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Analysis of Interval-Valued Intuitionistic Fuzzy Aczel–Alsina Geometric Aggregation Operators and Their Application to Multiple Attribute Decision-Making

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Abstract: When dealing with the haziness that is intrinsic in decision analysis-driven decision making procedures, interval-valued intuitionistic fuzzy sets (IVIFSs) can be quite effective. Our approach to solving the multiple attribute decision making (MADM) difficulties, where all of the evidence provided by the decision-makers is demonstrated as interval-valued intuitionistic fuzzy (IVIF) decision matrices, in which all of the components are distinguished by an IVIF number (IVIFN), is based on Aczel–Alsina operational processes. We begin by introducing novel IVIFN operations including the Aczel–Alsina sum, product, scalar multiplication, and exponential. We may then create IVIF aggregation operators, such as the IVIF Aczel–Alsina weighted geometric operator, the IVIF Aczel–Alsina ordered weighted geometric operator, and the IVIF Aczel–Alsina hybrid geometric operator, among others. We present a MADM approach that relies on the IVIF aggregation operators that have been developed. A case study is used to demonstrate the practical applicability of the strategies proposed in this paper. By contrasting the newly developed technique with existing techniques, the method is capable of demonstrating the advantages of the newly developed approach. A key result of this work is the discovery that some of the current IVIF aggregation operators are subsets of the operators reported in this article.

Keywords: MADM; Aczel–Alsina operations; IVIFNs; IVIF Aczel–Alsina geometric aggregation operators

MSC: 90B50; 47S40

1. Introduction

The intuitionistic fuzzy set [1] was extended by Atanassov and Gargov to the IVIFS [2], which is represented by membership and non-membership functions whose values are intervals rather than real numbers. Due to the advantages of IVIFS, several researchers have attempted to incorporate IVIF information generated by different kinds of operators to generate judgments [3,4]. For instance, Xu [5] constructed several aggregation operators for IVIFNs, including the IVIF weighted averaging (IVIFWA) operator and the IVIF hybrid averaging (IVIFHA) operator. Liu [6] presented two IVIF operators based on the power average and Heronian mean operators and then integrated IVIF information using them. Zhao and Xu [7] provided several novel IVIF aggregation operations. Yu et al. [8] created the IVIF prioritized weighted averaging/geometric operator. Chen and Han [9,10] provided a MADM approach that was built on the multiplication of IVIF values, in addition to the LP and NLP methodologies. The influenced IVIF weighted and ordered weighted geometric operators were invented by Wei et al. [11]. Li [12] suggested a MADM technique using IVIFSs based on TOPSIS-based nonlinear programming (NLP). Xu and Gou [13] discussed the IVIF aggregation operator in detail. Chen et al. [14] developed a variety of MADM techniques based on IVIFSs. The influenced IVIF hybrid Choquet integral operators developed by Meng et al. [15] were used in decision-making issues. Wang and Liu [16,17] recommended the IVIF Einstein weighted averaging and geometric operators. IVIF MADM has already been widely applied in a variety of fields, including hotel location selection [18], air quality evaluation [19], solid waste management [20], hotel location selection [18], sustainable supplier selection [21], potential partner selection [22], and weapon-group target analysis [23].

Schweizer and Sklar pioneered the idea of triangular norms in their theory of empirical metric spaces [24]. As it develops out, t-norms and their associated t-conorms are vital operations in fuzzification and other evolutionary computing, for instance, Lukasiewicz t-norm and t-conorm [25], Hamacher t-norm and t-conorm [26], Einstein t-norm and t-conorm [17], general continuous Archimedean t-norm, t-conorm [27], etc. Klement et al. [28] conducted a thorough examination of the characteristics and concept implications of triangular norms in the latest years.

1.1. Motivation of the Study

Generalizing the ideas of Menger [29] from 1942, Schweizer and Sklar [24] proposed in 1960 the concept of triangular norms, or t-norms. While their methodology was developed within the context of probabilistic metric spaces for the purpose of making generalizations the triangular inequality of metrics, however, within some years they have been considered in several other branches, most notably fuzzy set theory (there, t-norms generate the fuzzy conjunctions, generalizing the original proposal of Zadeh [30] considering the min operation when introducing the intersection of fuzzy sets). Already in the framework of probabilistic metric spaces, but later also to cover the fuzzy disjunctions, the dual operations to t-norms, namely t-conorms were considered [31]. Later, t-norms and t-conorms have been considered in several generalizations of the fuzzy set theory, including intuitionistic fuzzy set theory [1], interval-valued fuzzy set theory and fuzzy type-2 theory [32], IVIFS theory [2], etc. For more details concerning t-norms and t-conorms we highly suggest the monograph [28] due to Klement et al.

Let $F : [0,1]^2 \to [0,1]$ be a commutative, associative and monotone function. Then, if $e = 1$ is its neutral element, $F(x,1) = F(1,x) = x$ for all $x \in [0,1]$, F is called a triangular norm (t-norm in short). Similarly, if $e = 0$ is its neutral element, i.e., $F(x,0) = F(0,x) = x$ for all $x \in [0,1]$, then F is called a triangular t-conorm (t-conorm, in short).

To have a clear distinction for t-norms and t-conorms in notation, we will consider the traditional notation T for t-norms and S for t-conorms. Note that these two classes are dual, i.e., for any t-norm T, the function $S : [0,1]^2 \to [0,1]$ given by $S(x,y) = 1 - T(1-x, 1-y)$ is a t-conorm (also called a t-conorm dual to T), and for any t-conorm S, the function $T : [0,1]^2 \to [0,1]$ determined by $T(x,y) = 1 - S(1-x, 1-y)$ is a t-norm (t-norm dual to S).

It is not difficult to see that the strongest (greatest) t-norm is $T_M(x,y) = min(x,y)$ following the notation from [28], while the smallest t-norm is the drastic product T_D which is vanishing on $[0,1]^2$ (clearly, if $max(x,y) = 1$ then for any t-norm we have $T(x,y) = min(x,y)$). Two prototypical t-norms playing an important role both in theory and applications are the product t-norm T_P (standard product of reals), and the Lukasiewicz

t-norm T_L given by $T_L(x,y) = max(0, x+y-1)$. One of the most distinguished subclasses of the class of all t-norms is formed by the continuous Archimedean t-norms, i.e., t-norms generated by a continuous additive generator. Their importance is clearly visible when n-ary extensions of t-norms are considered. For deeper results and more details see [28]. In our paper, we will deal with some specially generated t-norms, namely with strict t-norms which are isomorphic to the product t-norm, and which are generated by decreasing bijective additive generators $t : [0,1] \to [0,\infty]$. In such a case, $T(x,y) = t^{-1}(t(x) + t(y))$, and, considering the n-array extension (which is unique due to the associativity of t-norms), $T(x_1,\ldots,x_n) = t^{-1}(\sum_{i=1}^{n} t(x_i))$. Recall that both extremal t-norms T_M and T_D, as well as the product t-norm T_P commute with the power functions, i.e., for any $\lambda > 0$, they satisfy the equality $T(x^\lambda, y^\lambda) = T(x,y)^\lambda$. Aczel and Alsina in the early 1980s [33] have characterized all other t-norm solutions of the above functional equation, showing that these are just strict t-norms generated by additive generators $t_ð$, $ð \in]0,\infty[$, given by $t_ð(x) = (-logx)^ð$. The related t-norms are denoted as $T_A^ð$ and called (strict) Aczel–Alsina t-norms, and given by

$$T_A^ð(x,y) = \begin{cases} T_D(x,y), & \text{if } ð = 0 \\ \min(x,y), & \text{if } ð = \infty \\ e^{-((-\log x)^ð + (-\log y)^ð)^{1/ð}}, & \text{otherwise.} \end{cases}$$

Observe that including the extremal t-norms, we obtain their Aczel–Alsina family $(T_A^ð)$, $ð \in [0,\infty]$ of t-norms, which is strictly increasing and continuous in parameter ð.

Due to the duality, similar notes and examples can be introduced for t-conorms. There, the smallest t-conorms is $S_M = max$ (dual to T_M), and the greatest t-conorm is the drastic product S_D, which is constant 1 on $[0,1]^2$. For any t-conorm S, if $min(x,y) = 0$, then $S(x,y) = max(x,y)$. Dual t-conorm S_L to T_L(Lukasiewicz t-conorm, also called a truncated sum) is given by $S_L(x,y) = min(1, x+y)$, and the dual t-conorm S_P to the product T_P (called a probabilistic sum) is given by $S_P(x,y) = x + y - xy$. Continuous Archimedean t-conorms are also generated by additive generators (which are increasing), and if S is dual to a continuous Archimedean t-norm T generated by an additive generator t, then S is generated by an additive generator s given by $s(x) = t(1-x)$. In particular, dual t-conorms $S_A^ð$ to strict Aczel–Alsina t-noms $T_A^ð$ are generated by additive generators $s_ð(x) = (-log(1-x))^ð$, and they are given by

$$S_A^ð(x,y) = \begin{cases} S_D(x,y), & \text{if } ð = 0 \\ \max(x,y), & \text{if } ð = \infty \\ 1 - e^{-((-\log(1-x))^ð + (-\log(1-y))^ð)^{1/ð}}, & \text{otherwise.} \end{cases}$$

Observe that including the extremal t-conorms, we obtain their Aczel–Alsina family $(S_A^ð)$, $ð \in [0,\infty]$ of t-conorms, which is strictly decreasing and continuous in parameter ð.

Aczel-Alsina [33] came up with two new operations called Aczel–Alsina t-norm and Aczel-Alsina t-conorm. These operations have a good relationship with the deployment of parameters. Wang et al. [34] used the Aczel-Alsina triangular norm (AA t-norm) to come up with a score level convolution neural network that increases the distance between imposters and legitimate at the same time. Senapati et al. [35–38] came up with Aczel—Alsina operations depending on intuitionistic fuzzy, IVIF, hesitant fuzzy, picture fuzzy aggregation operators, and they used them to solve MADM problems. The primary objective of this insightful article is to illustrate several geometric aggregation operators using IVIF data, known to as IVIF Aczel–Alsina geometric aggregations, for the purpose of identifying the successfully guide of decisions made utilizing decision-making techniques. Unaware of the previously existing unique ways that have been developed in this domain, we have fully examined every possibility to exhibit our proposed approach, in order for it to exceed all past attempts to apprehend the system assessment problem.

1.2. Structure of This Study

The framework of the study is presented in Figure 1. The following details are presented: The next section discusses several basic concepts relating to IVIFSs. Section 3 discusses the Aczel–Alsina operational laws governing the IVIFNs. Section 4 discusses the IVIF Aczel–Alsina weighted geometric (IVIFAAWG) operator, the IVIF Aczel–Alsina order weighted geometric (IVIFAAOWG) operator, and the IVIF Aczel–Alsina hybrid geometric (IVIFAAHG) operator, as well as a few particular instances. In Section 5, we demonstrate how to use the IVIFAAWG operator to construct particular approaches for resolving multiple attribute decision-making challenges in which support and understanding are represented as IVIF values. Section 6 shows the overall methodology with a genuine scenario. Section 7 investigates the effect of a parameter on the outcome of decision-making. Section 8 provides a comparison investigation of alternative important strategies to substantiate the suggested technique's sufficiency. Section 9 concludes this analysis and identifies potential future concerns.

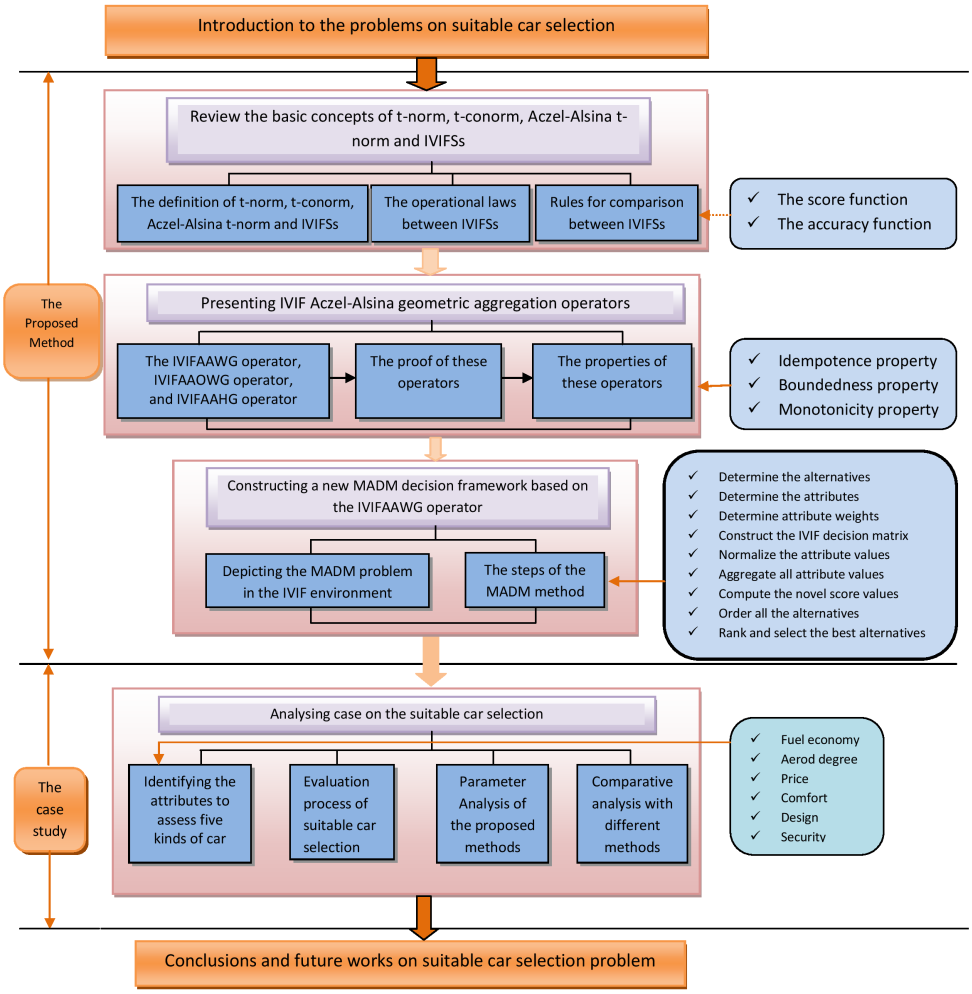

Figure 1. The framework of the study.

2. Preliminaries

This section will summarize some major themes that will be discussed throughout the remainder of this work.

Definition 1 ([2]). *Assuming F is a recognized universe of discourse, an IVIFS in F is an expression $\tilde{E}$ given by*

$$\tilde{E} = \{\langle f, \tilde{\beta}_E(f), \tilde{\delta}_E(f)\rangle : f \in F\} \quad (1)$$

where $\tilde{\beta}_E(f) : F \to D[0,1]$, $\tilde{\delta}_E(f) : F \to D[0,1]$ and $D[0,1]$ is the set of all subintervals of $[0,1]$. The intervals $\tilde{\beta}_E(f)$ and $\tilde{\delta}_E(f)$ denote the intervals of the degree of membership and degree of non-membership of the element f in the set $\tilde{E}$, where $\tilde{\beta}_E(f) = [\beta_E^L(f), \beta_E^U(f)]$ and $\tilde{\delta}_E(f) = [\delta_E^L(f), \delta_E^U(f)]$, for all $f \in F$, including the condition $0 \leq \beta_E^U(f) + \delta_E^U(f) \leq 1$. $\pi_E(f) = [\pi_E^L(f), \pi_E^U(f)]$ denotes the indeterminacy degree of element f that belongs to $\tilde{E}$, where $\pi_E^L(f) = 1 - \beta_E^U(f) - \delta_E^U(f)$ and $\pi_E^U(f) = 1 - \beta_E^L(f) - \delta_E^L(f)$.

Assume that $\tilde{E} = \{\langle f, \tilde{\beta}_E(f), \tilde{\delta}_E(f)\rangle : f \in F\}$ and $\tilde{W} = \{\langle f, \tilde{\beta}_W(f), \tilde{\delta}_W(f)\rangle : f \in F\}$ are two IFSs over the universe F. The next relations and operations concerning two IVIFSs were described as follows [2,25]:

(i) $\tilde{E} \subseteq \tilde{W}$, if $\beta_E^L(f) \leq \beta_W^L(f)$, $\beta_E^U(f) \leq \beta_W^U(f)$, $\delta_E^L(f) \geq \delta_W^L(f)$, and $\delta_E^U(f) \geq \delta_W^U(f)$ for all $f \in F$;
(ii) $\tilde{E} = \tilde{W}$ iff $\tilde{E} \subseteq \tilde{W}$ and $\tilde{W} \subseteq \tilde{E}$;
(iii) $\tilde{E}^C = \{\langle f, \tilde{\delta}_E(f), \tilde{\beta}_E(f)\rangle | f \in F\}$ for all $f \in F$;
(iv) $\tilde{E} \cap_{T,S} \tilde{W} = \{\langle f, [T\{\beta_E^L(f), \beta_W^L(f)\}, T\{\beta_E^U(f), \beta_W^U(f)\}], [S\{\delta_E^L(f), \delta_W^L(f)\}, S\{\delta_E^U(f), \delta_W^U(f)\}]\rangle | f \in F\}$;
(v) $\tilde{E} \cup_{S,T} \tilde{W} = \{\langle f, [S\{\beta_E^L(f), \beta_W^L(f)\}, S\{\beta_E^U(f), \beta_W^U(f)\}], [T\{\delta_E^L(f), \delta_W^L(f)\}, T\{\delta_E^U(f), \delta_W^U(f)\}]\rangle | f \in F\}$;

where any pair (T, S) can be utilized, T indicates a t-norm and S a so-called t-conorm dual to the t-norm T, characterized by $S(x, y) = 1 - T(1 - x, 1 - y)$.

For convenience, Xu [5] called $\tilde{\partial} = ([\beta_\partial^L, \beta_\partial^U], [\delta_\partial^L, \delta_\partial^U])$ an IVIFN, where $[\beta_\partial^L, \beta_\partial^U] \in D[0,1]$, $[\delta_\partial^L, \delta_\partial^U] \in D[0,1]$ and $\beta_\partial^U + \delta_\partial^U \leq 1$.

For any three IVIFNs $\tilde{\partial} = ([\beta_\partial^L, \beta_\partial^U], [\delta_\partial^L, \delta_\partial^U])$, $\tilde{\partial}_1 = ([\beta_{\partial_1}^L, \beta_{\partial_1}^U], [\delta_{\partial_1}^L, \delta_{\partial_1}^U])$ and $\tilde{\partial}_2 = ([\beta_{\partial_2}^L, \beta_{\partial_2}^U], [\delta_{\partial_2}^L, \delta_{\partial_2}^U])$, Xu [5] and Xu and Chen [39] stated a few operations as follows:

(i) $\tilde{\partial}_1 \cap \tilde{\partial}_2 = ([\min\{\beta_{\partial_1}^L, \beta_{\partial_2}^L\}, \min\{\beta_{\partial_1}^U, \beta_{\partial_2}^U\}], [\max\{\delta_{\partial_1}^L, \delta_{\partial_2}^L\}, \max\{\delta_{\partial_1}^U, \delta_{\partial_2}^U\}])$;
(ii) $\tilde{\partial}_1 \cup \tilde{\partial}_2 = ([\max\{\beta_{\partial_1}^L, \beta_{\partial_2}^L\}, \max\{\beta_{\partial_1}^U, \beta_{\partial_2}^U\}], [\min\{\delta_{\partial_1}^L, \delta_{\partial_2}^L\}, \min\{\delta_{\partial_1}^U, \delta_{\partial_2}^U\}])$;
(iii) $\tilde{\partial}_1 \oplus \tilde{\partial}_2 = ([\beta_{\partial_1}^L + \beta_{\partial_2}^L - \beta_{\partial_1}^L\beta_{\partial_2}^L, \beta_{\partial_1}^U + \beta_{\partial_2}^U - \beta_{\partial_1}^U\beta_{\partial_2}^U], [\delta_{\partial_1}^L\delta_{\partial_2}^L, \delta_{\partial_1}^U\delta_{\partial_2}^U])$;
(iv) $\tilde{\partial}_1 \otimes \tilde{\partial}_2 = ([\beta_{\partial_1}^L\beta_{\partial_2}^L, \beta_{\partial_1}^U\beta_{\partial_2}^U], [\delta_{\partial_1}^L + \delta_{\partial_2}^L - \delta_{\partial_1}^L\delta_{\partial_2}^L, \delta_{\partial_1}^U + \delta_{\partial_2}^U - \delta_{\partial_1}^U\delta_{\partial_2}])$;
(v) $\varphi \cdot \tilde{\partial} = ([1 - (1 - \beta_\partial^L)^\varphi, 1 - (1 - \beta_\partial^U)^\varphi], [(\delta_\partial^L)^\varphi, (\delta_\partial^U)^\varphi])$, $\varphi > 0$;
(vi) $\tilde{\partial}^\varphi = ([(\beta_\partial^L)^\varphi, (\beta_\partial^U)^\varphi], [1 - (1 - \delta_\partial^L)^\varphi, 1 - (1 - \delta_\partial^U)^\varphi])$, $\varphi > 0$.

Several indices [5,40] were used to characterize IVIFN.

Definition 2 ([40]). *For any IVIFN $\tilde{\partial} = ([\beta_\partial^L, \beta_\partial^U], [\delta_\partial^L, \delta_\partial^U])$, the score function $Sco(\tilde{\partial})$, accuracy function $Acc(\tilde{\partial})$, membership uncertainty index $Mui(\tilde{\partial})$ and hesitation uncertainty index $Hui(\tilde{\partial})$ of ∂ be defined as follows:*

$$Sco(\tilde{\partial}) = \frac{1}{2}(\beta_\partial^L + \beta_\partial^U - \delta_\partial^L - \delta_\partial^U), \tag{2}$$

$$Acc(\tilde{\partial}) = \frac{1}{2}(\beta_\partial^L + \beta_\partial^U + \delta_\partial^L + \delta_\partial^U), \tag{3}$$

$$Mui(\tilde{\partial}) = \beta_\partial^U + \delta_\partial^L - \beta_\partial^L - \delta_\partial^U, \tag{4}$$

$$Hui(\tilde{\partial}) = \beta_\partial^U + \delta_\partial^U - \beta_\partial^L - \delta_\partial^L. \tag{5}$$

Based on these indices of IVIFNs, the total ordering [40] on IVIFNs was defined as follows.

Definition 3. *Let $\tilde{\partial}_1 = ([\beta_{\partial_1}^L, \beta_{\partial_1}^U], [\delta_{\partial_1}^L, \delta_{\partial_1}^U])$ and $\tilde{\partial}_2 = ([\beta_{\partial_2}^L, \beta_{\partial_2}^U], [\delta_{\partial_2}^L, \delta_{\partial_2}^U])$ be two IVIFNs, then*

(1) *if $Sco(\tilde{\partial}_1) < Sco(\tilde{\partial}_2)$, then $\tilde{\partial}_1 < \tilde{\partial}_2$,*
(2) *if $Sco(\tilde{\partial}_1) = Sco(\tilde{\partial}_2)$, then*
    (a) *if $Acc(\tilde{\partial}_1) < Acc(\tilde{\partial}_2)$, then $\tilde{\partial}_1 < \tilde{\partial}_2$,*
    (b) *if $Acc(\tilde{\partial}_1) = Acc(\tilde{\partial}_2)$, then*

(I) *if* $Mui(\tilde{\partial}_1) < Mui(\tilde{\partial}_2)$, *then* $\tilde{\partial}_1 < \tilde{\partial}_2$,

(II) *if* $Mui(\tilde{\partial}_1) = Mui(\tilde{\partial}_2)$, *then*

  (i) *if* $Hui(\tilde{\partial}_1) < Hui(\tilde{\partial}_2)$, *then* $\tilde{\partial}_1 < \tilde{\partial}_2$,

  (ii) *if* $Hui(\tilde{\partial}_1) = Hui(\tilde{\partial}_2)$, *then* $\tilde{\partial}_1$ *and* $\tilde{\partial}_2$ *are same, i.e.,* $\beta^L_{\partial_1} = \beta^L_{\partial_2}$, $\beta^U_{\partial_1} = \beta^U_{\partial_2}$, $\delta^L_{\partial_1} = \delta^L_{\partial_2}$ *and* $\delta^U_{\partial_1} = \delta^U_{\partial_2}$, *denoted by* $\tilde{\partial}_1 = \tilde{\partial}_2$.

Definition 3 defines a way to compare two IVIFNs by prioritizing the functions of score, accuracy, membership uncertainty index, and hesitation uncertainty index. Because once two IVIFNs are analyzed, the sequencing is examined in the following order: general belonging degree, accuracy or hesitation level, membership uncertainty index, and hesitation uncertainty index. This comparative procedure is repeated unless one of the four functions defined in Definition 3 recognizes the two IVIFNs. When these two IVIFNs are distinguished at a particular level of severity, the computation is completed and functions with lower value levels are not computed.

Deschrijver et al. [41] designed the concept of the notion of non-empty intervals. They denoted by L the lattice of non-empty intervals $L = \{[m,n] | (m,n) \in [0,1]^2, m \leq n\}$ with the partial order $\leq_L$ determined as $[m,n] \leq_L [p,q] \Leftrightarrow m \leq p$ and $n \leq q$. The inferior and superior elements are denoted by the symbol $0_L = [0,0]$ and $1_L = [1,1]$, respectively.

In this specific situation, Wang and Liu [16,17] meant by $L^\star$ the lattice of non-empty IVIFNs $L^\star = \{\langle [m,n],[p,q]\rangle | [m,n],[p,q] \in D[0,1], n+q \leq 1\}$ with the partial order $\leq_{L^\star}$ characterized as $\langle [m_1,n_1],[p_1,q_1]\rangle \leq_{L^\star} \langle [m_2,n_2],[p_2,q_2]\rangle \Leftrightarrow [m_1,n_1] \leq_L [m_2,n_2] \& [p_2,q_2] \leq_L [p_1,q_1] \Leftrightarrow m_1 \leq m_2, n_1 \leq n_2, p_1 \geq p_2$ and $q_1 \geq q_2$, where the inferior and superior elements are $0_{L^\star} = \langle 0_L, 1_L\rangle = \langle [0,0],[1,1]\rangle$ and $1_{L^\star} = \langle 1_L, 0_L\rangle = \langle [1,1],[0,0]\rangle$, respectively.

Remark 1. *If* $\alpha \leq_{L^\star} \nu$, *then* $\alpha \leq \nu$, *i.e., the total order consists of the standard partial order on* $L^\star$.

Definition 4. $g_{L^\star} : (L^\star)^\hbar \to L^\star$ *is an aggregation function if it is monotone with respect to* $\leq_{L^\star}$ *and satisfies* $g_{L^\star}(0_{L^\star}, \ldots, 0_{L^\star}) = 0_{L^\star}$ *and* $g_{L^\star}(1_{L^\star}, \ldots, 1_{L^\star}) = 1_{L^\star}$.

Currently, a wide number of operators are now being developed for accumulating IVIF data in $L^\star$ [42,43]. The IVIF weighted geometric (IVIFWG) operator and the IVIF ordered weighted geometric (IVIFOWG) operator are probably the most frequently acknowledged operators for accumulating inputs, and they are discussed in details in the following.

Definition 5. *Let* $\tilde{\partial}_\zeta = ([\beta^L_{\partial_\zeta}, \beta^U_{\partial_\zeta}], [\delta^L_{\partial_\zeta}, \delta^U_{\partial_\zeta}])$ $(\zeta = 1,2,\ldots,\hbar)$ *be an accumulation of IVIFNs and* $\xi = (\xi_1, \xi_2, \ldots, \xi_\hbar)^T$ *is the weight vector of* ∂_ζ $(\zeta = 1,2,\ldots,\hbar)$ *so as* $\xi_\zeta \in [0,1]$, $\zeta = 1,2,\ldots,\hbar$ *and* $\sum_{\zeta=1}^{\hbar} \xi_\zeta = 1$. *Therefore, the IVIF weighted geometric (IVIFWG) operator of dimension* $\hbar$ *is a function* $IVIFWG : (L^\star)^\hbar \to L^\star$ *and* $IVIFWG(\tilde{\partial}_1, \tilde{\partial}_2, \ldots, \tilde{\partial}_\hbar) = \bigotimes_{\zeta=1}^{\hbar} (\tilde{\partial}_\zeta)^{\xi_\zeta}$

$$= \left(\left[\prod_{\zeta=1}^{\hbar} (\beta^L_{\partial_\zeta})^{\xi_\zeta}, \prod_{\zeta=1}^{\hbar} (\beta^U_{\partial_\zeta})^{\xi_\zeta} \right], \left[1 - \prod_{\zeta=1}^{\hbar} (1 - \delta^L_{\partial_\zeta})^{\xi_\zeta}, 1 - \prod_{\zeta=1}^{\hbar} (1 - \delta^U_{\partial_\zeta})^{\xi_\zeta} \right] \right).$$

Definition 6. *Let* $\tilde{\partial}_\zeta = ([\beta^L_{\partial_\zeta}, \beta^U_{\partial_\zeta}], [\delta^L_{\partial_\zeta}, \delta^U_{\partial_\zeta}])$ $(\zeta = 1,2,\ldots,\hbar)$ *be a collection of IVIFNs and* $\xi = (\xi_1, \xi_2, \ldots, \xi_\hbar)^T$ *is the weight vector of* ∂_ζ $(\zeta = 1,2,\ldots,\hbar)$ *so as* $\xi_\zeta \in [0,1]$, $\zeta = 1,2,\ldots,\hbar$ *and* $\sum_{\zeta=1}^{\hbar} \xi_\zeta = 1$. *Then, the IVIF ordered weighted geometric (IVIFOWG) operator of dimension* $\hbar$ *is a function* $IVIFOWG : (L^\star)^\hbar \to L^\star$ *and* $IVIFOWG(\tilde{\partial}_1, \tilde{\partial}_2, \ldots, \tilde{\partial}_\hbar) = \bigotimes_{\zeta=1}^{\hbar} (\tilde{\partial}_{\varrho(j)})^{\xi_\zeta}$

$$= \left(\left[\prod_{\zeta=1}^{\hbar} (\beta^L_{\partial_{\varrho(j)}})^{\xi_\zeta}, \prod_{\zeta=1}^{\hbar} (\beta^U_{\partial_{\varrho(j)}})^{\xi_\zeta} \right], \left[1 - \prod_{\zeta=1}^{\hbar} (1 - \delta^L_{\partial_{\varrho(j)}})^{\xi_\zeta}, 1 - \prod_{\zeta=1}^{\hbar} (1 - \delta^U_{\partial_{\varrho(j)}})^{\xi_\zeta} \right] \right).$$

3. Aczel–Alsina Operations of IVIFNs

This section will introduce the Aczel–Alsina operations on IVIFNs and discuss some of its fundamental properties.

If you let the t-norm T be the Aczel–Alsina product T_A and the t-conorm S be the Aczel–Alsina sum S_A, the generalized intersection and union over two IVIFNs E and W are the Aczel–Alsina product ($E \otimes W$) and Aczel–Alsina sum ($E \oplus W$) over two IVIFNs E and W, respectively, which can be seen:

$$\begin{aligned} E \otimes W &= \left\langle \left[T_A\{\beta_E^L, \beta_W^L\}, T_A\{\beta_E^U, \beta_W^U\}\right], \left[S_A\{\delta_E^L, \delta_W^L\}, S_A\{\delta_E^U, \delta_W^U\}\right] \right\rangle \\ E \oplus W &= \left\langle \left[S_A\{\beta_E^L, \beta_W^L\}, S_A\{\beta_E^U, \beta_W^U\}\right], \left[T_A\{\delta_E^L, \delta_W^L\}, T_A\{\delta_E^U, \delta_W^U\}\right] \right\rangle. \end{aligned}$$

Proposition 1. *Let* $\tilde{\eth}_1 = ([\beta_{\eth_1}^L, \beta_{\eth_1}^U], [\delta_{\eth_1}^L, \delta_{\eth_1}^U])$ *and* $\tilde{\eth}_2 = ([\beta_{\eth_2}^L, \beta_{\eth_2}^U], [\delta_{\eth_2}^L, \delta_{\eth_2}^U])$ *be two IVIFNs,* $\eth \in [0, \infty]$ *and* $\varphi > 0$*. Then, the Aczel–Alsina t-norm and t-conorm operations of IVIFNs are assigned as:*

(i) $\tilde{\eth}_1 \oplus \tilde{\eth}_2 = \left\langle \left[1 - e^{-((-\log(1-\beta_{\eth_1}^L))^{\eth} + (-\log(1-\beta_{\eth_2}^L))^{\eth})^{1/\eth}}, \right.\right.$
$\left. 1 - e^{-((-\log(1-\beta_{\eth_1}^U))^{\eth} + (-\log(1-\beta_{\eth_2}^U))^{\eth})^{1/\eth}} \right], \left[e^{-((-\log\delta_{\eth_1}^L)^{\eth} + (-\log\delta_{\eth_2}^L)^{\eth})^{1/\eth}}, \right.$
$\left.\left. e^{-((-\log\delta_{\eth_1}^U)^{\eth} + (-\log\delta_{\eth_2}^U)^{\eth})^{1/\eth}} \right] \right\rangle,$

(ii) $\tilde{\eth}_1 \otimes \tilde{\eth}_2 = \left\langle \left[e^{-((-\log\beta_{\eth_1}^L)^{\eth} + (-\log\beta_{\eth_2}^L)^{\eth})^{1/\eth}}, \right.\right.$
$\left. e^{-((-\log\beta_{\eth_1}^U)^{\eth} + (-\log\beta_{\eth_2}^U)^{\eth})^{1/\eth}} \right], \left[1 - e^{-((-\log(1-\delta_{\eth_1}^L))^{\eth} + (-\log(1-\delta_{\eth_2}^L))^{\eth})^{1/\eth}}, \right.$
$\left.\left. 1 - e^{-((-\log(1-\delta_{\eth_1}^U))^{\eth} + (-\log(1-\delta_{\eth_2}^U))^{\eth})^{1/\eth}} \right] \right\rangle,$

Definition 7. *Let* $\tilde{\eth} = ([\beta_{\eth}^L, \beta_{\eth}^U], [\delta_{\eth}^L, \beta_{\eth}^U])$ *be a IVIFN,* $\eth \in [0, \infty]$ *and* $\varphi > 0$*. Then, the following two operations of IVIFNs are defined as:*

(i) $\varphi\tilde{\eth} = \left\langle \left[1 - e^{-(\varphi(-\log(1-\beta_{\eth}^L))^{\eth})^{1/\eth}}, 1 - e^{-(\varphi(-\log(1-\beta_{\eth}^U))^{\eth})^{1/\eth}}\right], \right.$
$\left. \left[e^{-(\varphi(-\log\delta_{\eth}^L)^{\eth})^{1/\eth}}, e^{-(\varphi(-\log\delta_{\eth}^U)^{\eth})^{1/\eth}} \right] \right\rangle,$

(ii) $\tilde{\eth}^{\varphi} = \left\langle \left[e^{-(\varphi(-\log\beta_{\eth}^L)^{\eth})^{1/\eth}}, e^{-(\varphi(-\log\beta_{\eth}^U)^{\eth})^{1/\eth}} \right], \left[1 - e^{-(\varphi(-\log(1-\delta_{\eth}^L))^{\eth})^{1/\eth}}, \right.\right.$
$\left. 1 - e^{-(\varphi(-\log(1-\delta_{\eth}^U))^{\eth})^{1/\eth}} \right\rangle.$

Example 1. *Let* $\tilde{\eth} = ([0.55, 0.60], [0.35, 0.40])$*,* $\tilde{\eth}_1 = ([0.75, 0.80], [0.15, 0.20])$ *and* $\tilde{\eth}_2 = ([0.35, 0.45], [0.45, 0.50])$ *be three IVIFNs, then applying Aczel–Alsina operation on IVIFNs as specified in Proposition 1 and Definition 7 for* $\eth = 3$ *and* $\varphi = 2$*, we get*

(i) $\tilde{\eth}_1 \oplus \tilde{\eth}_2 = \left\langle \left[1 - e^{-((-\log(1-0.75))^3 + (-\log(1-0.35))^3)^{1/3}}, \right.\right.$
$\left. 1 - e^{-((-\log(1-0.80))^3 + (-\log(1-0.45))^3)^{1/3}} \right], \left[e^{-((-\log 0.15)^3 + (-\log 0.45)^3)^{1/3}}, \right.$
$\left.\left. e^{-((-\log 0.20)^3 + (-\log 0.50)^3)^{1/3}} \right] \right\rangle = ([0.75341, 0.80534], [0.14325, 0.19182]).$

(ii) $\tilde{\eth}_1 \otimes \tilde{\eth}_2 = \left\langle \left[e^{-((-\log 0.75)^3 + (-\log 0.35)^3)^{1/3}}, e^{-((-\log 0.80)^3 + (-\log 0.45)^3)^{1/3}} \right], \left[1 - \right.\right.$
$\left.\left. e^{-((-\log(1-0.15))^3 + (-\log(1-0.45))^3)^{1/3}}, 1 - e^{-((-\log(1-0.20))^3 + (-\log(1-0.50))^3)^{1/3}} \right] \right\rangle$
$= ([0.34751, 0.44741], [0.45218, 0.50380]).$

(iii) $2\tilde{\eth} = \left\langle \left[1 - e^{-(2(-\log(1-0.55))^3)^{1/3}}, 1 - e^{-(2(-\log(1-0.60))^3)^{1/3}}\right], \left[e^{-(2(-\log 0.35)^3)^{1/3}}, \right.\right.$
$\left.\left. e^{-(2(-\log 0.40)^3)^{1/3}} \right] \right\rangle = ([0.63434, 0.68477], [0.26642, 0.31523]).$

(iv) $\tilde{\eth}^2 = \left\langle \left[e^{-(2(-\log 0.55)^3)^{1/3}}, e^{-(2(-\log 0.60)^3)^{1/3}} \right], \left[1 - e^{-(2(-\log(1-0.35))^3)^{1/3}}, \right.\right.$
$\left.\left. 1 - e^{-(2(-\log(1-0.40))^3)^{1/3}} \right] \right\rangle = ([0.47084, 0.52540], [0.41885, 0.47460]).$

Theorem 1. *Let* $\tilde{\partial} = ([\beta^L_{\partial}, \beta^U_{\partial}], [\delta^L_{\partial}, \beta^U_{\partial}])$, $\tilde{\partial}_1 = ([\beta^L_{\partial_1}, \beta^U_{\partial_1}], [\delta^L_{\partial_1}, \delta^U_{\partial_1}])$, *and* $\tilde{\partial}_2 = ([\beta^L_{\partial_2}, \beta^U_{\partial_2}], [\delta^L_{\partial_2}, \delta^U_{\partial_2}])$ *be three IVIFNs, then we have*

(i) $\tilde{\partial}_1 \oplus \tilde{\partial}_2 = \tilde{\partial}_2 \oplus \tilde{\partial}_1$;
(ii) $\tilde{\partial}_1 \otimes \tilde{\partial}_2 = \tilde{\partial}_2 \otimes \tilde{\partial}_1$;
(iii) $\varphi(\tilde{\partial}_1 \oplus \tilde{\partial}_2) = \varphi\tilde{\partial}_1 \oplus \varphi\tilde{\partial}_2$, $\varphi > 0$;
(iv) $(\varphi_1 + \varphi_2)\tilde{\partial} = \varphi_1\tilde{\partial} \oplus \varphi_2\tilde{\partial}$, $\varphi_1, \varphi_2 > 0$;
(v) $(\tilde{\partial}_1 \otimes \tilde{\partial}_2)^{\varphi} = \tilde{\partial}_1^{\varphi} \otimes \tilde{\partial}_2^{\varphi}$, $\varphi > 0$;
(vi) $\tilde{\partial}^{\varphi_1} \otimes \tilde{\partial}^{\varphi_2} = \tilde{\partial}^{(\varphi_1+\varphi_2)}$, $\varphi_1, \varphi_2 > 0$.

Proof. For the three IVIFNs $\tilde{\partial}$, $\tilde{\partial}_1$ and $\tilde{\partial}_2$, $\eth \in [0, \infty]$, and $\varphi, \varphi_1, \varphi_2 > 0$, as stated in Proposition 1 and Definition 7, we can get

(i) $\tilde{\partial}_1 \oplus \tilde{\partial}_2 = \Big\langle \Big[1 - e^{-((-\log(1-\beta^L_{\partial_1}))^{\eth} + (-\log(1-\beta^L_{\partial_2}))^{\eth})^{1/\eth}}, 1 - e^{-((-\log(1-\beta^U_{\partial_1}))^{\eth} + (-\log(1-\beta^U_{\partial_2}))^{\eth})^{1/\eth}}\Big], \Big[e^{-((-\log\delta^L_{\partial_1})^{\eth} + (-\log\delta^L_{\partial_2})^{\eth})^{1/\eth}}, e^{-((-\log\delta^U_{\partial_1})^{\eth} + (-\log\delta^U_{\partial_2})^{\eth})^{1/\eth}}\Big]\Big\rangle = \Big\langle \Big[1 - e^{-((-\log(1-\beta^L_{\partial_2}))^{\eth} + (-\log(1-\beta^L_{\partial_1}))^{\eth})^{1/\eth}}, 1 - e^{-((-\log(1-\beta^U_{\partial_2}))^{\eth} + (-\log(1-\beta^U_{\partial_1}))^{\eth})^{1/\eth}}\Big], \Big[e^{-((-\log\delta^L_{\partial_2})^{\eth} + (-\log\delta^L_{\partial_1})^{\eth})^{1/\eth}}, e^{-((-\log\delta^U_{\partial_2})^{\eth} + (-\log\delta^U_{\partial_1})^{\eth})^{1/\eth}}\Big]\Big\rangle = \tilde{\partial}_2 \oplus \tilde{\partial}_1$.

(ii) It is simple.

(iii) Let $t = 1 - e^{-((-\log(1-\beta^L_{\partial_1}))^{\eth} + (-\log(1-\beta^L_{\partial_2}))^{\eth})^{1/\eth}}$.
Then, $\log(1-t) = -((-\log(1-\beta^L_{\partial_1}))^{\eth} + (-\log(1-\beta^L_{\partial_2}))^{\eth})^{1/\eth}$.
Using this, we get $\varphi(\tilde{\partial}_1 \oplus \tilde{\partial}_2) = \varphi\Big\langle \Big[1 - e^{-((-\log(1-\beta^L_{\partial_1}))^{\eth} + (-\log(1-\beta^L_{\partial_2}))^{\eth})^{1/\eth}}, 1 - e^{-((-\log(1-\beta^U_{\partial_1}))^{\eth} + (-\log(1-\beta^U_{\partial_2}))^{\eth})^{1/\eth}}\Big], \Big[e^{-((-\log\delta^L_{\partial_1})^{\eth} + (-\log\delta^L_{\partial_2})^{\eth})^{1/\eth}}, e^{-((-\log\delta^U_{\partial_1})^{\eth} + (-\log\delta^U_{\partial_2})^{\eth})^{1/\eth}}\Big]\Big\rangle = \Big\langle \Big[1 - e^{-(\varphi((-\log(1-\beta^L_{\partial_1}))^{\eth} + (-\log(1-\beta^L_{\partial_2}))^{\eth})^{1/\eth}}, 1 - e^{-(\varphi((-\log(1-\beta^U_{\partial_1}))^{\eth} + (-\log(1-\beta^U_{\partial_2}))^{\eth})^{1/\eth}}\Big], \Big[e^{-(\varphi((-\log\delta^L_{\partial_1})^{\eth} + (-\log\delta^L_{\partial_2})^{\eth}))^{1/\eth}}, e^{-(\varphi((-\log\delta^U_{\partial_1})^{\eth} + (-\log\delta^U_{\partial_2})^{\eth}))^{1/\eth}}\Big]\Big\rangle = \Big\langle \Big[1 - e^{-(\varphi(-\log(1-\beta^L_{\partial_1}))^{\eth})^{1/\eth}}, 1 - e^{-(\varphi(-\log(1-\beta^U_{\partial_1}))^{\eth})^{1/\eth}}\Big], \Big[e^{-(\varphi(-\log\delta^L_{\partial_1})^{\eth})^{1/\eth}}, e^{-(\varphi(-\log\delta^U_{\partial_1})^{\eth})^{1/\eth}}\Big]\Big\rangle \oplus \Big\langle \Big[1 - e^{-(\varphi(-\log(1-\beta^L_{\partial_2}))^{\eth})^{1/\eth}}, 1 - e^{-(\varphi(-\log(1-\beta^U_{\partial_2}))^{\eth})^{1/\eth}}\Big], \Big[e^{-(\varphi(-\log\delta^L_{\partial_2})^{\eth})^{1/\eth}}, e^{-(\varphi(-\log\delta^U_{\partial_2})^{\eth})^{1/\eth}}\Big]\Big\rangle = \varphi\tilde{\partial}_1 \oplus \varphi\tilde{\partial}_2$.

(iv) $\varphi_1\tilde{\partial} \oplus \varphi_2\tilde{\partial} = \Big\langle \Big[1 - e^{-(\varphi_1(-\log(1-\beta^L_{\partial}))^{\eth})^{1/\eth}}, 1 - e^{-(\varphi_1(-\log(1-\beta^U_{\partial}))^{\eth})^{1/\eth}}\Big], \Big[e^{-(\varphi_1(-\log\delta^L_{\partial})^{\eth})^{1/\eth}}, e^{-(\varphi_1(-\log\delta^U_{\partial})^{\eth})^{1/\eth}}\Big]\Big\rangle \oplus \Big\langle \Big[1 - e^{-(\varphi_2(-\log(1-\beta^L_{\partial}))^{\eth})^{1/\eth}}, 1 - e^{-(\varphi_2(-\log(1-\beta^U_{\partial}))^{\eth})^{1/\eth}}\Big], \Big[e^{-(\varphi_2(-\log\delta^L_{\partial})^{\eth})^{1/\eth}}, e^{-(\varphi_2(-\log\delta^U_{\partial})^{\eth})^{1/\eth}}\Big]\Big\rangle$
$= \Big\langle \Big[1 - e^{-((\varphi_1+\varphi_2)(-\log(1-\beta^L_{\partial}))^{\eth})^{1/\eth}}, 1 - e^{-((\varphi_1+\varphi_2)(-\log(1-\beta^U_{\partial}))^{\eth})^{1/\eth}}\Big], \Big[e^{-((\varphi_1+\varphi_2)(-\log\delta^L_{\partial})^{\eth})^{1/\eth}}, e^{-((\varphi_1+\varphi_2)(-\log\delta^U_{\partial})^{\eth})^{1/\eth}}\Big]\Big\rangle = (\varphi_1 + \varphi_2)\tilde{\partial}$.

(v) $(\tilde{\partial}_1 \otimes \tilde{\partial}_2)^{\varphi} = \Big\langle \Big[e^{-((-\log\beta^L_{\partial_1})^{\eth} + (-\log\beta^L_{\partial_2})^{\eth})^{1/\eth}}, e^{-((-\log\beta^U_{\partial_1})^{\eth} + (-\log\beta^U_{\partial_2})^{\eth})^{1/\eth}}\Big], \Big[1 - e^{-((-\log(1-\delta^L_{\partial_1}))^{\eth} + (-\log(1-\delta^L_{\partial_2}))^{\eth})^{1/\eth}}, 1 - e^{-((-\log(1-\delta^U_{\partial_1}))^{\eth} + (-\log(1-\delta^U_{\partial_2}))^{\eth})^{1/\eth}}\Big]\Big\rangle^{\varphi} = \Big\langle \Big[e^{-(\varphi((-\log\beta^L_{\partial_1})^{\eth} + (-\log\beta^L_{\partial_2})^{\eth}))^{1/\eth}}, e^{-(\varphi((-\log\beta^U_{\partial_1})^{\eth} + (-\log\beta^U_{\partial_2})^{\eth}))^{1/\eth}}\Big], \Big[1 - e^{-(\varphi((-\log(1-\delta^L_{\partial_1}))^{\eth} + (-\log(1-\delta^L_{\partial_2}))^{\eth})^{1/\eth}}, 1 - e^{-(\varphi((-\log(1-\delta^U_{\partial_1}))^{\eth} + (-\log(1-\delta^U_{\partial_2}))^{\eth})^{1/\eth}}\Big]\Big\rangle = \Big\langle \Big[e^{-(\varphi(-\log\beta^L_{\partial_1})^{\eth})^{1/\eth}},$

$$e^{-(\varphi(-\log\beta^U_{\eth_1})^{\eth})^{1/\eth}}\Big],\Big[1-e^{-(\varphi(-\log(1-\delta^L_{\eth_1}))^{\eth})^{1/\eth}},1-e^{-(\varphi(-\log(1-\delta^U_{\eth_1}))^{\eth})^{1/\eth}}\Big]\Big\rangle$$
$$\oplus\Big\langle\Big[e^{-(\varphi(-\log\beta^L_{\eth_2})^{\eth})^{1/\eth}},e^{-(\varphi(-\log\beta^U_{\eth_2})^{\eth})^{1/\eth}}\Big],\Big[1-e^{-(\varphi(-\log(1-\delta^L_{\eth_2}))^{\eth})^{1/\eth}},$$
$$1-e^{-(\varphi(-\log(1-\delta^U_{\eth_2}))^{\eth})^{1/\eth}}\Big]\Big\rangle=\tilde{\eth}_1^{\varphi}\otimes\tilde{\eth}_2^{\varphi}.$$

(vi) $$\tilde{\eth}^{\varphi_1}\otimes\tilde{\eth}^{\varphi_2}=\Big\langle\Big[e^{-(\varphi_1(-\log\beta^L_{\eth})^{\eth})^{1/\eth}},e^{-(\varphi_1(-\log\beta^U_{\eth})^{\eth})^{1/\eth}}\Big],\Big[1-$$
$$e^{-(\varphi_1(-\log(1-\delta^L_{\eth}))^{\eth})^{1/\eth}},1-e^{-(\varphi_1(-\log(1-\delta^U_{\eth}))^{\eth})^{1/\eth}}\Big]\Big\rangle\otimes\Big\langle\Big[e^{-(\varphi_2(-\log\beta^L_{\eth})^{\eth})^{1/\eth}},$$
$$e^{-(\varphi_2(-\log\beta^U_{\eth})^{\eth})^{1/\eth}}\Big],\Big[1-e^{-(\varphi_2(-\log(1-\delta^L_{\eth}))^{\eth})^{1/\eth}},1-e^{-(\varphi_2(-\log(1-\delta^U_{\eth}))^{\eth})^{1/\eth}}\Big]\Big\rangle$$
$$=\Big\langle\Big[e^{-((\varphi_1+\varphi_2)(-\log\beta^L_{\eth})^{\eth})^{1/\eth}},e^{-((\varphi_1+\varphi_2)(-\log\beta^U_{\eth})^{\eth})^{1/\eth}}\Big],\Big[1-$$
$$e^{-((\varphi_1+\varphi_2)(-\log(1-\delta^L_{\eth}))^{\eth})^{1/\eth}},1-e^{-((\varphi_1+\varphi_2)(-\log(1-\delta^U_{\eth}))^{\eth})^{1/\eth}}\Big]\Big\rangle=\tilde{\eth}^{(\varphi_1+\varphi_2)}.$$

□

4. IVIF Aczel–Alsina Geometric Aggregation Operators

We demonstrate some IVIF geometric aggregation operators throughout this section using the Aczel–Alsina operations.

Definition 8. *Let* $\tilde{\eth}_\zeta=([\beta^L_{\eth_\zeta},\beta^U_{\eth_\zeta}],[\delta^L_{\eth_\zeta},\delta^U_{\eth_\zeta}])$ $(\zeta=1,2,\ldots,\hbar)$ *be an accumulation of IVIFNs and* $\xi=(\xi_1,\xi_2,\ldots,\xi_\hbar)^T$ *be the weight vector associated with* $\eth_\zeta$ $(\zeta=1,2,\ldots,\hbar)$*, along with* $\xi_\zeta\in[0,1]$ *and* $\sum_{\zeta=1}^{\hbar}\xi_\zeta=1$*. In that case an IVIF Aczel–Alsina weighted geometric (IVIFAAWG) operator can be described as function* $IVIFAAWG:(L^\star)^\hbar\to L^\star$*, in which*

$$IVIFAAWG_\xi(\tilde{\eth}_1,\tilde{\eth}_2,\ldots,\tilde{\eth}_\hbar)=\bigotimes_{\zeta=1}^{\hbar}(\tilde{\eth}_\zeta)^{\xi_\zeta}=(\tilde{\eth}_1)^{\xi_1}\bigotimes(\tilde{\eth}_2)^{\xi_2}\bigotimes\cdots\bigotimes(\tilde{\eth}_\hbar)^{\xi_\hbar}.$$

Following that, we prove the associated theorem for the Aczel–Alsina operations on IVIFNs.

Theorem 2. *Let* $\tilde{\eth}_\zeta=([\beta^L_{\eth_\zeta},\beta^U_{\eth_\zeta}],[\delta^L_{\eth_\zeta},\delta^U_{\eth_\zeta}])$ $(\zeta=1,2,\ldots,\hbar)$ *be an accumulation of IVIFNs and* $\eth\in[0,\infty]$*, then aggregated value of them utilizing the IVIFAAWG operator is also a IVIFNs, and*

$$\begin{aligned}IVIFAAWG_\xi(\tilde{\eth}_1,\tilde{\eth}_2,\ldots,\tilde{\eth}_\hbar)&=\bigotimes_{\zeta=1}^{\hbar}(\tilde{\eth}_\zeta)^{\xi_\zeta}\\&=\left\langle\left[e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta^L_{\eth_\zeta})^{\eth}\right)^{1/\eth}},e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta^U_{\eth_\zeta})^{\eth}\right)^{1/\eth}}\right],\right.\\&\left.\left[1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta^L_{\eth_\zeta}))^{\eth}\right)^{1/\eth}},1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta^U_{\eth_\zeta}))^{\eth}\right)^{1/\eth}}\right]\right\rangle\end{aligned}\tag{6}$$

where $\xi=(\xi_1,\xi_2,\ldots,\xi_\hbar)$ *function as weight vector associated with* $\tilde{\eth}_\zeta$ $(\zeta=1,2,\ldots,\hbar)$ *so that* $\xi_\zeta\in[0,1]$*, and* $\sum_{\zeta=1}^{\hbar}\xi_\zeta=1$.

Proof. We may prove Theorem 2 using the following mathematical induction method: (i) When $\hbar = 2$, rely upon Aczel–Alsina operations of IVIFNs, we acquire

$$
\begin{aligned}
(\tilde{\eth}_1)^{\xi_1} &= \Big\langle \Big[e^{-(\xi_1(-\log \beta^L_{\eth_1})^{\eth})^{1/\eth}}, e^{-(\xi_1(-\log \beta^U_{\eth_1})^{\eth})^{1/\eth}} \Big], \\
&\quad \Big[1 - e^{-(\xi_1(-\log(1-\delta^L_{\eth_1}))^{\eth})^{1/\eth}}, 1 - e^{-(\xi_1(-\log(1-\delta^U_{\eth_1}))^{\eth})^{1/\eth}} \Big\rangle, \\
(\tilde{\eth}_2)^{\xi_2} &= \Big\langle \Big[e^{-(\xi_2(-\log \beta^L_{\eth_2})^{\eth})^{1/\eth}}, e^{-(\xi_2(-\log \beta^U_{\eth_2})^{\eth})^{1/\eth}} \Big], \\
&\quad \Big[1 - e^{-(\xi_2(-\log(1-\delta^L_{\eth_2}))^{\eth})^{1/\eth}}, 1 - e^{-(\xi_2(-\log(1-\delta^U_{\eth_2}))^{\eth})^{1/\eth}} \Big\rangle.
\end{aligned}
$$

Depending on Definition 7 and Proposition 1, we get
$IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2) = (\tilde{\eth}_1)^{\xi_1} \otimes (\tilde{\eth}_2)^{\xi_2} = \Big\langle \Big[e^{-(\xi_1(-\log \beta^L_{\eth_1})^{\eth})^{1/\eth}}, e^{-(\xi_1(-\log \beta^U_{\eth_1})^{\eth})^{1/\eth}} \Big],$
$\Big[1 - e^{-(\xi_1(-\log(1-\delta^L_{\eth_1}))^{\eth})^{1/\eth}}, 1 - e^{-(\xi_1(-\log(1-\delta^U_{\eth_1}))^{\eth})^{1/\eth}} \Big\rangle \otimes \Big\langle \Big[e^{-(\xi_2(-\log \beta^L_{\eth_2})^{\eth})^{1/\eth}},$
$e^{-(\xi_2(-\log \beta^U_{\eth_2})^{\eth})^{1/\eth}} \Big], \Big[1 - e^{-(\xi_2(-\log(1-\delta^L_{\eth_2}))^{\eth})^{1/\eth}}, 1 - e^{-(\xi_2(-\log(1-\delta^U_{\eth_2}))^{\eth})^{1/\eth}} \Big\rangle$
$= \Bigg\langle \Bigg[e^{-\left(\xi_1(-\log \beta^L_{\eth_1})^{\eth} + \xi_2(-\log \beta^L_{\eth_2})^{\eth}\right)^{1/\eth}}, e^{-\left(\xi_1(-\log \beta^U_{\eth_1})^{\eth} + \xi_2(-\log \beta^U_{\eth_2})^{\eth}\right)^{1/\eth}} \Bigg], \Bigg[1 -$
$e^{-\left(\xi_1(-\log(1-\delta^L_{\eth_1}))^{\eth} + \xi_2(-\log(1-\delta^L_{\eth_2}))^{\eth}\right)^{1/\eth}}, 1 - e^{-\left(\xi_1(-\log(1-\delta^U_{\eth_1}))^{\eth} + \xi_2(-\log(1-\delta^U_{\eth_2}))^{\eth}\right)^{1/\eth}} \Bigg] \Bigg\rangle$
$= \Bigg\langle \Bigg[e^{-\left(\sum_{\zeta=1}^{2} \xi_\zeta(-\log \beta^L_{\eth_\zeta})^{\eth}\right)^{1/\eth}}, e^{-\left(\sum_{\zeta=1}^{2} \xi_\zeta(-\log \beta^U_{\eth_\zeta})^{\eth}\right)^{1/\eth}} \Bigg], \Bigg[1 - e^{-\left(\sum_{\zeta=1}^{2} \xi_\zeta(-\log(1-\delta^L_{\eth_\zeta}))^{\eth}\right)^{1/\eth}},$
$1 - e^{-\left(\sum_{\zeta=1}^{2} \xi_\zeta(-\log(1-\delta^U_{\eth_\zeta}))^{\eth}\right)^{1/\eth}} \Bigg] \Bigg\rangle$. Hence, (6) is true for $\hbar = 2$.
(ii) Assume that (6) is true for $\hbar = k$, then we have

$$
\begin{aligned}
IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_k) &= \bigotimes_{\zeta=1}^{k} (\tilde{\eth}_\zeta)^{\xi_\zeta} \\
&= \Bigg\langle \Bigg[e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log \beta^L_{\eth_\zeta})^{\eth}\right)^{1/\eth}}, e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log \beta^U_{\eth_\zeta})^{\eth}\right)^{1/\eth}} \Bigg], \\
&\quad \Bigg[1 - e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log(1-\delta^L_{\eth_\zeta}))^{\eth}\right)^{1/\eth}}, 1 - e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log(1-\delta^U_{\eth_\zeta}))^{\eth}\right)^{1/\eth}} \Bigg] \Bigg\rangle.
\end{aligned}
$$

Now for $\hbar = k + 1$, then
$IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_k, \tilde{\eth}_{k+1}) = \bigotimes_{\zeta=1}^{k} (\tilde{\eth}_\zeta)^{\xi_\zeta} \otimes (\tilde{\eth}_{k+1})^{\xi_{k+1}}$
$= \Bigg\langle \Bigg[e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log \beta^L_{\eth_\zeta})^{\eth}\right)^{1/\eth}}, e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log \beta^U_{\eth_\zeta})^{\eth}\right)^{1/\eth}} \Bigg],$
$\Bigg[1 - e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log(1-\delta^L_{\eth_\zeta}))^{\eth}\right)^{1/\eth}}, 1 - e^{-\left(\sum_{\zeta=1}^{k} \xi_\zeta(-\log(1-\delta^U_{\eth_\zeta}))^{\eth}\right)^{1/\eth}} \Bigg] \Bigg\rangle$
$\otimes \Bigg\langle \Bigg[e^{-\left(\xi_{k+1}(-\log \beta^L_{\eth_{k+1}})^{\eth}\right)^{1/\eth}}, e^{-\left(\xi_{k+1}(-\log \beta^U_{\eth_{k+1}})^{\eth}\right)^{1/\eth}} \Bigg],$

$$\left[1-e^{-\left(\xi_{k+1}(-\log(1-\delta^{L}_{\eth_{k+1}}))^{\eth}\right)^{1/\eth}}, 1-e^{-\left(\xi_{k+1}(-\log(1-\delta^{U}_{\eth_{k+1}}))^{\eth}\right)^{1/\eth}}\right]\Bigg\rangle$$

$$=\left\langle\left[e^{-\left(\sum_{\zeta=1}^{k+1}\xi_{\zeta}(-\log\beta^{L}_{\eth_{\zeta}})^{\eth}\right)^{1/\eth}}, e^{-\left(\sum_{\zeta=1}^{k+1}\xi_{\zeta}(-\log\beta^{U}_{\eth_{\zeta}})^{\eth}\right)^{1/\eth}}\right],\right.$$

$$\left.\left[1-e^{-\left(\sum_{\zeta=1}^{k+1}\xi_{\zeta}(-\log(1-\delta^{L}_{\eth_{\zeta}}))^{\eth}\right)^{1/\eth}}, 1-e^{-\left(\sum_{\zeta=1}^{k+1}\xi_{\zeta}(-\log(1-\delta^{U}_{\eth_{\zeta}}))^{\eth}\right)^{1/\eth}}\right]\right\rangle.$$

Thus, (6) is true for $\hbar = k+1$.

Therefore, from (i) and (ii), we may conclude that (6) holds for any $\hbar$. □

Theorem 3. *(Idempotency) If all $\tilde{\eth}_{\zeta} = ([\beta^{L}_{\eth_{\zeta}}, \beta^{U}_{\eth_{\zeta}}], [\delta^{L}_{\eth_{\zeta}}, \delta^{U}_{\eth_{\zeta}}])$ $(\zeta = 1, 2, \ldots, \hbar)$ are equal, i.e., $\tilde{\eth}_{\zeta} = \tilde{\eth}$ for all ζ, then $IVIFAAWG_{\xi}$ $(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = \tilde{\eth}$.*

Proof. Since $\tilde{\eth}_{\zeta} = ([\beta^{L}_{\eth_{\zeta}}, \beta^{U}_{\eth_{\zeta}}], [\delta^{L}_{\eth_{\zeta}}, \delta^{U}_{\eth_{\zeta}}])$ $(\zeta = 1, 2, \ldots, \hbar)$, then we have by Equation (6),

$$IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = \bigotimes_{\zeta=1}^{\hbar}(\tilde{\eth}_{\zeta})^{\xi_{\zeta}} = \left\langle\left[e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_{\zeta}(-\log\beta^{L}_{\eth_{\zeta}})^{\eth}\right)^{1/\eth}},\right.\right.$$

$$\left.\left. e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_{\zeta}(-\log\beta^{U}_{\eth_{\zeta}})^{\eth}\right)^{1/\eth}}\right], \left[1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_{\zeta}(-\log(1-\delta^{L}_{\eth_{\zeta}}))^{\eth}\right)^{1/\eth}}, 1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_{\zeta}(-\log(1-\delta^{U}_{\eth_{\zeta}}))^{\eth}\right)^{1/\eth}}\right]\right\rangle$$

$$=\left\langle\left[e^{-\left((-\log\beta^{L}_{\eth})^{\eth}\right)^{1/\eth}}, e^{-\left((-\log\beta^{U}_{\eth})^{\eth}\right)^{1/\eth}}\right], \left[1-e^{-\left((-\log(1-\delta^{L}_{\eth}))^{\eth}\right)^{1/\eth}}, 1-\right.\right.$$

$$\left.\left. e^{-\left((-\log(1-\delta^{U}_{\eth}))^{\eth}\right)^{1/\eth}}\right]\right\rangle = \left\langle\left[e^{\log\beta^{L}_{\eth}}, e^{\log\beta^{U}_{\eth}}\right], \left[1-e^{\log(1-\delta^{L}_{\eth})}, 1-e^{\log(1-\delta^{U}_{\eth})}\right]\right\rangle$$

$= ([\beta^{L}_{\eth}, \beta^{U}_{\eth}], [\delta^{L}_{\eth}, \delta^{U}_{\eth}]) = \tilde{\eth}$. Thus, $IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = \tilde{\eth}$ holds. □

Theorem 4. *(Boundedness) Let $\tilde{\eth}_{\zeta} = ([\beta^{L}_{\eth_{\zeta}}, \beta^{U}_{\eth_{\zeta}}], [\delta^{L}_{\eth_{\zeta}}, \delta^{U}_{\eth_{\zeta}}])$ $(\zeta = 1, 2, \ldots, \hbar)$ be an accumulation of IVIFNs. Let $\tilde{\eth}^{-} = \min(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar})$ and $\tilde{\eth}^{+} = \max(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar})$. Then, $\tilde{\eth}^{-} \leq IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) \leq \tilde{\eth}^{+}$.*

Proof. Let $\tilde{\eth}_{\zeta} = ([\beta^{L}_{\eth_{\zeta}}, \beta^{U}_{\eth_{\zeta}}], [\delta^{L}_{\eth_{\zeta}}, \delta^{U}_{\eth_{\zeta}}])$ $(\zeta = 1, 2, \ldots, \hbar)$ be an accumulation of IVIFNs. Let $\tilde{\eth}^{-} = \min(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = ([\beta^{L-}_{\eth}, \beta^{U-}_{\eth}], [\delta^{L-}_{\eth}, \delta^{U-}_{\eth}])$ and $\tilde{\eth}^{+} = \max(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = ([\beta^{L+}_{\eth}, \beta^{U+}_{\eth}], [\delta^{L+}_{\eth}, \delta^{U+}_{\eth}])$. We have, $\beta^{L-}_{\eth} = \min_{\zeta}\{\beta^{L}_{\eth_{\zeta}}\}$, $\beta^{U-}_{\eth} = \min_{\zeta}\{\beta^{U}_{\eth_{\zeta}}\}$, $\delta^{L-}_{\eth} = \max_{\zeta}\{\delta^{L}_{\eth_{\zeta}}\}$, $\delta^{U-}_{\eth} = \max_{\zeta}\{\delta^{U}_{\eth_{\zeta}}\}$, $\beta^{L+}_{\eth} = \max_{\zeta}\{\beta^{L}_{\eth_{\zeta}}\}$, $\beta^{U+}_{\eth} = \max_{\zeta}\{\beta^{U}_{\eth_{\zeta}}\}$, $\delta^{L+}_{\eth} = \min_{\zeta}\{\delta^{L}_{\eth_{\zeta}}\}$, and $\delta^{U+}_{\eth} = \min_{\zeta}\{\delta^{U}_{\eth_{\zeta}}\}$. Hence, there have the subsequent inequalities,

$$e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta_{\eth}^{L-})^{\eth}\right)^{1/\eth}} \le e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta_{\eth_\zeta}^{L})^{\eth}\right)^{1/\eth}} \le e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta_{\eth}^{L+})^{\eth}\right)^{1/\eth}},$$

$$e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta_{\eth}^{U-})^{\eth}\right)^{1/\eth}} \le e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta_{\eth_\zeta}^{U})^{\eth}\right)^{1/\eth}} \le e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log\beta_{\eth}^{U+})^{\eth}\right)^{1/\eth}},$$

$$1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta_{\eth}^{L+}))^{\eth}\right)^{1/\eth}} \le 1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta_{\eth_\zeta}^{L}))^{\eth}\right)^{1/\eth}} \le 1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta_{\eth}^{L-}))^{\eth}\right)^{1/\eth}},$$

$$1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta_{\eth}^{U+}))^{\eth}\right)^{1/\eth}} \le 1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta_{\eth_\zeta}^{U}))^{\eth}\right)^{1/\eth}} \le 1-e^{-\left(\sum_{\zeta=1}^{\hbar}\xi_\zeta(-\log(1-\delta_{\eth}^{U-}))^{\eth}\right)^{1/\eth}}.$$

Therefore, $\tilde{\eth}^{-} \le IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) \le \tilde{\eth}^{+}$. □

Theorem 5. *(Monotonicity) Let $\tilde{\eth}_\zeta$ and $\tilde{\eth}'_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ be two sets of IVIFNs, if $\tilde{\eth}_\zeta \le \tilde{\eth}'_\zeta$ for all ζ, then $IVIFAAWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) \le IVIFAA\text{-}WG_{\xi}(\tilde{\eth}'_1, \tilde{\eth}'_2, \ldots, \tilde{\eth}'_{\hbar})$.*

Proof. The proof is straightforward. □

Now, we present IVIF Aczel–Alsina ordered weighted geometric (IVIFAAOWG) operator.

Definition 9. *Let $\tilde{\eth}_\zeta = ([\beta^L_{\eth_\zeta}, \beta^U_{\eth_\zeta}], [\delta^L_{\eth_\zeta}, \delta^U_{\eth_\zeta}])$ $(\zeta = 1, 2, \ldots, \hbar)$ be an accumulation of IVIFNs. An IVIF Aczel–Alsina ordered weighted geometric (IVIFAAOWG) operator of dimension $\hbar$ is a mapping $IVIFAAOWG : (L^\star)^{\hbar} \to L^\star$ with the corresponding vector $\xi = (\xi_1, \xi_2, \ldots, \xi_{\hbar})^T$ such that $\xi_\zeta \in [0,1]$, and $\sum_{\zeta=1}^{\hbar} \xi_\zeta = 1$, as*

$$\begin{aligned} IVIFAAOWG_{\xi}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) &= \bigotimes_{\zeta=1}^{\hbar} (\tilde{\eth}_{\varrho(\zeta)})^{\xi_\zeta} \\ &= (\tilde{\eth}_{\varrho(1)})^{\xi_1} \bigotimes (\tilde{\eth}_{\varrho(2)})^{\xi_2} \bigotimes \cdots \bigotimes (\tilde{\eth}_{\varrho(\hbar)})^{\xi_{\hbar}}, \end{aligned}$$

where $(\varrho(1), \varrho(2), \ldots, \varrho(\hbar))$ are the permutation of $(\zeta = 1, 2, \ldots, \hbar)$, for which $\tilde{\eth}_{\varrho(\zeta-1)} \ge \tilde{\eth}_{\varrho(\zeta)}$ for all $\zeta = 1, 2, \ldots, \hbar$.

We generate the following theorem on IVIFNs based on the Aczel–Alsina product.

Theorem 6. *Let $\tilde{\eth}_\zeta = ([\beta^L_{\eth_\zeta}, \beta^U_{\eth_\zeta}], [\delta^L_{\eth_\zeta}, \delta^U_{\eth_\zeta}])$ $(\zeta = 1, 2, \ldots, \hbar)$ be an accumulation of IVIFNs. An IVIF Aczel–Alsina ordered weighted geometric (IVIFAAOWG) operator of dimension $\hbar$ is a mapping $IVIFAAOWG : (L^\star)^{\hbar} \to L^\star$ with the associated vector $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_{\hbar})^T$ such that $\vartheta_\zeta \in [0,1]$, and $\sum_{\zeta=1}^{\hbar} \vartheta_\zeta = 1$. Then,*

$$IVIFAAOWG_{\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = \bigotimes_{\zeta=1}^{\hbar} (\tilde{\eth}_{\varrho(\zeta)})^{\vartheta_\zeta}$$
$$= \left\langle \left[e^{-\left(\sum_{\zeta=1}^{\hbar} \vartheta_\zeta \left(-\log \beta^L_{\eth_{\varrho(\zeta)}}\right)^{\eth}\right)^{1/\eth}}, e^{-\left(\sum_{\zeta=1}^{\hbar} \vartheta_\zeta \left(-\log \beta^U_{\eth_{\varrho(\zeta)}}\right)^{\eth}\right)^{1/\eth}} \right], \left[1 - e^{-\left(\sum_{\zeta=1}^{\hbar} \vartheta_\zeta \left(-\log \left(1-\delta^L_{\eth_{\varrho(\zeta)}}\right)\right)^{\eth}\right)^{1/\eth}}, 1 - e^{-\left(\sum_{\zeta=1}^{\hbar} \vartheta_\zeta \left(-\log \left(1-\delta^U_{\eth_{\varrho(\zeta)}}\right)\right)^{\eth}\right)^{1/\eth}} \right] \right\rangle$$

where $(\varrho(1), \varrho(2), \ldots, \varrho(\hbar))$ are the permutation of $(\zeta = 1, 2, \ldots, \hbar)$, for which $\tilde{\eth}_{\varrho(\zeta-1)} \geq \tilde{\eth}_{\varrho(\zeta)}$ for all $\zeta = 1, 2, \ldots, \hbar$.

Proof. Like Theorem 2, Theorem 6 is simply obtained. □

The following characteristics can be proven well by employing the IVIFAAOWG operator.

Property 1. *(Idempotency) If $\tilde{\eth}_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ are identical, i.e., $\tilde{\eth}_\zeta = \tilde{\eth}$ for every ζ, then $IVIFAAOWG_{\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = \tilde{\eth}$.*

Property 2. *(Boundedness) Let $\tilde{\eth}_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ be an accumulation of IVIFNs. Let $\tilde{\eth}^- = \min_s \tilde{\eth}_\zeta$, $\tilde{\eth}^+ = \max_s \tilde{\eth}_\zeta$. Then, $\tilde{\eth}^- \leq IVIFAAOWG_{\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) \leq \tilde{\eth}^+$.*

Property 3. *(Monotonicity) Suppose that $\tilde{\eth}_\zeta$ and $\tilde{\eth}'_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ are two sets of IVIFNs and $\tilde{\eth}_\zeta \leq \tilde{\eth}'_\zeta$ for every ζ, then $IVIFAAOWG_{\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) \leq IVIFAAOWG_{\vartheta}(\tilde{\eth}'_1, \tilde{\eth}'_2, \ldots, \tilde{\eth}'_{\hbar})$.*

Property 4. *(Commutativity) Let $\tilde{\eth}_\zeta$ and $\tilde{\eth}'_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ be two sets of IVIFNs, then $IVIFAAOWG_{\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) = IVIFAAOWG_{\vartheta}(\tilde{\eth}'_1, \tilde{\eth}'_2, \ldots, \tilde{\eth}'_{\hbar})$ where $\tilde{\eth}'_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ is any permutation of $\tilde{\eth}_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$.*

As defined in Definition 8, the IVIFAAWG operator measures only the IVIFNs, and as defined in Definition 9, the IVIFAAOWG operator measures only the IVIFNs' consistent positions. Following that, weights represent different aspects of both the IVIFAAWG and IVIFAAOWG operators. Nevertheless, both the operators think about just one of them. To overcome this disadvantage, in the following we will exhibit IVIF Aczel–Alsina hybrid geometric (IVIFAAHG) operator, which weights both the given IVIFN and its ordered position.

Definition 10. *Let $\tilde{\eth}_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ be an accumulation of IVIFNs. An IVIFAAHG operator of dimension $\hbar$ is a function $IVIFAAHG : (L^\star)^{\hbar} \to L^\star$, such that*

$$\begin{aligned} IVIFAAHG_{\xi,\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_{\hbar}) &= \bigotimes_{\zeta=1}^{\hbar} (\dot{\tilde{\eth}}_{\varrho(\zeta)})^{\vartheta_\zeta} \\ &= (\dot{\tilde{\eth}}_{\varrho(1)})^{\vartheta_1} \bigotimes (\dot{\tilde{\eth}}_{\varrho(2)})^{\vartheta_2} \bigotimes \cdots \bigotimes (\dot{\tilde{\eth}}_{\varrho(\hbar)})^{\vartheta_{\hbar}} \end{aligned}$$

where $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_{\hbar})^T$ is the weighting vector associated with the IVIFAAHG operator, with $\vartheta_\zeta \in [0,1]$ $(\zeta = 1, 2, \ldots, \hbar)$ and $\sum_{\zeta=1}^{\hbar} \vartheta_\zeta = 1$; $\dot{\tilde{\eth}}_\zeta = \tilde{\eth}_\zeta^{\hbar \xi_\zeta}$, $\zeta = 1, 2, \ldots, \hbar$, $(\dot{\tilde{\eth}}_{\varrho(1)}, \dot{\tilde{\eth}}_{\varrho(2)}, \ldots, \dot{\tilde{\eth}}_{\varrho(\hbar)})$ is any permutation of a collection of the weighted IVIFNs $(\dot{\tilde{\eth}}_1, \dot{\tilde{\eth}}_2, \ldots, \dot{\tilde{\eth}}_{\hbar})$, such that $\dot{\tilde{\eth}}_{\varrho(\zeta-1)} \geq \dot{\tilde{\eth}}_{\varrho(\zeta)}$ $(\zeta = 1, 2, \ldots, \hbar)$; $\xi = (\xi_1, \xi_2, \ldots, \xi_{\hbar})^T$ is the weight vector of $\tilde{\eth}_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$, with $\xi_\zeta \in [0,1]$ $(\zeta = 1, 2, \ldots, \hbar)$ and $\sum_{\zeta=1}^{\hbar} \xi_\zeta = 1$, and $\hbar$ is the balancing coefficient, which plays a role of balance.

The following theorem can be deduced using Aczel–Alsina operations on IVIFNs information.

Theorem 7. *Let* $\tilde{\eth}_\zeta = ([\beta^L_{\eth_\zeta}, \beta^U_{\eth_\zeta}], [\delta^L_{\eth_\zeta}, \delta^U_{\eth_\zeta}])$ $(\zeta = 1, 2, \ldots, \hbar)$ *be an accumulation of IVIFNs and* $\eth \in [0, \infty]$. *Their aggregated value by IVIFAAHG operator is still a IVIFN, and*

$$IVIFAAHG_\xi(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_\hbar) = \bigotimes_{\zeta=1}^{\hbar} (\dot{\tilde{\eth}}_{\varrho(\zeta)})^{\xi_\zeta} =$$

$$\left\langle \left[e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta \left(-\log \dot{\beta}^L_{\eth_{\varrho(\zeta)}}\right)^{\eth}\right)^{1/\eth}}, e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta \left(-\log \dot{\beta}^U_{\eth_{\varrho(\zeta)}}\right)^{\eth}\right)^{1/\eth}} \right], \left[1 - e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta \left(-\log \left(1-\dot{\delta}^L_{\eth_{\varrho(\zeta)}}\right)\right)^{\eth}\right)^{1/\eth}}, 1 - e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta \left(-\log \left(1-\dot{\delta}^U_{\eth_{\varrho(\zeta)}}\right)\right)^{\eth}\right)^{1/\eth}} \right] \right\rangle$$

where $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_\hbar)^T$ *is the weighting vector associated with the IVIFAAHG operator, with* $\vartheta_\zeta \in [0, 1]$ $(\zeta = 1, 2, \ldots, \hbar)$ *and* $\sum_{\zeta=1}^{\hbar} \vartheta_\zeta = 1$; $\dot{\tilde{\eth}}_\zeta = \tilde{\eth}_\zeta^{\hbar\xi_\zeta}$, $\zeta = 1, 2, \ldots, \hbar$, $(\dot{\tilde{\eth}}_{\varrho(1)}, \dot{\tilde{\eth}}_{\varrho(2)}, \ldots, \dot{\tilde{\eth}}_{\varrho(\hbar)})$ *is any permutation of a collection of the weighted IVIFNs* $(\dot{\tilde{\eth}}_1, \dot{\tilde{\eth}}_2, \ldots, \dot{\tilde{\eth}}_\hbar)$, *such that* $\dot{\tilde{\eth}}_{\varrho(\zeta-1)} \geq \dot{\tilde{\eth}}_{\varrho(\zeta)}$ $(\zeta = 1, 2, \ldots, \hbar)$; $\xi = (\xi_1, \xi_2, \ldots, \xi_\hbar)^T$ *is the weight vector of* $\tilde{\eth}_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$, *with* $\xi_\zeta \in [0, 1]$ $(\zeta = 1, 2, \ldots, \hbar)$ *and* $\sum_{\zeta=1}^{\hbar} \xi_\zeta = 1$, *and* $\hbar$ *is the balancing coefficient, which plays a role of balance.*

Proof. Like Theorem 2, Theorem 7 is simply obtained. □

Theorem 8. *The IVIFAAWG and IVIFAAOWG operators are both variants of the IVIFAAHG operator.*

Proof. (1) Assume $\vartheta = (1/\hbar, 1/\hbar, \ldots, 1/\hbar)^T$. Then,

$$\begin{aligned} IVIFAAHG_{\xi,\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_\hbar) &= (\dot{\tilde{\eth}}_{\varrho(1)})^{\vartheta_1} \bigotimes (\dot{\tilde{\eth}}_{\varrho(2)})^{\vartheta_2} \bigotimes \cdots \bigotimes (\dot{\tilde{\eth}}_{\varrho(\hbar)})^{\vartheta_\hbar} \\ &= (\dot{\tilde{\eth}}_{\varrho(1)})^{(1/\hbar)} \bigotimes (\dot{\tilde{\eth}}_{\varrho(2)})^{(1/\hbar)} \bigotimes \cdots \bigotimes (\dot{\tilde{\eth}}_{\varrho(\hbar)})^{(1/\hbar)} \\ &= (\tilde{\eth}_1)^{\xi_1} \bigotimes (\tilde{\eth}_2)^{\xi_2} \bigotimes \cdots \bigotimes (\tilde{\eth}_\hbar)^{\xi_\hbar} \\ &= IVIFAAWG_\xi(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_\hbar), \end{aligned}$$

(2) Let $\xi = (1/\hbar, 1/\hbar, \ldots, 1/\hbar)^T$. Then, $\dot{\tilde{\eth}}_\zeta = \tilde{\eth}_\zeta$ $(\zeta = 1, 2, \ldots, \hbar)$ and

$$\begin{aligned} IVIFAAHG_{\xi,\vartheta}(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_\hbar) &= (\dot{\tilde{\eth}}_{\varrho(1)})^{\vartheta_1} \bigotimes (\dot{\tilde{\eth}}_{\varrho(2)})^{\vartheta_2} \bigotimes \cdots \bigotimes (\dot{\tilde{\eth}}_{\varrho(\hbar)})^{\vartheta_\hbar} \\ &= (\tilde{\eth}_{\varrho(1)})^{\vartheta_1} \bigotimes (\tilde{\eth}_{\varrho(2)})^{\vartheta_2} \bigotimes \cdots \bigotimes (\tilde{\eth}_{\varrho(\hbar)})^{\vartheta_\hbar} \\ &= IVIFAAOWG_\vartheta(\tilde{\eth}_1, \tilde{\eth}_2, \ldots, \tilde{\eth}_\hbar), \end{aligned}$$

which completes the proof. □

5. MADM Methods Influenced by IVIFAAWG Operator

In this section, we shall take advantage of the IVIFAAWG operator to create a way for addressing MADM difficulties with IVIF information.

For a MADM issue, let $\Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_\psi\}$ function as the set of alternatives and $J = \{J_1, J_2, \ldots, J_\hbar\}$ function as the set of attributes, and attributes weight vector is $\xi = (\xi_1, \xi_2, \ldots, \xi_\hbar)^T$, fulfilling $\xi_\zeta \in [0, 1]$ and $\sum_{\zeta=1}^{\hbar} \xi_\zeta = 1$. We explicit the assessment information of the alternative $\Phi_\wp$ concerning the criterion J_ζ by $\widetilde{Y}_{\wp\zeta} = ([\beta^L_{\eth_{\wp\zeta}}, \beta^U_{\eth_{\wp\zeta}}], [\delta^L_{\eth_{\wp\zeta}}, \delta^U_{\eth_{\wp\zeta}}])$, and $\Gamma = \left(\widetilde{Y}_{\wp\zeta}\right)_{\psi\times\hbar}$ is definitely an IVIF decision matrix. Hence, the MADM issue with IVIFNs may be discussed in the following matrix form, acknowledged by Equation (7).

$$\Gamma = (\widetilde{Y}_{\wp\zeta})_{\psi\times\hbar} = \begin{array}{c} \\ \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_\psi \end{array} \begin{array}{c} J_1 \quad J_2 \cdots J_\hbar \\ \begin{pmatrix} ([\beta^L_{\eth_{11}}, \beta^U_{\eth_{11}}], [\delta^L_{\eth_{11}}, \delta^U_{\eth_{11}}]) & ([\beta^L_{\eth_{12}}, \beta^U_{\eth_{12}}], [\delta^L_{\eth_{12}}, \delta^U_{\eth_{12}}]) & \cdots & ([\beta^L_{\eth_{1\hbar}}, \beta^U_{\eth_{1\hbar}}], [\delta^L_{\eth_{1\hbar}}, \delta^U_{\eth_{1\hbar}}]) \\ ([\beta^L_{\eth_{21}}, \beta^U_{\eth_{21}}], [\delta^L_{\eth_{21}}, \delta^U_{\eth_{21}}]) & ([\beta^L_{\eth_{22}}, \beta^U_{\eth_{22}}], [\delta^L_{\eth_{22}}, \delta^U_{\eth_{22}}]) & \cdots & ([\beta^L_{\eth_{2\hbar}}, \beta^U_{\eth_{2\hbar}}], [\delta^L_{\eth_{2\hbar}}, \delta^U_{\eth_{2\hbar}}]) \\ \vdots & \vdots & \ddots & \vdots \\ ([\beta^L_{\eth_{\psi1}}, \beta^U_{\eth_{\psi1}}], [\delta^L_{\eth_{\psi1}}, \delta^U_{\eth_{\psi1}}]) & ([\beta^L_{\eth_{\psi2}}, \beta^U_{\eth_{\psi2}}], [\delta^L_{\eth_{\psi2}}, \delta^U_{\eth_{\psi2}}]) & \cdots & ([\beta^L_{\eth_{\psi\hbar}}, \beta^U_{\eth_{\psi\hbar}}], [\delta^L_{\eth_{\psi\hbar}}, \delta^U_{\eth_{\psi\hbar}}]) \end{pmatrix} \end{array} \quad (7)$$

where every one of the components $\widetilde{Y}_{\wp\zeta} = ([\beta^L_{\eth_{\wp\zeta}}, \beta^U_{\eth_{\wp\zeta}}], [\delta^L_{\eth_{\wp\zeta}}, \delta^U_{\eth_{\wp\zeta}}])$ is certainly an IVIFN, where $[\beta^L_{\eth_{\wp\zeta}}, \beta^U_{\eth_{\wp\zeta}}]$ is the positive membership degree because of which alternative $\Phi_\wp$ fulfills the attribute J_ζ that has been appropriated by the decision-makers, and $[\delta^L_{\eth_{\wp\zeta}}, \delta^U_{\eth_{\wp\zeta}}]$ gave the degree that the alternative $\Phi_\wp$ does not fulfill the attribute J_ζ that has been distributed by the decision-maker, where $[\beta^L_{\eth_{\wp\zeta}}, \beta^U_{\eth_{\wp\zeta}}] \subset D[0,1]$, $[\delta^L_{\eth_{\wp\zeta}}, \delta^U_{\eth_{\wp\zeta}}] \subset D[0,1]$ and $0 \le \beta^U_{\eth_{\wp\zeta}} + \delta^U_{\eth_{\wp\zeta}} \le 1$, $(\wp = 1, 2, \ldots, \psi)$.

The methodology dependent on IVIFAAWG operator to find out the MADM difficulties with IVIF data explicitly incorporates these steps:

Step 1. Modify decision matrix $\Gamma = \left(\widetilde{Y}_{\wp\zeta}\right)_{\psi\times\hbar}$ into the normalization matrix $\overline{\Gamma} = \left(\overline{\widetilde{Y}}_{\wp\zeta}\right)_{\psi\times\hbar}$.

$$\overline{\widetilde{Y}}_{\wp\zeta} = \begin{cases} \widetilde{Y}_{\wp\zeta} \text{ for benefit attribute } J_\zeta \\ (\widetilde{Y}_{\wp\zeta})^c \text{ for cost attribute } J_\zeta \end{cases} \quad (8)$$

where $(\widetilde{Y}_{\wp\zeta})^c$ is the complement of $\widetilde{Y}_{\wp\zeta}$, such that $(\widetilde{Y}_{\wp\zeta})^c = ([\delta^L_{\eth_{\wp\zeta}}, \delta^U_{\eth_{\wp\zeta}}], [\beta^L_{\eth_{\wp\zeta}}, \beta^U_{\eth_{\wp\zeta}}])$.

In fact, if all the attributes J_ζ $(\zeta = 1, 2, \ldots, \hbar)$ are the same type, then there is no need to normalize them, but if it is found that there are two types of attributes then we will convert cost attributes to benefit attributes. Then, $\Gamma = \left(\widetilde{Y}_{\wp\zeta}\right)_{\psi\times\hbar}$ will be transformed into IVIF decision matrix $\overline{\Gamma} = \left(\overline{\widetilde{Y}}_{\wp\zeta}\right)_{\psi\times\hbar}$.

Step 2. Make use of the decision data expressed in matrix Γ, and the operator IVIFAAWG to get the overall preference values $\widetilde{Y}_\wp$ $(\wp = 1, 2, \ldots, \psi)$ of the alternative $\Phi_\wp$, i.e.,

$$\widetilde{Y}_\wp = IVIFAAWG_\xi(\widetilde{Y}_{\wp1}, \widetilde{Y}_{\wp2}, \ldots, \widetilde{Y}_{\wp\hbar}) = \bigotimes_{\zeta=1}^{\hbar} (\widetilde{Y}_{\wp\zeta})^{\xi_\zeta}$$

$$= \left\langle \left[e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta(-\log \beta^L_{\eth_\zeta})^\eth\right)^{1/\eth}}, e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta(-\log \beta^U_{\eth_\zeta})^\eth\right)^{1/\eth}} \right], \left[1 - e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta(-\log(1-\delta^L_{\eth_\zeta}))^\eth\right)^{1/\eth}}, 1 - e^{-\left(\sum_{\zeta=1}^{\hbar} \xi_\zeta(-\log(1-\delta^U_{\eth_\zeta}))^\eth\right)^{1/\eth}} \right] \right\rangle. \quad (9)$$

Step 3. Rank all of the alternatives in order of preference. Make use of the method in Definition 3 to rank the entire rating values $\widetilde{Y}_\wp$ $(\wp = 1, 2, \ldots, \psi)$ and rank all the alternatives $\Phi_\wp$ $(\wp = 1, 2, \ldots, \psi)$ as per ~$\widetilde{Y}_\wp$ $(\wp = 1, 2, \ldots, \psi)$ in descending order. Lastly, we choose the advantageous alternative(s) with the highest rating value.

Step 4. End.

6. Numerical Example

This section contains an interesting explanation demonstrating the systematic methodology for choosing an appropriate car.

6.1. Problem Description

Consider a consumer who is considering purchasing a car. There are five distinct types of cars (alternatives) $\Phi_\wp$ $(\wp = 1, 2, \ldots, 5)$. The consumer considers six attributes while deciding which vehicle to buy (adapted from Herrera and Martinez [44]): J_1: Fuel economy; J_2: Aerod degree; J_3: Price; J_4: Comfort; J_5: Design; and J_6: Security. The weight vector of the attributes J_ζ $(\zeta = 1, 2, \ldots, 6)$ is $\xi = (0.15, 0.25, 0.14, 0.16, 0.20, 0.10)^T$. Expect that the features of the alternatives $\Phi_\wp$ $(\wp = 1, 2, \ldots, 5)$ are addressed by the IVIFNs, as demonstrated in the IVIF decision matrix $\Gamma = \left(\widetilde{Y}_{\wp\zeta}\right)_{6\times 5}$ (Table 1).

Table 1. IVIF decision matrix.

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5
J_1	([0.56,0.66],[0.26,0.31])	([0.38,0.47],[0.34,0.44])	([0.56,0.63],[0.23,0.32])	([0.64,0.73],[0.16,0.27])	([0.48,0.63],[0.26,0.36])
J_2	([0.78,0.88],[0.07,0.12])	([0.47,0.56],[0.27,0.37])	([0.51,0.57],[0.16,0.26])	([0.65,0.75],[0.13,0.20])	([0.64,0.69],[0.21,0.31])
J_3	([0.61,0.82],[0.11,0.18])	([0.79,0.84],[0.11,0.16])	([0.51,0.56],[0.36,0.44])	([0.54,0.64],[0.25,0.36])	([0.79,0.84],[0.08,0.16])
J_4	([0.82,0.91],[0.02,0.07])	([0.55,0.65],[0.22,0.32])	([0.63,0.74],[0.21,0.25])	([0.65,0.75],[0.20,0.25])	([0.60,0.73],[0.17,0.27])
J_5	([0.44,0.56],[0.32,0.42])	([0.68,0.78],[0.17,0.22])	([0.35,0.45],[0.35,0.45])	([0.59,0.69],[0.25,0.30])	([0.45,0.54],[0.35,0.45])
J_6	([0.70,0.83],[0.08,0.17])	([0.53,0.58],[0.31,0.36])	([0.76,0.83],[0.07,0.17])	([0.41,0.51],[0.36,0.42])	([0.56,0.66],[0.22,0.32])

6.2. The IFAAWG Operator-Based Technique

To determine one of most perfect car $\Phi_\wp$ $(\wp = 1, 2, \ldots, 5)$, we employ the IFAAWG operator to construct a MADM theory using intuitionistic fuzzy information, which is frequently evaluated as follows:

- **Step 1.** Because the attributes are classified into two types, we begin by converting the attribute of the cost type into the attribute of the benefit type by employing Equation (8). At that point, $\Gamma = \left(\widetilde{Y}_{\wp\zeta}\right)_{6\times 5}$ is changed into the normalized decision matrix $\overline{\Gamma} = \left(\overline{\widetilde{Y}}_{\wp\zeta}\right)_{6\times 5}$ (Table 2).
- Step 2. Assume that $\eth = 1$. The IVIFAAWG operator is used to know the overall alternative values $\widetilde{Y}_\wp$ for five alternatives $\Phi_\wp$ $(\wp = 1, 2, \ldots, 5)$,
$\widetilde{Y}_1 = ([0.637689, 0.762041], [0.154922, 0.224767])$,
$\widetilde{Y}_2 = ([0.547075, 0.633999], [0.237723, 0.318131])$,
$\widetilde{Y}_3 = ([0.516381, 0.595996], [0.241492, 0.328694])$,
$\widetilde{Y}_4 = ([0.591841, 0.691298], [0.212673, 0.286312])$,
$\widetilde{Y}_5 = ([0.574598, 0.669233], [0.226311, 0.324705])$.
- Step 3. We evaluate the score values $\hat{K}(\widetilde{Y}_\wp)$ $(\wp = 1, 2, \ldots, 5)$ of the universal IVIFNs $\widetilde{Y}_\wp$ $(\wp = 1, 2, \ldots, 5)$ utilizing Equation (2) as $\hat{K}(\widetilde{Y}_1) = 0.510021$, $\hat{K}(\widetilde{Y}_2) = 0.312610$, $\hat{K}(\widetilde{Y}_3) = 0.271095$, $\hat{K}(\widetilde{Y}_4) = 0.392077$, $\hat{K}(\widetilde{Y}_5) = 0.346408$.
- Step 4. Ranking these five alternatives $\Phi_\wp$ $(\wp = 1, 2, \ldots, 5)$ according to the score values $\hat{K}(\widetilde{Y}_\wp)$ $(\wp = 1, 2, \ldots, 5)$ of the overall IVIFNs as $\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$.
- Step 5. Thus, the best car is Φ_1.

Table 2. Normalized IVIF decision matrix.

	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5
J_1	([0.56,0.66],[0.26,0.31])	([0.38,0.47],[0.34,0.44])	([0.56,0.63],[0.23,0.32])	([0.64,0.73],[0.16,0.27])	([0.48,0.63],[0.26,0.36])
J_2	([0.78,0.88],[0.07,0.12])	([0.47,0.56],[0.27,0.37])	([0.51,0.57],[0.16,0.26])	([0.65,0.75],[0.13,0.20])	([0.64,0.69],[0.21,0.31])
J_3	([0.61,0.82],[0.11,0.18])	([0.79,0.84],[0.11,0.16])	([0.51,0.56],[0.36,0.44])	([0.54,0.64],[0.25,0.36])	([0.79,0.84],[0.08,0.16])
J_4	([0.82,0.91],[0.02,0.07])	([0.55,0.65],[0.22,0.32])	([0.63,0.74],[0.21,0.25])	([0.65,0.75],[0.20,0.25])	([0.60,0.73],[0.17,0.27])
J_5	([0.44,0.56],[0.32,0.42])	([0.68,0.78],[0.17,0.22])	([0.35,0.45],[0.35,0.45])	([0.59,0.69],[0.25,0.30])	([0.45,0.54],[0.35,0.45])
J_6	([0.70,0.83],[0.08,0.17])	([0.53,0.58],[0.31,0.36])	([0.76,0.83],[0.07,0.17])	([0.41,0.51],[0.36,0.42])	([0.56,0.66],[0.22,0.32])

7. The Impact of the Parameter $\eth$ in This Technique

To show how the different values of the parameter $\eth$ affect the alternatives, we use different values of the parameter $\eth$ to categorize the alternatives. The IVIFAAWG operator is used to rank the alternatives $\Phi_{\wp}$ $(t = 1, 2 \ldots, 5)$, and they are shown in Table 3 and shown in Figure 2. Clearly, when the value of $\eth$ for IVIFAAWG operator starts growing, the score values of the possible alternatives decrease, but the ranking stays the same: $\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$. Thus, the most important alternative is Φ_1.

Table 3. Ranking order of the alternatives with various parameter $\eth$ by IVIFAAWG operator.

$\eth$	$\hat{K}(\widetilde{Y}_1)$	$\hat{K}(\widetilde{Y}_2)$	$\hat{K}(\widetilde{Y}_3)$	$\hat{K}(\widetilde{Y}_4)$	$\hat{K}(\widetilde{Y}_5)$	**Ranking Order**
1	0.510021	0.312610	0.271095	0.392077	0.346408	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
2	0.432522	0.274768	0.232654	0.369262	0.315391	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
3	0.374209	0.245251	0.200569	0.344796	0.289366	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
4	0.332473	0.222023	0.174023	0.319924	0.267292	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
5	0.302133	0.203274	0.152149	0.295960	0.248489	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
6	0.279331	0.187738	0.134129	0.273905	0.232478	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
7	0.261630	0.174584	0.119238	0.254265	0.218860	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
8	0.247512	0.163268	0.106868	0.237122	0.207273	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
9	0.236003	0.153420	0.096520	0.222303	0.197390	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
10	0.226454	0.144778	0.087797	0.209530	0.188927	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$

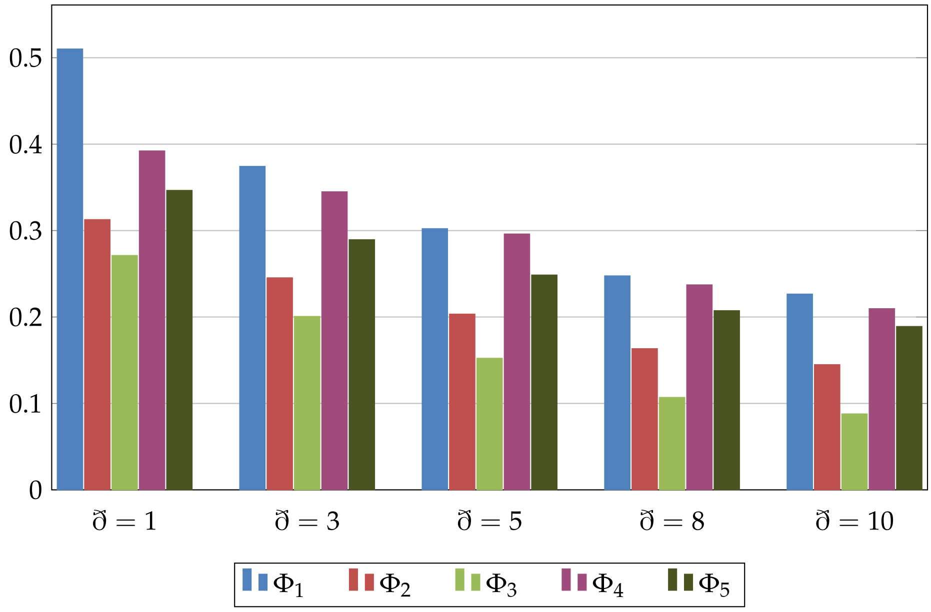

Figure 2. Score values belonging to the alternatives for various values $\eth$ by IVIFAAWG operator.

Additionally, Figure 2 reveals that when the value of $\eth$ is changed in the example, the ranking results remain identical, demonstrating the resilience of the IVIFAAWG operators.

8. Sensitivity Analysis (SA) of Criteria Weights

To investigate the effect of criteria weights on ranking order, we present a sensitivity investigation. This is done using 24 different weight sets, namely—$Q1, Q2, \ldots, Q24$ (Table 4) formed by considering all possible combinations of the criteria weights $\eta_1 = 0.15$, $\eta_2 = 0.25$, $\eta_3 = 0.14$, $\eta_4 = 0.16$, $\eta_5 = 0.20$, and $\eta_6 = 0.10$. This is especially valuable for achieving a more broad scope of criteria weights for taking a gander at the affectability of the created model. The scores of alternatives are accumulated in Figure 3, and their respective ranking orders are indexed in Table 5. Upon examining the ranking order of alternatives, it is seen that Φ_1 holds the first rank in 100% of the scenarios when the IVIFWG operator (taking

$\eth = 2$) is applied. Hence, the priority of alternatives acquired by utilizing our developed method is credible.

Table 4. Various weight sets of criteria.

Weight Sets	η_1	η_2	η_3	η_4	η_5	η_6	Weight Sets	η_1	η_2	η_3	η_4	η_5	η_6
Q1	0.15	0.25	0.14	0.16	0.20	0.10	Q13	0.16	0.20	0.25	0.15	0.14	0.10
Q2	0.15	0.14	0.16	0.20	0.10	0.25	Q14	0.16	0.25	0.15	0.14	0.10	0.20
Q3	0.15	0.16	0.20	0.10	0.25	0.14	Q15	0.16	0.15	0.14	0.10	0.20	0.25
Q4	0.15	0.20	0.10	0.25	0.14	0.16	Q16	0.16	0.14	0.10	0.20	0.25	0.15
Q5	0.25	0.15	0.14	0.20	0.10	0.16	Q17	0.20	0.25	0.10	0.14	0.15	0.16
Q6	0.25	0.14	0.20	0.10	0.16	0.15	Q18	0.20	0.10	0.14	0.15	0.16	0.25
Q7	0.25	0.20	0.10	0.16	0.15	0.14	Q19	0.20	0.14	0.15	0.16	0.25	0.10
Q8	0.25	0.10	0.16	0.15	0.14	0.20	Q20	0.20	0.15	0.16	0.25	0.10	0.14
Q9	0.14	0.16	0.15	0.20	0.10	0.25	Q21	0.10	0.14	0.15	0.16	0.20	0.25
Q10	0.14	0.15	0.20	0.10	0.25	0.16	Q22	0.10	0.15	0.16	0.20	0.25	0.14
Q11	0.14	0.20	0.10	0.25	0.16	0.15	Q23	0.10	0.16	0.20	0.25	0.14	0.15
Q12	0.14	0.10	0.25	0.16	0.15	0.20	Q24	0.10	0.20	0.25	0.14	0.15	0.16

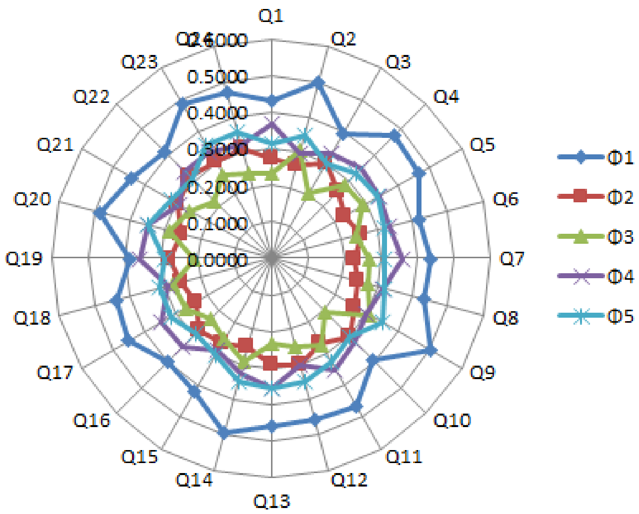

Figure 3. Utility values of alternatives for distinct sets of weighted criteria.

Table 5. Priority order of alternatives for diverse weight sets.

	Ranking Order		Ranking Order		Ranking Order
Q1	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$	Q9	$\Phi_1 \succ \Phi_5 \succ \Phi_3 \succ \Phi_4 \succ \Phi_2$	Q17	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_3 \succ \Phi_2$
Q2	$\Phi_1 \succ \Phi_5 \succ \Phi_3 \succ \Phi_4 \succ \Phi_2$	Q10	$\Phi_1 \succ \Phi_4 \succ \Phi_2 \succ \Phi_5 \succ \Phi_3$	Q18	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_3 \succ \Phi_2$
Q3	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$	Q11	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_3 \succ \Phi_2$	Q19	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
Q4	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_3 \succ \Phi_2$	Q12	$\Phi_1 \succ \Phi_5 \succ \Phi_2 \succ \Phi_4 \succ \Phi_3$	Q20	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_3 \succ \Phi_2$
Q5	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_3 \succ \Phi_2$	Q13	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_2 \succ \Phi_3$	Q21	$\Phi_1 \succ \Phi_5 \succ \Phi_2 \succ \Phi_4 \succ \Phi_3$
Q6	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$	Q14	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_3 \succ \Phi_2$	Q22	$\Phi_1 \succ \Phi_4 \succ \Phi_2 \succ \Phi_5 \succ \Phi_3$
Q7	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_3 \succ \Phi_2$	Q15	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_2 \succ \Phi_3$	Q23	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_2 \succ \Phi_3$
Q8	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_3 \succ \Phi_2$	Q16	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$	Q24	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_2 \succ \Phi_3$

9. Comparison Study

Following that, we will compare and contrast our proposed approach with some other conventional methods such as the IVIF weighted averaging (IVIFWA) operator [5], the IVIF weighted geometric (IVIFWG) operator [39], the IVIF Einstein weighted geometric ($IVIFWG^{\varepsilon}$) operator [16], and the IVIF Einstein weighted averaging ($IVIFWA^{\varepsilon}$) operator [17]. The comparison findings are given in Tables 6 and 7, and they are depicted in

Figure 4 visually. If you look at Tables 3 and 6, you can see that the IVIFWG operator is a special case of the IVIFAAWG operator, and that this happens when $\eth = 1$.

As a consequence, our recommended procedures for resolving IVIF MADM problems are frequently more extensive and adaptable than some of the techniques now in use.

Table 6. Comparative assessment using a few popular methodologies.

Techniques	$\hat{K}(\widetilde{Y}_1)$	$\hat{K}(\widetilde{Y}_2)$	$\hat{K}(\widetilde{Y}_3)$	$\hat{K}(\widetilde{Y}_4)$	$\hat{K}(\widetilde{Y}_5)$	Preference Order
Xu [5]	0.605185	0.370086	0.326785	0.375578	0.391143	$\Phi_1 \succ \Phi_5 \succ \Phi_4 \succ \Phi_2 \succ \Phi_3$
Xu & Chen [39]	0.510021	0.312610	0.271095	0.392077	0.346408	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
Wang & Liu [16]	0.523568	0.321157	0.279447	0.396147	0.352904	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
Wang & Liu [17]	0.597400	0.362337	0.319370	0.413345	0.385472	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$
Proposed method	0.226454	0.144778	0.087797	0.209530	0.188927	$\Phi_1 \succ \Phi_4 \succ \Phi_5 \succ \Phi_2 \succ \Phi_3$

Table 7. Qualitative evaluations of the current methods.

Techniques	Whether It Is More Straightforward to Express Ambiguous Data	Whether It Should Make Information Aggregation More Parameter-Adjustable
Xu [5]	Yes	No
Xu & Chen [39]	Yes	No
Wang & Liu [16]	Yes	No
Wang & Liu [17]	Yes	No
Proposed method	Yes	Yes

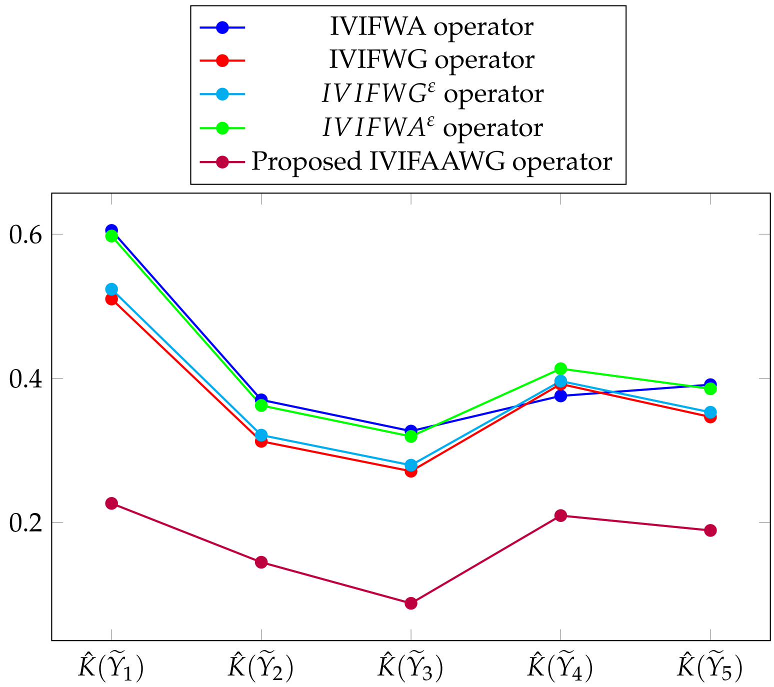

Figure 4. Comparison analysis with a few prevailing techniques.

10. Conclusions

We began this study by extending the Aczel–Alsina t-norm and t-conorm to IVIF scenarios, defining and examining a few additional working principles for IVIFNs. Then, in light of these new operating laws, different new aggregation operators, such as the IVIFAAWG operator, the IVIFAAOWG operator, and the IVIFAAHG operator, were devised to accommodate situations in which the specified assertions are IVIFNs. The fundamental characteristics of the recommended operators are examined, as well as their specific situations. We provide a realistic approach to MADM difficulties with IVIFNs depending on the IFAAWG operator. Furthermore, an exemplary scenario of choosing suitable cars is utilized to demonstrate the developed model, and a comparative study with some other methods is undertaken to show the recommended operators' distinct advantages. In future studies, we plan to extend the challenge further by introducing new characteristics, including the use of probabilistic aggregations. Additionally, we will discuss additional decision-making aspects like cluster analysis, performance analysis [45], sustainable city logistics [46], risk investment assessment [47], Wireless Sensor Networks [48], capital budgeting techniques [49], home buying process [50], and other domains in uncertain environment [51–58].

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List of Abbreviations

The following abbreviations are used in this manuscript:

IVIF	interval-valued intuitionistic fuzzy
IVIFS	interval-valued intuitionistic fuzzy set
IVIFN	interval-valued intuitionistic fuzzy number
MADM	multiple attribute decision making
IVIFWA	IVIF weighted averaging
IVIFHA	IVIF hybrid averaging
$IVIFWA^{\varepsilon}$	IVIF Einstein weighted averaging
$IVIFWG^{\varepsilon}$	IVIF Einstein weighted geometric
IVIFAAWG	IVIF Aczel-Alsina weighted geometric
IVIFAAOWG	IVIF Aczel-Alsina order weighted geometric
IVIFAAHG	IVIF Aczel-Alsina hybrid geometric

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