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Information Measures Based on T-Spherical Fuzzy Sets and Their Applications in Decision Making and Pattern Recognition

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Abstract: The T-spherical fuzzy set (TSFS) is a modification of the fuzzy set (FS), intuitionistic fuzzy set (IFS), Pythagorean fuzzy set (PyFS), q-rung orthopair fuzzy set (q-ROFS), and picture fuzzy set (PFS), with three characteristic functions: the membership degree (MD) denoted by S , the nonmembership degree (NMD) denoted by D , and the abstinence degree (AD) denoted by I . It can be used to solve problems of uncertain information with no restrictions. The distance measure (DM) is a tool that sums up the difference between points, while the similarity measure (SM) is a method applied to calculate the similarity between objects within an interval of $[0, 1]$. The current work aims to introduce novel DMs and SMs in the environment of TSFSs to show the limitations of the previously defined DMs and SMs. The suggested DMs and SMs provide more room for all three degrees to be selected without restriction. We investigated the effectiveness of the proposed DMs and SMs by applying a pattern-recognition technique, and we determined their applicability for multicriteria decision making (MCDM) using numerical examples. The newly proposed DMs and SMs are briefly compared to existing DMs and SMs, and appropriate conclusions are drawn.

Keywords: T-spherical fuzzy set (TSFS); distance measure (DM); similarity measure (SM); pattern recognition; multicriteria decision making (MCDM)



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1. Introduction

In 1965, Zadeh [1] was the first to establish the notion of an FS. S and D are the two degrees of measurement in an FS. An MD within the interval $[0, 1]$ defines the element of a set in S , whereas D can be generated by subtracting S from 1. Following Zadeh, Atanassov [2] improved this notion by suggesting the concept of the IFS, in which S and D are outlined independently but with the requirement that their total belongs to the scale $[0, 1]$. Furthermore, the term $1 - \text{sum}(S, D)$ was characterized as a hesitancy degree (HD). There are certain limits in the Atanassov model of IFS, as the total of S and D frequently falls outside of the interval $[0, 1]$. As a result, Yager [3] introduced a new notion known as the PyFS, in which the area for assigning the MD and NMD of IFSs is expanded, and the requirement of the PyFS becomes $\text{sum}(S^2, D^2) \in [0, 1]$. However, this is not always applicable, as squaring the sum sometimes results in it falling outside of the unit interval $[0, 1]$, showing the limitations of the PyFS. As a result, Yager [4] proposed the idea of the q-ROFS, in which the sum of the q -th power of MD and the q -th power of NMD is equal to or less than 1. The q-ROFS condition becomes $\text{sum}(S^q, D^q) \in [0, 1]$.

Without a doubt, Atanassov's model of IFSs improved Zadeh's concept of FSs; however, there are occasions when more than two choices exist, whereby IFSs fail to describe and solve such problems. Cuong and Kreinovich [5] proposed a novel model called the PFS, which includes three characteristic functions defined as S , I , and D , where I is the abstinence degree (AD), with the requirement that their sum must fall within the interval

[0, 1]. The term $1 - \text{sum}(S, I, D)$ refers to the refusal degree (RD) of an element of the PFS. PFSs broadened the scope of FSs and IFSs; however, the problem still exists that we cannot assign the values of the MD, NMD, and AD. After realizing certain limitations that exist in the structures of FSs, IFSs, and PFSs, Mahmood et al. [6] expanded this concept and presented the notion of the SFS as a modification of PFSs by extending their range. The sum of S , I , and D may fall outside of the unit interval in the structure of SFSs, but their square must fall within the unit interval, which is specified as $\text{sum}(S^2, I^2, D^2) \in [0, 1]$. SFSs have a larger domain than PFSs due to their new condition. Sometimes, such a situation occurs where the total of the squares of S , I , and D falls outside the unit interval, rendering this method insufficient. To cope with such situations, Mahmood et al. [6] developed an extension of SFSs called the TSFS, which has no restrictions. The TSFS contains the constraint that $\text{sum}(S^q, I^q, D^q) \in [0, 1]$, where $q \in \mathbb{Z}^+$. Considering the above-described research, it can be concluded that the TSFS is a modification of the IFS, PyFS, and PFS, with no limitations. TSFSs are more important than IFSs, PyFSs, and PFSs since they have greater space. As a result, in TSFSs, we can allocate the values of the MD, NMD, and AD to our liking due to the greater space available, which is infeasible in IFSs, PyFSs, and PFSs due to their constrained structure.

The DM is a tool for calculating the difference between distinct items on a scale of [0, 1], while the SM is a scheme that calculates the degree of similarity between two items on a scale of [0, 1]. Several DMs and SMs have been introduced in various extended versions of FSs. Wang and Xin [7] analyzed the links between the DMs and SMs of IFSs, and used their suggested DMs and SMs to recognize patterns. Xiao [8] presented a DM for IFSs based on the Jensen–Shannon divergence and used it as an algorithm for pattern classification, thus providing a solution to the interference concerns. Jiang et al. [9] suggested a DM and SM for IFSs based on a transformed isosceles triangle, using them to tackle pattern-recognition difficulties. Du and Hu [10] presented aggregation DMs and the induced SMs of IFSs and successfully utilized them in a variety of pattern-recognition applications. DMs and SMs of PyFSs were suggested by Zeng et al. [11] and applied to MCDM. They also presented a numerical example to test the suggested decision-making method's efficacy. On the basis of the Hausdorff metric, Hussain and Yang [12] presented the DMs and SMs of PyFSs. To test the validity of the suggested DMs and SMs, they were used to recognize several patterns based on linguistic variables. Wang et al. [13] designed DMs and SMs of q -ROFSs using cosine functions and applied them to pattern-recognition and scheme-selection issues. Liu et al. [14] suggested various cosine DMs and SMs of q -ROFSs and used them to tackle issues such as decision making from both a geometric and an algebraic perspective. Garg et al. [15] suggested generalized Dice SMs of complicated q -ROFSs and used them to solve issues such as pattern recognition and medical diagnosis. Khan et al. [16] suggested better cosine and cotangent function-based SMs of q -ROFSs and tested their validity using numerical examples. The notion of complex q -ROF variation coefficient SMs was presented by Liu et al. [17], and the suggested SMs were used for medical diagnosis and pattern-recognition challenges. Donyatalab et al. [18] presented DMs and SMs of q -ROFSs based on the square root cosine similarity measure and expanded them in two states: with and without HD. Khan et al. [19] proposed bi-parametric DMs and SMs of PFSs and used them in medical diagnosis. Cosine SMs of PFSs were developed by Wei [20], and the proposed SMs were applied to strategic decision-making issues. Wei [21] introduced SMs of PFSs and offered examples to demonstrate their validity for building material recognition and mineral field recognition. Jan et al. [22] presented a set of generalized DMs and SMs of PFSs and used them to tackle issues such as building material detection and MADM. Khan et al. [23] introduced DMs and SMs of SFSs and used them to solve data mining, medical diagnosis, and MADM problems. Rafiq et al. [24] presented cosine SMs of SFSs and used them to solve issues involving MADM. Wei et al. [25] proposed certain SMs of SFSs based on cosine functions and used them to solve pattern-recognition and medical diagnosis problems. Shishavan et al. [26] proposed various new SMs of SFSs and used them to solve problems such as medical diagnosis and green supplier selection.

Wu et al. [27] presented cosine-based SMs of TSFSs and used them for pattern-recognition issues. Ullah et al. [28] proposed SMs of TSFSs and used them for pattern-recognition tasks. Abid et al. [29] suggested a novel SM for TSFSs and discussed its applications in pattern recognition and decision making. Ibrahim et al. [30] discussed (3,2)-fuzzy sets and suggested their applications to topology and optimal choices. Al-shami et al. [31] suggested SR-fuzzy sets and discussed their applications to weighted aggregated operators in MADM.

Several SMs and DMs have been developed so far and discussed previously, such as the SMs suggested by Wang et al. [13], Liu et al. [14], Garg et al. [15], Khan et al. [16], Peng and Liu [32], Zeng et al. [33], and Zeng et al. [34]. All of these studies discussed only two aspects of imprecise and uncertain information and hence led towards information loss. In the context of TSFSs, we develop several SMs and DMs that address four characteristics of information that are ambiguous and imprecise. In handling issues with ambiguous knowledge, these SMs and DMs provide higher accuracy, reliability, and fascinating flexibility. We draw a comparison between the proposed and past SMs and DMs to see how adaptive the proposed method is. We discuss all four degrees of TSFSs, i.e., MD, AD, NMD, and RD, demonstrating the novelty of this concept. Because all of these degrees are present, the spectrum of TSFSs expands, revealing the limitations of all previously described notions such as IFSs, PyFSs, q-ROFSs, and PFSs. Existing conceptions are unable to handle the difficulties when discussed in a TSF environment due to their limited range and lack of degrees. When data are provided in a TSF environment, the proposed DMs and SMs can solve any real-life difficulty by analyzing the facts. The significance of the topic stems from its concept, which demonstrates that the previously defined DMs and SMs failed to solve challenges when information was delivered in a TSF setting due to their limited range and low adaptability. As a result of the constraints of existing DMs and SMs, the suggested DMs and SMs can alleviate problems when data are presented in a TSF environment.

The remaining portion of the paper is arranged based on some definitions of TSFSs, as presented in Section 2. Sections 3 and 4 define new DMs and SMs, respectively. Section 5 looks at some numerical examples and applications of presented DMs and SMs for developing an algorithm for pattern recognition and MCDM. Section 6 discusses some of the current work's consequences. Finally, in Section 7, we present our conclusion.

2. Preliminaries

In this section, we discuss some definitions of SFSs and TSFSs. The definitions discussed here provide the basis of our work. We consider X a universe of discourse throughout our study.

Definition 1. [4] A q -ROFS on X is written as:

$$P = \{x, S(x), D(x) : 0 \leq \text{sum}(S^q, D^q) \leq 1\}$$

such that S and D ranging from 0 to 1 denote the MD and NMD of $x \in X$, respectively, and for $q \geq 1$ and $r(x) = \sqrt[q]{1 - \text{sum}(S^q, D^q)}$ is the refusal degree (RD). (S, D) is termed as a q -rung orthopair fuzzy number (q -ROFN).

Definition 2. [6] A TSFS on X is written as

$$P = \{x, S(x), I(x), D(x) : 0 \leq \text{sum}(S^q, I^q, D^q) \leq 1\}$$

such that S , I , and D ranging from 0 to 1 denote the MD, AD, and NMD of $x \in X$, respectively, and $q \in \mathbb{Z}^+$ and $r(x) = \sqrt[q]{1 - \text{sum}(S^q, I^q, D^q)}$ is the RD. (S, I, D) is termed a T -spherical fuzzy number (TSFN). We denote the set of all TSFSs by $\text{TSFS}(X)$.

Definition 3. Let $M, N \in TSFS(X)$. Then

- (1) $M^c = \{ \langle x, S_M(x), I_M(x), D_M(x) \rangle | x \in X \}$
- (2) $M \subseteq N$ iff $\forall x \in X, S_M(x) \leq S_N(x)$, and $I_M(x) \geq I_N(x)$, and $D_M(x) \geq D_N(x)$
- (3) $M = N$ iff $\forall x \in X, S_M(x) = S_N(x)$, and $I_M(x) = I_N(x)$, and $D_M(x) = D_N(x)$
- (4) $M \cap N = \{ \langle x, S_M(x) \wedge S_N(x), I_M(x) \vee I_N(x), D_M(x) \wedge D_N(x) \rangle | x \in X \}$
- (5) $M \cup N = \{ \langle x, S_M(x) \vee S_N(x), I_M(x) \wedge I_N(x), D_M(x) \vee D_N(x) \rangle | x \in X \}$
- (6) $M \oplus N = \{ \langle x, \sqrt[3]{S_M^q(x) + S_N^q(x) - S_M^q(x)S_N^q(x)}, I_M(x) \wedge I_N(x), D_M(x) \wedge D_N(x) \rangle | x \in X \}$
- (7) $M \otimes N = \{ \langle x, S_M(x)S_N(x), \sqrt[3]{I_M^q(x) + I_N^q(x) - I_M^q(x)I_N^q(x)}, \sqrt[3]{D_M^q(x) + D_N^q(x) - D_M^q(x)D_N^q(x)} \rangle | x \in X \}$
- (8) $M \ominus N = \left\{ \langle x, \sqrt[3]{\frac{S_M^q(x) - S_N^q(x)}{1 - S_N^q(x)}}, \frac{I_M(x)}{I_N(x)}, \frac{D_M(x)}{D_N(x)} \rangle | x \in X \right\}$, if $S_M(x) \geq S_N(x)$,
 $I_M(x) \leq \min \left\{ I_N(x), \frac{I_N(x)\pi_M(x)}{\pi_N(x)} \right\}$
- (9) $M \oslash N = \left\{ \langle x, \frac{S_M(x)}{S_N(x)}, \sqrt[3]{\frac{I_M^q(x) - I_N^q(x)}{1 - I_N^q(x)}}, \sqrt[3]{\frac{D_M^q(x) - D_N^q(x)}{1 - D_N^q(x)}} \rangle | x \in X \right\}$, if $I_M(x) \geq I_N(x)$,
 $S_M(x) \leq \min \left\{ S_N(x), \frac{S_N(x)\pi_M(x)}{\pi_N(x)} \right\}$

3. Distance Measures for TSFSs

Here, we propose some DMs based on T-spherical fuzzy information. The proposed new DMs are supported by examples, and their exceptional cases are studied.

Definition 4. Let $M, N, O \in TSFS(X)$. A DM denoted by $D(M, N)$ is a mapping of $D: TSFSs(X) \times TSFSs(X) \rightarrow [0, 1]$, possessing the features listed below:

- (1) $0 \leq D(M, N) \leq 1$.
- (2) $D(M, N) = D(N, M)$.
- (3) $D(M, N) = 0$ iff $M = N$.
- (4) $D(M, M^c) = 1$.
- (5) If $M \subseteq N \subseteq O$, then $D(M, N) \leq D(M, O)$ and $D(N, O) \leq D(M, O)$.

Theorem 1. Let $M, N \in TSFS(X)$. Then $D_i(M, N) (i = 1, 2, \dots, 12)$ is a DM.

$$D_1(M, N) = \frac{1}{2|X|} \sum_{x \in X} \left(\frac{|S_M^q(x) - S_N^q(x)| + |I_M^q(x) - I_N^q(x)| + |D_M^q(x) - D_N^q(x)|}{|r_M^q(x) - r_N^q(x)|} \right) \quad (1)$$

$$D_2(M, N) = \frac{1}{2|X|} \sum_{x \in X} \left(|S_M^q(x) - S_N^q(x) - I_M^q(x) + I_N^q(x) - D_M^q(x) + D_N^q(x)| \right) \quad (2)$$

$$D_3(M, N) = \frac{1}{4|X|} \left(\left(\sum_{x \in X} \frac{|S_M^q(x) - S_N^q(x)| + |I_M^q(x) - I_N^q(x)|}{|D_M^q(x) - D_N^q(x)| + |r_M^q(x) - r_N^q(x)|} \right) + \sum_{x \in X} \left(\left| \frac{(S_M^q(x) - I_M^q(x) - D_M^q(x)) - (S_N^q(x) - I_N^q(x) - D_N^q(x))}{(S_M^q(x) - I_M^q(x) - D_M^q(x)) + (S_N^q(x) - I_N^q(x) - D_N^q(x))} \right| \right) \right) \quad (3)$$

$$D_4(M, N) = \frac{1}{|X|} \left(\sum_{x \in X} (|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|) \right) \quad (4)$$

$$D_5(M, N) = \frac{2}{|X|} \left(\sum_{x \in X} \frac{(|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|)}{1 + (|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|)} \right) \quad (5)$$

$$D_6(M, N) = \frac{2 \sum_{x \in X} \left(|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)| \right)}{\sum_{x \in X} \left(1 + \left(|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)| \right) \right)} \quad (6)$$

$$D_7(M, N) = 1 - \alpha \frac{\sum_{x \in X} \left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) - \beta \frac{\sum_{x \in X} \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right)}{\sum_{x \in X} \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)}, \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \in [0, 1] \quad (7)$$

$$D_8(M, N) = 1 - \frac{\alpha}{|X|} \sum_{x \in X} \frac{\left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right)}{\sum_{x \in X} \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right)} - \frac{\beta}{|X|} \sum_{x \in X} \frac{\left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)}{\sum_{x \in X} \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)}, \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \in [0, 1] \quad (8)$$

$$D_9(M, N) = 1 - \frac{1}{|X|} \sum_{x \in X} \frac{\left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)}{\sum_{x \in X} \left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)} \quad (9)$$

$$D_{10}(M, N) = 1 - \frac{\sum_{x \in X} \left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)}{\sum_{x \in X} \left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)} \quad (10)$$

$$D_{11}(M, N) = 1 - \frac{1}{|X|} \sum_{x \in X} \frac{\left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)}{\sum_{x \in X} \left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)} \quad (11)$$

$$D_{12}(M, N) = 1 - \frac{\sum_{x \in X} \left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)}{\sum_{x \in X} \left(\frac{S_M^q(x) \wedge S_N^q(x)}{S_M^q(x) \vee S_N^q(x)} \right) + \left(\frac{I_M^q(x) \wedge I_N^q(x)}{I_M^q(x) \vee I_N^q(x)} \right) + \left(\frac{D_M^q(x) \wedge D_N^q(x)}{D_M^q(x) \vee D_N^q(x)} \right)} \quad (12)$$

Proof of Theorem 1. We only give the proof of (2).

$$\begin{aligned} D_2(M, N) &= \frac{1}{2|X|} \sum_{x \in X} \left| S_M^q(x) - S_N^q(x) - \left(I_M^q(x) - I_N^q(x) \right) - \left(D_M^q(x) - D_N^q(x) \right) \right| \\ &= \frac{1}{2|X|} \sum_{x \in X} \left(I_M^q(x) - I_N^q(x) - \left(S_M^q(x) - S_N^q(x) \right) - \left(D_M^q(x) - D_N^q(x) \right) \right) \\ D_2(M, O) &= \frac{1}{2|X|} \sum_{x \in X} \left| S_M^q(x) - S_O^q(x) - \left(I_M^q(x) - I_O^q(x) \right) - \left(D_M^q(x) - D_O^q(x) \right) \right| \\ &= \frac{1}{2|X|} \sum_{x \in X} \left(I_M^q(x) - I_O^q(x) - \left(S_M^q(x) - S_O^q(x) \right) - \left(D_M^q(x) - D_O^q(x) \right) \right). \end{aligned}$$

Suppose $\triangle D_2 = D_2(M, O) - D_2(M, N)$;

$$\triangle D_2 = \frac{1}{2|X|} \sum_{x \in X} \left(S_O^q(x) - S_N^q(x) + I_N^q(x) - I_O^q(x) + D_N^q(x) - D_O^q(x) \right)$$

≥ 0 , i.e., $D_2(M, O) \geq D_2(M, N)$.

Hence, $D_2(M, O) \geq D_2(N, O)$.

(1) Similarly, from (1).

(2) From the formula, we obtain

$$\begin{aligned}
D_5(M, N) &= \frac{2}{|X|} \sum_{x \in X} \frac{|S_M^q(x) - S_O^q(x)| \vee |I_M^q(x) - I_O^q(x)| \vee |D_M^q(x) - D_O^q(x)|}{1 + |S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|} \\
&= \frac{2}{|X|} \sum_{x \in X} \frac{1}{\frac{1}{|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|} + 1} \\
D_5(M, O) &= \frac{2}{|X|} \sum_{x \in X} \frac{|S_M^q(x) - S_O^q(x)| \vee |I_M^q(x) - I_O^q(x)| \vee |D_M^q(x) - D_O^q(x)|}{1 + |S_M^q(x) - S_O^q(x)| \vee |I_M^q(x) - I_O^q(x)| \vee |D_M^q(x) - D_O^q(x)|} \\
&= \frac{2}{|X|} \sum_{x \in X} \frac{1}{\frac{1}{|S_M^q(x) - S_O^q(x)| \vee |I_M^q(x) - I_O^q(x)| \vee |D_M^q(x) - D_O^q(x)|} + 1}
\end{aligned}$$

Because $|S_M^q(x) - S_O^q(x)| \vee |I_M^q(x) - I_O^q(x)| \vee |D_M^q(x) - D_O^q(x)| \geq |S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|$, so $D_5(M, N) \leq D_5(M, O)$. Hence, $D_5(M, O) \geq D_5(N, O)$.

(3) Similarly, from (5).

(4) From the formula, we obtain

$$\begin{aligned}
D_7(M, N) &= 1 - \alpha \frac{\sum_{x \in X} (S_M^q(x) \wedge S_N^q(x))}{\sum_{x \in X} (S_M^q(x) \vee S_N^q(x))} - \beta \frac{\sum_{x \in X} (I_M^q(x) \wedge I_N^q(x))}{\sum_{x \in X} (I_M^q(x) \vee I_N^q(x))} - \gamma \frac{\sum_{x \in X} (D_M^q(x) \wedge D_N^q(x))}{\sum_{x \in X} (D_M^q(x) \vee D_N^q(x))} \\
&= 1 - \alpha \frac{\sum_{x \in X} S_M^q(x)}{\sum_{x \in X} S_N^q(x)} - \beta \frac{\sum_{x \in X} I_N^q(x)}{\sum_{x \in X} I_M^q(x)} - \gamma \frac{\sum_{x \in X} D_M^q(x)}{\sum_{x \in X} D_N^q(x)}, \\
D_7(M, O) &= 1 - \alpha \frac{\sum_{x \in X} (S_M^q(x) \wedge S_O^q(x))}{\sum_{x \in X} (S_M^q(x) \vee S_O^q(x))} - \beta \frac{\sum_{x \in X} (I_M^q(x) \wedge I_O^q(x))}{\sum_{x \in X} (I_M^q(x) \vee I_O^q(x))} - \gamma \frac{\sum_{x \in X} (D_M^q(x) \wedge D_O^q(x))}{\sum_{x \in X} (D_M^q(x) \vee D_O^q(x))} \\
&= 1 - \alpha \frac{\sum_{x \in X} S_M^q(x)}{\sum_{x \in X} S_O^q(x)} - \beta \frac{\sum_{x \in X} I_O^q(x)}{\sum_{x \in X} I_M^q(x)} - \gamma \frac{\sum_{x \in X} D_M^q(x)}{\sum_{x \in X} D_O^q(x)},
\end{aligned}$$

Because

$$\frac{\sum_{x \in X} S_M^q(x)}{\sum_{x \in X} S_N^q(x)} \geq \frac{\sum_{x \in X} S_M^q(x)}{\sum_{x \in X} S_O^q(x)}, \quad \frac{\sum_{x \in X} I_N^q(x)}{\sum_{x \in X} I_M^q(x)} \geq \frac{\sum_{x \in X} I_O^q(x)}{\sum_{x \in X} I_M^q(x)}$$

So $D_7(M, N) \leq D_7(M, O)$. Likewise, $D_7(M, O) \leq D_7(N, O)$. \square

Example 1. Let $M, N \in TSFS(X)$ as given in Table 1. Both M and N are TSFSs for $q = 3$.

Table 1. Two TSFSs, M and N .

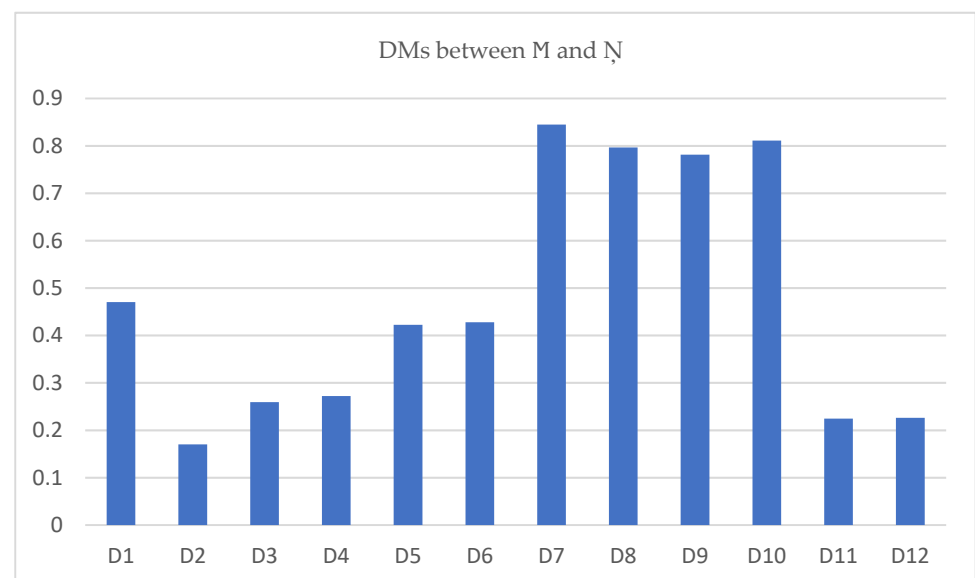
	M				N			
	MD	AD	NMD	RD	MD	AD	NMD	RD
x_1	0.56	0.47	0.22	0.892076	0.81	0.30	0.37	0.731180
x_2	0.11	0.11	0.11	0.998667	0.59	0.66	0.66	0.603342
x_3	0.35	0.45	0.61	0.856205	0.42	0.56	0.71	0.945407

We applied the DMs D_i ($i = 1, 2, 3, \dots, 12$) defined above to the two sets provided in Table 1, and the outcomes are portrayed in Table 2.

Table 2. DMS between M and N.

DMS	VALUES
$D_1(M, N)$	0.470220
$D_2(M, N)$	0.170083
$D_3(M, N)$	0.259433
$D_4(M, N)$	0.272106
$D_5(M, N)$	0.422257
$D_6(M, N)$	0.427803
$D_7(M, N)$	0.844794
$D_8(M, N)$	0.796716
$D_9(M, N)$	0.781236
$D_{10}(M, N)$	0.811193
$D_{11}(M, N)$	0.224520
$D_{12}(M, N)$	0.226194

The results of Table 2 are represented graphically in Figure 1 below.


Figure 1. DMS between M and N.

4. Similarity Measures for TSFSs

We present various SMs based on TSF information in this section. Examples are provided to support the suggested new SMs, and their special circumstances are investigated.

Definition 5. Let $M, N, O \in TSFS(X)$. An SM $S(M, N)$ is a mapping $S : TSFSs(X) \times TSFSs(X) \rightarrow [0, 1]$, possessing the features listed below:

- (1) $0 \leq S(M, N) \leq 1$;
- (2) $S(M, N) = S(N, M)$;
- (3) $S(M, N) = 1$ if $M = N$;
- (4) $S(M, M^c) = 0$ if M is a crisp set;
- (5) If $M \subseteq N \subseteq O$, then $S(M, O) \leq S(M, N)$ and $S(M, O) \leq S(N, O)$.

Theorem 2. Let $M, N, O \in TSFS(X)$. Then, $S_i(M, N) = (i = 1, 2, 3, \dots, 12)$ is an SM:

$$S_1(M, N) = 1 - \frac{1}{2|X|} \sum_{x \in X} \left(\left| S_M^q(x) - S_N^q(x) \right| + \left| I_M^q(x) - I_N^q(x) \right| + \left| D_M^q(x) - D_N^q(x) \right| + \left| r_M^q(x) - r_N^q(x) \right| \right) \quad (13)$$

$$S_2(M, N) = 1 - \frac{1}{2|X|} \left(\sum_{x \in X} |S_M^q(x) - S_N^q(x)| - (I_M^q(x) - I_N^q(x)) - (D_M^q(x) - D_N^q(x)) \right) \quad (14)$$

$$S_3(M, N) = \left(1 - \frac{1}{4|X|} \left(\sum_{x \in X} \left(|S_M^q(x) - S_N^q(x)| + |I_M^q(x) - I_N^q(x)| + |D_M^q(x) - D_N^q(x)| \right) + \sum_{x \in X} \left(|S_M^q(x) - S_N^q(x)| - (I_M^q(x) - I_N^q(x)) - (D_M^q(x) - D_N^q(x)) \right) \right) \right) \quad (15)$$

$$S_4(M, N) = 1 - \frac{1}{|X|} \sum_{x \in X} (|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|) \quad (16)$$

$$S_5(M, N) = 1 - \frac{1}{|X|} \sum_{x \in X} \frac{1 - \left(|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)| \right)}{1 + \left(|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)| \right)} \quad (17)$$

$$S_6(M, N) = \frac{\sum_{x \in X} 1 - (|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|)}{\sum_{x \in X} 1 + (|S_M^q(x) - S_N^q(x)| \vee |I_M^q(x) - I_N^q(x)| \vee |D_M^q(x) - D_N^q(x)|)} \quad (18)$$

$$S_7(M, N) = \alpha \frac{\sum_{x \in X} (S_M^q(x) \wedge S_N^q(x))}{\sum_{x \in X} (S_M^q(x) \vee S_N^q(x))} + \beta \frac{\sum_{x \in X} (I_M^q(x) \wedge I_N^q(x))}{\sum_{x \in X} (I_M^q(x) \vee I_N^q(x))} + \gamma \frac{\sum_{x \in X} (D_M^q(x) \wedge D_N^q(x))}{\sum_{x \in X} (D_M^q(x) \vee D_N^q(x))}, \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \in [0, 1] \quad (19)$$

$$S_8(M, N) = \frac{\alpha}{|X|} \frac{\sum_{x \in X} (S_M^q(x) \wedge S_N^q(x))}{\sum_{x \in X} (S_M^q(x) \vee S_N^q(x))} + \frac{\beta}{|X|} \frac{\sum_{x \in X} (I_M^q(x) \wedge I_N^q(x))}{\sum_{x \in X} (I_M^q(x) \vee I_N^q(x))} + \frac{\gamma}{|X|} \frac{\sum_{x \in X} (D_M^q(x) \wedge D_N^q(x))}{\sum_{x \in X} (D_M^q(x) \vee D_N^q(x))}, \alpha + \beta + \gamma = 1, \alpha, \beta, \gamma \in [0, 1] \quad (20)$$

$$S_9(M, N) = \frac{1}{|X|} \sum_{x \in X} \frac{(S_M^q(x) \wedge S_N^q(x)) + (I_M^q(x) \wedge I_N^q(x)) + (D_M^q(x) \wedge D_N^q(x))}{(S_M^q(x) \vee S_N^q(x)) + (I_M^q(x) \vee I_N^q(x)) + (D_M^q(x) \vee D_N^q(x))} \quad (21)$$

$$S_{10}(M, N) = \frac{\sum_{x \in X} ((S_M^q(x) \wedge S_N^q(x)) + (I_M^q(x) \wedge I_N^q(x)) + (D_M^q(x) \wedge D_N^q(x)))}{\sum_{x \in X} ((S_M^q(x) \vee S_N^q(x)) + (I_M^q(x) \vee I_N^q(x)) + (D_M^q(x) \vee D_N^q(x)))} \quad (22)$$

$$S_{11}(M, N) = \frac{1}{|X|} \sum_{x \in X} \frac{(S_M^q(x) \wedge S_N^q(x)) + (1 - I_M^q(x)) \wedge (1 - I_N^q(x)) + (1 - D_M^q(x)) \wedge (1 - D_N^q(x))}{(S_M^q(x) \vee S_N^q(x)) + (1 - I_M^q(x)) \vee (1 - I_N^q(x)) + (1 - D_M^q(x)) \vee (1 - D_N^q(x))} \quad (23)$$

$$S_{12}(M, N) = \frac{\sum_{x \in X} (S_M^q(x) \wedge S_N^q(x)) + (1 - I_M^q(x)) \wedge (1 - I_N^q(x)) + (1 - D_M^q(x)) \wedge (1 - D_N^q(x))}{\sum_{x \in X} (S_M^q(x) \vee S_N^q(x)) + (1 - I_M^q(x)) \vee (1 - I_N^q(x)) + (1 - D_M^q(x)) \vee (1 - D_N^q(x))} \quad (24)$$

Proof of Theorem 2. Straight forward. \square

Example 2. Let $M, N \in TSFS(X)$, as shown in Table 3. Both M and N are TSFSs for $q = 3$.

Table 3. Two TSFSs, \mathbf{M} and \mathbf{N} .

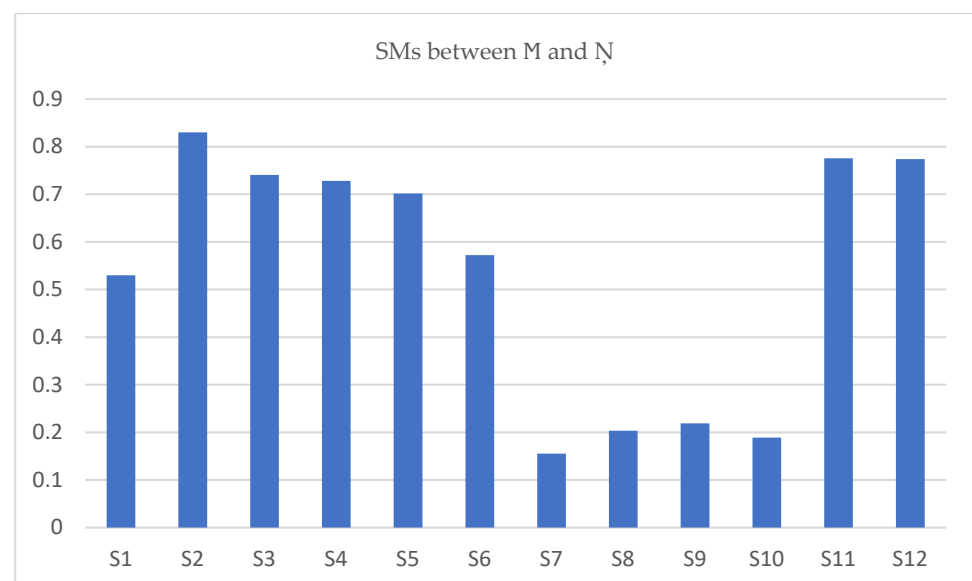
	\mathbf{M}				\mathbf{N}			
	MD	AD	NMD	RD	MD	AD	NMD	RD
x_1	0.56	0.47	0.22	0.892076	0.81	0.30	0.37	0.731180
x_2	0.11	0.11	0.11	0.998667	0.59	0.66	0.66	0.603342
x_3	0.35	0.45	0.61	0.856205	0.42	0.56	0.71	0.945407

We applied the SMs $S_i (i = 1, 2, 3 \dots 12)$ defined above to the two sets given in Table 3, and the outcomes are portrayed in Table 4.

Table 4. SMs between \mathbf{M} and \mathbf{N} .

SMs	Values
$S_1(\mathbf{M}, \mathbf{N})$	0.529780
$S_2(\mathbf{M}, \mathbf{N})$	0.829916
$S_3(\mathbf{M}, \mathbf{N})$	0.740567
$S_4(\mathbf{M}, \mathbf{N})$	0.727894
$S_5(\mathbf{M}, \mathbf{N})$	0.701391
$S_6(\mathbf{M}, \mathbf{N})$	0.572196
$S_7(\mathbf{M}, \mathbf{N})$	0.155205
$S_8(\mathbf{M}, \mathbf{N})$	0.203284
$S_9(\mathbf{M}, \mathbf{N})$	0.218764
$S_{10}(\mathbf{M}, \mathbf{N})$	0.188807
$S_{11}(\mathbf{M}, \mathbf{N})$	0.775480
$S_{12}(\mathbf{M}, \mathbf{N})$	0.773805

The results of Table 4 are graphically represented in Figure 2 below.

**Figure 2.** SMs between \mathbf{M} and \mathbf{N} .

Theorem 3. For $i = 1, 2, 3, \dots, 13$. If $\alpha = \beta = \frac{1}{2}$, so:

- (1) $S_i(M, N^c) = S_i(M, N^c), i \neq 11, 12;$
- (2) $S_i(M, N) = S_i(M \cap N, M \cup N);$
- (3) $S_i(M, M \cap N) = S_i(N, M \cup N);$
- (4) $S_i(M, M \cup N) = S_i(N, M \cap N);$

Proof of Theorem 3. Straight forward. \square

Theorem 4. For $i = 1, 2, 3, \dots, 6$, so:

- (1) $S_i(M, M \otimes N) = S_i(N, M \oplus N).$
- (2) $S_i(M, M \oplus N) = S_i(N, M \otimes N).$

Proof of Theorem 4. Straight forward. \square

Theorem 5. For $i = 1, 4, 5, 6$, and $\forall x \in X$, $S_M^q(x) - S_N^q(x) = 1, I_M^q(x) - I_N^q(x) = 1, D_M^q(x) - D_N^q(x) = 1$ we have

- (1) $S_i(M, M \otimes N) = S_i(N, N \oplus M),, S_M(x) \leq, S_N(x),, I_M(x) \geq, I_N(x);$
- (2) $S_i(M, M \oplus N) = S_i(N, N \otimes M),, S_M(x) \leq, S_N(x),, I_M(x) \geq, I_N(x);$

Proof of Theorem 5. Straight forward. \square

5. Consequences of the Current Work

This section aims to examine some consequences of the current work. An extensive amount of work has already been conducted on the theory of SMs and DMs. After looking at the limitations of the existing DMs and SMs, new DMs and SMs are presented. The newly presented DMs and SMs are the generalized form of the existing DMs and SMs.

1. If we have $q = 2$, then:
 - (a) The DMs $D_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of SFSs.
 - (b) The SMs $S_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of SFSs.
2. If we have $q = 1$, then:
 - (a) The DMs $D_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of PFSs.
 - (b) The SMs $S_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of PFSs.
3. If we neglect the AD, then:
 - (a) The DMs $D_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of q-ROFSs.
 - (b) The SMs $S_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of q-ROFSs.
4. If we have $q = 2$ and neglect the AD, then:
 - (a) The DMs $D_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of PyFSs.
 - (b) The SMs $S_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of PyFSs.
5. If we have $q = 1$ and neglect the AD, then:
 - (a) The DMs $D_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of IFSSs.
 - (b) The SMs $S_i (i = 1, 2, 3 \dots 12)$ reduce to the layout of IFSSs.

6. Algorithm and Applications

This section aims to check the effectiveness of the presented DMs and SMs. For this purpose, we design an algorithm for pattern recognition to check which pattern is useful. Moreover, we provide some applications of the newly introduced DMs and SMs in MADM to check which is the best selection for MADM while using the newly introduced DMs and SMs.

6.1. Algorithm in Pattern Recognition

Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite parameter; there are m patterns $N_j = \{(S_{N_j}(x_i), I_{N_j}(x_i), D_{N_j}(x_i)/x_i) \in X\}$ ($j = 1, 2, 3, \dots, m$) and a test sample $N = \{(S_N(x_i), I_N(x_i), D_N(x_i)/x_i) \in X\}$. To check the closeness, we ask to which pattern does the sample N belong to? The recognition steps are listed below:

Step 1. First, we have to evaluate the DM $D(N_j, N)$ ($j = 1, 2, \dots, m$) and SMs $S(N_j, N)$ ($j = 1, 2, \dots, m$) between N_j and N , respectively.

Step 2. Then, we have to select the minimum $D(N_{j_0}, N)$ from $D(N_j, N)$ ($j = 1, 2, \dots, m$) and maximum $S(N_{j_0}, N)$ from $S(N_j, N)$ ($j = 1, 2, \dots, m$), respectively, i.e., $D(N_{j_0}, N) = \min_{1 \leq j \leq m} \{D(N_j, N)\}$ and $S(N_{j_0}, N) = \max_{1 \leq j \leq m} \{S(N_j, N)\}$. Then, the test sample N is classified into pattern N_{j_0} according to the principle of the minimum of DMs and maximum of SMs.

Example 6.1. We use the DMs and SMs given in this research to tackle the challenge of recognizing construction materials presented by Ullah et al. [28]. Let us assume TSFNs N_i ($i = 1, 2, 3, 4$) which show building materials of four types. Let us consider $X = \{x_i : i = 1, 2, 3, \dots, 7\}$ as the attributes. There is another unknown material N . Using some DMs and SMs defined for TSFSs, we will define the class from four materials of unknown material denoted by N_i ($i = 1, 2, 3, 4$). Then, we have to evaluate class N_i to N .

Step 1. All the information provided in Table 5 is about the type of TSFNs. It is worth noting that all of the numbers in Table 5 are TSFNs for $n = 4$, signaling that neither IFS nor PFS methods can handle this type of data.

Step 2. The DM of each TSFN given in Table 5 is evaluated with N using the newly defined DMs in Section 3.

Step 3. Examining Table 6, for DMs the result obtained is

$$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$$

Hence, the pattern N_1 is closer to N as the DM of (N_1, N) is smaller than all the remaining pairs. Therefore, it is said that the unknown pattern N belongs to an N_1 -type pattern.

The results obtained in Table 6 are shown in Figure 3 below, which indicates that the distance of the unknown pattern N is closer to N_1 .

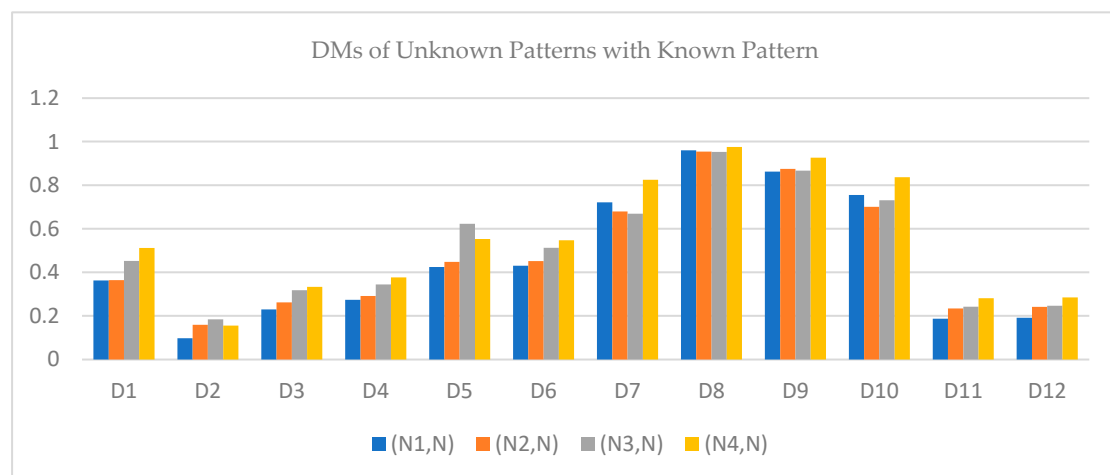
Comparative Study: This section aims to compare the results achieved using DM for TSFSs in the current work with results obtained using DM for TSFSs, as proposed by Ullah et al. [28] and Wu and Chen [27]. Here, we also present the limitations of the previously defined DMs of IFSs and PFSs. A brief discussion of the calculated results of this paper compared with other papers is given in Table 7 below.

Table 5. Data on building materials.

	N ₁			N ₂			N ₃			N ₄			N		
x ₁	0.56	0.47	0.22	0.81	0.30	0.37	0.43	0.43	0.55	0.57	0.51	0.39	0.34	0.56	0.78
x ₂	0.11	0.11	0.11	0.59	0.66	0.66	0.91	0.34	0.68	0.56	0.76	0.31	0.47	0.38	0.84
x ₃	0.35	0.45	0.61	0.42	0.56	0.71	0.81	0.41	0.35	0.27	0.59	0.72	0.55	0.44	0.65
x ₄	0.33	0.54	0.31	0.59	0.45	0.9	0.44	0.55	0.77	0.46	0.46	0.45	0.76	0.46	0.85
x ₅	0.35	0.2	0.64	0.16	0.33	0.42	0.55	0.44	0.77	0.57	0.66	0.91	0.13	0.35	0.57
x ₆	0.47	0.37	0.68	0.68	0.46	0.88	0.47	0.66	0.75	0.41	0.73	0.41	0.24	0.54	0.45
x ₇	0.78	0.55	0.03	0.49	0.54	0.39	0.58	0.34	0.23	0.21	0.43	0.13	0.82	0.46	0.69

Table 6. DM of N_1 with N .

DMs	(N_1, N)	(N_2, N)	(N_3, N)	(N_4, N)
D_1	0.362768	0.363894	0.452119	0.511623
D_2	0.096853	0.159078	0.183696	0.155122
D_3	0.229811	0.261486	0.317907	0.333373
D_4	0.273785	0.291556	0.344432	0.376600
D_5	0.423967	0.447694	0.622836	0.552754
D_6	0.429876	0.451480	0.512383	0.547146
D_7	0.720886	0.679505	0.668698	0.824774
D_8	0.960127	0.954215	0.952671	0.974968
D_9	0.861950	0.874685	0.866928	0.926015
D_{10}	0.754929	0.700429	0.730597	0.836401
D_{11}	0.186668	0.233649	0.242184	0.280998
D_{12}	0.191473	0.241229	0.246095	0.284689

**Figure 3.** Results of DMs of unknown patterns with a known pattern.

The results shown in Table 7 support our hypothesis that the DM of TSFSs can manage large amounts of data. The benefit of the introduced DMs is that there are no restrictions on giving values to membership degrees in the environment of TSFSs as it includes all four degrees. There are restrictions when assigning values to membership degrees in IFSs, PyFSs, and PFSs. All the existing DMs failed to solve the issues due to the absence of RD and the restricted range when the data are given in the TSF environment. It is also concluded from Table 8 that N_1 is close to N as the distance of (N_1, N) is smaller than all the remaining pairs. Therefore, it is concluded that the unknown pattern N belongs to an N_1 -type pattern, which supports our results.

Figure 4 depicts a comparison of our work with that of others, and explains that our results are more convenient and accurate than those of previous work.

The SM of each TSFN given in Table 5 is evaluated with N using the newly defined SMs in Section 4.

Examining Table 8, for SMs the result obtained is

$$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$$

Table 7. Comparative study.

DM	Environment	Results
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_4, N) < (N_2, N) < (N_3, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_2, N) < (N_4, N) < (N_3, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_3, N) < (N_2, N) < (N_1, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_3, N) < (N_2, N) < (N_4, N) < (N_1, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_3, N) < (N_2, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_2, N) < (N_3, N) < (N_1, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$
DM for TSFSs (Current Work)	TSFSs	$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$
DM for TSFSs Ullah et al. [28]	TSFSs	$(N_2, N) < (N_3, N) < (N_4, N) < (N_1, N)$
DM for TSFSs Ullah et al. [28]	TSFSs	$(N_2, N) < (N_1, N) < (N_3, N) < (N_4, N)$
DM for TSFSs Ullah et al. [28]	TSFSs	$(N_3, N) < (N_2, N) < (N_1, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_3, N) < (N_1, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_1, N) < (N_3, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_1, N) < (N_3, N) < (N_2, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_3, N) < (N_1, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_3, N) < (N_1, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_3, N) < (N_1, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_1, N) < (N_3, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_1, N) < (N_2, N) < (N_3, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_1, N) < (N_3, N) < (N_4, N)$
DM for TSFSs Wu and Chen [27]	TSFSs	$(N_2, N) < (N_1, N) < (N_3, N) < (N_4, N)$

Table 8. SM of N_i with N .

SMs	(N_1, N)	(N_2, N)	(N_3, N)	(N_4, N)
S_1	0.637232	0.636106	0.547881	0.518229
S_2	0.903147	0.840922	0.816304	0.844878
S_3	0.770189	0.738514	0.682093	0.666627
S_4	0.726215	0.708444	0.655568	0.623400
S_5	0.619747	0.595512	0.570856	0.498807
S_6	0.570124	0.548520	0.487617	0.452854
S_7	0.279114	0.320495	0.331302	0.175226
S_8	0.039873	0.045785	0.047329	0.025032
S_9	0.138050	0.125315	0.133072	0.073985
S_{10}	0.245071	0.299571	0.269403	0.163599
S_{11}	0.813332	0.766351	0.757816	0.719002
S_{12}	0.808527	0.758771	0.753905	0.715311

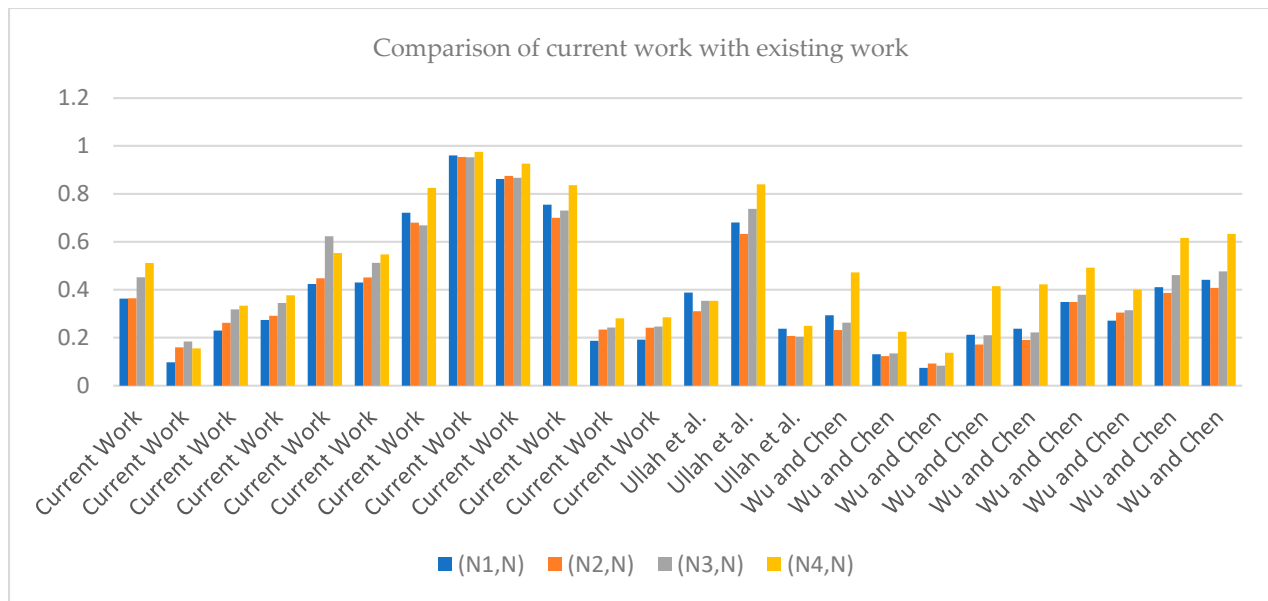


Figure 4. Comparison of the current work with the existing work.

Hence, the pattern N_1 is close to N , as the SM of (N_1, N) is greater than all the remaining pairs. Therefore, it is said that the unknown pattern N belongs to an N_1 -type pattern.

The results obtained in Table 8 are shown in Figure 5 below, which indicates that the similarity of the unknown pattern N is close to N_1 .

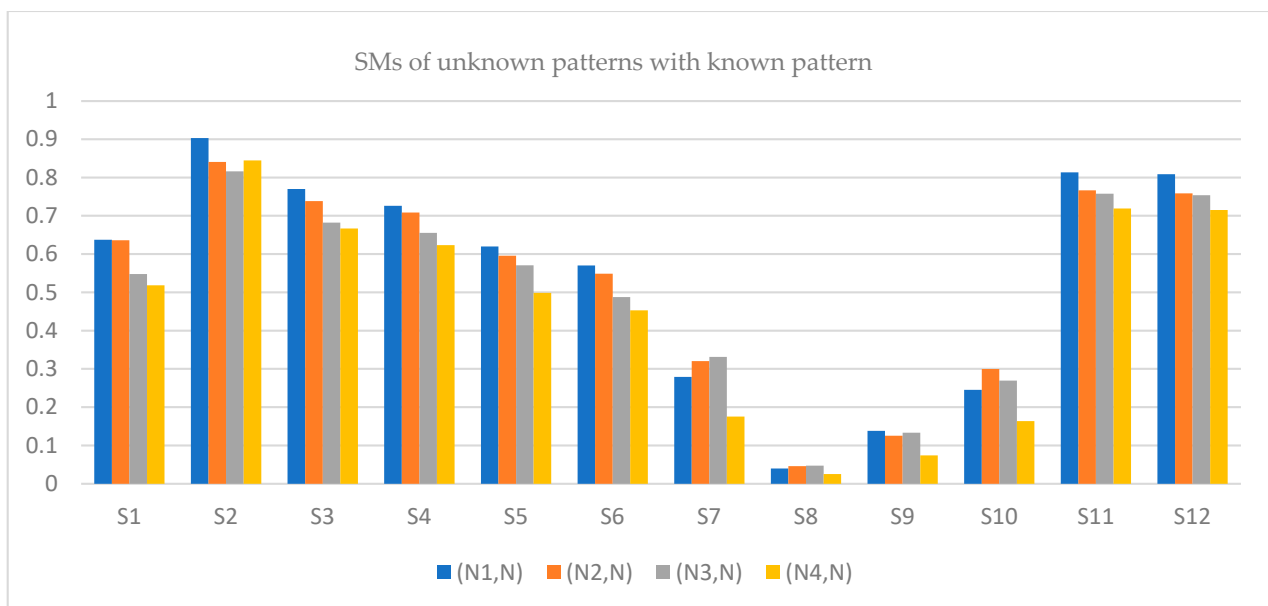


Figure 5. Results of SMs of unknown patterns with a known pattern.

Comparative Study: This section aims to compare the results achieved using SM for TSFSs in the current work with results obtained using SM for TSFs, as proposed by Ullah et al. [28] and Wu and Chen [27]. Here, we also present the restrictions of the previously defined SMs of IFSs and PFSs. A brief discussion of the calculated results of this paper compared with other papers is given in Table 9 below.

Table 9. Comparative study.

SM	Environment	Results
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_3, N) < (N_2, N) < (N_4, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_1, N) < (N_2, N) < (N_3, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_1, N) < (N_2, N) < (N_3, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_2, N) < (N_3, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_1, N) < (N_3, N) < (N_2, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs (Current Work)	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs Ullah et al. [28]	TSFSs	$(N_4, N) < (N_1, N) < (N_3, N) < (N_2, N)$
SM for TSFSs Ullah et al. [28]	TSFSs	$(N_4, N) < (N_3, N) < (N_1, N) < (N_2, N)$
SM for TSFSs Ullah et al. [28]	TSFSs	$(N_4, N) < (N_1, N) < (N_2, N) < (N_3, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_1, N) < (N_3, N) < (N_2, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_3, N) < (N_1, N) < (N_2, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_2, N) < (N_3, N) < (N_1, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_1, N) < (N_3, N) < (N_2, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_1, N) < (N_3, N) < (N_2, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_3, N) < (N_1, N) < (N_2, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_3, N) < (N_2, N) < (N_1, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_3, N) < (N_1, N) < (N_2, N)$
SM for TSFSs Wu and Chen [27]	TSFSs	$(N_4, N) < (N_3, N) < (N_1, N) < (N_2, N)$

The results in Table 9 support our hypothesis that the SM of TSFSs can manage large amounts of data. The benefit of the introduced SMs is that there are no restrictions on giving values to membership degrees in the environment of TSFSs as it includes all four degrees, but there are restrictions on assigning values to membership degrees in IFSs, PyFSs, and PFSs. All existing SMs failed to solve the issues due to the absence of RD and the restricted range, especially when the data are given in the environment of TSFSs. It is also concluded from Table 9 that N_1 is close to N as the similarity of (N_1, N) is more significant than all the remaining pairs. Therefore, it is said that the unknown pattern N belongs to an N_1 -type pattern, which supports our results.

Figure 6 depicts a comparison of our work with that of others and also explains that our results are suitable and more accurate than those of previous work.

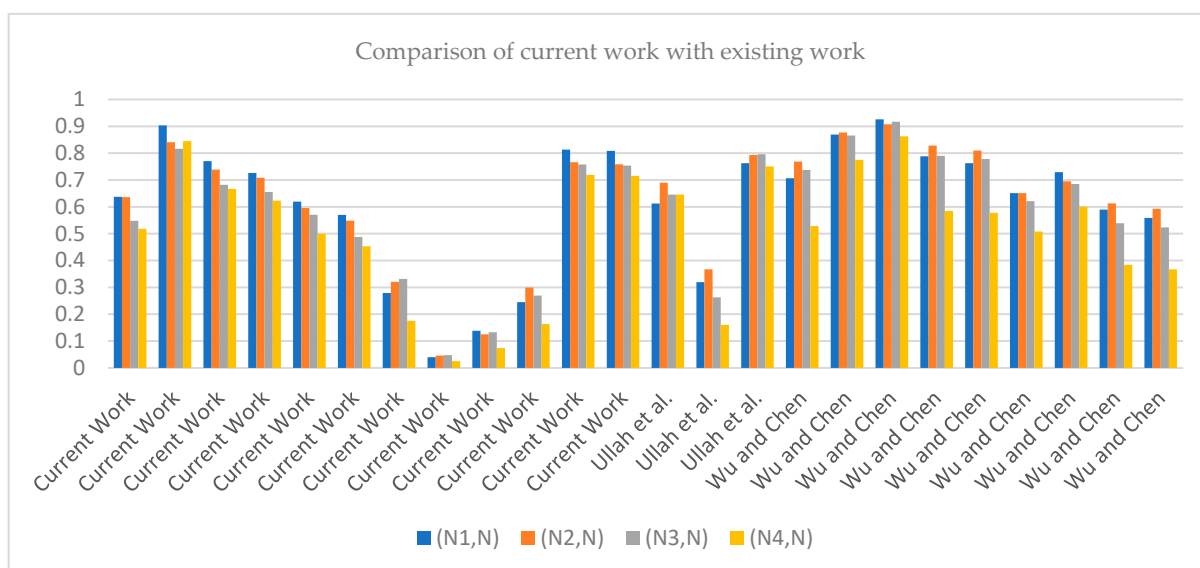


Figure 6. Comparison of the current work with the existing work.

6.2. Applications in MCDM

Here, we will present the applications of the introduced DMs and SMs for MCDM.

Example 6.2. We use the DMs and SMs given in this research to tackle the challenge of MCDM presented by Ullah et al. [35]. Islamabad, Pakistan's capital, is regarded as one of the world's most beautiful cities. A large number of people regularly frequent Islamabad's parks and picnic areas. The Metropolitan Corporation of Islamabad (MCI), is in charge of the city's administration. To maintain its appeal, the MCI decided to refurbish all of the parks and picnic areas. The MCI needed to recruit some private contractors to do so. The MCI chose four private companies for further consideration after some preliminary screening. N_1 : Bilawal Builders, N_2 : Hussain Estate and Builders, N_3 : Nimo Engineering, Construction and Interiors, and N_4 : Scholar Builders. The MCI's specialists devised five-point criteria for selecting the best corporation or company: M_1 : cost, M_2 : previous performance, M_3 : time constraints, M_4 : quality assurance, and M_5 : labor quantity. The decision-making committee provided all of the information regarding TSFNs, which is given in Table 10. The evaluation steps for the MCDM algorithm are given as:

Table 10. Data for MCDM of the alternatives.

	M ₁				M ₂				M ₃				M ₄				M ₅			
N ₁	0.1	0.7	0.4	0.5	0.8	0.9	0.8	0.8	0.8	0.6	0.7	0.8	0.3	0.5	0.7					
N ₂	0.2	0.7	0.6	0.6	0.7	0.8	0.3	0.7	0.7	0.1	0.7	0.9	0.4	0.6	0.8					
N ₃	0.5	0.6	0.6	0.5	0.6	0.7	0.5	0.7	0.1	0.9	0.6	0.2	0.5	0.6	0.9					
N ₄	0.5	0.6	0.8	0.8	0.7	0.4	0.8	0.7	0.3	0.6	0.6	0.1	0.8	0.4	0.4					
M	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0					

Step 1. Table 10 contains the decision maker's suggestions in the form of TSFNs. Assuming that all of the values in Table 10 are simply TSFNs for $n = 5$, it is demonstrated that neither IFS nor PFS operators can solve this problem.

Step 2. The DM of each TSFN given in Table 10 is calculated with $M(1,0,0)$, which describes an ideal TSFN using newly introduced DMs, as shown in Section 3.

Step 3. Analyzing Table 11, for DMs the result obtained is

$$N_4 < N_3 < N_1 < N_2$$

Table 11. DMs of N_i with M .

DMs	(N_1, M)	(N_2, M)	(N_3, M)	(N_4, M)
D_1	0.228043	0.245462	0.214225	0.195397
D_2	0.175194	0.178774	0.140006	0.119011
D_3	0.197735	0.218285	0.187124	0.163446
D_4	0.228043	0.245462	0.214225	0.195397
D_5	0.237325	0.247650	0.225878	0.217792
D_6	0.949301	0.990603	0.903515	0.871168
D_7	0.982434	0.996369	0.971380	0.956318
D_8	0.995608	0.999092	0.992845	0.989079
D_9	0.978043	0.995462	0.642255	0.945397
D_{10}	0.946347	0.987729	0.870932	0.805294
D_{11}	0.866796	0.869182	0.843337	0.829340
D_{12}	0.467185	0.476730	0.373351	0.316872

Therefore, N_4 is the best choice as the DM of (N_4, N) is smaller than all the remaining pairs. Furthermore, due to the complex nature of TSFNs, the DMs of IFSs and PFSs cannot be applied to this sort of data.

The outcomes of Table 11 are portrayed in Figure 7, which shows that after applying the proposed DMs, N_4 is the best choice for MCDM purposes.

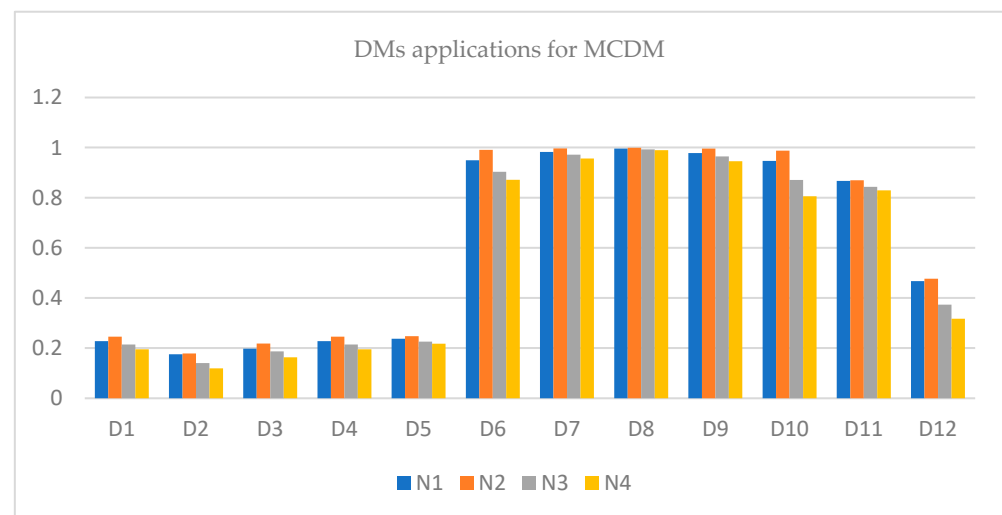


Figure 7. Results of DMs for MCDM.

Comparative Study: This section aims to compare the results achieved using DM for TSFNs in the current work with results obtained using DM for TSFs, as proposed by Ullah et al. [28]. Here, we also present the limitations of the previously defined DMs of IFSs and PFSs. A brief discussion of the calculated results of this paper compared with other papers is given in Table 12 below.

Table 12. Comparative study.

DM	Environment	Results
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs (Current Work)	TSFSs	$N_4 < N_3 < N_1 < N_2$
DM for TSFSs Ullah et al. [28]	TSFSs	$N_4 < N_3 < N_1 < N_2$

The results in Table 12 support our hypothesis that the DM of TSFSs can manage large amounts of data. The benefit of the introduced DMs is that there are no restrictions on giving values to membership degrees in the environment of TSFSs as it includes all four degrees, but there are restrictions on giving values to membership degrees in IFSs, PyFSs, and PFSs. All the existing DMs failed to solve the issues due to the absence of RD and the restricted range when the data are given in the TSF environment. It is also concluded from Table 13 that N_4 is the best choice as the DM of (N_4, N) is smaller than all the remaining pairs. Therefore, it is said that N_4 is the best choice for MCDM purposes, which supports our results.

Table 13. SMs of N_i with M.

SMs	(N_1, M)	(N_2, M)	(N_3, M)	(N_4, M)
S_1	0.771957	0.754538	0.785775	0.804603
S_2	0.824806	0.821226	0.859993	0.880989
S_3	0.802265	0.781715	0.812876	0.836553
S_4	0.771957	0.754538	0.785775	0.804603
S_5	0.012675	0.002349	0.024121	0.032208
S_6	0.050698	0.009396	0.096485	0.128831
S_7	0.017565	0.003630	0.028620	0.043682
S_8	0.004391	0.000908	0.007155	0.010921
S_9	0.021957	0.004538	0.035775	0.054603
S_{10}	0.053653	0.012271	0.129067	0.194706
S_{11}	0.133203	0.130817	0.156662	0.170659
S_{12}	0.532815	0.523269	0.626649	0.683127

Figure 8 depicts a comparison of our study with others, and it also explains that our results are suitable and more accurate than those of previous work.

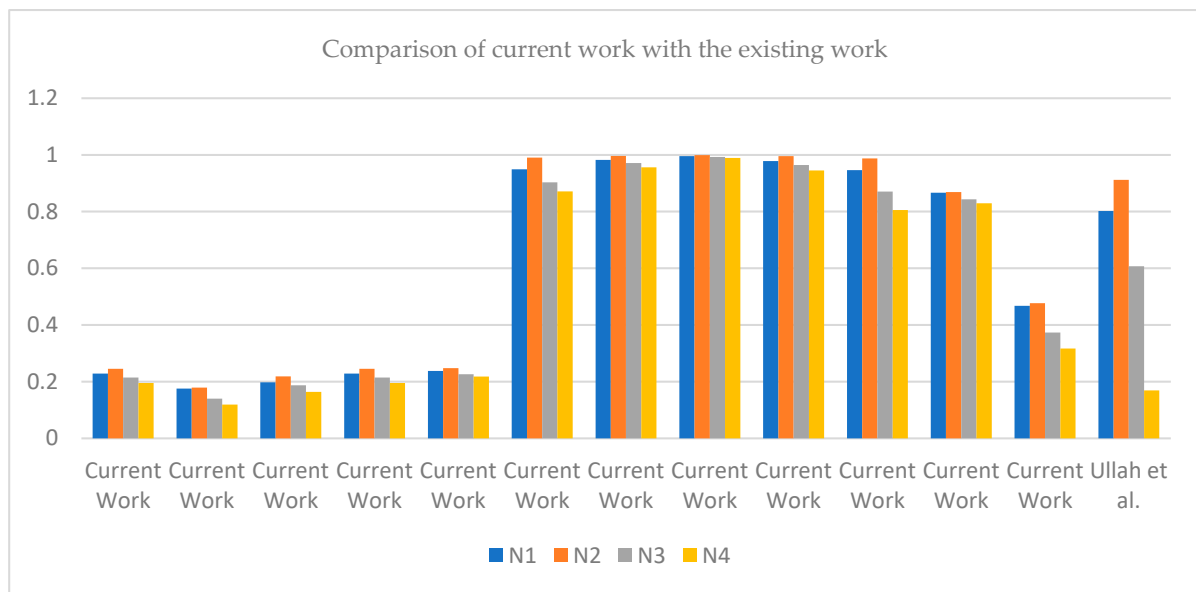


Figure 8. Comparison of the current work with the existing work.

The SM of each TSFN given in Table 10 is calculated with $M(1,0,0)$, which describes an ideal TSFN, using newly introduced SMs, as shown in Section 4.

Analyzing Table 13, for SMs the result obtained is

$$N_2 < N_1 < N_3 < N_4$$

Hence, N_4 is the best choice as the SM of (N_4, N) is greater than all the remaining pairs. Furthermore, due to the complex nature of TSFNs, the SMs of IFs and PFs cannot be applied to this sort of data.

The outcomes of Table 13 are portrayed in Figure 9, which shows that after applying the proposed SMs, N_4 is the best choice for MCDM purposes.

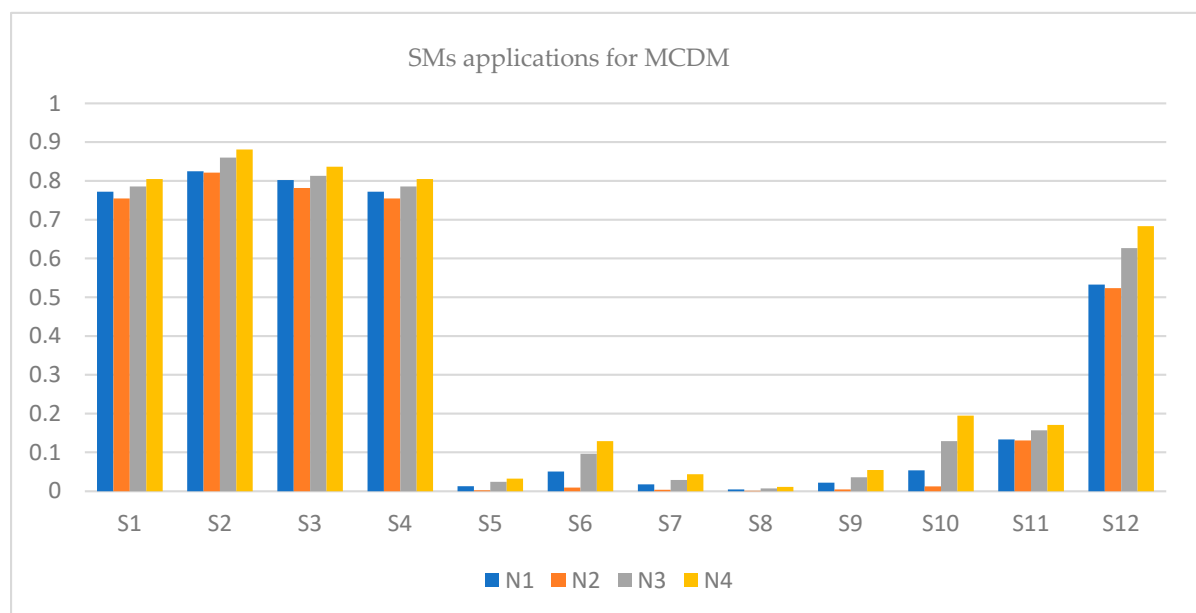


Figure 9. Results of SMs for MCDM.

Comparative Study: This section aims to compare the results achieved using SM for TSFSs in the current work with results obtained using SM for TSFSs, as proposed by Ullah et al. [28]. Here, we also present the limitations of the previously defined SMs of IFSs and PFSs. A brief discussion of the calculated results of this paper compared with other papers is given in Table 14 below.

Table 14. Comparative study.

SM	Environment	Results
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs (Current Work)	TSFSs	$N_2 < N_1 < N_3 < N_4$
SM for TSFSs Ullah et al. [28]	TSFSs	$N_2 < N_1 < N_3 < N_4$

The results in Table 14 support our hypothesis that the SM of TSFSs can manage large amounts of data. The benefit of the introduced SMs is that there are no restrictions on giving values to membership degrees in the environment of TSFSs as it includes all four degrees but there are restrictions on giving values to membership degrees in IFSs, PyFSs, and PFSs. All existing SMs failed to solve the issues due to the absence of RD and restricted range, especially when the data are given in the environment of TSFSs. It is also concluded from Table 14 that N_4 is the best choice as the SM of (N_4, N) is greater than all the remaining pairs. Therefore, it is said that N_4 is the best choice, which supports our claim.

Figure 10 depicts a comparison of our study with others, and it also explains that our results are suitable and more accurate than those of previous work.

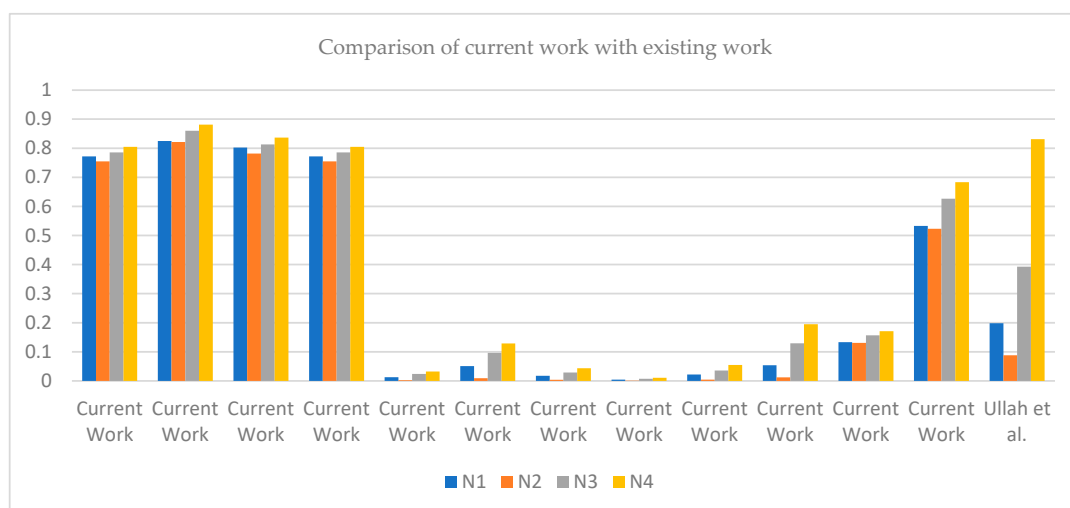


Figure 10. Comparison of the current work with the existing work.

7. Conclusions

The background of FSs, IFSs, PyFSs, q-ROFSs, and PFSs is briefly presented in this work, along with their possible limitations. It is also shown how the TSFS framework modifies all known theories and provides a strong foundation with no limitations at all. The primary goal of this work is to describe several new DMs and SMs, keeping in view the restrictions of previously defined DMs and SMs. Here, in this paper, we propose some new formulas for DMs and SMs. Because of their more comprehensive range, the proposed DMs and SMs are more essential and innovative than previously defined ones. We cannot assign values to MD, NMD, and AD independently because the previously specified DMs and SMs have a limited structure. However, because the range of TSFSs is so large, we can assign values to MD, NMD, and AD without any restrictions or limitations. When data are presented in a TSF setting, it can be determined that the recommended DMs and SMs are more suitable and accurate for solving problems. The proposed DMs and SMs are used to construct pattern-recognition and MCDM algorithms, which are then applied to real-life issues. A comparison of existing and developing notions and the shortcomings of existing theories is established. We also prove that the proposed DMs and SMs are more suitable to use when the data are provided in a TSF environment with no restrictions at all. The effectiveness of the proposed DMs and SMs is also evaluated, and it is proved that the proposed DMs and SMs are more effective than the previously defined DMs and SMs due to the limitations of the latter. In the future, the proposed work could be used in several other ambiguous situations, such as interval-valued TSFSs, complex TSFSs, medical diagnosis, and manufacturing industry digital reforms [34]. A limitation of our current work is that when the data are provided in an interval-valued TSF environment and complex TSF environment, then our proposed DMs and SMs fail to solve such challenges. In the near future, we aim to extend our theory to the environment of complex TSFSs [35], bipolar soft sets [36], picture FSs [37], and bipolar hesitant FSs [38,39].

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