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Characterizations of Matrix Equalities for Generalized Inverses of Matrix Products

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Abstract: This paper considers how to construct and describe matrix equalities that are composed of algebraic operations of matrices and their generalized inverses. We select a group of known and new reverse-order laws for generalized inverses of several matrix products and derive various necessary and sufficient conditions for them to hold using the matrix rank method and the block matrix method.

Keywords: generalized inverse; matrix product; matrix equality; reverse-order law; set inclusion; rank; range; block matrix

MSC: 15A09; 15A24; 47A05; 47A50

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1. Introduction

Throughout, let $\mathbb{C}^{m\times n}$ denote the collections of all $m\times n$ matrices with complex numbers; A^* denote the conjugate transpose; $r(A)$ denote the rank of A, i.e., the maximum order of the invertible submatrix of A; $\mathscr{R}(A)=\{Ax\,|\,x\in\mathbb{C}^n\}$ and $\mathscr{N}(A)=\{x\in\mathbb{C}^n\,|\,Ax=0\}$ denote the range and the null space of a matrix $A\in\mathbb{C}^{m\times n}$, respectively; I_m denote the identity matrix of order m; and $[\,A,\,B]$ denote a columnwise partitioned matrix consisting of two submatrices A and B. The Moore–Penrose generalized inverse of $A\in\mathbb{C}^{m\times n}$, denoted by $A^{\dagger}$, is the unique matrix $X\in\mathbb{C}^{n\times m}$ that satisfies the four Penrose equations:

$$(1)\ AXA=A,\ \ (2)\ XAX=X,\ \ (3)\ (AX)^*=AX,\ \ (4)\ (XA)^*=XA, \tag{1}$$

see [1]. Starting with Penrose himself, a matrix X is called a $\{i,\ldots,j\}$-generalized inverse of A, denoted by $A^{(i,\ldots,j)}$, if it satisfies the ith,…, jth equations in (1). The collection of all $\{i,\ldots,j\}$-generalized inverses of A is denoted by $\{A^{(i,\ldots,j)}\}$. There are in all 15 types of $\{i,\ldots,j\}$-generalized inverses of A by definition, but matrix X is called an inner inverse of A if it satisfies $AXA=A$ and is denoted by $A^{(1)}=A^-$.

In this paper, we focus our attention on $\{1\}$-generalized inverses of matrices. As usual, we denote matrix qualities composed of $\{1\}$-generalized inverses by

$$f(A_1^-,\,A_2^-,\ldots,A_p^-)=g(B_1^-,\,B_2^-,\ldots,B_q^-), \tag{2}$$

where $A_1,\,A_2,\ldots,A_p,\,B_1,\,B_2,\ldots,B_q$ are given matrices of appropriate sizes. For a given general algebraic matrix equality, a primary task that we are confronted with is to determine clear and intrinsic identifying conditions for it to hold. However, there do not exist effective and useful rules and techniques of characterizing a given algebraic equality by means of ordinary operations of matrices and their generalized inverses because of the noncommutativity of matrix algebra and the singularity of matrices. In view of this fact, few of (2) can be described with satisfactory conclusions in the theory of generalized inverses except some kinds of special cases with simple and reasonable forms. As well-known examples of (2), we mention the following two matrix equalities:

$$(AB)^-=B^-A^-,\quad (ABC)^-=C^-B^-A^-, \tag{3}$$

where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Obviously, they can be viewed as the direct extensions of the two ordinary reverse-order laws $(AB)^{-1} = B^{-1}A^{-1}$ and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ for the products of two or three invertible matrices of the same size, and therefore, they are usually called the reverse-order laws for generalized inverses of the matrix products AB and ABC, respectively. Apparently, the two reverse-order laws in (3) and their special forms, such as $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and $(ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger}$, seem simple and neat in comparison with many other complicated matrix equalities that involve generalized inverses. On the other hand, since $MM^{\dagger} = I_m$ and $M^{\dagger}M = I_n$ do not necessarily hold for a singular matrix M, the two reverse-order laws in (3) do not necessarily hold for singular matrices. Therefore, it is a primary work to determine the necessary and sufficient conditions for the two reverse-order laws in (3) to hold before we can utilize them in dealing with calculations related to matrices and their generalized inverses. In fact, they were well known as classic objects in the theory of generalized inverses of matrices and have been studied by many authors since the 1960s; see, e.g., [2–14] for the historical perspective and development on the subject area of reverse-order laws.

In addition to ordinary forms of reverse-order laws in (3), there are many other kinds of simple and complicated algebraic equalities that are composed of mixed reverse-order products of given matrices and their generalized inverses, such as

$$(AB)^{-} = B^{-}(ABB^{-})^{-}, \ \ (AB)^{-} = (A^{-}AB)^{-}A^{-}, \tag{4}$$

$$(ABC)^{-} = (BC)^{-}B(AB)^{-}, \ \ (ABC)^{-} = C^{-}(A^{-}ABCC^{-})^{-}A^{-}, \tag{5}$$

$$(ABCD)^{-} = (CD)^{-}C(BC)^{-}B(AB)^{-}, \tag{6}$$

$$(ABCDE)^{-} = (CDE)^{-}CD(BCD)^{-}BC(ABC)^{-}. \tag{7}$$

These equalities are usually called the mixed or nested reverse-order laws for generalized inverses of matrices. Clearly, these reverse-order laws of special kinds are all constructed from the ordinary algebraic operations of the given matrices and their generalized inverses, and each of them has certain reasonable interpretations; in particular, they can be reduced to the reverse-order laws for standard inverses of matrix products when the given matrices in them are all invertible. Admittedly, knowing how to deal with a given matrix equality composed by matrices and their generalized inverses is a difficult problem. In fact, these kinds of problems have no uniformly acceptable solutions, and no algebraists and algebraic techniques can accurately tell people what to do with complicated matrix operations and matrix equalities.

The rest of this paper is organized as follows. In Section 2, the author introduces a group of known formulas, facts, and results about ranks, ranges, and generalized inverses. In Section 3, the author derives several groups of equivalent facts related to the matrix equalities in (3)–(7) and gives some of their consequences. Section 4 gives some remarks and further research problems pertaining to characterizations of matrix equalities for generalized inverses of matrix products.

2. Some Preliminaries

We begin with presentations and expositions of a series of known facts and results regarding matrices and their ordinary operations, which can be found in various reference books about linear algebra and matrix theory (cf. [2,15–17]).

Note from the definitions of generalized inverses of a matrix that they are in fact defined to be (common) solutions of some matrix equations. Thus, analytical expressions of generalized inverses of matrices, as shown below, can be written as certain matrix-valued functions with one or more variable matrices.

Lemma 1 ([1])**.** *Let $A \in \mathbb{C}^{m\times n}$. Then, the general expression of A^{-} of A can be written as*

$$A^{-} = A^{\dagger} + F_AU + VE_A, \tag{8}$$

where $E_A = I_m - AA^{\dagger}$, $F_A = I_n - A^{\dagger}A$, *and* $U, V \in \mathbb{C}^{n\times m}$ *are arbitrary.*

There is much good to be said about equalities and inequalities for ranks of matrices. In what follows, we present a series of well-known or established results and facts concerning ranks of matrices, which we shall use to deal with matrix equality problems and matrix set inclusion problems with regard to the generalized inverses of matrix products described above.

Lemma 2. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $A_1 \in \mathbb{C}^{m\times n_1}$, $A_2 \in \mathbb{C}^{m\times n_2}$, $B_1 \in \mathbb{C}^{m\times p_1}$, *and* $B_2 \in \mathbb{C}^{m\times p_2}$. *Then,*

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \text{ and } r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B), \tag{9}$$

$$\mathscr{R}(A_1) = \mathscr{R}(A_2) \text{ and } \mathscr{R}(B_1) = \mathscr{R}(B_2) \Rightarrow r[A_1,\, B_1] = r[A_2,\, B_2]. \tag{10}$$

Lemma 3 ([18])**.** *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, *and* $D \in \mathbb{C}^{l\times k}$. *Then,*

$$r[A,\, B] = r(A) + r(E_A B) = r(B) + r(E_B A), \tag{11}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{12}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C), \tag{13}$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r\begin{bmatrix} 0 & E_A B \\ CF_A & D - CA^{\dagger}B \end{bmatrix}. \tag{14}$$

In particular, the following results hold:

(a) $r\,[A,\, B] = r(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A) \Leftrightarrow AA^{\dagger}B = B \Leftrightarrow E_A B = 0.$

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*) \Leftrightarrow CA^{\dagger}A = C \Leftrightarrow CF_A = 0.$

(c) $r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) \Leftrightarrow E_B A F_C = 0.$

(d) $r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A),\ \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*)$, *and* $CA^{\dagger}B = D.$

Lemma 4 ([8])**.** *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, *and* $D \in \mathbb{C}^{l\times k}$. *Then,*

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} - r(A). \tag{15}$$

In particular,

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} AA^* & B \\ CA^* & D \end{bmatrix} - r(A) \ \text{ if } \ \mathscr{R}(B) \subseteq \mathscr{R}(A), \tag{16}$$

$$r(D - CA^{\dagger}B) = r\begin{bmatrix} A^*A & A^*B \\ C & D \end{bmatrix} - r(A) \ \text{ if } \ \mathscr{R}(C^*) \subseteq \mathscr{R}(A^*). \tag{17}$$

Lemma 5 ([18])**.** *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$. *Then,*

$$r(AB) = r(A) + r(B) - n + r((I_n - BB^-)(I_n - A^-A)), \tag{18}$$

$$r(ABC) = r(AB) + r(BC) - r(B) + r((I_n - (BC)(BC)^-)B(I_p - (AB)^-(AB))) \tag{19}$$

hold for all A^-, B^-, $(AB)^-$, *and* $(BC)^-$. *In particular, the following results hold*:

(a) *The rank of* AB *satisfies the following inequalities:*

$$\max\{0,\ r(A) + r(B) - n\} \le r(A) + r(B) - r[A^*,\, B] \le r(AB) \le \min\{r(A),\ r(B)\}. \tag{20}$$

(b) *The rank of ABC satisfies the following inequalities:*

$$\begin{aligned} r(ABC) &\le \min\{r(AB),\ r(BC)\} \le \min\{r(A),\ r(B),\ r(C)\}, && (21)\\ r(ABC) &\ge \max\{0,\ r(AB)+r(BC)-r(B)\}\\ &\ge \max\{0,\ r(A)+r(B)+r(C)-r[\,A^*,\ B\,]-r[\,B^*,\ C\,]\}\\ &\ge \max\{0,\ r(A)+r(B)+r(C)-n-p\}, && (22)\\ r(ABC) &\ge r(AB)+r(C)-r[\,(AB)^*,\ C\,]\\ &\ge \max\{0,\ r(AB)+r(C)-p\}\\ &\ge \max\{0,\ r(A)+r(B)+r(C)-n-p\}, && (23)\\ r(ABC) &\ge r(A)+r(BC)-r[A^*,\ BC]\\ &\ge \max\{0,\ r(A)+r(BC)-n\}\\ &\ge \max\{0,\ r(A)+r(B)+r(C)-n-p\}. && (24) \end{aligned}$$

(c) $r(ABC) = r(B) \Leftrightarrow r(AB) = r(BC) = r(B)$.

(d) $r(ABC) = r(A)+r(B)+r(C)-n-p \Leftrightarrow r(ABC) = r(AB)+r(C)-p$ *and* $r(AB) = r(A)+r(B)-n$.

(e) $r(ABC) = r(A)+r(B)+r(C)-n-p \Leftrightarrow r(ABC) = r(A)+r(BC)-n$ *and* $r(BC) = r(B)+r(C)-p$.

Lemma 6 ([19,20])**.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$ be given. Then,*

$$\max_{A^- \in \{A^-\}} r(D - CA^-B) = \min\left\{ r[C,\ D],\ \ r\begin{bmatrix} B\\ D\end{bmatrix},\ \ r\begin{bmatrix} A & B\\ C & D\end{bmatrix} - r(A)\right\}. \tag{25}$$

Therefore,

$$CA^-B = D \text{ for all } A^- \Leftrightarrow [C,\ D] = 0 \ \text{ or } \begin{bmatrix} B\\ D\end{bmatrix} = 0 \ \text{ or } \begin{bmatrix} A & B\\ C & D\end{bmatrix} = r(A). \tag{26}$$

There is no doubt that analytical formulas for calculating ranks of matrices can be used to establish and analyze various complicated matrix expressions and matrix equalities. Specifically, the rank equalities and their consequences in the above four lemmas are understandable in elementary linear algebra. When the matrices are given in various concrete forms, these established results can be simplified further by usual computations of matrices, so that we can employ them to describe a variety of concrete matrix equalities that involve products of matrices and their generalized inverses in matrix analysis and applications.

At the end of this section, we give a known result regarding a matrix equality composed of six matrices and their generalized inverses.

Lemma 7 ([21])**.** *Let $A_1 \in \mathbb{C}^{m_1\times m_2}$, $A_2 \in \mathbb{C}^{m_3\times m_2}$, $A_3 \in \mathbb{C}^{m_3\times m_4}$, $A_4 \in \mathbb{C}^{m_5\times m_4}$, $A_5 \in \mathbb{C}^{m_5\times m_6}$, and $A \in \mathbb{C}^{m_1\times m_6}$ be given. Then, the following five statements are equivalent*:

(a) *The equality $A_1A_2^-A_3A_4^-A_5 = A$ holds for all A_2^- and A_4^-.*

(b) *The product $A_1A_2^-A_3A_4^-A_5$ is invariant with respect to the choices of A_2^- and A_4^-, and $A_1A_2^\dagger A_3A_4^\dagger A_5 = A$.*

(c) *One of the following six conditions holds*:

  (i) $A_1 = 0$ *and* $A = 0$.

  (ii) $A_3 = 0$ *and* $A = 0$.

  (iii) $A_5 = 0$ *and* $A = 0$.

  (iv) $A = 0$, $A_1A_2^\dagger A_3 = 0$, $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(A_2^*)$, *and* $\mathscr{R}(A_3) \subseteq \mathscr{R}(A_2)$.

  (v) $A = 0$, $A_3A_4^\dagger A_5 = 0$, $\mathscr{R}(A_3^*) \subseteq \mathscr{R}(A_4^*)$, *and* $\mathscr{R}(A_5) \subseteq \mathscr{R}(A_4)$.

  (vi) $A = A_1A_2^\dagger A_3A_4^\dagger A_5$, $\mathscr{R}(A_1^*) \subseteq \mathscr{R}(A_2^*)$, $\mathscr{R}(A_5) \subseteq \mathscr{R}(A_4)$, $\mathscr{R}((A_1A_2^\dagger A_3)^*) \subseteq \mathscr{R}(A_4^*)$, $\mathscr{R}(A_3A_4^\dagger A_5) \subseteq \mathscr{R}(A_2)$, *and* $E_{A_2}A_3F_{A_4} = 0$.

(d) *One of the following six conditions holds*:

(i) $A_1 = 0$ *and* $A = 0$.

(ii) $A_3 = 0$ *and* $A = 0$.

(iii) $A_5 = 0$ *and* $A = 0$.

(iv) $A = 0$ *and* $r\begin{bmatrix} A_2 & A_3 \\ A_1 & 0 \end{bmatrix} = r(A_2)$.

(v) $A = 0$ *and* $r\begin{bmatrix} A_4 & A_5 \\ A_3 & 0 \end{bmatrix} = r(A_4)$.

(vi) $A = A_1A_2^{\dagger}A_3A_4^{\dagger}A_5$, $\mathscr{R}([0,\, A_1]^*) \subseteq \mathscr{R}\left(\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix}\right)^*$, $\mathscr{R}\begin{bmatrix} 0 \\ A_5 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix}$, *and* $r\begin{bmatrix} A_3 & A_2 \\ A_4 & 0 \end{bmatrix} = r(A_2) + r(A_4)$.

(e) *One of the following six conditions holds*:

(i) $A_1 = 0$ *and* $A = 0$.

(ii) $A_3 = 0$ *and* $A = 0$.

(iii) $A_5 = 0$ *and* $A = 0$.

(iv) $A = 0$ *and* $r\begin{bmatrix} A_2 & A_3 \\ A_1 & 0 \end{bmatrix} = r(A_2)$.

(v) $A = 0$ *and* $r\begin{bmatrix} A_4 & A_5 \\ A_3 & 0 \end{bmatrix} = r(A_4)$.

(vi) $r\begin{bmatrix} -A & 0 & A_1 \\ 0 & A_3 & A_2 \\ A_5 & A_4 & 0 \end{bmatrix} = r(A_2) + r(A_4)$.

Obviously, all the preceding formulas and facts belong to mathematical competencies and conceptions in matrix algebra. Specifically, the rank equalities for block matrices in Lemma 7 are easy to understand and grasp, and thereby, they can technically and perspicuously be utilized to establish and describe many kinds of concrete matrix expressions and equalities consisting of matrices and their generalized inverses. As a matter of fact, the matrix rank method has been highly regarded as the ultimate manifestation of the characterizations of algebraic matrix equalities in comparison with other algebraic tools in matrix theory.

3. Set Inclusions for Generalized Inverses of Matrix Products

The formulas and facts in Lemma 7 are explicit in form and easily manageable for the different choices of the given matrices, and thereby, they are readily used to solve a wide range of problems for establishing algebraic equalities for matrices and generalized inverses. In this section, we propose a rich variety of matrix set inclusions that are originated from the reverse-order laws in (3)–(7) and derive several groups of equivalent statements associated with these matrix set inclusions through the use of formulas and facts prepared in Section 2.

Referring to Lemma 7, we can perspicuously illustrate how to describe matrix set inclusions for generalized inverses of different matrices.

Theorem 1. *Let* $A_1 \in \mathbb{C}^{m_1\times m_2}$, $A_2 \in \mathbb{C}^{m_3\times m_2}$, $A_3 \in \mathbb{C}^{m_3\times m_4}$, $A_4 \in \mathbb{C}^{m_5\times m_4}$, $A_5 \in \mathbb{C}^{m_5\times m_6}$, *and* $A \in \mathbb{C}^{m_6\times m_1}$. *Then, we have the following results*:

(a) *The following five statements are equivalent*:

(i) $\{A_1A_2^-A_3A_4^-A_5\} \subseteq \{A^-\}$, *namely*, $AA_1A_2^-A_3A_4^-A_5A = A$ *holds for all* A_2^- *and* A_4^-.

(ii) $\{AA_1A_2^-A_3A_4^-A_5\} \subseteq \{AA^-\}$.

(iii) $\{A_1A_2^-A_3A_4^-A_5A\} \subseteq \{A^-A\}$.

(iv) $A = 0$ *or* $r\begin{bmatrix} -A & 0 & AA_1 \\ 0 & A_3 & A_2 \\ A_5A & A_4 & 0 \end{bmatrix} = r(A_2) + r(A_4)$.

(v) $A = 0$ or $r\begin{bmatrix} A_3 & A_2 \\ A_4 & A_5AA_1 \end{bmatrix} = r(A_2) + r(A_4) - r(A)$.

(b) *The following four statements are equivalent*:

(i) $\{A_2^- A_3 A_4^-\} \subseteq \{A^-\}$, *namely* $AA_2^- A_3 A_4^- A = A$ *holds for all* A_2^- *and* A_4^-.

(ii) $\{AA_2^- A_3 A_4^-\} \subseteq \{AA^-\}$.

(iii) $\{A_2^- A_3 A_4^- A\} \subseteq \{A^- A\}$.

(iv) $A = 0$ or $r\begin{bmatrix} A_3 & A_2 \\ A_4 & A \end{bmatrix} = r(A_2) + r(A_4) - r(A)$.

(c) *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times m}$ *and* $C \in \mathbb{C}^{p\times n}$. *Then, the following four statements are equivalent*:

(i) $\{A^- B^-\} \subseteq \{C^-\}$, *namely* $CA^-B^-C = C$ *holds for all* C^-.

(ii) $\{CA^- B^-\} \subseteq \{CC^-\}$.

(iii) $\{A^- B^- C\} \subseteq \{C^- C\}$.

(iv) $C = 0$ or $r(BA - C) = r(A) + r(B) - r(C) - m$.

Proof. By definition, the set inclusion $\{A_1A_2^-A_3A_4^-A_5\} \subseteq \{A^-\}$ is equivalent to $AA_1A_2^-A_3A_4^-A_5A = A$ holds for all A_2^- and A_4^-. In this case, replacing A_1 with AA_1 and A_5 with A_5A in Lemma 7(a) and (e), and then simplifying lead to the equivalence of (i) and (iv) in (a) of this theorem.

Furthermore, it is easy to verify by elementary block matrix operations that the following rank equality:

$$r\begin{bmatrix} -A & 0 & AA_1 \\ 0 & A_3 & A_2 \\ A_5A & A_4 & 0 \end{bmatrix} = r\begin{bmatrix} -A & 0 & 0 \\ 0 & A_3 & A_2 \\ 0 & A_4 & A_5AA_1 \end{bmatrix} = r(A) + r\begin{bmatrix} A_3 & A_2 \\ A_4 & A_5AA_1 \end{bmatrix}$$

holds. Substituting it into (vi) of Lemma 7(e) leads to the equivalence of (iv) and (v) in (a) of this theorem.

Pre- and post-multiplying both sides of the set inclusion in (i) of (a) of this theorem with A, respectively, lead to (ii) and (iii) in (a) of this theorem. Conversely, post- and pre-multiplying both sides of the set inclusions in (ii) and (iii) of (a) of this theorem with A, respectively, lead to (i) in (a) of this theorem.

Results (b) and (c) are direct consequences of (a) under the given assumptions. □

Mindful of the differences of both sides of the matrix set inclusions in the above theorem, we may say that the statements in Theorem 1 in fact provide some useful strategies and techniques of describing matrix set inclusions via matrix rank equalities, and thereby, they can be utilized to construct and solve various equality problems that appear in matrix theory and its applications with regard to products of matrices and their generalized inverses.

In the following, we present some applications of the above results in the characterizations of reverse-order laws for generalized inverses of two or more matrix products. Recall that there were plenty of classic discussions in the literature on the construction and characterization of reverse-order laws for generalized inverses of the matrix product AB, which motivated from time to time deep-going considerations and explorations of various universal algebraic methods to deal with reverse-order law problems. The first reverse-order law in (3) was proposed and well approached in the theory of generalized inverses of matrices; see, e.g., [7,9,14,22–24]. In view of this fact, we first derive from Theorem 1(c) a group of equivalent facts concerning the matrix set inclusion $\{(AB)^-\} \supseteq \{B^-A^-\}$ and its variation forms.

Theorem 2. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$ *be given. Then, the following 23 statements are equivalent*:

(i) $\{(AB)^-\} \supseteq \{B^-A^-\}$.

(ii) $\{(AB)^-\} \supseteq \{B^*(BB^*)^-(A^*A)^-A^*\}$.

(iii) $\{(AB)^-\} \supseteq \{B^-(BB^-)^-(A^-A)^-A^-\}$.
(iv) $\{AB(AB)^-\} \supseteq \{ABB^-A^-\}$.
(v) $\{AB(AB)^-\} \supseteq \{ABB^*(BB^*)^-(A^*A)^-A^*\}$.
(vi) $\{AB(AB)^-\} \supseteq \{ABB^-(BB^-)^-(A^-A)^-A^-\}$.
(vii) $\{(AB)^-AB\} \supseteq \{B^-A^-AB\}$.
(viii) $\{(AB)^-AB\} \supseteq \{B^*(BB^*)^-(A^*A)^-A^*AB\}$.
(ix) $\{(AB)^-AB\} \supseteq \{B^-(BB^-)^-(A^-A)^-A^-AB\}$.
(x) $\{B(AB)^-A\} \supseteq \{BB^-A^-A\}$.
(xi) $\{B(AB)^-A\} \supseteq \{BB^-(BB^-)^-(A^-A)^-A^-A\}$.
(xii) $\{(A^-ABB^-)^-\} \supseteq \{(BB^-)^-(A^-A)^-\}$.
(xiii) $\{(B^*A^*)^-\} \supseteq \{(A^*)^-(B^*)^-\}$.
(xiv) $\{(A^*ABB^*)^-\} \supseteq \{(BB^*)^-(A^*A)^-\}$.
(xv) $\{(BB^*A^*A)^-\} \supseteq \{(A^*A)^-(BB^*)^-\}$.
(xvi) $\{((A^*A)^{1/2}(BB^*)^{1/2})^-\} \supseteq \{((BB^*)^{1/2})^-((A^*A)^{1/2})^-\}$.
(xvii) $\{((BB^*)^{1/2}(A^*A)^{1/2}))^-\} \supseteq \{((A^*A)^{1/2})^-((BB^*)^{1/2})^-\}$.
(xviii) $\{(AA^*ABB^*B)^-\} \supseteq \{(BB^*B)^-(AA^*A)^-\}$.
(xix) $\{(B^*BB^*A^*AA^*)^-\} \supseteq \{(A^*AA^*)^-(B^*BB^*)^-\}$.
(xx) $AB = 0$ *or* $r(AB) = r(A) + r(B) - n$.
(xxi) $AB = 0$ *or* $(I_n - BB^-)(I_n - A^-A) = 0$ *for some/all* A^- *and* B^-.
(xxii) $\mathscr{N}(A) \supseteq \mathscr{R}(B)$ *or* $\mathscr{N}(A) \subseteq \mathscr{R}(B)$.
(xxiii) $\mathscr{R}(A^*) \supseteq \mathscr{N}(B^*)$ *or* $\mathscr{R}(A^*) \subseteq \mathscr{N}(B^*)$.

Proof. Replacing C with AB in (i) and (iv) of Theorem 1(c), we see that (i) in this theorem holds if and only if $AB = 0$ or

$$r(A) + r(B) - r(AB) = r\begin{bmatrix} I_n & B \\ A & AB \end{bmatrix} = r\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = n,$$

establishing the equivalence of (i) and (xx).

By (i) and (v) in Theorem 1(a), (ii) in this theorem holds if and only if $AB = 0$ or

$$r(A) + r(B) - r(AB) = r(A^*A) + r(BB^*) - r(AB) = r\begin{bmatrix} I_n & BB^* \\ A^*A & A^*ABB^* \end{bmatrix} = r\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = n,$$

establishing the equivalence of (ii) and (xx).

By (i) and (v) in Theorem 1(a), (iii) in this theorem holds if and only if $AB = 0$ or

$$r(A) + r(B) - r(AB) = r(A^-A) + r(BB^-) - r(AB) = r\begin{bmatrix} I_n & BB^- \\ A^-A & A^-ABB^- \end{bmatrix} = r\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} = n,$$

establishing the equivalence of (iii) and (xx).

The equivalences of (i) and (xii)–(xx) follow from the following rank equalities:

$$r(A) = r(AA^-) = r(AA^*) = r(AA^*A), \tag{27}$$

$$r(B) = r(B^-B) = r(B^*B) = r(BB^*B), \tag{28}$$

$$\begin{aligned} r(AB) &= r(B^*A^*) = r(A^-ABB^-) = r(A^*ABB^*) = r(BB^*A^*A) \\ &= r((A^*A)^{1/2}(BB^*)^{1/2}) = r((BB^*)^{1/2}(A^*A)^{1/2}) \\ &= r(AA^*ABB^*B) = r(B^*BB^*A^*AA^*). \end{aligned} \tag{29}$$

The equivalences of (i)–(xi) in this theorem follow from (i), (ii), and (iii) in Theorem 1(a). The equivalences of (i) and (xx)–(xxiii) in this theorem were proven in [14]. □

Theorem 3. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$. *Then, the following 16 statements are equivalent*:

(i) $\{(AB)^-\} \ni B^\dagger A^\dagger$, *i.e.*, $ABB^\dagger A^\dagger AB = AB$.
(ii) $\{(AB)^-\} \supseteq \{(B^*B)^-B^*A^*(AA^*)^-\}$.
(iii) $\{AB(AB)^-\} \ni ABB^\dagger A^\dagger$.
(iv) $\{AB(AB)^-\} \supseteq \{AB(B^*B)^-B^*A^*(AA^*)^-\}$.
(v) $\{(AB)^-AB\} \ni B^\dagger A^\dagger AB$.
(vi) $\{(AB)^-AB\} \supseteq \{(B^*B)^-B^*A^*(AA^*)^-AB\}$.
(vii) $\{B(AB)^-A\} \ni BB^\dagger A^\dagger A$.
(viii) $\{B(AB)^-A\} \supseteq \{B(B^*B)^-B^*A^*(AA^*)^-A\}$.
(ix) $\{(A^-ABB^-)^-\} \ni (BB^-)^\dagger(A^-A)^\dagger$.
(x) $\{(A^*ABB^*)^-\} \ni (BB^*)^\dagger(A^*A)^\dagger$.
(xi) $\{(BB^*A^*A)^-\} \ni (A^*A)^\dagger(BB^*)^\dagger$.
(xii) $\{((A^*A)^{1/2}(BB^*)^{1/2})^-\} \ni ((BB^*)^{1/2})^\dagger((A^*A)^{1/2})^\dagger$.
(xiii) $\{((BB^*)^{1/2}(A^*A)^{1/2})^-\} \ni ((A^*A)^{1/2})^\dagger((BB^*)^{1/2})^\dagger$.
(xiv) $\{(AA^*ABB^*B)^-\} \ni (BB^*B)^\dagger(AA^*A)^\dagger$.
(xv) $\{(B^*BB^*A^*AA^*)^-\} \ni (A^*AA^*)^\dagger(B^*BB^*)^\dagger$.
(xvi) $r(AB) = r(A) + r(B) - r[A^*,\, B]$.

Proof. The equivalence of (i) and (xvi) follows from the well-known rank formula:

$$r(AB - ABB^\dagger A^\dagger AB) = r[A^*,\, B] - r(A) - r(B) + r(AB);$$

see [25,26].

By (i) and (iv) in Theorem 1(b), (ii) in this theorem holds if and only if $AB = 0$ or

$$\begin{aligned} r(A) + r(B) - r(AB) &= r(AA^*) + r(B^*B) - r(AB) = r\begin{bmatrix} B^*A^* & B^*B \\ AA^* & AB \end{bmatrix} \\ &= r([A^*,\, B]^*[A^*,\, B]) = r[A^*,\, B]. \end{aligned}$$

Note also that $AB = 0$ is a special case of the above matrix rank equality, thus establishing the equivalence of (ii) and (xvi).

The equivalences of (i) and (ix)–(xvi) in this theorem follow from (27)–(29), and

$$r[A^*,\, B] = r[(A^-A)^*,\, BB^-] = r[A^*A,\, BB^*] = r[(A^*A)^{1/2},\, (BB^*)^{1/2}] = r[A^*AA^*,\, BB^*B].$$

The equivalences of (i)–(viii) in this theorem follow from (i), (ii), and (iii) in Theorem 1(b). □

In the following, the author presents two groups of results on set inclusions associated with the two reverse-order laws $(ABC)^- = (BC)^-B(AB)^-$ and $(ABC)^- = C^-B^-A^-$ and their variation forms for a triple matrix product ABC.

Theorem 4. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$ be given, and denote $M = ABC$. Then, the following 36 statements are equivalent*:

(i) $\{M^-\} \supseteq \{(BC)^-B(AB)^-\}$.
(ii) $\{M^-\} \supseteq \{C^*(BCC^*)^-B(A^*AB)^-A^*\}$.
(iii) $\{M^-\} \supseteq \{(B^*BC)^-B^*BB^*(ABB^*)^-\}$.
(iv) $\{M^-\} \supseteq \{C^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.
(v) $\{M^-\} \supseteq \{C^-(BCC^-)^-B(A^-AB)^-A^-\}$.
(vi) $\{M^-\} \supseteq \{(B^-BC)^-B^-BB^-(ABB^-)^-\}$.
(vii) $\{M^-\} \supseteq \{C^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.
(viii) $\{MM^-\} \supseteq \{M(BC)^-B(AB)^-\}$.
(ix) $\{MM^-\} \supseteq \{MC^*(BCC^*)^-B(A^*AB)^-A^*\}$.
(x) $\{MM^-\} \supseteq \{M(B^*BC)^-B^*BB^*(ABB^*)^-\}$.
(xi) $\{MM^-\} \supseteq \{MC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.

(xii) $\{MM^-\} \supseteq \{MC^-(BCC^-)^-B(A^-AB)^-A^-\}$.
(xiii) $\{MM^-\} \supseteq \{M(B^-BC)^-B^-BB^-(ABB^-)^-\}$.
(xiv) $\{MM^-\} \supseteq \{MC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.
(xv) $\{M^-M\} \supseteq \{(BC)^-B(AB)^-M\}$.
(xvi) $\{M^-M\} \supseteq \{C^*(BCC^*)^-B(A^*AB)^-A^*M\}$.
(xvii) $\{M^-M\} \supseteq \{(B^*BC)^-B^*BB^*(ABB^*)^-M\}$.
(xviii) $\{M^-M\} \supseteq \{C^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*M\}$.
(xix) $\{M^-M\} \supseteq \{C^-(BCC^-)^-B(A^-AB)^-A^-M\}$.
(xx) $\{M^-M\} \supseteq \{(B^-BC)^-B^-BB^-(ABB^-)^-M\}$.
(xxi) $\{M^-M\} \supseteq \{C^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-M\}$.
(xxii) $\{CM^-A\} \supseteq \{C(BC)^-B(AB)^-A\}$.
(xxiii) $\{CM^-A\} \supseteq \{CC^*(BCC^*)^-B(A^*AB)^-A^*A\}$.
(xxiv) $\{CM^-A\} \supseteq \{C(B^*BC)^-B^*BB^*(ABB^*)^-A\}$.
(xxv) $\{CM^-A\} \supseteq \{CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*A\}$.
(xxvi) $\{CM^-A\} \supseteq \{CC^-(BCC^-)^-B(A^-AB)^-A^-A\}$.
(xxvii) $\{CM^-A\} \supseteq \{C(B^-BC)^-B^-BB^-(ABB^-)^-A\}$.
(xxviii) $\{CM^-A\} \supseteq \{CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-A\}$.
(xxix) $\{(A^-MC^-)^-\} \supseteq \{(BCC^-)^-B(A^-AB)^-\}$.
(xxx) $\{(A^*MC^*)^-\} \supseteq \{(BCC^*)^-B(A^*AB)^-\}$.
(xxxi) $\{(AA^*MC^*C)^-\} \supseteq \{(BCC^*C)^-B(AA^*AB)^-\}$.
(xxxii) $\{((AB)^-M(BC)^-)^-\} \supseteq \{((BC)(BC)^-)^-B((AB)^-(AB))^-\}$.
(xxxiii) $\{((AB)^*M(BC)^*)^-\} \supseteq \{((BC)(BC)^*)^-B((AB)^*(AB))^-\}$.
(xxxiv) $\{(((AB)^*(AB))^{1/2}B^-((BC)(BC)^*)^{1/2})^-\} \supseteq \{(((BC)(BC)^*)^{1/2})^-B(((AB)^*(AB))^{1/2})^-\}$.
(xxxv) $M = 0$ *or* $(I_n - (BC)(BC)^-)B\big(I_p - (AB)^-(AB)\big) = 0$ *for some/all* $(AB)^-$ *and* $(BC)^-$.
(xxxvi) $M = 0$ *or* $r(M) = r(AB) + r(BC) - r(B)$.

Proof. By (i) and (iv) in Theorem 1(b), (i) in this theorem holds if and only if

$$M = 0 \ \text{ or } \ r(AB) + r(BC) - r(M) = r\begin{bmatrix} B & BC \\ AB & ABC \end{bmatrix} = r\begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} = r(B),$$

establishing the equivalence of (i) and (xxxvi). The equivalences of (i)–(xxviii) in this theorem can also be shown by Theorem 1(b) and (c). The details of the proofs are omitted here due to space limitation.

The equivalences of (i), (xxix)–(xxxiv), and (xxxvi) follow from the following basic facts:

$$\begin{aligned} r(AB) &= r(A^-AB) = r(A^*B) = r(AA^*AB), \\ r(BC) &= r(BCC^-) = r(BCC^*) = r(BCC^*C), \\ r(M) &= r(A^-MC^-) = r(A^*MC^*) = r(AA^*MC^*C) \\ &= r((AB)^-M(BC)^-) = r((AB)^*M(BC)^*). \end{aligned}$$

The equivalence of (xxxv) and (xxxvi) follows from (19). □

Theorem 5. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$ *be given, and denote* $M = ABC$. *Then, the following 27 statements are equivalent*:
(i) $\{M^-\} \supseteq \{C^-B^-A^-\}$.
(ii) $\{M^-\} \supseteq \{C^*(CC^*)^-B^-(A^*A)^-A^*\}$.
(iii) $\{M^-\} \supseteq \{C^-(CC^-)^-B^-(A^-A)^-A^-\}$.
(iv) $\{M^-\} \supseteq \{(BC)^-A^-\}$ *and* $\{(BC)^-\} \supseteq \{C^-B^-\}$.

(v) $\{M^-\} \supseteq \{C^-(AB)^-\}$ *and* $\{(AB)^-\} \supseteq \{B^-A^-\}$.
(vi) $\{MM^-\} \supseteq \{MC^-B^-A^-\}$.
(vii) $\{MM^-\} \supseteq \{MC^*(CC^*)^-B^-(A^*A)^-A^*\}$.
(viii) $\{MM^-\} \supseteq \{MC^-(CC^-)^-B^-(A^-A)^-A^-\}$.
(ix) $\{MM^-\} \supseteq \{M(BC)^-A^-\}$ *and* $\{BC(BC)^-\} \supseteq \{BCC^-B^-\}$.
(x) $\{MM^-\} \supseteq \{MC^-(AB)^-\}$ *and* $\{AB(AB)^-\} \supseteq \{ABB^-A^-\}$.
(xi) $\{M^-M\} \supseteq \{C^-B^-A^-M\}$.
(xii) $\{M^-M\} \supseteq \{C^*(CC^*)^-B^-(A^*A)^-A^*M\}$.
(xiii) $\{M^-M\} \supseteq \{C^-(CC^-)^-B^-(A^-A)^-A^-M\}$.
(xiv) $\{M^-M\} \supseteq \{(BC)^-A^-M\}$ *and* $\{(BC)^-BC\} \supseteq \{C^-B^-BC\}$.
(xv) $\{M^-M\} \supseteq \{C^-(AB)^-M\}$ *and* $\{(AB)^-AB\} \supseteq \{B^-A^-AB\}$.
(xvi) $\{CM^-A\} \supseteq \{CC^-B^-A^-A\}$.
(xvii) $\{CM^-A\} \supseteq \{CC^*(CC^*)^-B^-(A^*A)^-A^*A\}$.
(xviii) $\{CM^-A\} \supseteq \{CC^-(CC^-)^-B^-(A^-A)^-A^-A\}$.
(xix) $\{BCM^-A\} \supseteq \{BC(BC)^-A^-A\}$ *and* $\{C(BC)^-B\} \supseteq \{CC^-B^-B\}$.
(xx) $\{CM^-AB\} \supseteq \{CC^-(AB)^-AB\}$ *and* $\{B(AB)^-A\} \supseteq \{BB^-A^-A\}$.
(xxi) $\{(C^*B^*A^*)^-\} \supseteq \{(A^*)^-(B^*)^-(C^*)^-\}$.
(xxii) $\{(A^-MC^-)^-\} \supseteq \{(CC^-)^-B^-(A^-A)^-\}$.
(xxiii) $\{(A^*MC^*)^-\} \supseteq \{(CC^*)^-B^-(A^*A)^-\}$.
(xxiv) $\{(AA^*MC^*C)^-\} \supseteq \{(CC^*C)^-B^-(AA^*A)^-\}$.
(xxv) $M = 0$ *or* $r(M) = r(A) + r(B) + r(C) - n - p$.
(xxvi) $M = 0$ *or* $\{r(M) = r(A) + r(BC) - n$ *and* $r(BC) = r(B) + r(C) - p\}$.
(xxvii) $M = 0$ *or* $\{r(M) = r(AB) + r(C) - p$ *and* $r(AB) = r(A) + r(B) - n\}$.

Proof. We first obtain from Lemma 5(b) the following inequalities:

$$p - r(C) + r(M) \geq r(M) \geq 0, \tag{30}$$
$$n - r(A) + r(M) \geq r(M) \geq 0, \tag{31}$$
$$r(M) \geq r(A) + r(B) + r(C) - n - p \geq 0, \tag{32}$$

which we shall use in the sequel. By (i) and (iv) in Theorem 1(b), (i) in this theorem holds if and only if

$$M = 0 \ \text{ or } \ r\begin{bmatrix} B^- & C \\ A & M \end{bmatrix} - r(A) - r(C) + r(M) = 0 \tag{33}$$

holds for all B^-. Applying (25) to the block matrix in (33), we obtain

$$\begin{aligned}
\max_{B^-} r\begin{bmatrix} B^- & C \\ A & M \end{bmatrix} &= \max_{B^-} r\left(\begin{bmatrix} I_p \\ 0 \end{bmatrix} B^- [I_n, 0] + \begin{bmatrix} 0 & C \\ A & M \end{bmatrix} \right) \\
&= \min\left\{ r\begin{bmatrix} I_p & 0 & C \\ 0 & A & M \end{bmatrix},\ r\begin{bmatrix} I_n & 0 \\ 0 & C \\ A & M \end{bmatrix},\ r\begin{bmatrix} -B & I_n & 0 \\ I_p & 0 & C \\ 0 & A & M \end{bmatrix} - r(B) \right\} \\
&= \min\left\{ r\begin{bmatrix} I_p & 0 & 0 \\ 0 & A & 0 \end{bmatrix},\ r\begin{bmatrix} I_n & 0 \\ 0 & C \\ 0 & 0 \end{bmatrix},\ r\begin{bmatrix} 0 & I_n & 0 \\ I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - r(B) \right\} \\
&= \min\{p + r(A),\ \ n + r(C),\ \ n + p - r(B)\}.
\end{aligned} \tag{34}$$

Substituting this result into the second equality in (33) and simplifying by (30)–(32) lead to

$$\begin{aligned}
&\min\{p - r(C) + r(M),\ n - r(A) + r(M),\ n + p - r(A) - r(B) - r(C) + r(M)\} \\
&= n + p - r(A) - r(B) - r(C) + r(M) = 0.
\end{aligned} \tag{35}$$

Combining (35) with the first condition $M = 0$ in (33) leads to the equivalence of (i) and (xxv).

The equivalences of (i)–(xx) in this theorem can be shown from Theorem 1(a) and (b) by similar approaches, and therefore, their proofs are omitted here.

From Lemma 5(b) also, we obtain the following inequalities:

$$r(M) \geq r(A) + r(BC) - n \geq r(A) + r(B) + r(C) - n - p,$$
$$r(M) \geq r(AB) + r(C) - p \geq r(A) + r(B) + r(C) - n - p.$$

Therefore,

$$\begin{aligned} & r(M) = r(A) + r(B) + r(C) - n - p \\ & \Leftrightarrow r(M) = r(A) + r(BC) - n \ \text{ and } \ r(BC) = r(B) + r(C) - p \\ & \Leftrightarrow r(M) = r(AB) + r(C) - p \ \text{ and } \ r(AB) = r(A) + r(B) - n. \end{aligned}$$

These facts imply that (xxv), (xxvi), and (xxvii) are equivalent.

The equivalences of (i) and (xxi)–(xxiv) follow from the basic rank equalities $r(M) = r(A^-M) = r(A^*M) = r(AA^*M)$, $r(M) = r(MC^-) = r(MC^*) = r(MC^*C)$, and $r(M) = r(A^-MC^-) = r(A^*MC^*) = r(AA^*MC^*C)$. $\square$

Theorem 6. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{p\times q}$, and $D \in \mathbb{C}^{q\times s}$ be given, and denote $N = ABCD$. Then, the following 36 statements are equivalent*:

(i) $\{N^-\} \supseteq \{(CD)^-C(BC)^-B(AB)^-\}$.
(ii) $\{N^-\} \supseteq \{(C^*CD)^-C^*C(BC)^-BB^*(ABB^*)^-\}$.
(iii) $\{N^-\} \supseteq \{(CD)^-CC^*(B^*BCC^*)^-B^*B(AB)^-\}$.
(iv) $\{N^-\} \supseteq \{D^*(C^*CDD^*)^-C^*C(BC)^-BB^*(A^*ABB^*)^-A^*\}$.
(v) $\{N^-\} \supseteq \{(C^*CD)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(ABB^*)^-\}$.
(vi) $\{N^-\} \supseteq \{D^*(C^*CDD^*)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.
(vii) $\{N^-\} \supseteq \{(C^-CD)^-C^-C(BC)^-BB^-(ABB^-)^-\}$.
(viii) $\{N^-\} \supseteq \{(CD)^-CC^-(B^-BCC^-)^-B^-B(AB)^-\}$.
(ix) $\{N^-\} \supseteq \{D^-(C^-CDD^-)^-C^-C(BC)^-BB^-(A^-ABB^-)^-A^-\}$.
(x) $\{N^-\} \supseteq \{(C^-CD)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(ABB^-)^-\}$.
(xi) $\{N^-\} \supseteq \{D^-(C^-CDD^-)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.
(xii) $\{NN^-\} \supseteq \{N(CD)^-C(BC)^-B(AB)^-\}$.
(xiii) $\{NN^-\} \supseteq \{N(C^*CD)^-C^*C(BC)^-BB^*(ABB^*)^-\}$.
(xiv) $\{NN^-\} \supseteq \{N(CD)^-CC^*(B^*BCC^*)^-B^*B(AB)^-\}$.
(xv) $\{NN^-\} \supseteq \{ND^*(C^*CDD^*)^-C^*C(BC)^-BB^*(A^*ABB^*)^-A^*\}$.
(xvi) $\{NN^-\} \supseteq \{N(C^*CD)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(ABB^*)^-\}$.
(xvii) $\{NN^-\} \supseteq \{ND^*(C^*CDD^*)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*\}$.
(xviii) $\{NN^-\} \supseteq \{N(C^-CD)^-C^-C(BC)^-BB^-(ABB^-)^-\}$.
(xix) $\{NN^-\} \supseteq \{N(CD)^-CC^-(B^-BCC^-)^-B^-B(AB)^-\}$.
(xx) $\{NN^-\} \supseteq \{ND^-(C^-CDD^-)^-C^-C(BC)^-BB^-(A^-ABB^-)^-A^-\}$.
(xxi) $\{NN^-\} \supseteq \{N(C^-CD)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(ABB^-)^-\}$.
(xxii) $\{NN^-\} \supseteq \{ND^-(C^-CDD^-)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-\}$.
(xxiii) $\{N^-N\} \supseteq \{(CD)^-C(BC)^-B(AB)^-N\}$.
(xxiv) $\{N^-N\} \supseteq \{(C^*CD)^-C^*C(BC)^-BB^*(ABB^*)^-N\}$.
(xxv) $\{N^-N\} \supseteq \{(CD)^-CC^*(B^*BCC^*)^-B^*B(AB)^-N\}$.
(xxvi) $\{N^-N\} \supseteq \{D^*(C^*CDD^*)^-C^*C(BC)^-BB^*(A^*ABB^*)^-A^*N\}$.
(xxvii) $\{N^-N\} \supseteq \{(C^*CD)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(ABB^*)^-N\}$.
(xxviii) $\{N^-N\} \supseteq \{D^*(C^*CDD^*)^-C^*CC^*(B^*BCC^*)^-B^*BB^*(A^*ABB^*)^-A^*N\}$.
(xxix) $\{N^-N\} \supseteq \{(C^-CD)^-C^-C(BC)^-BB^-(ABB^-)^-N\}$.

(xxx) $\{N^-N\} \supseteq \{(CD)^-CC^-(B^-BCC^-)^-B^-B(AB)^-N\}$.
(xxxi) $\{N^-N\} \supseteq \{D^-(C^-CDD^-)^-C^-C(BC)^-BB^-(A^-ABB^-)^-A^-N\}$.
(xxxii) $\{N^-N\} \supseteq \{(C^-CD)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(ABB^-)^-N\}$.
(xxxiii) $\{N^-N\} \supseteq \{D^-(C^-CDD^-)^-C^-CC^-(B^-BCC^-)^-B^-BB^-(A^-ABB^-)^-A^-N\}$.
(xxxiv) $N = 0$ *or* $r(N) = r(AB) + r(BC) + r(CD) - r(B) - r(C)$.
(xxxv) $N = 0$ *or* $\{r(N) = r(ABC) + r(CD) - r(C)$ *and* $r(ABC) = r(AB) + r(BC) - r(B)\}$.
(xxxvi) $N = 0$ *or* $\{r(N) = r(AB) + r(BCD) - r(B)$ *and* $r(BCD) = r(BC) + r(CD) - r(C)\}$.

Proof. We first obtain from Lemma 5(b) the following inequalities:

$$r(N) + r(C) - r(CD) \geq r(N) \geq 0, \tag{36}$$
$$r(N) + r(B) - r(AB) \geq r(N) \geq 0, \tag{37}$$
$$r(N) - r(AB) - r(BC) - r(CD) + r(B) + r(C) \geq 0. \tag{38}$$

By (i) and (iv) in Theorem 1(b), (i) in this theorem holds if and only if

$$N = 0 \ \text{ or } \ r\begin{bmatrix} C(BC)^-B & CD \\ AB & N \end{bmatrix} - r(AB) - r(CD) + r(N) = 0 \tag{39}$$

holds for all $(BC)^-$, where by (25), the maximum rank of the block matrix in (39) is

$$\begin{aligned}
&\max_{(BC)^-} r\begin{bmatrix} C(BC)^-B & CD \\ AB & ABCD \end{bmatrix} \\
&= \max_{(BC)^-} r\left(\begin{bmatrix} C \\ 0 \end{bmatrix} (BC)^-[B, 0] + \begin{bmatrix} 0 & CD \\ AB & ABCD \end{bmatrix} \right) \\
&= \min\left\{ r\begin{bmatrix} C & 0 & CD \\ 0 & AB & ABCD \end{bmatrix}, \ r\begin{bmatrix} B & 0 \\ 0 & CD \\ AB & ABCD \end{bmatrix}, \ r\begin{bmatrix} -BC & B & 0 \\ C & 0 & CD \\ 0 & AB & ABCD \end{bmatrix} - r(BC) \right\} \\
&= \min\left\{ r(AB) + r(C), \ \ r(CD) + r(B), \ \ r\begin{bmatrix} 0 & B & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - r(BC) \right\} \\
&= \min\{r(AB) + r(C), \ \ r(CD) + r(B), \ \ r(B) + r(C) - r(BC)\}.
\end{aligned} \tag{40}$$

Substituting this result into the second equality in (39) and simplifying by (36)–(38) lead to

$$\begin{aligned}
&\min\{r(N) + r(C) - r(CD), \ \ r(N) + r(B) - r(AB), \\
&\qquad r(N) - r(AB) - r(BC) - r(CD) + r(B) + r(C)\} \\
&= r(N) - r(AB) - r(BC) - r(CD) + r(B) + r(C) = 0,
\end{aligned} \tag{41}$$

Combining (41) with the first condition $N = 0$ in (39) leads to the equivalence of (i) and (xxxiv). The equivalences of (i)–(xxxiii) can be shown by similar approaches, and therefore, the details are omitted.

The equivalences of (xxxiv), (xxxv), and (xxxvi) in this theorem follow from Lemma 5(b). □

Given the above results and their derivations, we believe intuitively that there exist many possible variations and extensions of the matrix set inclusion problems. We conclude this section with direct applications of the preceding results to some specified operations of matrices.

Corollary 1. *Let* $A \in \mathbb{C}^{m\times m}$ *be given. Then, the following matrix set inclusions always hold:*

$$\{(A-A^2)^-\} \supseteq \{A^-(I_m - A)^-\}, \tag{42}$$
$$\{(A-A^2)^-\} \supseteq \{(I_m - A)^- A^-\}, \tag{43}$$
$$\{(I_m-A^2)^-\} \supseteq \{(I_m + A)^-(I_m - A)^-\}, \tag{44}$$
$$\{(I_m-A^2)^-\} \supseteq \{(I_m - A)^-(I_m + A)^-\}, \tag{45}$$
$$\{(A-A^3)^-\} \supseteq \{A^-(I_m + A)^-(I_m - A)^-\}, \tag{46}$$
$$\{(A-A^3)^-\} \supseteq \{(I_m + A)^- A^-(I_m - A)^-\}, \tag{47}$$
$$\{(A-A^3)^-\} \supseteq \{(I_m + A)^-(I_m - A)^- A^-\}. \tag{48}$$

Proof. Recall that the following three rank formulas:

$$r(A-A^2) = r(A) + r(I_m - A) - m,$$
$$r(I_m-A^2) = r(I_m + A) + r(I_m - A) - m,$$
$$r(A-A^3) = r(A) + r(I_m + A) + r(I_m - A) - 2m$$

are well known in elementary linear algebra. In this situation, applying Theorem 2(i) and (xx), Theorem 5(i) and (xxv), and the above three rank formulas to the matrix products $A - A^2 = A(I_m - A) = (I_m - A)A$, $I_m - A^2 = (I_m + A)(I_m - A) = (I_m - A)(I_m + A)$, and $A - A^3 = A(I_m + A)(I_m - A) = (I_m + A)A(I_m - A) = (I_m + A)(I_m - A)A$ lead to (42)–(48). □

Theorem 7. *Let* $A,\ B \in \mathbb{C}^{m\times n}$ *be given. Then, the following six statements are equivalent*:

(i) $\{(A+B)^-\} \supseteq \left\{ \begin{bmatrix} A \\ B \end{bmatrix}^- \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} [A,\ B]^- \right\}.$

(ii) $\{(A+B)^-\} \supseteq \left\{ \begin{bmatrix} A^*A \\ B^*B \end{bmatrix}^- \begin{bmatrix} A^*AA^* & 0 \\ 0 & B^*BB^* \end{bmatrix} [AA^*,\ BB^*]^- \right\}.$

(iii) $\{(A+B)^-\} \supseteq \left\{ \begin{bmatrix} A^-A \\ B^-B \end{bmatrix}^- \begin{bmatrix} A^-AA^- & 0 \\ 0 & B^-BB^- \end{bmatrix} [AA^-,\ BB^-]^- \right\}.$

(iv) $\{(A+B)^-\} \supseteq \left\{ [I_n,\ I_n] \begin{bmatrix} A^-A & A^-A \\ B^-B & B^-B \end{bmatrix}^- \begin{bmatrix} A^-AA^- & 0 \\ 0 & B^-BB^- \end{bmatrix} \begin{bmatrix} AA^- & BB^- \\ AA^- & BB^- \end{bmatrix}^- \begin{bmatrix} I_m \\ I_m \end{bmatrix} \right\}.$

(v) $\{(A+B)^-\} \supseteq \left\{ [I_n,\ I_n] \begin{bmatrix} A^*A & A^*A \\ B^*B & B^*B \end{bmatrix}^- \begin{bmatrix} A^*AA^* & 0 \\ 0 & B^*BB^* \end{bmatrix} \begin{bmatrix} AA^* & BB^* \\ AA^* & BB^* \end{bmatrix}^- \begin{bmatrix} I_m \\ I_m \end{bmatrix} \right\}.$

(vi) $A + B = 0$ *or* $r(A+B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\ B] - r(A) - r(B).$

Proof. Writing the sum $A + B$ as $A + B = [I_m,\ I_m] \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_n \\ I_n \end{bmatrix}$ and applying Theorem 4 to this triple matrix product yield the desired results. □

4. Concluding Remarks

The author collected and proposed a series of known and novel equalities for products of matrices and their generalized inverses, including a wide range of reverse-order laws for generalized inverses (matrix set inclusions associated with generalized inverses), and also presented various necessary and sufficient conditions for these matrix equalities to hold through the skillful use of various equalities and inequalities for ranks of matrices. Clearly, this study is a critical manifestation of how to construct reasonable matrix equalities that involve generalized inverses and how to describe these equalities by means of the cogent matrix rank method.

Finally, the author gives some additional remarks about relevant research problems regarding reverse-order laws. It has been recognized that the construction and characterization of reverse-order laws for generalized inverses of multiple matrix products are a huge algebraic work in matrix theory and applications, which mainly includes the following research topics:

(I) Construct and classify different types of reverse-order laws.
(II) Establish necessary and sufficient conditions for each reverse-order law to hold through the use of various matrix analysis methods and techniques.

Furthermore, the author points out that this kind of research problems can reasonably be proposed and studied for generalized inverses of elements in other algebraic systems, in which many different kinds of generalized inverses can properly be defined; see, e.g., [22,27–41] for their expositions.

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