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Article

Some Generalizations of the Jensen-Type Inequalities with Applications

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Abstract: Motivated by some results about reverses of the Jensen inequality for positive measure, in this paper we give generalizations of those results for real Stieltjes measure $d\lambda$ which is not necessarily positive using several Green functions. Utilizing these results we define some new mean value theorems of Lagrange and Cauchy types, and derive some new Cauchy-type means.

Keywords: Jensen's inequality; converse Jensen's inequality; Green function; means; exponential convexity

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1. Introduction

First papers about convex functions date from the end of the nineteenth century, but the real meaning of the convex functions has been developed with the works of the Danish mathematician J. L. W. V. Jensen (1859–1925) from 1905 and 1906 [1], and his famous inequality. In the many years after that, the Jensen inequality was considered under the weakened conditions, further improved, generalized, refined, reversed, etc., and it has been (and it is still) a constant inspiration for further investigation. By weakening of the conditions of the Jensen inequality, the Jensen-Stefensen inequality was derived. H. D. Brunk generalized it further in [2] and this result is known as the Jensen-Brunk inequality. Another generalization is the Jensen-Boas inequality (see [3]). These and many more results can be found in [4], an excellent book about convex functions. Also, many other famous inequalities are derived using the Jensen inequality, like the Cauchy inequality, the Hölder inequality, the Young inequality, inequalities between means, to name just a few. The applications of all of these inequalities are widely spread in different mathematical areas and so the influence and importance of the Jensen inequality are immeasurable. An interested reader can examine several old and brand new papers which use this inequality (for example [5–13]).

Though, the results that consider the Jensen inequality, its variants, reverses, converses, and refinements, consider the case when the measure is positive. Therefore it is of great interest to get the results where it is allowed that the measure can also be negative. In order to do that, we used the following set of the Green functions $G_p : [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ ($p = 1, 2, 3, 4$; $[\alpha, \beta] \subseteq \mathbb{R}$) defined by

$$G_1(t,s) = \begin{cases} \alpha - s, & \text{for } \alpha \le s \le t, \\ \alpha - t, & \text{for } t \le s \le \beta, \end{cases} \tag{1}$$

$$G_2(t,s) = \begin{cases} t - \beta, & \text{for } \alpha \le s \le t, \\ s - \beta, & \text{for } t \le s \le \beta, \end{cases} \tag{2}$$

$$G_3(t,s) = \begin{cases} t - \alpha, & \text{for } \alpha \le s \le t, \\ s - \alpha, & \text{for } t \le s \le \beta, \end{cases} \tag{3}$$

$$G_4(t,s) = \begin{cases} \beta - s, & \text{for } \alpha \leq s \leq t, \\ \beta - t, & \text{for } t \leq s \leq \beta. \end{cases} \quad (4)$$

These functions have certain nice properties. Due to them, we already got some new results for the inequalities of the Jensen-type and the inequalities of the converse Jensen-type (see for example [14,15]) and there are more to come.

Taking into account the properties of the functions G_p ($p = 1,2,3,4$), and motivated by the results from S. S. Dragomir in [16,17] with reverses of the Jensen inequality in the case when the measure is positive, here in this paper we will give the generalization of some of his results on the Jensen-type inequalities and the converse Jensen-type inequalities, but now allowing that the measure can also be negative. The results that are presented here, represent the continuation of the research presented in [18,19].

This paper is structured in the following way. After this introduction, the section with the main results follows. There are given three theorems where the Jensen-type (or converse Jensen-type) inequality for the continuous convex function is connected to the similar inequality for the Green function G_p ($p = 1,2,3,4$). In the next, third, section these theorems are then used to define new Lagrange and Cauchy type mean-value theorems, again connecting our function with the Green functions G_p ($p = 1,2,3,4$). The fourth section presents applications of our results in the direction of constructing exponentially convex functions and deriving new Cauchy means.

2. Main Results

As we already mentioned in the introduction, the functions G_p ($p = 1,2,3,4$) have some interesting and very useful properties, and they will also be very important for deriving our results in this paper. It is not difficult to see that for every fixed value $s \in [\alpha, \beta]$ our functions $G_p(\cdot, s)$ ($p = 1,2,3,4$) are continuous and convex on $[\alpha, \beta]$. Furthermore, every function $\phi : [\alpha, \beta] \to \mathbb{R}$ such that $\phi \in C^2([\alpha, \beta])$ can be expressed using these functions G_p ($p = 1,2,3,4$) as follows:

$$\phi(x) = \phi(\alpha) + (x-\alpha)\phi'(\beta) + \int_\alpha^\beta G_1(x,s)\phi''(s)ds, \quad (5)$$

$$\phi(x) = \phi(\beta) + (x-\beta)\phi'(\alpha) + \int_\alpha^\beta G_2(x,s)\phi''(s)ds, \quad (6)$$

$$\phi(x) = \phi(\beta) - (\beta-\alpha)\phi'(\beta) + (x-\alpha)\phi'(\alpha) + \int_\alpha^\beta G_3(x,s)\phi''(s)ds, \quad (7)$$

$$\phi(x) = \phi(\alpha) + (\beta-\alpha)\phi'(\alpha) - (\beta-x)\phi'(\beta) + \int_\alpha^\beta G_4(x,s)\phi''(s)ds. \quad (8)$$

These representations can be easily proved after a short calculation by integrating by parts, but the interested reader can also consult [14] (Lemma 1.1).

We will now give our main results, and for the sake of the clearer notation we will use

$$\overline{g} = \frac{\int_a^b g(x)d\lambda(x)}{\int_a^b d\lambda(x)}.$$

To begin with, we give an improvement of the Jensen inequality.

Theorem 1. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function and let $Im(g) \subseteq [m,M] \subseteq (\alpha,\beta)$. Suppose $\lambda : [a,b] \to \mathbb{R}$ is a continuous function or a function of bounded variation where $\lambda(a) \neq \lambda(b)$ and let $\overline{g} \in [\alpha,\beta]$. For every continuous convex function $\phi : [\alpha,\beta] \to \mathbb{R}$ holds*

$$\frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\overline{g}) \leq \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left(\phi'_-(M) - \phi'_+(m)\right), \quad (9)$$

if and only if for every $s \in [\alpha, \beta]$ holds

$$\frac{\int_a^b G_p(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} - G_p(\overline{g}, s) \leq \frac{(M - \overline{g})(\overline{g} - m)}{M - m} \Big((G_p)'_x(M, s) - (G_p)'_x(m, s) \Big), \quad (10)$$

where the functions G_p $(p = 1, 2, 3, 4)$ are defined in (1)–(4).

Moreover, this equivalence holds also in the case if we reverse the inequality sign in both (9) and (10).

Proof. Let for every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$ inequality (9) hold. As the functions $G_p(\cdot, s)$ $(p = 1, 2, 3, 4,\ s \in [\alpha, \beta])$ are continuous and convex on $[\alpha, \beta]$ for every $s \in [\alpha, \beta]$, the inequality (9) holds also for them. That is: it holds (10).
Let us now prove the other implication. We will first prove it in the case when $\phi \in C^2([\alpha, \beta])$. Suppose that for every $s \in [\alpha, \beta]$ inequality (10) holds. In the case when $\phi \in C^2([\alpha, \beta])$ we can use the representations (5)–(8), and we have

$$\phi'(x) = \phi'(\beta) + \int_\alpha^\beta (G_1)'_x(x, s)\phi''(s) ds, \quad (11)$$

$$\phi'(x) = \phi'(\alpha) + \int_\alpha^\beta (G_2)'_x(x, s)\phi''(s) ds, \quad (12)$$

$$\phi'(x) = \phi'(\alpha) + \int_\alpha^\beta (G_3)'_x(x, s)\phi''(s) ds, \quad (13)$$

$$\phi'(x) = \phi'(\beta) + \int_\alpha^\beta (G_4)'_x(x, s)\phi''(s) ds. \quad (14)$$

After some calculation, for every function $G_p(p = 1, 2, 3, 4)$ we obtain the following relation

$$\begin{aligned} &\frac{\int_a^b \phi(g(x)) d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\overline{g}) - \frac{(M - \overline{g})(\overline{g} - m)}{M - m} (\phi'(M) - \phi'(m)) \\ &\quad = \int_\alpha^\beta \Bigg[\frac{\int_a^b G_p(g(x), s) d\lambda(x)}{\int_a^b d\lambda(x)} - G_p(\overline{g}, s) \\ &\qquad - \frac{(M - \overline{g})(\overline{g} - m)}{M - m} \Big((G_p)'_x(M, s) - (G_p)'_x(m, s) \Big) \Bigg] \phi''(s) ds. \end{aligned} \quad (15)$$

If ϕ is convex function, then for every $s \in [\alpha, \beta]$ we have that $\phi''(s) \geq 0$. Further, if for every $s \in [\alpha, \beta]$ holds (10), then the term in the square brackets in (15) is less then or equal to zero. That means that the left hand side of (15) also has to be less then or equal to zero, which means that for every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, such that $\phi \in C^2([\alpha, \beta])$, holds inequality (9). After all, also notice that the existence of the second derivative of the function ϕ is not necessary, because it is possible to uniformly approximate a continuous convex function by convex polynomials (see also [4], p. 172).

We will skip the proof of the last sentence of our theorem since it can be conducted in the same way. □

Remark 1. *Note that the condition $[m, M] \subseteq (\alpha, \beta)$ assures that the one sided derivatives are finite. If $\phi'_+(\alpha)$ and $\phi'_-(\beta)$ are finite, then we can also allow that $Im(g)$ can be the whole interval $[\alpha, \beta]$.*

Remark 2. *The previous theorem considers the case when the function ϕ is convex. Concluding in a similar way, we can also get the results that hold for concave function. Under the conditions of the previous theorem, we have the following.*

For every continuous concave function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds (9) *if and only if for every $s \in [\alpha, \beta]$ holds* (10) *with reversed inequality sign.*

Also, for every continuous concave function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds (9) with reversed inequality sign if and only if for every $s \in [\alpha, \beta]$ holds (10).

Our next theorem states a similar result for the converse Jensen inequality, in the literature also known as the Lah-Ribarič or Edmundson-Lah-Ribarič inequality.

Theorem 2. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function and let $Im(g) \subseteq [m, M] \subseteq (\alpha, \beta)$. Suppose that $\lambda : [a, b] \to \mathbb{R}$ is a continuous function or a function of bounded variation where $\lambda(a) \neq \lambda(b)$. For every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds*

$$\frac{M-\overline{g}}{M-m}\phi(m) + \frac{\overline{g}-m}{M-m}\phi(M) - \frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \leq \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left(\phi'_-(M) - \phi'_+(m)\right), \tag{16}$$

if and only if for every $s \in [\alpha, \beta]$ holds

$$\frac{M-\overline{g}}{M-m}G_p(m,s) + \frac{\overline{g}-m}{M-m}G_p(M,s) - \frac{\int_a^b G_p(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} \leq \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left((G_p)'_x(M,s) - (G_p)'_x(m,s)\right), \tag{17}$$

where the functions G_p $(p = 1, 2, 3, 4)$ are defined in (1)–(4).

Moreover, this equivalence holds also in the case if we reverse the inequality sign in both (16) and (17).

Proof. We can carry out this proof in the same way as the previous one.

Let for every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$ inequality (16) hold. As the functions $G_p(\cdot, s)$ $(p = 1, 2, 3, 4, s \in [\alpha, \beta])$ are continuous and convex on $[\alpha, \beta]$ for every $s \in [\alpha, \beta]$, the inequality (16) also holds for them, i.e., (17) holds.

We prove now the other implication, and we do that first in the case when $\phi \in C^2([\alpha, \beta])$. Suppose that for every $s \in [\alpha, \beta]$ holds (17). When $\phi \in C^2([\alpha, \beta])$, using the representations (5)–(8) and relations (11)–(14), after some calculation for every function G_p $(p = 1, 2, 3, 4)$ we obtain

$$\begin{aligned}
&\frac{M-\overline{g}}{M-m}\phi(m) + \frac{\overline{g}-m}{M-m}\phi(M) - \frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left(\phi'(M) - \phi'(m)\right) \\
&= \int_\alpha^\beta \left[\frac{M-\overline{g}}{M-m}G_p(m,s) + \frac{\overline{g}-m}{M-m}G_p(M,s) - \frac{\int_a^b G_p(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)}\right. \\
&\qquad \left. - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left((G_p)'_x(M,s) - (G_p)'_x(m,s)\right)\right]\phi''(s)ds.
\end{aligned} \tag{18}$$

If ϕ is convex function, then for every $s \in [\alpha, \beta]$ we have that $\phi''(s) \geq 0$. Further, if for every $s \in [\alpha, \beta]$ holds (17), then the term in the square brackets in (18) is less then or equal to zero. This leads to conclusion that also the left hand side of (18) has to be less than or equal to zero, i.e., that for every contionuos convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, such that $\phi \in C^2([\alpha, \beta])$, holds (16).

As in the proof of Theorem 1, the same remark about the differentiability condition here also holds. And the comment about proving the last sentence of this theorem holds also here. □

Remark 3. *Note that here we don't need the condition that $\overline{g} \in [\alpha, \beta]$.*

Remark 4. *Also, in this case, concluding in a similar way, we can get the results that hold for the concave function. Under the conditions of the previous theorem, we have the following.*

For every continuous concave function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds (16) if and only if for every $s \in [\alpha, \beta]$ holds (17) with reversed inequality sign.

Also, for every continuous concave function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds (16) with reversed inequality sign if and only if for every $s \in [\alpha, \beta]$ holds (17).

In our third theorem we have again an improvement of the Jensen inequality, but now without derivatives.

Theorem 3. *Let $g : [a, b] \to \mathbb{R}$ be a continuous function and let $Im(g) \subseteq [m, M] \subseteq [\alpha, \beta]$. Suppose that $\lambda : [a, b] \to \mathbb{R}$ is a continuous function or a function of bounded variation where $\lambda(a) \neq \lambda(b)$ and let $\bar{g} \in [\alpha, \beta]$. For every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds*

$$\frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\bar{g}) \leq \max\left\{\frac{M-\bar{g}}{M-m}, \frac{\bar{g}-m}{M-m}\right\} \cdot \left[\phi(m) + \phi(M) - 2\phi\left(\frac{m+M}{2}\right)\right], \tag{19}$$

if and only if for every $s \in [\alpha, \beta]$ holds

$$\frac{\int_a^b G_p(g(x), s)d\lambda(x)}{\int_a^b d\lambda(x)} - G_p(\bar{g}, s) \leq \max\left\{\frac{M-\bar{g}}{M-m}, \frac{\bar{g}-m}{M-m}\right\} \cdot \left[G_p(m, s) + G_p(M, s) - 2G_p\left(\frac{m+M}{2}, s\right)\right], \tag{20}$$

where the functions G_p $(p = 1, 2, 3, 4)$ are defined in (1)–(4).

Moreover, this equivalence holds also in the case if we reverse the inequality sign in both (19) and (20).

Proof. This proof goes similarly to the previous two. The first implication is obvious.

In order to prove the other implication, we prove it first when $\phi \in C^2([\alpha, \beta])$. Suppose that for every $s \in [\alpha, \beta]$ holds (20). When $\phi \in C^2([\alpha, \beta])$, using the representations (5)–(8) we obtain

$$\begin{aligned}
&\frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\bar{g}) - \max\left\{\frac{M-\bar{g}}{M-m}, \frac{\bar{g}-m}{M-m}\right\} \cdot \left[\phi(m) + \phi(M) - 2\phi\left(\frac{m+M}{2}\right)\right] \\
&= \int_\alpha^\beta \left[\frac{\int_a^b G_p(g(x), s)d\lambda(x)}{\int_a^b d\lambda(x)} - G_p(\bar{g}, s)\right. \\
&\quad \left. - \max\left\{\frac{M-\bar{g}}{M-m}, \frac{\bar{g}-m}{M-m}\right\} \cdot \left[G_p(m, s) + G_p(M, s) - 2G_p\left(\frac{m+M}{2}, s\right)\right]\right]\phi''(s)ds.
\end{aligned}$$

If ϕ is convex, then for every $s \in [\alpha, \beta]$ we have that $\phi''(s) \geq 0$, and if for every $s \in [\alpha, \beta]$ holds (20), we conclude again that for every continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, such that $\phi \in C^2([\alpha, \beta])$, holds (19). As already explained in the previous proofs, also in this case the existence of ϕ'' is not necessary.

The proof of the last sentence of this theorem is analogous. □

Remark 5. *Under the conditions of the previous theorem, we have the following.*

For every continuous concave function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds (19) if and only if for every $s \in [\alpha, \beta]$ holds (20) with reversed inequality sign.

Also, for every continuous concave function $\phi : [\alpha, \beta] \to \mathbb{R}$ holds (19) with reversed inequality sign if and only if for every $s \in [\alpha, \beta]$ holds (20).

3. Mean-Value Theorems

Let $g : [a,b] \to \mathbb{R}$ be a continuous function, $Im(g) \subseteq [m,M] \subseteq (\alpha,\beta)$, and suppose that $\lambda : [a,b] \to \mathbb{R}$ is a continuous function or a function of bounded variation with $\lambda(a) \neq \lambda(b)$. The previous three theorems and inequalities (9), (16) and (19) offer us the possibility of formulating new Lagrange-type and Cauchy-type mean value theorems, and these mean value theorems can give us a possibility to derive some new means.

Firstly, for easier notation and formulation of these results, for functions g and λ and for continuous convex function $\phi : [\alpha,\beta] \to \mathbb{R}$, we will define three functionals by subtraction of the right-hand from the left-hand side of inequalities (9), (16) and (19):

$$A_1(g,\lambda,\phi) = \frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\overline{g}) - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}(\phi'(M)-\phi'(m)),$$

$$\begin{aligned} A_2(g,\lambda,\phi) = & \frac{M-\overline{g}}{M-m}\phi(m) + \frac{\overline{g}-m}{M-m}\phi(M) - \frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} \\ & - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}(\phi'(M)-\phi'(m)), \end{aligned}$$

$$\begin{aligned} A_3(g,\lambda,\phi) = & \frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\overline{g}) \\ & - \max\left\{\frac{M-\overline{g}}{M-m}, \frac{\overline{g}-m}{M-m}\right\} \cdot \left[\phi(m)+\phi(M)-2\phi\left(\frac{m+M}{2}\right)\right], \end{aligned}$$

with a remark that for $A_1(g,\lambda,\phi)$ and $A_3(g,\lambda,\phi)$ the value of $\overline{g}$ has to be in $[\alpha,\beta]$.

From Theorems 1–3 we obtain that:

- $A_1(g,\lambda,\phi) \leq 0$ if for any $p \in \{1,2,3,4\}$ we have that for every $s \in [\alpha,\beta]$ holds (10), and $A_1(g,\lambda,\phi) \geq 0$ if for any $p \in \{1,2,3,4\}$ we have that for every $s \in [\alpha,\beta]$ holds (10) with reversed inequality sign;
- $A_2(g,\lambda,\phi) \leq 0$ if for any $p \in \{1,2,3,4\}$ we have that for every $s \in [\alpha,\beta]$ holds (17), and $A_2(g,\lambda,\phi) \geq 0$ if for any $p \in \{1,2,3,4\}$ we have that for every $s \in [\alpha,\beta]$ holds (17) with reversed inequality sign;
- $A_3(g,\lambda,\phi) \leq 0$ if for any $p \in \{1,2,3,4\}$ we have that for every $s \in [\alpha,\beta]$ holds (20), and $A_3(g,\lambda,\phi) \geq 0$ if for any $p \in \{1,2,3,4\}$ we have that for every $s \in [\alpha,\beta]$ holds (20) with reversed inequality sign.

For each of these functionals we will now derive the Lagrange-type mean value theorem and after that also Cauchy-type mean value theorem.

Theorem 4. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function, $Im(g) \subseteq [m,M] \subseteq (\alpha,\beta)$, and let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^2([\alpha,\beta])$. Suppose that $\lambda : [a,b] \to \mathbb{R}$ is a continuous function or a function of bounded variation where $\lambda(a) \neq \lambda(b)$ and let $\overline{g} \in [\alpha,\beta]$.*

If for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha,\beta]$ holds (10) or if for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha,\beta]$ holds (10) with reversed inequality sign, then there exists $\xi \in [\alpha,\beta]$ such that

$$A_1(g,\lambda,\phi) = \frac{1}{2}\phi''(\xi)A_1(g,\lambda,\phi_0),$$

where $\phi_0(t) = t^2$.

Proof. Suppose that for some $p \in \{1,2,3,4\}$ for every $s \in [\alpha, \beta]$ holds inequality (10), or holds (10) with reversed inequality sign, i.e., that for that $p \in \{1,2,3,4\}$ the term

$$\frac{\int_a^b G_p(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} - G_p(\overline{g},s) - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left((G_p)'_x(M,s) - (G_p)'_x(m,s)\right)$$

retains the same sign on the whole $[\alpha, \beta]$.

As for ϕ the relation (15) is valid, we can apply the integral mean-value theorem on it, and obtain that for that $p \in \{1,2,3,4\}$ there exists $\xi \in [\alpha, \beta]$ such that

$$\begin{aligned}
&\frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\overline{g}) - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left(\phi'(M) - \phi'(m)\right) \\
&= \phi''(\xi)\int_\alpha^\beta \left[\frac{\int_a^b G_p(g(x),s)d\lambda(x)}{\int_a^b d\lambda(x)} - G_p(\overline{g},s)\right. \\
&\qquad\left. - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left((G_p)'_x(M,s) - (G_p)'_x(m,s)\right)\right]ds. \qquad (21)
\end{aligned}$$

Now we have to calculate the integral on the right side. Suppose that $p = 1$. We have that

$$\int_\alpha^\beta G_1(t,s)ds = \int_\alpha^t (\alpha - s)ds + \int_t^\beta (\alpha - t)ds = \frac{1}{2}(t-\alpha)(t+\alpha-2\beta)$$

and

$$\int_\alpha^\beta (G_1)'_x(t,s)ds = \int_\alpha^t 0ds + \int_t^\beta (-1)ds = t - \beta.$$

Calculating the right side of (21) we obtain

$$\begin{aligned}
&\frac{\int_a^b \phi(g(x))d\lambda(x)}{\int_a^b d\lambda(x)} - \phi(\overline{g}) - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\left(\phi'(M) - \phi'(m)\right) \\
&= \phi''(\xi)\left[\frac{\int_a^b \int_\alpha^\beta G_1(g(x),s)dsd\lambda(x)}{\int_a^b d\lambda(x)} - \int_\alpha^\beta G_1(\overline{g},s)ds\right. \\
&\qquad\left. - \frac{(M-\overline{g})(\overline{g}-m)}{M-m}\int_\alpha^\beta \left((G_1)'_x(M,s) - (G_1)'_x(m,s)\right)ds\right] \\
&= \frac{1}{2}\phi''(\xi)\left[\frac{\int_a^b (g(x))^2 d\lambda(x)}{\int_a^b d\lambda(x)} - \overline{g}^{\,2} - 2(M-\overline{g})(\overline{g}-m)\right] \\
&= \frac{1}{2}\phi''(\xi)A_1(g,\lambda,\phi_0),
\end{aligned}$$

what proves the statement of our theorem.

For other p we proceed the same way. For $p = 2$ we have that

$$\int_\alpha^\beta G_2(t,s)ds = \int_\alpha^t (t-\beta)ds + \int_t^\beta (s-\beta)ds = \frac{1}{2}(t-\beta)(t+\beta-2\alpha)$$

and

$$\int_\alpha^\beta (G_2)'_x(t,s)ds = \int_\alpha^t 1ds + \int_t^\beta 0ds = t - \alpha,$$

for $p = 3$

$$\int_\alpha^\beta G_3(t,s)ds = \int_\alpha^t (t-\alpha)ds + \int_t^\beta (s-\alpha)ds = \frac{1}{2}\Big[(t-\alpha)^2 + (\beta-\alpha)^2\Big]$$

and

$$\int_\alpha^\beta (G_3)'_x(t,s)ds = \int_\alpha^t 1ds + \int_t^\beta 0ds = t-\alpha,$$

for $p = 4$

$$\int_\alpha^\beta G_4(t,s)ds = \int_\alpha^t (\beta-s)ds + \int_t^\beta (\beta-t)ds = \frac{1}{2}\Big[(\beta-t)^2 + (\beta-\alpha)^2\Big]$$

and

$$\int_\alpha^\beta (G_4)'_x(t,s)ds = \int_\alpha^t 0ds + \int_t^\beta (-1)ds = t-\beta.$$

In all these cases direct calculation brings us to the same conclusion which proves our theorem. □

In the next two theorems we have the Lagrange mean-value theorems for the functionals $A_2(g,\lambda,\phi)$ and $A_3(g,\lambda,\phi)$. We give these results here without the proofs as these proofs are conducted analogously.

Theorem 5. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function, $Im(g) \subseteq [m,M] \subseteq (\alpha,\beta)$, and let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^2([\alpha,\beta])$. Suppose that $\lambda : [a,b] \to \mathbb{R}$ is a continuous function or a function of bounded variation where $\lambda(a) \neq \lambda(b)$.*

If for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha,\beta]$ holds (17) or if for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha,\beta]$ holds (17) with reversed inequality sign, then there exists $\xi \in [\alpha,\beta]$ such that

$$A_2(g,\lambda,\phi) = \frac{1}{2}\phi''(\xi)A_2(g,\lambda,\phi_0),$$

where $\phi_0(t) = t^2$.

Theorem 6. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function, $Im(g) \subseteq [m,M] \subseteq (\alpha,\beta)$, and let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi \in C^2([\alpha,\beta])$. Suppose that $\lambda : [a,b] \to \mathbb{R}$ is a continuous function or a function of bounded variation where $\lambda(a) \neq \lambda(b)$ and let $\overline{g} \in [\alpha,\beta]$.*

If for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha,\beta]$ holds (20) or if for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha,\beta]$ holds (20) with reversed inequality sign, then there exists $\xi \in [\alpha,\beta]$ such that

$$A_3(g,\lambda,\phi) = \frac{1}{2}\phi''(\xi)A_3(g,\lambda,\phi_0),$$

where $\phi_0(t) = t^2$.

The Cauchy-type mean value theorem for all three functionals $A_i(g,\lambda,\phi)$, $i = 1,2,3$, is given in the following result.

Theorem 7. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function, $Im(g) \subseteq [m,M] \subseteq (\alpha,\beta)$, and let $\phi,\psi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi,\psi \in C^2([\alpha,\beta])$. Suppose that $\lambda : [a,b] \to \mathbb{R}$ is a continuous function or a function of bounded variation where $\lambda(a) \neq \lambda(b)$.*

If for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha, \beta]$ holds (10) or if for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha, \beta]$ holds (10) with reversed inequality sign, then there exists $\xi \in [\alpha, \beta]$ such that

$$\frac{A_1(g,\lambda,\phi)}{A_1(g,\lambda,\psi)} = \frac{\phi''(\xi)}{\psi''(\xi)},$$

under condition that the denominator of the fraction on the left side is not equal to zero, and assuming that $\overline{g} \in [\alpha, \beta]$.

If for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha, \beta]$ holds (17) or if for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha, \beta]$ holds (17) with reversed inequality sign, then there exists $\xi \in [\alpha, \beta]$ such that

$$\frac{A_2(g,\lambda,\phi)}{A_2(g,\lambda,\psi)} = \frac{\phi''(\xi)}{\psi''(\xi)},$$

under condition that the denominator of the fraction on the left side is not equal to zero.

If for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha, \beta]$ holds (20) or if for any $p \in \{1,2,3,4\}$ for every $s \in [\alpha, \beta]$ holds (20) with reversed inequality sign, then there exists $\xi \in [\alpha, \beta]$ such that

$$\frac{A_3(g,\lambda,\phi)}{A_3(g,\lambda,\psi)} = \frac{\phi''(\xi)}{\psi''(\xi)},$$

under condition that the denominator of the fraction on the left side is not equal to zero, and assuming that $\overline{g} \in [\alpha, \beta]$.

Proof. We will prove this theorem only for the functional A_1, as the other cases can be proved analogously.

Let us define the function χ as follows:

$$\chi(t) = A_1(g,\lambda,\psi) \cdot \phi(t) - A_1(g,\lambda,\phi) \cdot \psi(t).$$

On this new function we can also apply Theorem 4, because it is the linear combination of the functions ϕ and ψ. After short calculation we obtain that there exists $\xi \in [\alpha, \beta]$ such that

$$\left(A_1(g,\lambda,\psi)\frac{\phi''(\xi)}{2} - A_1(g,\lambda,\phi)\frac{\psi''(\xi)}{2} \right) A_1(g,\lambda,\phi_0) = 0,$$

where $\phi_0(t) = t^2$. The term $A_1(g,\lambda,\phi_0)$ has to be different from zero, because, otherwise, we would have a contradiction with the condition that the denominator of the left side is not equal to zero, and consequently, we get the statement of our theorem. $\square$

Remark 6. *If our functions ϕ and ψ are such that there exists the inverse function of ϕ''/ψ'', then we have*

$$\xi_i = \left(\frac{\phi''}{\psi''}\right)^{-1} \left(\frac{A_i(g,\lambda,\phi)}{A_i(g,\lambda,\psi)}\right) \in [\alpha, \beta] \quad (i = 1,2,3),$$

what brings us to new Cauchy means.

4. Applications

In order to round off this paper, we would like to present some applications. When we speak about the mean-value theorems it is somehow most natural to derive some means. In order to get the Cauchy-type means with certain nice properties, we will use the method from the paper [20], which will firstly help us to define some new exponentially convex functions, and then to define the new means.

At the very beginning of this section, before stating our results, we have to recall some of the very basic definitions and facts about exponential convexity. Throughout this section, with I we will denote an open interval in $\mathbb{R}$.

Definition 1. *A function $f : I \to \mathbb{R}$ is $n-$exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \rho_i \rho_j f\left(\frac{x_i + x_j}{2}\right) \geq 0$$

for all $\rho_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$. A function $f : I \to \mathbb{R}$ is $n-$exponentially convex if it is $n-$exponentially convex in the Jensen sense and continuous on I.

Remark 7. *Note that $1-$exponentially convex functions in the Jensen sense are non-negative functions, as we can see from the definition. Further, $n-$exponentially convex functions in the Jensen sense are $k-$exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.*

Definition 2. *A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I, if it is $n-$exponentially convex in the Jensen sense for all $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.*

Remark 8. *Here we mention also some examples of exponentially convex functions from [20], as we will need them in the process of constructing our means:*

(i) $f : I \to \mathbb{R}$ *defined by* $f(x) = ce^{rx}$, *where* $c \geq 0$ *and* $r \in \mathbb{R}$.
(ii) $f : \mathbb{R}^+ \to \mathbb{R}$ *defined by* $f(x) = x^{-r}$, *where* $r > 0$.
(iii) $f : \mathbb{R}^+ \to \mathbb{R}^+$ *defined by* $f(x) = e^{-r\sqrt{x}}$, *where* $r > 0$.

Remark 9. *A positive function $f : I \to \mathbb{R}^+$ is log-convex in the Jensen sense on I if and only if it is $2-$exponentially convex in the Jensen sense on I, i.e., if and only if for every $\rho_1, \rho_2 \in \mathbb{R}$ and $x_1, x_2 \in I$ holds*

$$\rho_1^2 f(x_1) + 2\rho_1\rho_2 f\left(\frac{x_1 + x_2}{2}\right) + \rho_2^2 f(x_2) \geq 0.$$

If such function is also continuous on I, it follows that it is log-convex on I.

We also recall the following two lemmas from [4], p. 2.

Lemma 1. *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0\,.$$

Lemma 2. *If $f : I \to \mathbb{R}$ is a convex function and $x_1, x_2, y_1, y_2 \in I$ are such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}\,.$$

We have to recall also the definition of the divided difference of the second order.

Definition 3. *The divided difference of the second order of a function $f : I \to \mathbb{R}$ at mutually different points $x_0, x_1, x_2 \in I$ is defined recursively by*

$$\begin{aligned}
[x_i]f &= f(x_i), \qquad i = 0, 1, 2, \\
[x_i, x_{i+1}]f &= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}, \qquad i = 0, 1, \\
[x_0, x_1, x_2]f &= \frac{[x_1, x_2]f - [x_0, x_1]f}{x_2 - x_0}.
\end{aligned} \tag{22}$$

Remark 10. *The value $[x_0, x_1, x_2]f$ is independent of the order of the points x_0, x_1 and x_2. Also, we can extended this definition to include the case when some or all of the points are equal ([4], p. 14). Taking the limit $x_1 \to x_0$ in (22), we obtain*

$$\lim_{x_1 \to x_0} [x_0, x_1, x_2]f = [x_0, x_0, x_2]f = \frac{f(x_2) - f(x_0) - f'(x_0)(x_2 - x_0)}{(x_2 - x_0)^2}, \quad x_2 \neq x_0,$$

assuming that f' exists. Further, taking the limits $x_i \to x_0$, $i = 1, 2$, in (22), we obtain

$$\lim_{x_2 \to x_0} \lim_{x_1 \to x_0} [x_0, x_1, x_2]f = [x_0, x_0, x_0]f = \frac{f''(x_0)}{2}$$

assuming that f'' exists.

A function $f : I \to \mathbb{R}$ is convex if and only if for every choice of three mutually different points $x_0, x_1, x_2 \in I$ holds $[x_0, x_1, x_2]f \geq 0$.

Now, we can start with our results. We will take s certain family of functions, then apply our functionals to it, and in this way, we will construct $n-$exponentially convex and exponentially convex functions.

Before we proceed, using the functionals A_i $(i = 1, 2, 3)$ we will define three new functionals to assure that the functionals we use in this process are always non-negative, whenever they are defined. For continuous function $g : [a, b] \to \mathbb{R}$ with $Im(g) \subseteq [m, M] \subseteq (\alpha, \beta)$, a continuous function or a function of bounded variation $\lambda : [a, b] \to \mathbb{R}$ with $\lambda(a) \neq \lambda(b)$, and continuous convex function $\phi : [\alpha, \beta] \to \mathbb{R}$, we are now defining new functionals F_i $(i = 1, 2, 3)$ by:

$$\begin{aligned}
F_1(g, \lambda, \phi) &= -A_1(g, \lambda, \phi), \text{ if for any } p \in \{1, 2, 3, 4\} \text{ for every } s \in [\alpha, \beta] \text{ holds (10);} \\
F_1(g, \lambda, \phi) &= A_1(g, \lambda, \phi), \text{ if for any } p \in \{1, 2, 3, 4\} \text{ for every } s \in [\alpha, \beta] \text{ holds (10) with} \\
&\qquad\qquad\qquad \text{reversed inequality sign;} \\
F_2(g, \lambda, \phi) &= -A_2(g, \lambda, \phi), \text{ if for any } p \in \{1, 2, 3, 4\} \text{ for every } s \in [\alpha, \beta] \text{ holds (17);} \\
F_2(g, \lambda, \phi) &= A_2(g, \lambda, \phi), \text{ if for any } p \in \{1, 2, 3, 4\} \text{ for every } s \in [\alpha, \beta] \text{ holds (17) with} \\
&\qquad\qquad\qquad \text{reversed inequality sign;} \\
F_3(g, \lambda, \phi) &= -A_3(g, \lambda, \phi), \text{ if for any } p \in \{1, 2, 3, 4\} \text{ for every } s \in [\alpha, \beta] \text{ holds (20);} \\
F_3(g, \lambda, \phi) &= A_3(g, \lambda, \phi), \text{ if for any } p \in \{1, 2, 3, 4\} \text{ for every } s \in [\alpha, \beta] \text{ holds (20) with} \\
&\qquad\qquad\qquad \text{reversed inequality sign.}
\end{aligned}$$

It is now $F_i(g, \lambda, \phi) \geq 0$ $(i = 1, 2, 3)$ always when these functionals are defined.

Theorem 8. *Let $\Gamma = \{\phi_u : u \in I\}$ be a family of functions $\phi_u : [\alpha, \beta] \to \mathbb{R}$ where $\phi_u \in C([\alpha, \beta])$, such that for every three mutually different points $x_0, x_1, x_2 \in [\alpha, \beta]$ the function $u \mapsto [x_0, x_1, x_2]\phi_u$ is $n-$exponentially convex in the Jensen sense on I. Then the functions $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1, 2, 3)$ are also $n-$exponentially convex in the Jensen sense on I. If the function $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1, 2, 3)$ is also continuous on I, then it is $n-$exponentially convex on I.*

Proof. Let us define the function χ by

$$\chi(x) = \sum_{j,k=1}^{n} \rho_j \rho_k \phi_{\frac{u_j + u_k}{2}}(x),$$

where $\rho_j, \rho_k \in \mathbb{R}$, $u_j, u_k \in I$, $\phi_{\frac{u_j+u_k}{2}} \in \Gamma$, for $1 \leq j,k \leq n$. As the linear combination of continuous functions from the set Γ, this function is also continuous on $[\alpha, \beta]$. The fact that the mapping $u \mapsto [x_0, x_1, x_2]\phi_u$ is $n-$exponentially convex in the Jensen sense on I implies that

$$[x_0, x_1, x_2]\chi = \sum_{j,k=1}^{n} \rho_j \rho_k [x_0, x_1, x_2]\phi_{\frac{u_j+u_k}{2}} \geq 0,$$

for every three mutually different points $x_0, x_1, x_2 \in [\alpha, \beta]$, and this means that also our function χ is convex on $[\alpha, \beta]$. This implies

$$F_i(g, \lambda, \chi) \geq 0,$$

and therefore

$$\sum_{j,k=1}^{n} \rho_j \rho_k F_i\left(g, \lambda, \phi_{\frac{u_j+u_k}{2}}\right) \geq 0,$$

which means that $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1,2,3)$ are $n-$exponentially convex in the Jensen sense on I.

If $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1,2,3)$ is additionally also continuous on I, then it is n-exponentially convex by definition. □

As an immediate consequence of this theorem, we have the following corollary.

Corollary 1. *Let $\Gamma = \{\phi_u : u \in I\}$ be a family of functions $\phi_u : [\alpha, \beta] \to \mathbb{R}$ where $\phi_u \in C([\alpha, \beta])$, such that for every three mutually different points $x_0, x_1, x_2 \in [\alpha, \beta]$ the function $u \mapsto [x_0, x_1, x_2]\phi_u$ is exponentially convex in the Jensen sense on I. Then the functions $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1,2,3)$ are also exponentially convex in the Jensen sense on I. If the function $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1,2,3)$ is also continuous on I, then it is exponentially convex on I.*

We also have the following corollary.

Corollary 2. *Let $\Gamma = \{\phi_u : u \in I\}$ be a family of functions $\phi_u : [\alpha, \beta] \to \mathbb{R}$ where $\phi_u \in C([\alpha, \beta])$, such that for every three mutually different points $x_0, x_1, x_2 \in [\alpha, \beta]$ the function $u \mapsto [x_0, x_1, x_2]\phi_u$ is $2-$exponentially convex in the Jensen sense on I.*

(i) *If the function $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1,2,3)$ is continuous on I, then it is $2-$exponentially convex on I. Further, if $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1,2,3)$ is also strictly positive, then it is also log-convex on I, and it holds*

$$(F_i(g, \lambda, \phi_{u_2}))^{u_3-u_1} \leq (F_i(g, \lambda, \phi_{u_1}))^{u_3-u_2} (F_i(g, \lambda, \phi_{u_3}))^{u_2-u_1} \quad (i = 1,2,3), \tag{23}$$

for $u_1, u_2, u_3 \in I$ such that $u_1 < u_2 < u_3$.

(ii) *If the function $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1,2,3)$ is strictly positive and differentiable on I, then it holds*

$$\mu_{u,v}(g, F_i, \Gamma) \leq \mu_{y,z}(g, F_i, \Gamma) \qquad (i = 1,2,3) \tag{24}$$

for every $u, v, y, z \in I$ such that $u \leq y$ and $v \leq z$, where

$$\mu_{u,v}(g, F_i, \Gamma) = \begin{cases} \left(\dfrac{F_i(g, \lambda, \phi_u)}{F_i(g, \lambda, \phi_v)}\right)^{\frac{1}{u-v}}, & u \neq v, \\ \exp\left(\dfrac{\frac{d}{du} F_i(g, \lambda, \phi_u)}{F_i(g, \lambda, \phi_u)}\right), & u = v, \end{cases} \tag{25}$$

for $\phi_u, \phi_v \in \Gamma$.

Proof. (i) From Theorem 8 we get that the first sentence is valid, and the log-convexity follows from Remark 9. We still have to prove (25). We have that the function $u \mapsto F_i(g, \lambda, \phi_u)$ is strictly positive, and we can apply Lemma 1 on the function $f(x) = \log F_i(g, \lambda, \phi_x)$. We have that

$$(u_3 - u_2)F_i(g, \lambda, \phi_{u_1}) + (u_1 - u_3)F_i(g, \lambda, \phi_{u_2}) + (u_2 - u_1)F_i(g, \lambda, \phi_{u_3}) \geq 0$$

for $u_1, u_2, u_3 \in I$ $(u_1 < u_2 < u_3)$ and therefore inequality (23) holds.

(ii) From (i) we have that the function $u \mapsto F_i(g, \lambda, \phi_u)$ $(i = 1, 2, 3)$ is log-convex on I, and that means that the function $u \mapsto \log F_i(g, \lambda, \phi_u)$ $(i = 1, 2, 3)$ is convex on I. Applying Lemma 2 we get

$$\frac{\log F_i(g, \lambda, \phi_u) - \log F_i(g, \lambda, \phi_v)}{u - v} \leq \frac{\log F_i(g, \lambda, \phi_y) - \log F_i(g, \lambda, \phi_z)}{y - z} \tag{26}$$

where $u \leq y, \quad v \leq z, \quad u \neq v, \quad y \neq z$.

This implies

$$\mu_{u,v}(g, F_i, \Gamma) \leq \mu_{y,z}(g, F_i, \Gamma).$$

We get the cases $u = v$ and $y = z$ as limit cases from (26). □

Remark 11. *When two or all of the points $x_0, x_1, x_2 \in [\alpha, \beta]$ are equal, the results from* Theorem 8, Corollaries 1 *and* 2 *are also valid. The proofs for that can be obtained using* Remark 10 *and adequate characterization of convexity.*

Now we will look at some families of functions that fulfill the assumptions of Theorem 8, Corollaries 1 and 2, and using them we will get some Cauchy-type means.

Example 1. *Let us define a family of functions*

$$\Gamma_1 = \{\psi_u : \mathbb{R} \to [0, +\infty) : u \in \mathbb{R}\}$$

by

$$\psi_u(x) = \begin{cases} \frac{1}{u^2} e^{ux}, & u \neq 0; \\ \frac{1}{2} x^2, & u = 0. \end{cases}$$

It is $\frac{d^2}{dx^2}\psi_u(x) = e^{ux} > 0$ for $x \in \mathbb{R}$, and so for every $u \in \mathbb{R}$ the function ψ_u is convex on $\mathbb{R}$. Remark 8 *gives us that $u \mapsto \frac{d^2}{dx^2}\psi_u(x)$ is exponentially convex, and from [20] we also have that $u \mapsto [x_0, x_1, x_2]\psi_u$ is exponentially convex and therefore exponentially convex in the Jensen sense. That means that the family Γ_1 of functions ψ_u fulfills the assumptions from* Corollary 1, *and therefore we have that for $i = 1, 2, 3$ functions $u \mapsto F_i(g, \lambda, \psi_u)$ are exponentially convex in the Jensen sense. Although $u \mapsto \psi_u$ is not continuous at $u = 0$, these functions are continuous, and it follows that they are exponentially convex.*

Applying Corollary 2 *on Γ_1, we obtain*

$$\mu_{u,v}(g, F_i, \Gamma_1) = \begin{cases} \left(\frac{F_i(g,\lambda,\psi_u)}{F_i(g,\lambda,\psi_v)}\right)^{\frac{1}{u-v}}, & u \neq v; \\ \exp\left(\frac{F_i(g,\lambda,id\cdot\psi_u)}{F_i(g,\lambda,\psi_u)} - \frac{2}{u}\right), & u = v \neq 0; \\ \exp\left(\frac{1}{3}\frac{F_i(g,\lambda,id\cdot\psi_0)}{F_i(g,\lambda,\psi_0)}\right), & u = v = 0; \end{cases}$$

and from (24) we conclude that they are monotone in parameters u and v.

Using the Cauchy-type theorem from Section 3, applied for $\phi = \psi_u \in \Gamma_1$ and $\psi = \psi_v \in \Gamma_1$, we get that for

$$\mathcal{M}_{u,v}(g, F_i, \Gamma_1) = \log \mu_{u,v}(g, F_i, \Gamma_1) \quad (i = 1, 2, 3)$$

holds

$$\alpha \leq \mathcal{M}_{u,v}(g, F_i, \Gamma_1) \leq \beta, \ \text{for } i = 1, 2, 3.$$

If we set that $Im(g) = [\alpha, \beta]$, we get that

$$\alpha = \min_{t \in [a,b]} \{g(t)\} \leq \mathcal{M}_{u,v}(g, F_i, \Gamma_1) \leq \max_{t \in [a,b]} \{g(t)\} = \beta, \ \text{for } i = 1, 2, 3,$$

which means that then $\mathcal{M}_{u,v}(g, F_i, \Gamma_1)$ are means of the function g. From (24) we have that $\mathcal{M}_{u,v}(g, F_i, \Gamma_1)$ are also monotone.

Example 2. *Let us define a family of functions*

$$\Gamma_2 = \{\phi_u : \mathbb{R}^+ \to \mathbb{R} : u \in \mathbb{R}\}$$

by

$$\phi_u(x) = \begin{cases} \frac{x^u}{u(u-1)}, & u \neq 0, 1; \\ -\log x, & u = 0; \\ x \log x, & u = 1. \end{cases}$$

We have that $\frac{d^2}{dx^2}\phi_u(x) = x^{u-2} = e^{(u-2)\log x} > 0$, and so for every $u \in \mathbb{R}$ the function ϕ_u is convex for $x > 0$. From Remark 8 *we have that $u \mapsto \frac{d^2}{dx^2}\phi_u(x)$ is exponentially convex, and from [20] we have that $u \mapsto [x_0, x_1, x_2]\phi_u$ is exponentially convex and therefore exponentially convex in the Jensen sense. This means that Γ_2 fulfills the assumptions from* Corollary 1.

Now we will assume that $[\alpha, \beta]$ from Corollaries 1 *and* 2 *is a subset of $\mathbb{R}^+$, and we obtain that*

$$\mu_{u,v}(g, F_i, \Gamma_2) = \begin{cases} \left(\frac{F_i(g,\lambda,\phi_u)}{F_i(g,\lambda,\phi_v)}\right)^{\frac{1}{u-v}}, & u \neq v; \\ \exp\left(\frac{1-2u}{u(u-1)} - \frac{F_i(g,\lambda,\phi_0\phi_u)}{F_i(g,\lambda,\phi_u)}\right), & u = v \neq 0, 1; \\ \exp\left(1 - \frac{F_i(g,\lambda,\phi_0{}^2)}{2F_i(g,\lambda,\phi_0)}\right), & u = v = 0; \\ \exp\left(-1 - \frac{F_i(g,\lambda,\phi_0\phi_1)}{2F_i(g,\lambda,\phi_1)}\right), & u = v = 1. \end{cases}$$

As in the previous example, we have that functions $u \mapsto F_i(g, \lambda, \phi_u)$ ($i = 1, 2, 3$) are exponentially convex, and that $\mu_{u,v}(g, F_i, \Gamma_2)$ are monotone.

Using the Cauchy-type theorem from Section 3, applied for $\phi = \phi_u \in \Gamma_2$ and $\psi = \phi_v \in \Gamma_2$ we have that there exist

$$\xi_i \in [\alpha, \beta] \quad (i = 1, 2, 3),$$

such that

$$\xi_i^{u-v} = \frac{F_i(g, \lambda, \phi_u)}{F_i(g, \lambda, \phi_v)}.$$

As $\xi \mapsto \xi^{u-v}$ is invertible for $u \neq v$, in that case we obtain

$$\alpha \leq \left(\frac{F_i(g, \lambda, \phi_u)}{F_i(g, \lambda, \phi_v)}\right)^{\frac{1}{u-v}} \leq \beta, \ \text{where } i = 1, 2, 3.$$

As before, if we set $Im(g) = [\alpha, \beta]$, *we get*

$$\alpha = \min_{t\in[a,b]} \{g(t)\} \leq \left(\frac{F_i(g, \lambda, \phi_u)}{F_i(g, \lambda, \phi_v)} \right)^{\frac{1}{u-v}} \leq \max_{t\in[a,b]} \{g(t)\} = \beta \tag{27}$$

and that shows us that then $\mu_{u,v}(g, F_i, \Gamma_2)$ *are means of function* g.

We can also add here one additional parameter, we will denote it by r. *In that case, for* $r \neq 0$ *we use the substitution* $g \to g^r$, $u \to \frac{u}{r}$ *and* $v \to \frac{v}{r}$ *in* (27), *and we obtain*

$$\min_{t\in[a,b]} \left\{ (g(t))^r \right\} \leq \left(\frac{F_i(g^r, \lambda, \phi_u)}{F_i(g^r, \lambda, \phi_v)} \right)^{\frac{r}{u-v}} \leq \max_{t\in[a,b]} \left\{ (g(t))^r \right\}, \quad \textit{for } i = 1, 2, 3.$$

We can define new generalized mean by

$$\mu_{u,v;r}(g, F_i, \Gamma_2) = \begin{cases} \left(\mu_{\frac{u}{r},\frac{v}{r}}(g^r, F_i, \Gamma_2) \right)^{\frac{1}{r}}, & r \neq 0; \\ \mu_{u,v}(\log g, F_i, \Gamma_2), & r = 0. \end{cases}$$

Such means are also monotone. Namely, for $u, v, y, z \in \mathbb{R}$, $r \neq 0$ *such that* $u \leq y$, $v \leq z$, *we have*

$$\mu_{u,v;r}(g, F_i, \Gamma_2) \leq \mu_{y,z;r}(g, F_i, \Gamma_2), \quad \textit{where} \quad i = 1, 2, 3,$$

as

$$\mu_{\frac{u}{r},\frac{v}{r}}(g^r, F_i, \Gamma_2) = \left(\frac{F_i(g^r, \lambda, \phi_{\frac{u}{r}})}{F_i(g^r, \lambda, \phi_{\frac{v}{r}})} \right)^{\frac{r}{u-v}} \leq \left(\frac{F_i(g^r, \lambda, \phi_{\frac{y}{r}})}{F_i(g^r, \lambda, \phi_{\frac{z}{r}})} \right)^{\frac{r}{y-z}} = \mu_{\frac{y}{r},\frac{z}{r}}(g^r, F_i, \Gamma_2),$$

for $u, v, y, z \in \mathbb{R}$, $r \neq 0$, *such that* $\frac{u}{r} \leq \frac{y}{r}$, $\frac{v}{r} \leq \frac{z}{r}$, *and because* $\mu_{u,v}(g, F_i, \Gamma_2)$ *for* $i = 1, 2, 3$ *are monotone in both parameters. The result when* $r = 0$ *can be derived by taking the limit* $r \to 0$.

Example 3. *Let us define a family of functions*

$$\Gamma_3 = \{\theta_u : \mathbb{R}^+ \to \mathbb{R}^+ : u \in \mathbb{R}^+\}$$

by

$$\theta_u(x) = \frac{e^{-x\sqrt{u}}}{u}.$$

We have that $\frac{d^2}{dx^2}\theta_u(x) = e^{-x\sqrt{u}} > 0$, *and so for every* $u \in \mathbb{R}$ *the function* θ_u *is convex for* $x > 0$. Remark 8 *gives us that* $u \mapsto \frac{d^2}{dx^2}\theta_u(x)$ *is exponentially convex, and from* [20] *we then also have that* $u \mapsto [x_0, x_1, x_2]\theta_u$ *is exponentially convex. This means that* Γ_3 *fulfills the assumptions of* Corollary 1.

If we set that $[\alpha, \beta] \subset \mathbb{R}^+$, *we get*

$$\mu_{u,v}(g, F_i, \Gamma_3) = \begin{cases} \left(\frac{F_i(g,\lambda,\theta_u)}{F_i(g,\lambda,\theta_v)} \right)^{\frac{1}{u-v}}, & u \neq v; \\ \exp\left(-\frac{F_i(g,\lambda,id\cdot\theta_u)}{2\sqrt{u}F_i(g,\lambda,\theta_u)} - \frac{1}{u} \right), & u = v. \end{cases}$$

As before, we have also here that $u \mapsto F_i(g, \lambda, \theta_u)$ $(i = 1, 2, 3)$ *are exponentially convex, and that* $\mu_{u,v}(g, F_i, \Gamma_3)$ *are monotone.*

Using the Cauchy-type theorem from Section 3, *applied for* $\phi = \theta_u \in \Gamma_3$ *and* $\psi = \theta_v \in \Gamma_3$, *we get that for*

$$\mathcal{N}_{u,v}(g, F_i, \Gamma_3) = -(\sqrt{u} + \sqrt{v}) \log \mu_{u,v}(g, F_i, \Gamma_3)$$

holds

$$\alpha \leq \mathcal{N}_{u,v}(g, F_i, \Gamma_3) \leq \beta, \quad \text{for } i = 1,2,3.$$

If we set $Im(g)$ is $[\alpha, \beta]$, then $\mathcal{N}_{u,v}(g, F_i, \Gamma_3)$ are means of the function g.

Example 4. *Let us define a family of functions*

$$\Gamma_4 = \{\phi_u : \mathbb{R}^+ \to \mathbb{R}^+ : u \in \mathbb{R}^+\}$$

by

$$\phi_u(x) = \begin{cases} \frac{u^{-x}}{(\log u)^2}, & u \neq 1; \\ \frac{x^2}{2}, & u = 1. \end{cases}$$

We have that $\frac{d^2}{dx^2}\phi_u(x) = u^{-x} > 0$, and so for every $u \in \mathbb{R}$ ϕ_u is convex.

Remark 8 *gives us that $u \mapsto \frac{d^2}{dx^2}\phi_u(x)$ is exponentially convex, and from [20] we also have that $u \mapsto [x_0, x_1, x_2]\phi_u$ is exponentially convex. This means that Γ_4 fulfills the assumptions from* Corollary 1. *Assuming that $[\alpha, \beta] \subset \mathbb{R}^+$, we have*

$$\mu_{u,v}(g, F_i, \Gamma_4) = \begin{cases} \left(\frac{F_i(g,\lambda,\phi_u)}{F_i(g,\lambda,\phi_v)}\right)^{\frac{1}{u-v}}, & u \neq v; \\ \exp\left(-\frac{F_i(g,\lambda,id\cdot\phi_u)}{uF_i(g,\lambda,\phi_u)} - \frac{2}{u\log u}\right), & u = v \neq 1; \\ \exp\left(-\frac{2}{3}\frac{F_i(g,\lambda,id\cdot\phi_1)}{F_i(g,\lambda,\phi_1)}\right), & u = v = 1. \end{cases}$$

As before, we conclude that the functions $u \mapsto F_i(g, \lambda, \phi_v)$ ($i = 1,2,3$) are exponentially convex, and that $\mu_{u,v}(g, F_i, \Gamma_4)$ are monotone.

Using our Cauchy-type theorem applied for $\phi = \phi_u \in \Gamma_4$ and $\psi = \phi_v \in \Gamma_4$, we get that for

$$\mathcal{L}_{u,v}(g, F_i, \Gamma_4) = -L(u,v)\log \mu_{u,v}(g, F_i, \Gamma_4)$$

holds

$$\alpha \leq \mathcal{L}_{u,v}(g, F_i, \Gamma_4) \leq \beta, \quad \text{for } i = 1,2,3,$$

where $L(u,v)$ is the logarithmic mean defined by

$$L(u,v) = \begin{cases} \dfrac{v-u}{\log v - \log u}, & \text{for } u \neq v; \\ u, & \text{for } u = v. \end{cases}$$

If we set that $Im(g) = [\alpha, \beta]$, then $\mathcal{L}_{u,v}(g, F_i, \Gamma_4)$ are means of the function g.

5. Conclusions

The results of the Jensen inequality, its variants, reverses, converses and refinements, always consider the case when the measure is positive. Therefore, it is very interesting to get the results where it is allowed that the measure can also be negative, and that is done in this paper. This paper represents a further continuation of the research already published in papers [18,19]. Motivated by the results from [16,17] about the reverses of the Jensen inequality and converse Jensen inequality for positive measure, in this paper we used the Green functions G_p ($p = 1,2,3,4$) (defined in (1)–(4)) to give generalizations of these results for real Stieltjes measure $d\lambda$ which does not necessarily have to be positive. These results are then used for defining new mean-value theorems of Lagrange and Cauchy-type. As an application, these theorems are then used for constructing some new Cauchy means.

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