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New Inequalities and Generalizations for Symmetric Means Induced by Majorization Theory

Huan-Nan Shi [1] and Wei-Shih Du [2,*]

[1] Department of Electronic Information, Teacher's College, Beijing Union University, Beijing 100011, China; sfthuannan@buu.edu.cn
[2] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
* Correspondence: wsdu@mail.nknu.edu.tw

Abstract: In this paper, the authors study new inequalities and generalizations for symmetric means and give new proofs for some known results by applying majorization theory.

Keywords: majorization; inequality; log-concave sequence; symmetric function; symmetric mean

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1. Introduction and Preliminaries

Convex analysis has wide applications to many areas of mathematics and science. In the past nearly 80 years, convex analysis has reached a high level of maturity, and an increasing number of connections have been identified between mathematics, physics, economics and finance, automatic control systems, estimation and signal processing, communications and networks and so forth. Several authors have studied a large number of new concepts of generalized convexity and concavity; see, for example, [1–7] and the references therein. Majorization theory has contributed greatly to many branches of pure and applied mathematics, especially in the field of inequalities; for more details, one can refer to [4–6,8–13] and the references therein.

Definition 1 (see [5] (p. 4))**.** *A finite sequence $\{x_k\}_{k=1}^n$ or an infinite sequence $\{x_k\}_{k=1}^\infty$ of non-negative real numbers is said to be*

(i) logarithmically convex (abbreviated as log-convex) if

$$x_k^2 \le x_{k-1}x_{k+1}$$

for all $k = 2, \dots, n-1$ or for all $k \ge 2$; and

(ii) logarithmically concave (abbreviated as log-concave) if

$$x_k^2 \ge x_{k-1}x_{k+1}$$

for all $k = 2, \dots, n-1$ or for all $k \ge 2$.

The following characterizations of logarithmic convexity are crucial to our proofs.

Lemma 1 (see [5] (p. 4))**.** *Let*

$$\mathbb{N}_0^n = \underbrace{\{0,1,2,\dots\} \times \{0,1,2,\dots\} \times \cdots \times \{0,1,2,\dots\}}_{n}.$$

The necessary and sufficient condition for a non-negative sequence $\{a_k\}$ to be log-convex is that, for any $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, $\boldsymbol{q} = (q_1, q_2, \ldots, q_n) \in \mathbb{N}_0^n$ with $\boldsymbol{p} \prec \boldsymbol{q}$, we have

$$\prod_{i=1}^{n} a_{p_i} \leq \prod_{i=1}^{n} a_{q_i}.$$

Corollary 1. *Let $\{a_k\}$ be a positive sequence. If $\{a_k\}$ is log-concave, then, for any $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$, $\boldsymbol{q} = (q_1, q_2, \ldots, q_n) \in \mathbb{N}_0^n$ with $\boldsymbol{p} \prec \boldsymbol{q}$, we have*

$$\prod_{i=1}^{n} a_{p_i} \geq \prod_{i=1}^{n} a_{q_i}.$$

Proof. Since $\{a_k\}$ is a positive log-concave sequence, $\{\frac{1}{a_k}\}$ is a positive log-convex sequence. According to Lemma 1, we have $\prod_{i=1}^{n} \frac{1}{a_{q_i}} \leq \prod_{i=1}^{n} \frac{1}{a_{p_i}}$, this is $\prod_{i=1}^{n} a_{p_i} \geq \prod_{i=1}^{n} a_{q_i}$, so that and Corollary 1 holds. $\square$

Definition 2 (see [10,12]). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. A vector $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \prec \boldsymbol{y}$, if*

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for } 1 \leq k \leq n-1,$$

and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

We now recall the concepts of symmetric function and symmetric mean as follows.

Definition 3 (see, e.g., [9,11]). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.*

(i) *The kth symmetric function $s_k(\boldsymbol{x})$ for $1 \leq k \leq n$ is defined by*

$$s_k(\boldsymbol{x}) = s_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}.$$

In particular, $s_n(\boldsymbol{x}) = \prod_{i=1}^{n} x_i$ and $s_1(\boldsymbol{x}) = \sum_{i=1}^{n} x_i$. We assume that $s_0(\boldsymbol{x}) = 1$ and $s_k(\boldsymbol{x}) = 0$ for $k < 0$ or $k > n$.

(ii) *The kth symmetric mean is defined by*

$$B_k(\boldsymbol{x}) = \frac{s_k(\boldsymbol{x})}{\binom{n}{k}} \quad \text{for } k = 0, 1, \ldots, n.$$

The following lemma is important and will be used for proving our main results.

Lemma 2 (see [9] (p. 458) or [11] (p. 95)). *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \geq 0$ for $i = 1, 2, \ldots, n$. Then,*

$$B_{k+1}(\boldsymbol{x}) B_{k-1}(\boldsymbol{x}) \leq B_k^2(\boldsymbol{x})$$

for all $1 \leq k \leq n$. Equivalently speaking, the sequence $\{B_k(\boldsymbol{x})\}$ is log-concave.

Remark 1. *(i)* *In particular, for $n \geq 2$ and $x_i > 0$ with $i = 1, 2, \ldots, n$, we have*

$$B_1(\boldsymbol{x}) = \frac{s_1(\boldsymbol{x})}{\binom{n}{1}} = A_n(\boldsymbol{x}) = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

$$\sqrt[n]{B_n(\boldsymbol{x})} = \sqrt[n]{\frac{s_n(\boldsymbol{x})}{\binom{n}{n}}} = G_n(\boldsymbol{x}) = \sqrt[n]{x_1x_2\cdots x_n}$$

and

$$\frac{B_n(\boldsymbol{x})}{B_{n-1}(\boldsymbol{x})} = H_n(\boldsymbol{x}) = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}},$$

where $A_n(\boldsymbol{x})$, $G_n(\boldsymbol{x})$ and $H_n(\boldsymbol{x})$ denote the arithmetic mean, geometric mean and harmonic mean of the n positive numbers $x_i > 0$ for $i = 1, 2, \ldots, n$, respectively. See the famous monograph [8].

(ii) *Let $x_i > 0$ for $i = 1, 2, \ldots, n$. When $n \geq 2$, the double inequality between the arithmetic, geometric and harmonic means reads that*

$$A_n(\boldsymbol{x}) \geq G_n(\boldsymbol{x}) \geq H_n(\boldsymbol{x}). \tag{1}$$

The double inequality (1) is fundamental and important in all areas of mathematical sciences. There have been over one hundred proofs for the double inequality (1). See the related texts and references in the paper [2], for example.

In the history of the research process of inequality theory, many important generalization studies have come from simple inequalities that have wide applications. In 1995, by virtue of the Lagrange multiplier method, Zhu [13] proved the following interesting inequality

$$n^{n-2}(x_1x_2\cdots x_{n-1} + x_2x_3\cdots x_n + \cdots + x_nx_1\cdots x_{n-2}) \leq (x_1 + x_2 + \cdots + x_n)^{n-1},$$

which is equivalent to

$$B_{n-1}(\boldsymbol{x}) \leq B_1^{n-1}(\boldsymbol{x}), \tag{2}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i \geq 0$ for $i = 1, 2, \ldots, n$ with $n \geq 3$. In 2022, Hu [14] established the following inequality by mathematical induction:

$$(s_{n-1}(\boldsymbol{x}))^{n-1} \geq n^{n-2}(s_n(\boldsymbol{x}))^{n-2}s_1(\boldsymbol{x}), \tag{3}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \ldots, n$. It is easy to see that inequality (3) is equivalent to

$$B_{n-1}^{n-1}(\boldsymbol{x}) \geq B_n^{n-2}(\boldsymbol{x})B_1(\boldsymbol{x}).$$

Motivated by the works mentioned above, in this paper, we study and investigate new inequalities and generalizations for symmetric means by applying majorization theory. We also give new proofs for some known results proven by difficult typical elementary or analytical methods. Our new proofs given in this paper are novel and concise.

2. Main Results

In this section, we establish the following new inequalities for symmetric means.

Theorem 1. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \ldots, n$. Then, the following hold:*

(i) $B_{n-k}^{n-k}(\boldsymbol{x}) \geq B_n^{n-2k}(\boldsymbol{x})B_k^k(\boldsymbol{x})$ *for all* $2 \leq 2k \leq n$;
(ii) $B_{n-k}^{n-k+1}(\boldsymbol{x}) \geq B_{n-k+1}^{n-k}(\boldsymbol{x})$ *for all* $1 \leq k \leq n$;
(iii) $B_{n-k}^{n-k}(\boldsymbol{x})B_n^{2k}(\boldsymbol{x}) \geq B_k^k(\boldsymbol{x})B_n^n(\boldsymbol{x})$ *for all* $2 \leq 2k \leq n$;
(iv) $B_{k_1}^{1/k_1}(\boldsymbol{x}) \geq B_{k_2}^{1/k_2}(\boldsymbol{x})$ *for all* $1 \leq k_1 < k_2 \leq n$.

Proof. (i) Let $2 \leq 2k \leq n$. It is easy to verify that

$$\Big(\underbrace{n-k, n-k, \ldots, n-k}_{n-k}\Big) \prec \Big(\underbrace{n, n, \ldots, n}_{n-2k}, \underbrace{k, k, \ldots, k}_{k}\Big).$$

According to Lemma 2, the sequence $\{B_k(\boldsymbol{x})\}$ is log-concave, and by Corollary 1, it follows that

$$B_{n-k}^{n-k}(\boldsymbol{x}) \geq B_n^{n-2k}(\boldsymbol{x}) B_k^k(\boldsymbol{x}).$$

(ii) Let $1 \leq k \leq n$. Since

$$\Big(\underbrace{n-k,\ldots,n-k}_{n-k+1},\underbrace{0,\ldots,0}_{k-1}\Big) \prec \Big(\underbrace{n-k+1,\ldots,n-k+1}_{n-k},\underbrace{0,\ldots,0}_{k}\Big),$$

by the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, from Corollary 1, we have

$$B_{n-k}^{n-k+1}(\boldsymbol{x}) = B_{n-k}^{n-k+1}(\boldsymbol{x}) B_0^{k-1}(\boldsymbol{x}) \geq B_{n-k+1}^{n-k}(\boldsymbol{x}) B_0^k(\boldsymbol{x}) = B_{n-k+1}^{n-k}(\boldsymbol{x}).$$

(iii) Let $2 \leq 2k \leq n$. Since

$$\Big(\underbrace{n-k,\ldots,n-k}_{n-k},\underbrace{n,\ldots,n}_{2k}\Big) \prec \Big(\underbrace{k,\ldots,k}_{k},\underbrace{n,\ldots,n}_{n}\Big),$$

by the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, we obtain

$$B_{n-k}^{n-k}(\boldsymbol{x}) B_n^{2k}(\boldsymbol{x}) \geq B_k^k(\boldsymbol{x}) B_n^n(\boldsymbol{x}).$$

(iv) Let $1 \leq k_1 < k_2 \leq n$. Since

$$\Big(\underbrace{k_1,\ldots,k_1}_{k_2},\underbrace{0,\ldots,0}_{k_1-k_2}\Big) \prec \Big(\underbrace{k_2,\ldots,k_2}_{k_1}\Big),$$

from the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, we obtain

$$B_{k_1}^{k_2}(\boldsymbol{x}) B_0^{k_1-k_2}(\boldsymbol{x}) = \left[\frac{s_{k_1}(\boldsymbol{x})}{\binom{n}{k_1}}\right]^{k_2} \left[\frac{s_0(\boldsymbol{x})}{\binom{n}{0}}\right]^{k_1-k_2} \geq B_{k_2}^{k_1}(\boldsymbol{x}) = \left[\frac{s_{k_2}(\boldsymbol{x})}{\binom{n}{k_2}}\right]^{k_1},$$

which implies

$$B_{k_1}^{1/k_1}(\boldsymbol{x}) \geq B_{k_2}^{1/k_2}(\boldsymbol{x}).$$

The proof is completed. □

Remark 2. *When $k_1 = k$ and $k_2 = k+1$ in (iv) of Theorem 1, the inequality*

$$B_{k_1}^{1/k_1}(\boldsymbol{x}) \geq B_{k_2}^{1/k_2}(\boldsymbol{x})$$

becomes the famous Maclaurin's inequality

$$\left[\frac{s_k(\boldsymbol{x})}{\binom{n}{k}}\right]^{1/k} \leq \left[\frac{s_{k-1}(\boldsymbol{x})}{\binom{n}{k-1}}\right]^{1/(k-1)}.$$

In terms of the symmetric means $B_1(\boldsymbol{x})$, $B_{n-1}(\boldsymbol{x})$ and $B_n(\boldsymbol{x})$, the double inequality (1) can be reformulated as follows.

Theorem 2. *Let $n \geq 2$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \ldots, n$. Then,*

$$B_1^n(\boldsymbol{x}) \geq B_n(\boldsymbol{x}) \tag{4}$$

and

$$B_{n-1}^n(\boldsymbol{x}) \geq B_n^{n-1}(\boldsymbol{x}). \tag{5}$$

Proof. According to Lemma 2, the sequence $\{B_k(\boldsymbol{x})\}$ is log-concave. Since the majorization relation

$$\left(\underbrace{1,1,1,\ldots,1}_{n}\right) \prec \left(n,\underbrace{0,0,\ldots,0}_{n-1}\right)$$

is valid, by Corollary 1, we acquire

$$B_1^n(\boldsymbol{x}) \geq B_n(\boldsymbol{x})B_0^{n-1}(\boldsymbol{x}) = B_n(\boldsymbol{x}).$$

Next, we varify $B_{n-1}^n(\boldsymbol{x}) \geq B_n^{n-1}(\boldsymbol{x})$. It is not difficult to see that

$$\left(\underbrace{n-1,n-1,\ldots,n-1}_{n}\right) \prec \left(\underbrace{n,n,\ldots,n}_{n-1},0\right).$$

From the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, we have

$$B_{n-1}^n(\boldsymbol{x}) \geq B_n^{n-1}(\boldsymbol{x})B_0(\boldsymbol{x}) = B_n^{n-1}(\boldsymbol{x}).$$

The proof is completed. □

The following result is a generalization of inequality (2).

Theorem 3. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \ldots, n$. Then, for $n \geq 2k \geq 2$,*

$$B_k^{n-k}(\boldsymbol{x}) \geq B_{n-k}^k(\boldsymbol{x}). \tag{6}$$

Proof. According to Lemma 2, the sequence $\{B_k(\boldsymbol{x})\}$ is log-concave. Note that, for $n \geq 2k$, this is, for $k \leq n-k$,

$$\left(\underbrace{k,k,\ldots,k}_{n-k}\right) \prec \left(\underbrace{n-k,n-k,\ldots,n-k}_{k},\underbrace{0,0,\ldots,0}_{n-2k}\right).$$

By Corollary 1, we obtain

$$B_k^{n-k}(\boldsymbol{x}) = \left[\frac{s_k(\boldsymbol{x})}{\binom{n}{k}}\right]^{n-k} \geq B_{n-k}^k(\boldsymbol{x})B_0^{n-2k}(\boldsymbol{x}) = \left[\frac{s_{n-k}(\boldsymbol{x})}{\binom{n}{n-k}}\right]^k \left[\frac{s_0(\boldsymbol{x})}{\binom{n}{0}}\right]^{n-2k}.$$

The inequality (6) is thus proved. □

Theorem 4. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \ldots, n$. Then, we have*

$$\sqrt[n]{\prod_{k=1}^{n} B_{2k}(\boldsymbol{x})} \geq \sqrt[n+1]{\prod_{k=1}^{n} B_{2k-1}(\boldsymbol{x})}. \tag{7}$$

Proof. The majorization relation

$$\left(\underbrace{2,\ldots,2}_{n+1},\underbrace{4,\ldots,4}_{n+1},\ldots,\underbrace{2n,\ldots,2n}_{n+1}\right) \prec \left(\underbrace{1,\ldots,1}_{n},\underbrace{3,\ldots,3}_{n},\ldots,\underbrace{2n+1,\ldots,2n+1}_{n}\right)$$

is shown in [4] (p. 40). By the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, we obtain

$$\prod_{k=1}^{n} B_{2k}^{n+1}(\boldsymbol{x}) \geq \prod_{k=0}^{n} B_{2k+1}^{n}(\boldsymbol{x}),$$

which deduces (11). □

The arithmetic mean $A_n(\boldsymbol{x})$, geometric mean $G_n(\boldsymbol{x})$ and harmonic mean $H_n(\boldsymbol{x})$ also satisfy Sierpinski's inequality [9] (p. 62) below. In this paper, we give a new proof for Sierpinski's inequality via majorization.

Theorem 5 (Sierpinski's inequality [9] (p. 62))**.** *Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \dots, n$ with $n \geq 2$. Then,*

$$A_n(\boldsymbol{x})H_n^{n-1}(\boldsymbol{x}) \leq G_n^n(\boldsymbol{x}) \leq A_n^{n-1}(\boldsymbol{x})H_n(\boldsymbol{x}). \tag{8}$$

Proof. In terms of symmetric means $B_1(\boldsymbol{x})$, $B_{n-1}(\boldsymbol{x})$ and $B_n(\boldsymbol{x})$, the left and right inequalities in the double inequality (8) can be reformulated as

$$B_1(\boldsymbol{x})B_n^{n-1}(\boldsymbol{x}) \leq B_n(\boldsymbol{x})B_{n-1}^{n-1}(\boldsymbol{x})$$

and

$$B_n(\boldsymbol{x})B_{n-1}(\boldsymbol{x}) \leq B_1^{n-1}(\boldsymbol{x})B_n(\boldsymbol{x}).$$

By the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, the above two inequalities can be obtained from two majorization relations

$$\left(n, \underbrace{n-1, n-1, \dots, n-1}_{n-1}, 0\right) \prec \left(\underbrace{n, n, \dots, n}_{n-1}, 1\right)$$

and

$$\left(n, \underbrace{1, 1, \dots, 1}_{n-1}\right) \prec \left(n, n-1, \underbrace{0, 0, \dots, 0}_{n-2}\right),$$

respectively. This proves the inequality (8). $\square$

Theorem 6 ([9] (p. 260))**.** *Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \dots, n$. Then,*

$$s_n(\boldsymbol{x}) \leq \frac{s_1(\boldsymbol{x})s_{n-1}(\boldsymbol{x})}{n^2} \leq \frac{s_1^n(\boldsymbol{x})}{n^2}. \tag{9}$$

Proof. It is clear that $(n-1, 1) \prec (n, 0)$ and

$$\left(\underbrace{1, 1, 1, \dots, 1}_{n}\right) \prec \left(n-1, 1, \underbrace{0, \dots, 0}_{n-2}\right).$$

From the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, it follows that

$$B_{n-1}(\boldsymbol{x})B_1(\boldsymbol{x}) = \frac{s_{n-1}(\boldsymbol{x})}{\binom{n}{n-1}} \frac{s_1(\boldsymbol{x})}{\binom{n}{1}} \geq B_n(\boldsymbol{x})B_0(\boldsymbol{x}) = \frac{s_n(\boldsymbol{x})}{\binom{n}{1}} \frac{s_0(\boldsymbol{x})}{\binom{n}{0}}$$

and

$$B_1^n(\boldsymbol{x}) = \left[\frac{s_1(\boldsymbol{x})}{\binom{n}{1}}\right]^n \geq B_{n-1}(\boldsymbol{x})B_1(\boldsymbol{x})B_0^{n-2}(\boldsymbol{x}) = \frac{s_{n-1}(\boldsymbol{x})}{\binom{n}{n-1}} \frac{s_1(\boldsymbol{x})}{\binom{n}{1}} \left[\frac{s_0(\boldsymbol{x})}{\binom{n}{0}}\right]^{n-2}.$$

This proves the double inequality (9). $\square$

Theorem 7. *Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $x_i > 0$ for $i = 1, 2, \dots, n$. Then,*

$$s_k(\boldsymbol{x})s_{n-k}(\boldsymbol{x}) \geq \binom{n}{k}^2 s_n(\boldsymbol{x}) \tag{10}$$

and

$$(s_k(\boldsymbol{x}))^n \geq \binom{n}{k}^n (s_n(\boldsymbol{x}))^k \tag{11}$$

for $k = 1, 2, \ldots, n-1$.

Proof. It is clear that $(n-k, k) \prec (n, 0)$. From the logarithmic concavity of the sequence $\{B_k(\boldsymbol{x})\}$, we find

$$\frac{s_k(\boldsymbol{x})}{\binom{n}{k}} \frac{s_{n-k}(\boldsymbol{x})}{\binom{n}{n-k}} = B_k(\boldsymbol{x}) B_{n-k}(\boldsymbol{x}) \geq B_n(\boldsymbol{x}) B_0(\boldsymbol{x}) = \frac{s_n(\boldsymbol{x})}{\binom{n}{n}} \frac{s_0(\boldsymbol{x})}{\binom{n}{0}},$$

which show inequality (10). From the majorization relation

$$\Bigl(\underbrace{k, k, \ldots, k}_{n}, 0\Bigr) \prec \Bigl(\underbrace{n, n, \ldots, n}_{k}, \underbrace{0, 0, \ldots, 0}_{n-k}\Bigr),$$

it follows that inequality (11) holds. □

3. Conclusions

As a discrete form of logarithmic convex (concave) functions, logarithmic convex (concave) sequences play an important role in mathematical analysis and inequality theory. Lemma 1 and Corollary 1 are important conclusion about logarithmic convex (concave) sequences in majorization theory. In this paper, in view of the logarithmic concavity of symmetric mean sequences, we use Corollary 1 and various majorization relations to establish new inequalities and generalizations for symmetric means and give concise, novel and unique proofs for some known results.

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