mu)$ *be the time scale measure space with the positive* σ*-finite measure on* $\mathcal{B}([a, b]_{\mathbb{T}})$*, and let* $g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *be a* $\nabla\mu$*-integrable function. For given* $c \in (a, b)_{\mathbb{T}}$ *and* $q \neq 0$*, denote* $\mu([a, a + \sigma_1]_{\mathbb{T}}) = q \int_{[a,c]_{\mathbb{T}}} g(t)\nabla\mu(t)$ *and* $\mu((b - \sigma_2, b]_{\mathbb{T}}) = q \int_{[c,b]_{\mathbb{T}}} g(t)\nabla\mu(t)$*. Assume* (29), (30), *and* (37) *hold. If* $f : [a, b] \to \mathbb{R}$ *is a convex function, then* (38) *holds.*
If $f : [a, b] \to \mathbb{R}$ *is a concave function, the inequality in* (38) *is reversed.*

Proof. Taking $\alpha = 0$ in Corollary 12, we obtain the statement of this corollary. □

Remark 19. *If* $\mathbb{T} = \mathbb{R}$*, taking the Lebesgue scale measure* ∇x *in Corollary 14, the inequality* (38) *reduces to the results obtained in [25].*

Another diamond-α-dynamic Steffensen-type inequality for the class of convex functions on a general time scale is given in the following corollary.

Corollary 15. *Let* $([a, b]_{\mathbb{T}}, \mathcal{B}([a, b]_{\mathbb{T}}), \mu)$ *be the time scale measure space with the positive* σ*-finite measure on* $\mathcal{B}([a, b]_{\mathbb{T}})$*, and let* $g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *be a* $\diamondsuit_\alpha\mu$*-integrable function. For given* $c \in (a, b)_{\mathbb{T}}$ *and* $q \neq 0$*, denote* $\mu((c - \sigma_1, c]_{\mathbb{T}}) = q \int_{[a,c]_{\mathbb{T}}} g(t)\diamondsuit_\alpha\mu(t)$ *and* $\mu([c, c + \sigma_2]_{\mathbb{T}}) = q \int_{[c,b]_{\mathbb{T}}} g(t)\diamondsuit_\alpha\mu(t)$*. Assume* (39), (40), *and* (41) *hold. If* $f : [a, b] \to \mathbb{R}$ *is a convex function, then* (42) *holds.*
If $f : [a, b] \to \mathbb{R}$ *is a concave function, the inequality in* (42) *is reversed.*

Proof. Apply Theorem 9 using the same reasoning as in the proof of Corollary 12. □

Remark 20. *If* $\mathbb{T} = \mathbb{R}$*, taking the Lebesgue scale measure* $\diamondsuit_\alpha x$ *in Corollary 15, the inequality* (42) *reduces to the results obtained in [25].*

Corollary 16. *Let* $([a, b]_{\mathbb{T}}, \mathcal{B}([a, b]_{\mathbb{T}}), \mu)$ *be the time scale measure space with the positive* σ*-finite measure on* $\mathcal{B}([a, b]_{\mathbb{T}})$*, and let* $g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *be a* $\Delta\mu$*-integrable function. For given* $c \in (a, b)_{\mathbb{T}}$ *and* $q \neq 0$*, denote* $\mu((c - \sigma_1, c]_{\mathbb{T}}) = q \int_{[a,c]_{\mathbb{T}}} g(t)\Delta\mu(t)$ *and* $\mu([c, c + \sigma_2]_{\mathbb{T}}) = q \int_{[c,b]_{\mathbb{T}}} g(t)\Delta\mu(t)$*. Assume* (39), (40), *and* (45) *hold. If* $f : [a, b] \to \mathbb{R}$ *is a convex function, then* (46) *holds.*
If $f : [a, b] \to \mathbb{R}$ *is a concave function, the inequality in* (46) *is reversed.*

Proof. Taking $\alpha = 1$ in Corollary 15, we obtain the statement of this corollary. □

Remark 21. *If* $\mathbb{T} = \mathbb{R}$*, taking the Lebesgue scale measure* Δx *in Corollary 16, the inequality* (46) *reduces to the results obtained in [25].*

Corollary 17. *Let* $([a, b]_{\mathbb{T}}, \mathcal{B}([a, b]_{\mathbb{T}}), \mu)$ *be the time scale measure space with the positive* σ*-finite measure on* $\mathcal{B}([a, b]_{\mathbb{T}})$*, and let* $g : [a, b]_{\mathbb{T}} \to \mathbb{R}$ *be a* $\nabla\mu$*-integrable function. For given* $c \in (a, b)_{\mathbb{T}}$ *and* $q \neq 0$*, denote* $\mu((c - \sigma_1, c]_{\mathbb{T}}) = q \int_{[a,c]_{\mathbb{T}}} g(t)\nabla\mu(t)$ *and* $\mu([c, c + \sigma_2]_{\mathbb{T}}) = q \int_{[c,b]_{\mathbb{T}}} g(t)\nabla\mu(t)$*. Assume* (39), (40), *and* (47) *hold. If* $f : [a, b] \to \mathbb{R}$ *is a convex function, then* (48) *holds.*
If $f : [a, b] \to \mathbb{R}$ *is a concave function, the inequality in* (48) *is reversed.*

Proof. Taking $\alpha = 0$ in Corollary 15, we obtain the statement of this corollary. □

Remark 22. *If $\mathbb{T} = \mathbb{R}$, taking the Lebesgue scale measure ∇x in Corollary 17, the inequality (48) reduces to the results obtained in [25].*

5. Conclusions

In this paper, we extended Steffensen-type inequalities given in [12] utilizing the general time scale measure space with a positive σ-finite measure and diamond-α integral. Besides that, we obtained some new Steffensen-type inequalities for the class of functions from $\mathcal{M}_1^c[a,b]$, and we used these results to prove new Steffensen-type inequalities for convex functions. As a special case, we recaptured some known Steffensen-type inequalities in differential calculus on $\mathbb{R}$.

Our results can be used to obtain the discrete Steffensen's inequality and new discrete Steffensen-type inequalities in difference calculus on $\mathbb{Z}$. Further, as a special case of our results, one can also acquire new Steffensen-type inequalities in measure theoretic settings, as in [26,27].

In [28], Brito da Cruz et al. introduced a more general type of integral on time scales, called the new diamond integral. This integral is a refined version of the diamond-α integral, and when $\mathbb{T} = \mathbb{R}$, it is equal to the $\diamondsuit_{\frac{1}{2}}$-integral. Using the new diamond integral instead of the diamond-α integral, one can obtain some other Steffensen-type inequalities applying the technique described in this paper. Further, some new results dealing with this type of integral on time scales can be found in [19,29].

Let us also draw the reader's attention to complementary aspects of the study of functions and their applications in practical problems, which has been pointed out in recent papers [30–32].

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