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On the Generalized Gaussian Fibonacci Numbers and Horadam Hybrid Numbers: A Unified Approach

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Abstract: In this paper, we consider an approach based on the elementary matrix theory. In other words, we take into account the generalized Gaussian Fibonacci numbers. In this context, we consider a general tridiagonal matrix family. Then, we obtain determinants of the matrix family via the Chebyshev polynomials. Moreover, we consider one type of tridiagonal matrix, whose determinants are Horadam hybrid polynomials, i.e., the most general form of hybrid numbers. Then, we obtain its determinants by means of the Chebyshev polynomials of the second kind. We provided several illustrative examples, as well.

1. Introduction

The second-order homogeneous linear recurrence $\{w_n \equiv w_n(a, b; p, q)\}$ is defined by A. F. Horadam as below:

$$w_n = p w_{n-1} - q w_{n-2}, \quad \text{for } n \geqslant 2, \tag{1}$$

with initial conditions

$$w_0 = a \text{ and } w_1 = b,$$

for arbitrary integers a and b (Section 3, [1]). This is one of the possible extensions of some well-known number sequences and some of them have been listed in Table 1.

Table 1. Some subsequences of the Horadam numbers with OEIS Code.

p, q	a, b	Sequence	OEIS Code
1,1	0,1	Fibonacci	A000045
1,1	2,1	Lucas	A000032
2,1	0,1	Pell (or (2,1)-Fibonacci)	A000129
2,1	2,2	Pell-Lucas	A002203
3,1	0,1	Bronze Fibonacci (or (3,1)-Fibonacci)	A006190
3,1	2,3	Bronze Lucas	A006497
1,2	0,1	Jacobsthal (or (1,2)-Fibonacci)	A001045
1,2	2,1	Jacobsthal-Lucas	A014551
1,3	1,1	Nickel Fibonacci (or (1,3)-Fibonacci)	A006130
1,3	2,1	Nickel Lucas	A075118

Recently, by inspiring the Horadam sequence, the generalized Gaussian Fibonacci sequence $Gf_{n+1}(p,q;a,b)$ is defined by the following way:

$$Gf_{n+1} = pGf_n + qGf_{n-1}, \quad Gf_0 = a \text{ and } Gf_1 = b,$$

where a and b are initial conditions, see [2,3]. Some well-known special kind of the generalized Gaussian type sequences can be given in the Table 2.

Table 2. Some kind of Gaussian type sequences.

Notation	a	b	p	q	Number
GF_n	i	1	1	1	Gaussian Fibonacci
GP_n	i	1	2	1	Gaussian Pell
GJ_n	$i/2$	1	1	2	Gaussian Jacobsthal
GB_n	i	1	3	1	Gaussian Bronze
GN_n	$i/3$	1	1	3	Gaussian Nickel
GM_n	$-i/2$	1	3	-2	Gaussian Mersenne
GL_n	$2-i$	$1+2i$	1	1	Gaussian Lucas
Gp_n	$2-2i$	$2+2i$	2	1	Gaussian Pell-Lucas
Gj_n	$2-i/2$	$1+2i$	1	2	Gaussian Jacobsthal-Lucas

In the literature, there are a huge amount of papers that investigate distinct types of the subsequences of the generalized Gaussian Fibonacci numbers. For example, in [4], the authors define the nth generalized complex Fibonacci number and obtain some of their identities. In [5], the authors introduce the concept of the complex Fibonacci numbers and establish some quite general identities concerning them. In [6], the authors define and study the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials. Moreover, they give the generating function, Binet formula, explicit formula, Q matrix, determinantal representations, and partial derivation of these polynomials. In [7], the authors consider the generalized Gaussian Fibonacci and Lucas sequences and define the Gaussian Pell and Gaussian Pell–Lucas sequences. Moreover they give the generating functions and Binet formulas for them. The authors, in [8], introduce a new class of quaternions associated with the well-known Mersenne numbers. They extend the usual definitions into a wider structure by using arbitrary Mersenne numbers. In [9], the authors extend the Bronze Fibonacci number to the Gaussian Bronze Fibonacci number. Moreover, they obtain Binet formula, generating function and some identities for them. In [10], the authors define generalized k-Mersenne numbers and give a formula of generalized Mersenne polynomials and further they define Gaussian Mersenne numbers and obtain some identities.

Table 3 presents recurrence relations and some values of some kinds of the generalized Gaussian Fibonacci numbers.

On the other hand, the connection between particular cases of the Horadam sequence and the Chebyshev polynomials of the several kinds, has been a topic of research for decades. Note that,

$$T_n = \begin{pmatrix} p & 1 & & & \\ q & \ddots & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & q & p \end{pmatrix}_{n \times n},$$

we have (cf., e.g., [11,12])

$$\det T_n = (\sqrt{q})^n \, U_n\left(\frac{p}{2\sqrt{q}}\right),$$

and the first explicit formula to the Horadam sequence in terms of Chebyshev polynomials of the second kind is given below [13]:

$$w_n = (\sqrt{q})^n \left(\frac{b}{\sqrt{q}} U_{n-1}\left(\frac{p}{2\sqrt{q}} \right) - a\, U_{n-1}\left(\frac{p}{2\sqrt{q}} \right) \right), \tag{2}$$

where $\{U_n(x)\}_{n \geqslant 0}$ are the Chebyshev polynomials of the second kind satisfying the three-term recurrence relations

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0, \quad \text{for } n = 0, 1, 2, \dots,$$

with initial conditions $U_{-1}(x) = 0$ and $U_0(x) = 1$. One of the most well-known explicit formulas for these polynomials is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

for all $n = 0, 1, 2\dots$. The Chebyshev polynomials of the third kind can be stated by means of the Chebyshev polynomials of the second kind as below:

$$V_n(x) = U_n(x) - U_{n-1}(x).$$

The rest of the paper is organized as follows. In Section 2, we take into account the generalized Gaussian Fibonacci numbers. In this context, we consider a general form of tridiagonal matrix. Then, we obtain their determinants by exploiting the Chebyshev polynomials of the second kind. Subsequently, we give relationships between the determinants of some special forms of these matrices and some types of generalized Gaussian Fibonacci numbers. In Section 3, we consider the Horadam hybrid polynomials which is the most general form of the hybrid numbers. In this circumstance, it is known that the Horadam hybrid polynomials can be obtained by means of the determinants of tridiagonal matrices. By considering these type of matrices, we give the Horadam hybrid polynomials by means of the Chebyshev polynomials of the second kind. In other words, we represent good relationships between tridiagonal matrices, Chebyshev polynomials and the generalized Gaussian Fibonacci numbers and the Horadam hybrid polynomials. Finally, we give some illustrative examples.

Table 3. Some recurrences and their values.

Recurrence Relation	0	1	2	3	4
$GF_n = GF_n + GF_{n-1}$	i	1	$1+i$	$2+i$	$3+2i$
$GL_n = GL_n + GL_{n-1}$	$2-i$	$1+2i$	$3+i$	$4+3i$	$7+4i$
$GP_n = 2GP_n + GP_{n-1}$	i	1	$2+i$	$5+2i$	$12+5i$
$Gp_n = 2Gp_n + Gp_{n-1}$	$2-2i$	$2+2i$	$6+2i$	$14+6i$	$34+14i$
$GJ_n = GJ_n + 2GJ_{n-1}$	$\frac{i}{2}$	1	$1+i$	$3+i$	$5+3i$
$Gj_n = Gj_n + 2Gj_{n-1}$	$2-\frac{i}{2}$	$1+2i$	$1+i$	$3+i$	$5+3i$
$GM_n = 3GM_n - 2GM_{n-1}$	$-\frac{i}{2}$	1	$3+i$	$7+3i$	$15+7i$
$GB_n = 3GB_n + GB_{n-1}$	i	1	$3+i$	$10+3i$	$33+10i$
$GN_n = GN_{n-1} + 3GN_{n-2}$	$\frac{i}{3}$	1	$1+i$	$4+i$	$7+4i$

2. On Generalized Gaussian Fibonacci Numbers

At this section, we consider the generalized Gaussian Fibonacci numbers. Let us define a tridiagonal matrix in the following form:

$$T_n(a, b, c) = \begin{pmatrix} 1 & a & & & \\ -1 & c & b & & \\ & -1 & c & \ddots & \\ & & \ddots & \ddots & b \\ & & & -1 & c \end{pmatrix}.$$

Here our first aim is to obtain the determinants of the matrix $T_n(a, b, c)$ by exploiting the Chebyshev polynomials and give its relationship with generalized Gaussian Fibonacci numbers.

Lemma 1 ([14]). *Let $\{H_n, n = 1, 2, \ldots\}$ be a sequence of tridiagonal matrices of the form:*

$$
H_n = \begin{pmatrix}
h_{11} & h_{12} & & & & \\
h_{21} & h_{22} & h_{23} & & & \\
& h_{32} & h_{33} & \ddots & & \\
& & \ddots & \ddots & h_{n-1,1} & \\
& & & h_{n,n-1} & h_{n,n}
\end{pmatrix}.
$$

Then, the successive determinants of H_n are given by the recursive formula:

$$
\begin{aligned}
|H_1| &= h_{11}, \\
|H_2| &= h_{11}h_{22} - h_{12}h_{21}, \\
|H_n| &= h_{n,n}|H_{n-1}| - h_{n-1,n}h_{n,n-1}|H_{n-2}|.
\end{aligned}
$$

Theorem 1. *The determinant of the matrix $T_n(a, b, c)$ is*

$$
\det T_n(a, b, c) = (-i\sqrt{b})^{n-1}\left(U_{n-1}\left(\frac{ic}{2\sqrt{b}}\right) + i\sqrt{b}U_{n-2}\left(\frac{ic}{2\sqrt{b}}\right)\frac{a}{b}\right).
$$

Proof. Taking into account the Lemma 1 and exploiting some perturbations of tridiagonal matrices, see [11,15,16], the proof can be verified. □

Theorem 2. *For $n > 1$,*

$$
GF_n = (-i)^{n-1}V_{n-1}\left(\frac{i}{2}\right),
$$

$$
GP_n = (-i)^{n-1}V_{n-1}(i),
$$

$$
GB_n = (-i)^{n-1}V_{n-1}\left(\frac{3i}{2}\right),
$$

where V_n is the nth Chebyshev polynomial of third kind.

Proof. Taking into account (1), (2) and substituting some values for a, b, and c in Theorem 1, one can see the determinants as below:

$$
\det T_n(i, 1, 1) = (-i)^{n-1}\left(U_{n-1}\left(\frac{i}{2}\right) - U_{n-2}\left(\frac{i}{2}\right)\right) = (-i)^{n-1}V_{n-1}\left(\frac{i}{2}\right) = GF_n,
$$

$$
\det T_n(i, 1, 2) = (-i)^{n-1}(U_{n-1}(i) - U_{n-2}(i)) = (-i)^{n-1}V_{n-1}(i) = GP_n,
$$

$$
\det T_n(i, 1, 3) = (-i)^{n-1}\left(U_{n-1}\left(\frac{3i}{2}\right) - U_{n-2}\left(\frac{3i}{2}\right)\right) = (-i)^{n-1}V_{n-1}\left(\frac{3i}{2}\right) = GB_n. \quad \square
$$

Corollary 1. *For $n > 2$,*

$$
\det T_n(i, -2, 3) = (\sqrt{2})^{n-1}\left(U_{n-1}\left(\frac{3}{2\sqrt{2}}\right) + \frac{i}{2}\sqrt{2}U_{n-2}\left(\frac{3}{2\sqrt{2}}\right)\right) = GM_n,
$$

$$
\det T_n(i, 2, 1) = (-i\sqrt{2})^{n-1}\left(U_{n-1}\left(\frac{i}{2\sqrt{2}}\right) - \frac{1}{\sqrt{2}}U_{n-2}\left(\frac{i}{2\sqrt{2}}\right)\right) = GJ_n.
$$

Proof. It is clear from (1), (2) and Theorem 1. □

As an illustration of the Theorem 2, we analyze some particular cases of the matrix. Let us consider the matrix $T_n(i, 1, 1)$. In other words, it corresponds the following matrix:

Example 1. *Let us consider* $T_6(i, 1, 1)$,

$$T_6(i, 1, 1) = \begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

then

$$\det T_6(i, 1, 1) = (-i)^5 V_5\left(\frac{i}{2}\right)$$

$$= 32\left(\frac{i}{2}\right)^5 - 16\left(\frac{i}{2}\right)^4 - 32\left(\frac{i}{2}\right)^3 + 12\left(\frac{i}{2}\right)^2 + 6\left(\frac{i}{2}\right) - 1$$

$$= 8 + 5i = GF_6,$$

where GF_6 *denotes the 6th Gaussian Fibonacci number.*

Example 2. *Let us consider* $T_5(i, 1, 2)$,

$$T_5(i, 1, 2) = \begin{pmatrix} 1 & i & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

then

$$\det T_5(i, 1, 2) = (-i)^4 V_4(i) = 16i^4 - 8i^3 - 12i^2 - 4i + 1 = 29 + 12i = GP_5,$$

where GP_5 *denotes the 5th Gaussian Pell number.*

Example 3. *Let us consider* $T_5(i, -2, 3)$,

$$T_5(i, -2, 3) = \begin{pmatrix} 1 & i & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & 0 \\ 0 & -1 & 3 & -2 & 0 \\ 0 & 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}$$

then

$$det T_5(i, 1, 3) = (-i)^4 V_4\left(\frac{3i}{2}\right)$$

$$= 16\left(\frac{3i}{2}\right)^4 - 8\left(\frac{3i}{2}\right)^3 - 12\left(\frac{3i}{2}\right)^2 - 4\left(\frac{3i}{2}\right) + 1$$

$$= 109 + 33i = GB_5,$$

where GB_5 *denotes the 5th Gaussian Bronze number.*

3. A General Identity for Horadam Hybrid Numbers

Recently, the geometric and physical applications of complex, hyperbolic, and dual numbers have been thoroughly studied. In [17], M. Özdemir introduced the hybrid numbers as

$$Z = a + bi + c\epsilon + dh,$$

where $a, b, c \in \mathbb{R}$ with i, ϵ, h satisfying

$$i^2 = -1, \ \epsilon^2 = 0, \ h^2 = 1, \ ih = -hi = \epsilon + i.$$

Considering this extension, new results have been published in the literature. The reader is referred to [18–24] and the references therein.

In [25], the nth Horadam hybrid number, denoted by H_n, was introduced and several properties discussed. Its definition is

$$H_n = W_n + iW_{n+1} + \epsilon W_{n+2} + hW_{n+3},$$

where W_n is the nth Horadam number.

Finally, in [26], the author introduced the so-called Horadam hybrinomials. For $n \geq 1$, the nth Horadam hybrinomial is defined by

$$\mathbb{H}_n(x) = h_n(x) + h_{n+1}(x)i + h_{n+2}(x)\epsilon + h_{n+3}(x)h.$$

Moreover, Horadam hybrinomials stated via determinant of matrices, as below:

$$\det M_n(x) = \mathbb{H}_{n+1}(x),$$

where

$$M_n(x) = \begin{pmatrix} \mathbb{H}_2(x) & \mathbb{H}_1(x) & & & & \\ -q & px & 1 & & & \\ & -q & px & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -q & px & 1 \\ & & & & -q & px \end{pmatrix}.$$

The connection between particular cases of the Horadam sequence and the Chebyshev polynomials are well-known. The reader is referred, for example, to [13,27]. Here, we consider the matrix $M_n(x)$, we can represent the Horadam hybrid polynomials, by means of the Chebyshev polynomials of the second kind, as

$$\mathbb{H}_{n+1}(x) = (-i\sqrt{q})^{n-1}\left(U_{n-1}\left(\frac{ipx}{2\sqrt{q}}\right)\mathbb{H}_2(x) + i\sqrt{q}U_{n-2}\left(\frac{ipx}{2\sqrt{q}}\right)\mathbb{H}_1(x)\right),$$

where

$$\mathbb{H}_1(x) = a + bxi + (pbx^2 + qa)\epsilon + (p^2bx^3 + [pqa + qb]x)h,$$
$$\mathbb{H}_2(x) = bx + (pbx^2 + qa)i + (p^2bx^3 + [pqa + qb]x)\epsilon$$
$$+ ([p^2bx^3 + (pqa + qb)x^2]px + q[pbx^2 + qa])h.$$

Some special cases of Horadam hybrinomials are given in the Table 4.

Table 4. Some special cases for $\mathbb{H}_n(x)$.

	a	b	p	q
$FH_n(x)$	1	1	1	1
$LH_{n-1}(x)$	2	1	1	1
$PH_n(x)$	1	2	2	1
$QH_{n-1}(x)$	2	2	2	1

Notice that if $x = 1$, then

(i) $FH_n(1)$ is the nth Fibonacci hybrid number,
(ii) $LH_{n-1}(1)$ is the $(n-1)$th Lucas hybrid number,
(iii) $PH_n(1)$ is the nth Pell hybrid number,
(iv) $QH_{n-1}(1)$ is the $(n-1)$th Pell-Lucas hybrid number.

As an illustrative example, setting $n = 15$, $a = 1$, $b = 1$, $p = 1$, and $q = 1$, we obtain the Fibonacci hybrid polynomial $FH_{15}(x)$. In other words,

$$FH_{15}(x) = \mathbb{H}_{16}(x)$$
$$= (-i\sqrt{q})^{14}\left(U_{14}\left(\frac{ipx}{2\sqrt{q}}\right)\mathbb{H}_2(x) + i\sqrt{q}U_{13}\left(\frac{ipx}{2\sqrt{q}}\right)\mathbb{H}_1(x)\right)$$
$$= (-i)^{14}\left(U_{14}\left(\frac{ix}{2}\right)\mathbb{H}_2(x) + iU_{13}\left(\frac{ix}{2}\right)\mathbb{H}_1(x)\right),$$

where

$$\mathbb{H}_1(x) = 1 + xi + (x^2 + 1)\epsilon + (x^3 + 2x)h,$$
$$\mathbb{H}_2(x) = x + (x^2 + 1)i + (x^3 + 2x)\epsilon + (x^4 + 3x^2 + 1)h.$$

If we make $x = 1$, then the above equation gives the 15th Fibonacci hybrid number. By following the same way, for $n = 5$, $a = 1$, $b = 2$, $p = 2$, and $q = 1$, the Pell hybrid polynomials are represented by

$$PH_5(x) = \mathbb{H}_6(x)$$
$$= (-i)^4\left(U_4\left(\frac{2ix}{2}\right)\mathbb{H}_2(x) + iU_3\left(\frac{2ix}{2}\right)\mathbb{H}_1(x)\right)$$
$$= (1 + 20x^2 + 32x^4) + (6x + 48x^3 + 64x^5)i + (1 + 32x^2 + 128x^4 + 128x^6)\epsilon$$
$$+ (112x^3 + 8x + 320x^5 + 256x^7)h,$$

where

$$\mathbb{H}_1(x) = 1 + 2xi + (4x^2 + 1)\epsilon + (8x^3 + 4x)h,$$
$$\mathbb{H}_2(x) = 2x + (4x^2 + 1)i + (8x^3 + 4x)\epsilon + (16x^4 + 12x^2 + 1)h.$$

4. Conclusions

In this paper, we initially focus on general form of tridiagonal matrices and express their determinants by exploiting Chebyshev polynomials of second kind. Then, we consider the most general form of the several Gaussian Fibonacci numbers that is the generalized Gaussian Fibonacci numbers and the most general form of hybrid numbers that is the Horadam hybrid polynomials. Substituting various values into the statements of the matrices $T_n(a, b, c)$ and $M_n(x)$, one can obtain distinct subsequences of the generalized Gaussian Fibonacci numbers and Horadam hybrid numbers. Finally, we give some illustrative examples. To sum up, we obtain relationships between general types of tridiagonal matrices and the generalized Gaussian Fibonacci numbers and the Horadam hybrid polynomials which are the most general forms of their well-known subsequences.

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