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Solving Integral Equations Using Weakly Compatible Mappings via D^*-Metric Spaces

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Abstract: We introduce a new pair of mappings (S,T) on D^*-metric spaces called D^*_S-W.C. and $D^*_{R_S}$-W.C. Many examples are presented to show the difference between these mappings and other types of mappings in the literature. Moreover, we obtain several common fixed point results by using these types of mappings and the (E.A) property. We then employ the fixed point results to establish the existence and uniqueness of a solution for a class of nonlinear integral equations.

Keywords: metric space; D^*-metric space; common fixed points; weakly commuting mappings; (E.A) property

MSC: 47H10; 54H25

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1. Introduction and Preliminaries

Fixed point theory is one of the most popular tools in topology, functional analysis and nonlinear analysis. This structure has attracted considerable attention from mathematicians due to the development of fixed point theory in such spaces. Many scholars generalize the usual notion of a metric space and obtain various fixed point results. Some of these generalizations can be noted in [1–7].

In 2007, Sedghi et al. introduced the concept of D^*-metric space and several authors proved the existence of some fixed point results satisfying some contractive conditions, see [8,9].

Definition 1 ([8]). *A D^*-metric space is a pair (E, D^*) where E is a nonempty set, and D^* is a nonnegative real-valued function defined on $E \times E \times E$ such that for all $\lambda, \kappa, \varpi, a \in E$ we have:*
(D^*1) $D^*(\lambda,\kappa,\varpi) \geq 0$,
(D^*2) $D^*(\lambda,\kappa,\varpi) = 0$ *iff* $\lambda = \kappa = \varpi$;
(D^*3) $D^*(\lambda,\kappa,\varpi) = D^*(\lambda,\varpi,\kappa) = D^*(\kappa,\varpi,\lambda) = \ldots$, *(symmetry in all three variables); and*
(D^*4) $D^*(\lambda,\kappa,\varpi) \leq D^*(\lambda,\kappa,a) + D^*(a,\varpi,\varpi)$, *for all* $\lambda,\kappa,\varpi,a \in E$.
*The mapping D^* is called a D^*-*metric *on E.*

One can easily verify that every D^*-metric on E defines a metric d_{D^*} on E by

$$d_{D^*}(\lambda,\kappa) = D^*(\lambda,\kappa,\kappa) \quad \text{for all } \lambda,\kappa \in E. \tag{1}$$

Example 1 ([8]). *Let (E,d) be a metric space. The function $D^* : E \times E \times E \to [0,+\infty)$, defined by*

$$D^*(\lambda,\kappa,\varpi) = \max\{d(\lambda,\kappa), d(\kappa,\varpi), d(\varpi,\lambda)\},$$

or

$$D^*(\lambda,\kappa,\varpi) = d(\lambda,\kappa) + d(\kappa,\varpi) + d(\varpi,\lambda),$$

for all $\lambda, \kappa, \varpi \in E$ is a D^*-metric on E.

Example 2 ([8])**.** *(1) Let $E = R^+$ define*

$$D^*(\lambda,\kappa,\varpi) = \begin{cases} 0, & \text{if } \lambda = \kappa = \varpi; \\ \max\{\lambda,\kappa,\varpi\}, & \text{otherwise.} \end{cases}$$

for every $\lambda, \kappa, \varpi \in E$, then (E, D^*) is a D^*-metric space.
(2) If $E = R$, then we define $D^*(\lambda,\kappa,\varpi) = |\lambda + \kappa - 2\varpi| + |\kappa + \varpi - 2\lambda| + |\varpi + \lambda - 2\kappa|$ for every $\lambda, \kappa, \varpi \in R$. Then (R, D^*) is a D^*-metric space.

Lemma 1 ([8])**.** *Let (E, D^*) be a D^*-metric space. Then $D^*(\lambda,\kappa,\kappa) = D^*(\lambda,\lambda,\kappa)$.*

Definition 2 ([8])**.** *Let (E, D^*) be a D^*-metric space, then:*

1. *A sequence (λ_n) is said to converge if there exists $\lambda \in E$ such that $\lim_{n,m\to+\infty} D^*(\lambda_n,\lambda_m,\lambda) = \lim_{n,m\to+\infty} D^*(\lambda,\lambda,\lambda_n) = 0$.*
2. *(λ_n) is said to be D^*-Cauchy if given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $D^*(\lambda_n,\lambda_n,\lambda_m) < \epsilon$, for all $n, m \geq N$, that is $D^*(\lambda_n,\lambda_n,\lambda_m) \to 0$ as $n, m \to +\infty$.*
3. *A D^*-metric space (E, D^*) is said to be complete if every D^*-Cauchy sequence in E is D^*-convergent in E.*

Lemma 2 ([8])**.** *Let E be a D^*-metric space, then the function $D^*(\lambda,\kappa,\varpi)$ is jointly continuous on $E \times E \times E$.*

Definition 3 ([10])**.** *For two self mappings S and T on a set E, μ and $t \in E$ are called coincidence point and point of coincidence of S and T, (respectively), if $t = S\mu = T\mu$.*

A pair of self mappings is called weakly compatible if they commute at their coincidence points.

Aamri and El Moutawakil [11] introduced the following definition which is a generalization of the concept of compatible mappings.

Definition 4 ([11])**.** *Two self mapping S and T on a metric space (E, d) satisfy the (E.A) property if there exists a sequence (λ_n) such that*

$$\lim_{n\to+\infty} S\lambda_n = \lim_{n\to+\infty} T\lambda_n = t,$$

for some $t \in E$.

In [12,13], the authors show that some fixed point theorems in symmetric G-metrics can be deduced from fixed point theorems on metric or quasi-metric spaces. In [14], Sedghi et al. claimed that every G-metric space is D^*-metric, but in [15] Z. Mustafa et al. proved that D^*-metric need not be G-metric as well as the G-metric need not be D^*-metric.

In this paper, we introduce a new pair of mappings (S, T) on D^*-metric spaces called D^*_S-W.C. and $D^*_{R_S}$-W.C. Many examples are presented to show the difference between these mappings and other types of mappings in the literature. Furthermore, we obtain several common fixed point results by using these types of mappings and the (E.A) property. Moreover, we present application on integral equation using the main results.

2. Main Results

This section will be divided into two subsections.

2.1. New Definitions and Their Properties

Definition 5. *A pair of self mappings (S,T) of a D^*-metric space (E,D^*) is said to be D^*_S-weakly commuting (D^*_S-W.C.) if*

$$D^*(ST\zeta, TS\zeta, SS\zeta) \leq D^*(S\zeta, S\zeta, T\zeta), \quad \text{for all} \quad \zeta \in E. \tag{2}$$

Definition 6. *A pair of self mappings (S,T) of a D^*-metric space (E,D^*) is said to be $D^*_{R_S}$-weakly commuting ($D^*_{R_S}$-W.C.) if*

$$D^*(ST\zeta, TS\zeta, SS\zeta) \leq R\ D^*(S\zeta, S\zeta, T\zeta), \quad \text{for all} \quad \zeta \in E \tag{3}$$

where R is a positive real number.

Remark 1. *The D^*_S-W.C. are $D^*_{R_S}$-W.C. Reciprocally, if $R \leq 1$, then $D^*_{R_S}$-W.C. is D^*_S-W.C.*
Swapping S and T in (2) and (3), then (S,T) are called D^*_T-W.C. and $D^*_{R_T}$-W.C., respectively.

Example 3. *Let $E = [0,2]$ and $D^*(\lambda,\kappa,\varpi) = |\lambda+\kappa-2\varpi| + |\kappa+\varpi-2\lambda| + |\lambda+\varpi-2\kappa|$, for all $\lambda,\kappa,\varpi \in E$. Define $S(\lambda) = 2-\lambda,\ T(\lambda) = \lambda$, then by an easy calculation, one can show that $D^*(ST\lambda, TS\lambda, SS\lambda) = 8|1-\lambda|$ and $D^*(S\lambda, T\lambda, S\lambda) = 8|1-\lambda|$. Then, the pair (S,T) is D^*_S-W.C. and $D^*_{R_S}$-W.C.*

Example 4. *Let $E = [1,3]$ be endowed with the D^*-metric $D^*(\lambda,\kappa,\varpi) = |\lambda-\kappa| + |\kappa-\varpi| + |\lambda-\varpi|$, for all $\lambda,\kappa,\varpi \in E$. Define $S(\lambda) = \frac{1}{2}\lambda+1,\ T(\lambda) = \frac{2}{3}\lambda+1$, then $D^*(ST\lambda, TS\lambda, SS\lambda) = \frac{\lambda}{6} + \frac{1}{3} \leq RD^*(S\lambda, T\lambda, S\lambda) = R\frac{\lambda}{3}$ for $R \geq \frac{3}{2}$. But*
for $\lambda = 1$ we see that $D^*(ST\lambda, TS\lambda, SS\lambda) = \frac{1}{2}$ and $D^*(S\lambda, T\lambda, S\lambda) = \frac{1}{3}$. Therefore, the pair (S,T) is not D^*_S-W.C., but it is $D^*_{R_S}$-W.C.

A D^*_S-W.C. need not be D^*_T-W.C. as in the following example.

Example 5. *Let $E = [0,1]$ be endowed with the D^*-metric $D^*(\lambda,\kappa,\varpi) = \max\{|\lambda-\kappa, |\kappa-\varpi|, |\lambda-\varpi|\}$, for all $\lambda,\kappa,\varpi \in E$. Define $S(\lambda) = \frac{1}{5}\lambda^3,\ T(\lambda) = \lambda^3$, then we see that $D^*(ST\lambda, TS\lambda, SS\lambda) = \frac{124}{625}\lambda^9$ and $D^*(S\lambda, T\lambda, S\lambda) = \frac{4}{5}\lambda^3$, while by an easy calculation, one can show that for $\lambda = 1$ we have*

$$D^*(TS(\lambda), ST(\lambda), TT(\lambda)) = 0.992 \nleq D^*(T(\lambda), S(\lambda), T(\lambda)) = 0.8.$$

Therefore, the pair (S,T) is not D^*_T-W.C., but it is D^*_S-W.C. .

Lemma 3. *If S and T are D^*_S-W.C. or $D^*_{R_S}$-W.C., then S and T are weakly compatible.*

Proof. Let λ be a coincidence point of S and T, i.e., $S(\lambda) = T(\lambda)$, then if the pair (S,T) is D^*_S-W.C., we have

$$\begin{aligned} D^*(S(T(\lambda)), T(S(\lambda)), S(T(\lambda))) &= D^*(S(T(\lambda)), T(S(\lambda)), S(S(\lambda))) \\ &\leq D^*(S(\lambda), T(\lambda), S(\lambda)) \\ &= D^*(S(\lambda), S(\lambda), S(\lambda)) = 0. \end{aligned}$$

It follows that $S(T(\lambda)) = T(S(\lambda))$, that is S and T are commute at their coincidence point.

Similarly, if the pair (S,T) is $D^*_{R_S}$-W.C., we have

$$\begin{aligned} D^*(S(T(\lambda)), T(S(\lambda)), S(T(\lambda))) &= D^*(S(T(\lambda)), T(S(\lambda)), S(S(\lambda))) \\ &\leq RD^*(S(\lambda), T(\lambda), S(\lambda)) \\ &= RD^*(S(\lambda), S(\lambda), S(\lambda)) = 0. \end{aligned}$$

Thus $S(T(\lambda)) = T(S(\lambda))$, then the pair (S, T) is weakly compatible. $\square$

The converse of Lemma 3 need not be true. The following examples confirm this statement.

Example 6. *Let $E = [0, 1]$ and $D^*(\lambda, \kappa, \varpi) = |\lambda - \kappa| + |\kappa - \varpi| + |\lambda - \varpi|$. Define $S, T : \lambda \to \lambda$ by $S(\lambda) = \lambda^3$ and $T(\lambda) = \frac{1}{5}\lambda^3$, $\lambda \in E$. We see that $\lambda = 0$ is the only coincidence point, also $S(T(0)) = S(0) = 0$ and $T(S(0)) = T(0) = 0$, so S and T are weakly compatible. One can see that for $\lambda = 1$, we have*

$$D^*(S(T(\lambda)), T(S(\lambda)), S(S(\lambda))) = 1.984 \nleq 1.6 = D^*(S(\lambda), T(\lambda), S(\lambda)).$$

Therefore, S and T are not D^*_S-W.C.

The following is an example of a pair (S, T) where S and T are commuting mappings and D^*_S-W.C. (and D^*_T-W.C.).

Example 7. *Let $E = [0, +\infty]$ and $D^*(\lambda, \kappa, \varpi) = \max\{|\lambda - \kappa|, |\kappa - \varpi|, |\lambda - \varpi|\}$ for all $\lambda, \kappa, \varpi \in \lambda$. Define the mappings $S, T : E \to E$ by*

$$S(\lambda) = \begin{cases} \lambda - \frac{\lambda^2}{2}, & \text{if } 0 \leq \lambda \leq 1, \\ \frac{1}{2}, & \text{if } \lambda > 1. \end{cases}, \quad T(\lambda) = \begin{cases} \lambda, & \text{if } 0 \leq \lambda \leq 1, \\ 1, & \text{if } \lambda > 1. \end{cases}$$

It is easy to see that

$$S(T(\lambda)) = T(S(\lambda)) = \begin{cases} \lambda - \frac{\lambda^2}{2}, & \text{if } \quad 0 \leq \lambda \leq 1 \\ \frac{1}{2}, & \text{if } \quad \lambda > 1. \end{cases}$$

Now we shall show that S and T are D^*_T-W.C. and D^*_S-W.C. First, we see that

$$S(S(\lambda)) = \begin{cases} \lambda - \lambda^2 + \frac{\lambda^3}{2} - \frac{\lambda^4}{8}, & \text{if } \quad 0 \leq \lambda \leq 1 \\ \frac{3}{8}, & \text{if } \quad \lambda > 1. \end{cases}$$

and

$$T(T(\lambda)) = \begin{cases} \lambda & \text{if } \quad 0 \leq \lambda \leq 1 \\ 1 & \text{if } \quad \lambda > 1. \end{cases}$$

Moreover,

$$|S(\lambda) - T(\lambda)| = \begin{cases} \frac{\lambda^2}{2}, & \text{if } \quad 0 \leq \lambda \leq 1 \\ \frac{1}{2}, & \text{if } \quad \lambda > 1, \end{cases}$$

$$|S(T(\lambda)) - T(S(\lambda))| = 0 \quad \text{for all} \quad \lambda \in E,$$

$$|S(T(\lambda)) - S(S(\lambda))| = |T(S(\lambda)) - S(S(\lambda))| = \begin{cases} \frac{\lambda^2}{2} - \frac{\lambda^3}{2} + \frac{\lambda^4}{8}, & \text{if } \quad 0 \leq \lambda \leq 1 \\ \frac{1}{8}, & \text{if } \quad \lambda > 1. \end{cases}$$

and

$$|S(T(\lambda)) - T(T(\lambda))| = |T(S(\lambda)) - T(T(\lambda))| = \begin{cases} \frac{\lambda^2}{2}, & \text{if } \quad 0 \leq \lambda \leq 1 \\ \frac{1}{2}, & \text{if } \quad \lambda > 1. \end{cases}$$

Now, if $0 \le \lambda \le 1$, *we have*

$$\begin{aligned} D^*(S(T(\lambda)),T(S(\lambda)),S(S(\lambda))) =& |S(T(\lambda)) - S(S(\lambda))| \\ =& |\frac{\lambda^2}{2} - \frac{\lambda^3}{2} + \frac{\lambda^4}{8}| \\ \le& \frac{\lambda^2}{2} \\ =& |S(\lambda) - T(\lambda)| = D^*(S(\lambda),T(\lambda),S(\lambda)). \end{aligned}$$

If $\lambda > 1$, *then*

$$\begin{aligned} D^*(S(T(\lambda)),T(S(\lambda)),S(S(\lambda))) =& |S(T(\lambda)) - S(S(\lambda))| \\ =& \frac{1}{8} \le \frac{1}{2} = |S(\lambda) - T(\lambda)| = D^*(S(\lambda),T(\lambda),S(\lambda)). \end{aligned}$$

Moreover, if $0 \le \lambda \le 1$, *we have*

$$\begin{aligned} D^*(S(T(\lambda)),T(S(\lambda)),T(T(\lambda))) =& |S(T(\lambda)) - T(T(\lambda))| \\ =& \frac{\lambda^2}{2} = |S(\lambda) - T(\lambda)| = D^*(S(\lambda),T(\lambda),S(\lambda)). \end{aligned}$$

Finally, if $\lambda > 1$, *then*

$$\begin{aligned} D^*(S(T(\lambda)),T(S(\lambda)),S(S(\lambda))) =& |S(T(\lambda)) - S(S(\lambda))| \\ =& \frac{1}{2} = |S(\lambda) - T(\lambda)| = D^*(S(\lambda),T(\lambda),S(\lambda)). \end{aligned}$$

Therefore, S and T are D^*_S*-W.C. and* D^*_T*-W.C. Moreover, they are commutes.*

The following example shows that (S,T) are commutes but not D^*_S-W.C. and D^*_T-W.C.

Example 8. *Let* $E = [0,+\infty)$ *and* $D^*(\lambda,\kappa,\varpi) = \begin{cases} 0, & \text{if } \lambda = \kappa = \varpi; \\ \max\{\lambda,\kappa,\varpi\}, & \text{otherwise.} \end{cases}$

Define $S,T : E \to E$ *by* $S(\lambda) = \frac{\lambda+1}{2}$ *and* $T(\lambda) = \frac{\lambda+5}{6}$, $\lambda \in E$. *We see that* $S(T\lambda) = \frac{\lambda+11}{12} = T(S\lambda)$, *but for* $\lambda = \frac{1}{2}$ *we have* $D^*(S(T(\frac{1}{2})),T(S(\frac{1}{2})),S(S(\frac{1}{2}))) = \frac{23}{24} \not\le \frac{11}{12} = D^*(S(\frac{1}{2}),T(\frac{1}{2}),S(\frac{1}{2}))$, *and* $D^*(T(S(\frac{1}{2})),S(T(\frac{1}{2})),T(T(\frac{1}{2}))) = \frac{71}{72} \not\le \frac{11}{12} = D^*(S(\frac{1}{2}),T(\frac{1}{2}), S(\frac{1}{2}))$. *Thus S and T are not* D^*_S*-W.C. and* D^*_T*-W.C.*

The following example shows that (S,T) are commutes and D^*_T-W.C. but not D^*_S-W.C.

Example 9. *Let* $E = [0,+\infty)$ *and* $D^*(\lambda,\kappa,\varpi) = \begin{cases} 0, & \text{if } \lambda = \kappa = \varpi; \\ \max\{\lambda,\kappa,\varpi\}, & \text{otherwise.} \end{cases}$ *Define* $S,T : E \to E$ *by* $S(\lambda) = 2\lambda$ *and* $T(\lambda) = \frac{1}{5}\lambda$, $\lambda \in E$. *We see that* $S(T\lambda) = \frac{2\lambda}{5} = T(S\lambda)$, *but for* $\lambda = 1$ *we have* $D^*(S(T(1)),T(S(1)),S(S(1))) = 4 \not\le 2 = D^*(S(1),T(1),S(1))$, *so S and T are not* D^*_S*-W.C.*

Example 10. *Let* $E = [1,+\infty)$ *and* $D^*(\lambda,\kappa,\varpi) = \begin{cases} 0, & \text{if } \lambda = \kappa = \varpi; \\ \max\{\lambda,\kappa,\varpi\}, & \text{otherwise.} \end{cases}$ *Define* $S,T : E \to E$ *by* $S(\lambda) = \lambda^4$ *and* $T(\lambda) = \lambda^3$, $\lambda \in E$. *We see that* $\lambda = 1$ *is the only coincidence point and* $S(T(1)) = S(1) = 1$ *and* $T(S(1)) = T(1) = 1$, *so S and T are weakly compatible. But, there is no* $R > 0$ *such that,*

$$D^*(S(T(\lambda)),T(S(\lambda)),S(S(\lambda))) = \lambda^{12} \le R\,\lambda^4 = R\,D^*(S(\lambda),T(\lambda),S(\lambda)).$$

for all $\lambda \in (1,+\infty)$. *Therefore, S and T are not* $D^*_{R_S}$*-W.C.*

Now, we rewrite Definition 4 in the setting of D^*-metric spaces.

Definition 7. *Two self mappings S and T on a D^*-metric space (E, D^*) satisfy the (E.A) property if there exists a sequence (λ_n) such that $(S\lambda_n)$ and $(T\lambda_n)$ are D^*-converge to t for some $t \in E$, that is*

$$\lim_{n\to+\infty} D^*(S\lambda_n, S\lambda_n, t) = \lim_{n\to+\infty} D^*(T\lambda_n, T\lambda_n, t) = 0.$$

In the following examples, we show that if S and T satisfy the (E.A) property then (S, T) need not be $D^*_{R_S}$-W.C. or D^*_S-W.C.

Example 11. *We return to Example 6. Let $\lambda_n = \frac{1}{n}$. We have $\lim_{n\to+\infty} S(\lambda_n) = \lim_{n\to+\infty} (\frac{1}{n})^3 = 0$, and $\lim_{n\to+\infty} T(\lambda_n) = \lim_{n\to+\infty} \frac{1}{5}(\frac{1}{n})^3 = 0$, therefore, $\lim_{n\to+\infty} S(\lambda_n) = \lim_{n\to+\infty} T(\lambda_n) = 0 \in [0, 1]$. Then S and T satisfy the (E.A) property, but as shown in Example 6, (S, T) is not D^*_S-W.C.*

Example 12. *We return to Example 10. Let $\lambda_n = 1 + \frac{1}{n}$. We have $\lim_{n\to+\infty} S(\lambda_n) = \lim_{n\to+\infty} (1 + \frac{1}{n})^4 = 1$, and $\lim_{n\to+\infty} T(\lambda_n) = \lim_{n\to+\infty} (1 + \frac{1}{n})^3 = 1$, therefore, $\lim_{n\to+\infty} S(\lambda_n) = \lim_{n\to+\infty} T(\lambda_n) = 1 \in [1, +\infty)$. Then S and T satisfy the (E.A) property, but as shown in Example 10, (S, T) is not $D^*_{R_S}$-W.C.*

2.2. Common Fixed Point Results

According to Matkowski [16], let Ψ be the set of functions $\psi : [0, +\infty) \to [0, +\infty)$ that satisfy:

1. ψ is a nondecreasing function,
2. $\lim_{n\to+\infty} \psi^n(\lambda) = 0$ for all $\lambda \in (0, +\infty)$.

$\psi \in \Psi$ is called a $\Psi-$map and one can easily see that:

1. $\psi(\lambda) < \lambda$ for all $\lambda \in (0, +\infty)$.
2. $\psi(0) = 0$.

Now, we present the first result as follows:

Theorem 1. *Let (E, D^*) be a D^*-metric space and suppose $S, T : E \to E$ are two mappings satisfying the following conditions:*

1. *S and T are D^*_S-W.C.*
2. *$S(E) \subseteq T(E)$.*
3. *$T(E)$ is a D^*-complete subspace of E.*
4. *$D^*(S(\lambda), S(\kappa), S(\varpi)) \leq \psi(M(\lambda, \kappa, \varpi))$, for all $\lambda, \kappa, \varpi \in E$, where,*

$$M(\lambda, \kappa, \varpi) = \max \left\{ \begin{array}{c} D^*(T(\lambda), T(\kappa), T(\varpi)), D^*(T(\lambda), S(\lambda), T(\lambda)), \\ D^*(T(\kappa), S(\lambda), T(\kappa)), D^*(T(\varpi), S(\lambda), T(\varpi))), \\ D^*(T(\varpi), S(\kappa), T(\varpi)), D^*(T(\kappa), T(\varpi), T(\kappa)), \\ D^*(T(\kappa), S(\kappa), T(\kappa)) \end{array} \right\}. \tag{4}$$

Then S and T have a unique common fixed point.

Proof. Let $\lambda_0 \in E$, then there is $\lambda_1 \in E$ such that $S(\lambda_0) = T(\lambda_1)$ and $\lambda_2 \in E$ where $S(\lambda_1) = T(\lambda_2)$, then by induction we can define a sequence $(\kappa_n) \in E$ as follows:

$$\kappa_n = S(\lambda_n) = T(\lambda_{n+1}),\ n \in \mathbf{N} \cup \{0\}.$$

We will show that the sequence (κ_n) is D^*-Cauchy. Now

$$D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) = D^*(S(\lambda_n), S(\lambda_{n+1}), S(\lambda_{n+1})) \leq \psi(M(\lambda_n, \lambda_{n+1}, \lambda_{n+1})) \tag{5}$$

where

$$M(\lambda_n, \lambda_{n+1}, \lambda_{n+1}) = \max\left\{\begin{array}{l} D^*(T(\lambda_n), T(\lambda_{n+1}), T(\lambda_{n+1}), \\ D^*(T(\lambda_n), S(\lambda_n), T(\lambda_n)), \\ D^*(T(\lambda_{n+1}), S(\lambda_n), T(\lambda_{n+1})), \\ D^*(T(\lambda_{n+1}), S(\lambda_n), T(\lambda_{n+1})), \\ D^*(T(\lambda_n), S(\lambda_{n+1}), T(\lambda_{n+1})), \\ D^*(T(\lambda_{n+1}), T(\lambda_{n+1}), T(\lambda_{n+1})), \\ D^*(T(\lambda_{n+1}), S(\lambda_{n+1}), T(\lambda_{n+1})) \end{array}\right\} \tag{6}$$

$$= \max\left\{\begin{array}{l} D^*(\kappa_{n-1}, \kappa_n, \kappa_n), D^*(\kappa_{n-1}, \kappa_n, \kappa_{n-1}), \\ D^*(\kappa_n, \kappa_n, \kappa_n), D^*(\kappa_n, \kappa_n, \kappa_n), \\ D^*(\kappa_n, \kappa_{n+1}, \kappa_n), D^*(\kappa_n, \kappa_n, \kappa_n), \\ D^*(\kappa_n, \kappa_{n+1}, \kappa_n) \end{array}\right\} \tag{7}$$

$$= \max\{D^*(\kappa_{n-1}, \kappa_n, \kappa_{n-1}), D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1})\}. \tag{8}$$

If $M(\lambda_n, \lambda_{n+1}, \lambda_{n+1}) = D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1})$, then from (5) and the properties of ψ we have

$$D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) \leq \psi(D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1})) < D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1})$$

which is contradiction. Thus, $M(\lambda_n, \lambda_{n+1}, \lambda_{n+1}) = D^*(\kappa_{n-1}, \kappa_n, \kappa_{n-1}) = D^*(\kappa_{n-1}, \kappa_n, \kappa_n)$. Therefore, for $n \in \mathbf{N} \cup \{0\}$ and from (5) we have,

$$\begin{aligned} D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) &\leq \psi(D^*(\kappa_{n-1}, \kappa_n, \kappa_n)) \\ &\leq \psi^2(D^*(\kappa_{n-2}, \kappa_{n-1}, \kappa_{n-1})) \\ &\;\;\vdots \\ &\leq \psi^n(D^*(\kappa_0, \kappa_1, \kappa_1)). \end{aligned} \tag{9}$$

Given $\epsilon > 0$, since $\lim_{n\to+\infty} \psi^n(D^*(\kappa_0, \kappa_1, \kappa_1)) = 0$, and $\psi(\epsilon) < \epsilon$, there is an integer $n_0 \in \mathbf{N}$, such that

$$\psi^n(D^*(\kappa_0, \kappa_1, \kappa_1)) < \epsilon - \psi(\epsilon), \quad \text{for all } n \geq n_0.$$

Hence, for $n \geq n_0$ we have

$$D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) \leq \psi^n(D^*(\kappa_0, \kappa_1, \kappa_1)) < \epsilon - \psi(\epsilon). \tag{10}$$

Now,

$$D^*(\kappa_{n+1}, \kappa_{k+1}, \kappa_{k+1}) = D^*(S(\lambda_{n+1}), S(\lambda_{k+1}), S(\lambda_{k+1})) \leq \psi(M(\lambda_{n+1}, \lambda_{k+1}, \lambda_{k+1})) \tag{11}$$

where

$$M(\lambda_{n+1}, \lambda_{k+1}, \lambda_{k+1}) = \max\left\{\begin{array}{l} D^*(T(\lambda_{n+1}), T(\lambda_{k+1}), T(\lambda_{k+1}), \\ D^*(T(\lambda_{n+1}), S(\lambda_{n+1}), T(\lambda_{n+1})), \\ D^*(T(\lambda_{k+1}), S(\lambda_{n+1}), T(\lambda_{k+1})), \\ D^*(T(\lambda_{k+1}), S(\lambda_{n+1}), T(\lambda_{k+1})), \\ D^*(T(\lambda_{k+1}), S(\lambda_{k+1}), T(\lambda_{k+1})), \\ D^*(T(\lambda_{k+1}), T(\lambda_{k+1}), T(\lambda_{k+1})), \\ D^*(T(\lambda_{k+1}), S(\lambda_{k+1}), T(\lambda_{k+1})) \end{array}\right\} \tag{12}$$

$$= \max \begin{Bmatrix} D^*(\kappa_n, \kappa_k, \kappa_k), D^*(\kappa_n, \kappa_{n+1}, \kappa_n), \\ D^*(\kappa_k, \kappa_{n+1}, \kappa_k), D^*(\kappa_k, \kappa_{n+1}, \kappa_k), \\ D^*(\kappa_k, \kappa_{k+1}, \kappa_k), D^*(\kappa_k, \kappa_k, \kappa_k), \\ D^*(\kappa_k, \kappa_{k+1}, \kappa_k) \end{Bmatrix}$$

$$= \max \begin{Bmatrix} D^*(\kappa_n, \kappa_k, \kappa_k), D^*(\kappa_n, \kappa_{n+1}, \kappa_n), \\ D^*(\kappa_k, \kappa_{n+1}, \kappa_k), D^*(\kappa_k, \kappa_{k+1}, \kappa_k) \end{Bmatrix}. \tag{13}$$

Now, for $m, n \in \mathbf{N}$; $m > n$, we claim that

$$D^*(\kappa_n, \kappa_m, \kappa_m) < \epsilon, \qquad \text{for all } m \geq n \geq n_o. \tag{14}$$

We will prove (14) by induction on m. Inequality (14) holds for $m = n + 1$, by using (10) and the fact that $\epsilon - \psi(\epsilon) < \epsilon$.

Assume inequality (14) holds for $m \leq k$. Now, we prove (14) for $m = k + 1$. According to (13) we have four different cases.

If $M(\lambda_{n+1}, \lambda_{k+1}, \lambda_{k+1}) = D^*(\kappa_n, \kappa_k, \kappa_k)$, then using triangle inequality, (10), (11) and (13)

$$\begin{aligned} D^*(\kappa_n, \kappa_{k+1}, \kappa_{k+1}) &\leq D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + D^*(\kappa_{n+1}, \kappa_{k+1}, \kappa_{k+1}) \\ &< \epsilon - \psi(\epsilon) + \psi(D^*(\kappa_n, \kappa_k, \kappa_k)) \\ &< \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon. \end{aligned}$$

If $M(\lambda_{n+1}, \lambda_{k+1}, \lambda_{k+1}) = D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1})$, then from (10) and properties of ψ we have

$$\begin{aligned} D^*(\kappa_n, \kappa_{k+1}, \kappa_{k+1}) &\leq D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + D^*(\kappa_{n+1}, \kappa_{k+1}, \kappa_{k+1}) \\ &< \epsilon - \psi(\epsilon) + \psi(D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1})) \\ &< \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon. \end{aligned}$$

If $M(\lambda_{n+1}, \lambda_{k+1}, \lambda_{k+1}) = D^*(\kappa_k, \kappa_{k+1}, \kappa_{k+1})$, then from (10) and properties of ψ we have

$$\begin{aligned} D^*(\kappa_n, \kappa_{k+1}, \kappa_{k+1}) &\leq D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + D^*(\kappa_{n+1}, \kappa_{k+1}, \kappa_{k+1}) \\ &< \epsilon - \psi(\epsilon) + \psi(D^*(\kappa_k, \kappa_{k+1}, \kappa_{k+1})) \\ &< \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon. \end{aligned}$$

If $M(\lambda_{n+1}, \lambda_{k+1}, \lambda_{k+1}) = D^*(\kappa_k, \kappa_k, \kappa_{n+1})$, then by triangle inequality, (10), (11) and properties of ψ we get

$$\begin{aligned} D^*(\kappa_n, \kappa_{k+1}, \kappa_{k+1}) &\leq D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + D^*(\kappa_{n+1}, \kappa_{k+1}, \kappa_{k+1}) \\ &\leq D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + \psi(M(\lambda_{n+1}, \lambda_{k+1}, \lambda_{k+1})) \\ &= D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + \psi(D^*(\kappa_k, \kappa_k, \kappa_{n+1})) \\ &\leq D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + D^*(\kappa_k, \kappa_k, \kappa_{n+1}) \\ &\leq D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + \psi(M(\lambda_k, \lambda_k, \lambda_{n+1})) \\ &= D^*(\kappa_n, \kappa_{n+1}, \kappa_{n+1}) + \psi\left(\max \begin{Bmatrix} D^*(\kappa_{k-1}, \kappa_{k-1}, \kappa_n), \\ D^*(\kappa_{k-1}, \kappa_k, \kappa_{k-1}), \\ D^*(\kappa_n, \kappa_k, \kappa_k) \end{Bmatrix} \right) \\ &< \epsilon - \psi(\epsilon) + \psi(\epsilon) = \epsilon. \end{aligned}$$

Hence, $(\kappa_n) = (T(\lambda_{n+1}))$ is a D^*-cauchy sequence in $T(E)$. Since $T(E)$ is D^*-complete, then there exists $t \in T(E)$ such that $\lim_{n \to +\infty} T(\lambda_n) = t = \lim_{n \to +\infty} S(\lambda_n)$. Having $t \in T(E)$, there exists $\mu \in E$ such that $T(\mu) = t$, also

$$\lim_{n \to +\infty} T(\lambda_n) = T(\mu) = \lim_{n \to +\infty} S(\lambda_n). \tag{15}$$

We will show that $S(\mu) = T(\mu)$. Suppose that $S(\mu) \neq T(\mu)$, then condition (4) implies that

$$D^*(S(\mu), S(\mu), S(\lambda_n)) \leq \psi\left(\max \begin{Bmatrix} D^*(T(\mu), T(\mu), T(\lambda_n)), D^*(T(\mu), S(\mu), T(\mu)), \\ D^*(T(\mu), S(\mu), T(\mu)), D^*(T(\lambda_n), S(\mu), T(\lambda_n)), \\ D^*(T(\lambda_n), S(\mu), T(\lambda_n)), D^*(T(\mu), T(\lambda_n), T(\mu)) \\ D^*(T(\mu), S(\mu), T(\mu)) \end{Bmatrix} \right). \tag{16}$$

Taking the lim sup as $n \to +\infty$ and using (15) and the fact that the function D^* is continuous, we get

$$
\begin{aligned}
D^*(S(\mu),S(\mu),T(\mu)) &\leq \limsup \psi\left(\max\left\{\begin{array}{l} D^*(T(\mu),T(\mu),T(\lambda_n)), D^*(T(\mu),S(\mu),T(\mu)), \\ D^*(T(\mu),S(\mu),T(\mu)), D^*(T(\lambda_n),S(\mu),T(\lambda_n)), \\ D^*(T(\lambda_n),S(\mu),T(\lambda_n)), D^*(T(\mu),T(\lambda_n),T(\mu)), \\ D^*(T(\mu),S(\mu),T(\mu)) \end{array}\right\}\right) \\
&\leq \psi\left(\limsup \max\left\{\begin{array}{l} D^*(T(\mu),T(\mu),T(\lambda_n)), D^*(T(\mu),S(\mu),T(\mu)), \\ D^*(T(\mu),S(\mu),T(\mu)), D^*(T(\lambda_n),S(\mu),T(\lambda_n)), \\ D^*(T(\lambda_n),S(\mu),T(\lambda_n)), D^*(T(\mu),T(\lambda_n),T(\mu)), \\ D^*(T(\mu),S(\mu),T(\mu)) \end{array}\right\}\right)
\end{aligned}
\tag{17}
$$

and so,

$$
\begin{aligned}
D^*(S(\mu),S(\mu),T(\mu)) \leq \psi(D^*(T(\mu),S(\mu),T(\mu))) &< D^*(T(\mu),S(\mu),T(\mu)) \\
&= D^*(S(\mu),S(\mu),T(\mu)),
\end{aligned}
\tag{18}
$$

which is a contradiction, hence $S(\mu) = T(\mu)$. Since the pair (S,T) is D^*_S-W.C., then

$$D^*(S(T(\mu)),T(S(\mu)),S(S(\mu))) \leq D^*(S(\mu),T(\mu),S(\mu)) = 0.$$

Thus, $SS(\mu) = ST(\mu) = TS(\mu) = TT(\mu)$, and so

$$S(t) = ST(\mu) = TS(\mu) = T(t).$$

To this end, we shall show that $t = T(\mu)$ is a common fixed point of S and T. Assume that $St \neq t$, then

$$D^*(S(t),S(\mu),S(\mu)) \leq \psi\left(\max\left\{\begin{array}{l} D^*(T(t),T(\mu),T(\mu)), D^*(T(t),S(t),T(t)), \\ D^*(T(\mu),S(t),T(\mu)), D^*(T(\mu),S(t),T(\mu)), \\ D^*(T(\mu),S(\mu),T(\mu)), D^*(T(t),T(\mu),T(\mu)), \\ D^*(T(\mu),S(\mu),T(\mu)) \end{array}\right\}\right). \tag{19}$$

Since $T(t) = S(t)$ and $T(\mu) = S(\mu)$, then (19) implies that

$$D^*(S(t),S(\mu),S(\mu)) \leq \psi(D^*(S(t),S(\mu),S(\mu))) < D^*(S(t),S(\mu),S(\mu)), \tag{20}$$

which is a contradiction. Hence $S(t) = S(\mu) = t$ and so, t is a common fixed point of S and T.

To prove the uniqueness, suppose there are two common fixed points, u_o and v_o such that $u_o \neq v_o$. Then, by condition (4) we have

$$D^*(u_o,v_o,v_o) \leq \psi(D^*(u_o,v_o,v_o)) < D^*(u_o,v_o,v_o), \tag{21}$$

a contradiction, so $u_o = v_o$. Then, t is the unique common fixed point. □

Corollary 1. *Theorem 1 remains true if we replace D^*_S-W.C. with D^*_T-W.C. or $D^*_{R_S}$-W.C. (retaining the rest of hypothesis).*

Theorem 2. *Let (E,D^*) be a complete D^*-metric space and suppose $S,T : E \to E$ are two mappings satisfying the following conditions:*

1. $S(T\lambda) = T(S\lambda)$ *for all* $\lambda \in E$.
2. $S(E) \subseteq T(E)$.

3. $D^*(S(\lambda), S(\kappa), S(\varpi)) \leq \alpha M(\lambda, \kappa, \varpi)$, *for all* $\lambda, \kappa, \varpi \in E$, *where* $\alpha \in [0, 1)$ *and*

$$M(\lambda, \kappa, \varpi) = \max \left\{ \begin{array}{c} D^*(T(\lambda), T(\kappa), T(\varpi)), D^*(T(\lambda), S(\lambda), T(\lambda)), \\ D^*(T(\kappa), S(\lambda), T(\kappa)), D^*(T(\varpi), S(\lambda), T(\varpi))), \\ D^*(T(\varpi), S(\kappa), T(\varpi)), D^*(T(\kappa), T(\varpi), T(\kappa)) \\ D^*(T(\kappa), S(\kappa), T(\kappa)) \end{array} \right\}. \tag{22}$$

Then, S and T have a unique common fixed point.

Proof. The proof is the same argument as in Theorem (1). □

Now, we give an example to support Theorem 1.

Example 13. *Let* $E = [0, 1]$, $D^*(\lambda, \kappa, \varpi) = \begin{cases} 0, & if\ \lambda = \kappa = \varpi; \\ \max\{\lambda, \kappa, \varpi\}, & otherwise. \end{cases}$

$S(\lambda) = \frac{1}{5}\lambda^3$, $T(\lambda) = \frac{4}{5}\lambda^3$ *and* $\psi(t) = \frac{1}{3}t$ *for all* $\lambda, \kappa, \varpi \in E$ *and* $t \geq 0$. *Thus, we have* $S(E) = [0, \frac{1}{5}]$ *and* $T(E) = [0, \frac{4}{5}]$ *is a* D^**-complete subspace of E. Note that,*

$$\begin{aligned} D^*(ST\lambda, TS\lambda, SS\lambda) &= \max\{\frac{64}{625}\lambda^9, \frac{4}{625}\lambda^9, \frac{1}{625}\lambda^9\} \\ &= \frac{64}{625}\lambda^9 \\ &\leq \frac{4}{5}\lambda^3 \\ &= \max\{\frac{1}{5}\lambda^3, \frac{4}{5}\lambda^3\} \\ &= D^*(S\lambda, T\lambda, S\lambda). \end{aligned}$$

Hence, S and T are D^*_S*-W.C. Moreover, for each* $\lambda \neq \kappa \neq \varpi$ *we have*

$$\begin{aligned} D^*(S\lambda, fy, fz) &= \frac{1}{5}\max\{\lambda^3, \kappa^3, \varpi^3\} \\ &\leq \frac{4}{15}\max\{\lambda^3, \kappa^3, \varpi^3\} \\ &= \psi(D^*(T\lambda, gy, gz)) \\ &\leq \psi(M(\lambda, \kappa, \varpi)). \end{aligned}$$

Hence, $u_o = 0$ *is the unique common fixed point of S and T, while the conditions of Theorem 1 are satisfied.*

Note that, S does not commute with T. Indeed, $S(T(\lambda)) = \frac{64}{625}\lambda^9 \neq T(S(\lambda)) = \frac{4}{625}\lambda^9$ *for any* $\lambda \neq 0$ *in E. Thus, Theorem 2 is not applicable.*

Theorem 3. *Let* (E, D^*) *be a* D^**-metric space. Suppose the mappings* $S, T : E \to E$ *are* D^*_S*-W.C. satisfying the following conditions:*

1. *S and T satisfy the (E.A) property.*
2. $T(E)$ *is a closed subspace of E.*
3. $D^*(S(\lambda), S(\kappa), S(\varpi)) \leq \psi(M(\lambda, \kappa, \varpi))$, *where,*

$$M(\lambda, \kappa, \varpi) = \max \left\{ \begin{array}{l} D^*(T(\lambda), S(\kappa), T(\kappa)), D^*(T(\lambda), S(\varpi), T(\varpi)), \\ D^*(T(\kappa), S(\lambda), T(\lambda)), D^*(T(\varpi), S(\lambda), T(\lambda)), \\ D^*(T(\varpi), S(\kappa), T(\kappa)), D^*(T(\kappa), S(\varpi), T(\varpi)) \end{array} \right\} \tag{23}$$

for all $\lambda, \kappa, \varpi \in E$, *then S and T have a unique common fixed point.*

Proof. Since the mappings S and T satisfy the (E.A) property, then there exists in E a sequence (λ_n) satisfying $\lim_{n\to+\infty} S(\lambda_n) = \lim_{n\to+\infty} T(\lambda_n) = t$ for some $t \in E$.

Since, $T(E)$ is a closed subspace of E and $\lim_{n\to+\infty} T(\lambda_n) = t$, hence there exists $\mu \in E$ such that $T(\mu) = t$. Moreover,

$$\lim_{n\to+\infty} S(\lambda_n) = t = T(\mu) = \lim_{n\to+\infty} T(\lambda_n).$$

We will show that $S(\mu) = T(\mu)$. Suppose to the contrary that $S(\mu) \neq T(\mu)$. The condition (3) implies that

$$D^*(S(\mu), S(\mu), S(\lambda_n)) \leq \psi(M(\mu, \mu, \lambda_n)) = \psi\left(\max\begin{Bmatrix} D^*(T(\mu), S(\lambda_n), T(\lambda_n)), \\ D^*(T(\lambda_n), S(\mu), T(\mu)), \\ D^*(T(\mu), S(\mu), T(\mu)) \end{Bmatrix}\right). \quad (24)$$

Taking the lim sup as $n \to +\infty$ and using the fact that the functions D^* is jointly continuous, we get

$$\begin{aligned} 0 < D^*(S(\mu), S(\mu), T(\mu)) &\leq \psi(D^*(T(\mu), S(\mu), T(\mu))) \\ &< D^*(T(\mu), S(\mu), T(\mu)) = D^*(S(\mu), S(\mu), T(\mu)), \end{aligned} \quad (25)$$

which is contradiction, so $S(\mu) = T(\mu)$. Since S and T are D^*_S-W.C. , and so

$$D^*(ST(\mu), TS(\mu), SS(\mu)) \leq D^*(S(\mu), T(\mu), S(\mu)) = 0,$$

therefore $SS(\mu) = ST(\mu) = TS(\mu) = TT(\mu)$, then

$$S(t) = ST(\mu) = TS(\mu) = TT(\mu) = T(t).$$

Now, we shall show that $t = S(\mu)$ is a common fixed point of S and T. Assume, $S(t) \neq t$, then

$$D^*(S(t), t, t) = D^*(S(t), S(\mu), S(\mu)) < \psi(M(t, \mu, \mu)) \quad (26)$$

where

$$\begin{aligned} M(t, \mu, \mu) &= \max\{D^*(T(t), S(\mu), T(\mu)), D^*(T(\mu), S(t), T(t)), D^*(T(\mu), S(\mu), T(\mu))\} \\ &= \max\{D^*(T(t), S(\mu), T(\mu)), D^*(T(\mu), S(t), T(t))\} \\ &= \max\{D^*(S(t), t, t), D^*(t, S(t), T(t))\} \\ &= \max\{D^*(S(t), t, t), D^*(t, S(t), S(t))\} \\ &= D^*(S(t), t, t). \end{aligned}$$

Thus,

$$D^*(S(t), t, t) \leq \psi(D^*(S(t), t, t)) < D^*(S(t), t, t),$$

a contradiction, so that $t = S(t) = T(t)$, then t is a common fixed point of S and T.

To prove uniqueness, suppose we have u_o and v_o such that $u_o \neq v_o$, $S(u_o) = T(u_o) = u_o$ and $S(v_o) = T(v_o) = v_o$, then

$$D^*(u_o, v_o, v_o) = D^*(S(u_o), S(v_o), S(v_o)) \leq \psi\left(\max\begin{Bmatrix} D^*(T(u_o), S(v_o), T(v_o)), \\ D^*(T(v_o), S(v_o), T(v_o)), \\ D^*(T(v_o), S(u_o), T(u_o)) \end{Bmatrix}\right) \quad (27)$$

$$< D^*(u_o, v_o, v_o)$$

which is a contradiction, so $u_o = v_o$. Then t is a unique common fixed point. □

Theorem 4. *Let (E, D^*) be a D^*-metric space. Suppose the mappings $S, T : E \to E$ are weakly compatible satisfying the following conditions:*

1. *S and T satisfy the (E.A) property,*
2. *$T(E)$ is a closed subspace of E,*
3. *$D^*(S(\lambda), S(\kappa), S(\varpi)) \leq$*

$$\psi\left(\max\left\{\begin{array}{l} D^*(T(\lambda), T(\kappa), T(\varpi)), D^*(T(\lambda), S(\lambda), T(\varpi)), \\ D^*(T(\varpi), S(\varpi), T(\varpi)), D^*(T(\kappa), S(\kappa), T(\varpi)) \end{array}\right\}\right) \tag{28}$$

for all $\lambda, \kappa, \varpi \in E$, then S and T have a unique common fixed point.

Proof. Since S and T satisfy the (E.A) property, there exists in E a sequence (λ_n) satisfying $\lim_{n\to+\infty} S(\lambda_n) = \lim_{n\to+\infty} T(\lambda_n) = t$ for some $t \in E$.

Since $T(E)$ is a closed subspace, then there exists $\mu \in E$ such that $T(\mu) = t$. Also, $\lim_{n\to+\infty} S(\lambda_n) = T(\mu) = \lim_{n\to+\infty} T(\lambda_n)$.

We shall show that $S(\mu) = T(\mu)$. Suppose that $S(\mu) \neq T(\mu)$, then the condition (3) implies that

$$D^*(S(\mu), S(\mu), S(\lambda_n)) \leq \psi\left(\max\left\{\begin{array}{l} D^*(T(\mu), T(\mu), T(\lambda_n)), \\ D^*(T(\mu), S(\mu), T(\lambda_n)), \\ D^*(T(\lambda_n), S(\lambda_n), T(\lambda_n)) \end{array}\right\}\right). \tag{29}$$

Taking the lim sup as $n \to +\infty$, we get

$$\begin{aligned} D^*(S(\mu), S(\mu), T(\mu)) &\leq \psi(\max\{D^*(T(\mu), T(\mu), T(\mu)), D^*(T(\mu), S(\mu), T(\mu))\}) \\ &= \psi(D^*(T(\mu), S(\mu), T(\mu))). \end{aligned}$$

Therefore,

$$\begin{aligned} D^*(S(\mu), S(\mu), T(\mu)) &\leq \psi(D^*(T(\mu), S(\mu), T(\mu))) \\ &< D^*(T(\mu), S(\mu), T(\mu)) = D^*(T(\mu), S(\mu), S(\mu)). \end{aligned} \tag{30}$$

A contradiction, hence $S(\mu) = T(\mu)$. Since S and T are weakly compatible, then $TS(\mu) = ST(\mu)$, and therefore, $SS(\mu) = ST(\mu) = TS(\mu) = TT(\mu)$. It follows that $S(t) = ST(\mu) = TS(\mu) = TT(\mu) = T(t)$.

Finally, we will show that $t = T(\mu)$ is a common fixed point of S and T. Suppose that $S(t) \neq t$, then

$$\begin{aligned} D^*(S(t), t, t) &= D^*(S(t), S(\mu), S(\mu)) \leq \psi\left(\max\left\{\begin{array}{l} D^*(T(t), T(\mu), T(\mu)), \\ D^*(T(t), S(t), T(\mu)), \\ D^*(T(\mu), S(\mu), T(t)) \end{array}\right\}\right) \\ &= \psi\left(\max\left\{\begin{array}{l} D^*(S(t), t, t), \\ D^*(S(t), S(t), t) \end{array}\right\}\right) < \max\left\{\begin{array}{l} D^*(S(t), t, t), \\ D^*(S(t), S(t), t) \end{array}\right\} \\ &= D^*(S(t), S(t), t), \end{aligned}$$

which is a contradiction, so $t = S(t) = T(t)$. Then t is a common fixed point.

To prove uniqueness, suppose we have u_o and v_o such that $u_o \neq v_o$, $S(u_o) = T(u_o) = u_o$ and $S(v_o) = T(v_o) = v_o$, then an easy calculation leads to

$$D^*(u_o, v_o, v_o) < D^*(u_o, v_o, v_o),$$

which is a contradiction. Hence, $u_o = v_o$. Then t is a unique common fixed point. □

Now, we give some examples to support Theorem 4.

Example 14. *Let* $E = [0, +\infty)$, $D^*(\lambda, \kappa, \varpi) = \begin{cases} 0, & \text{if } \lambda = \kappa = \varpi; \\ \max\{\lambda, \kappa, \varpi\}, & \text{otherwise.} \end{cases}$,

$S(\lambda) = \frac{1}{4}\lambda$, $T(\lambda) = \frac{1}{2}\lambda$ *and* $\psi(t) = \frac{2}{3}t$ *for all* $\lambda, \kappa, \varpi \in E$ *and* $t \geq 0$.

Note that $\lambda = 0$ *is the only coincidence point of S and T. Moreover,* $S(T(0)) = T(S(0)) = 0$, *therefore S and T are weakly compatible.*

Let $\lambda_n = \frac{1}{n}$, *then* $\lim_{n\to+\infty} D^*(S\lambda_n, S\lambda_n, 0) = \lim_{n\to+\infty} D^*(T\lambda_n, T\lambda_n, 0) = 0$, *so S and T satisfy the (E.A.) property. Moreover, for* $\lambda \neq \kappa \neq \varpi$ *we have*

$$\begin{aligned} D^*(S\lambda, fy, fz) &= \frac{1}{4}\max\{\lambda, \kappa, \varpi\} \\ &\leq \frac{1}{3}\max\{\lambda, \kappa, \varpi\} \\ &= \frac{2}{3}(\frac{1}{2}\max\{\lambda, \kappa, \varpi\} \\ &= \psi(D^*(T\lambda, gy, gz)) \\ &\leq \psi\left(\max\left\{\begin{matrix} D^*(T(\lambda), T(\kappa), T(\varpi)), D^*(T(\lambda), S(\lambda), T(\varpi)), \\ D^*(T(\varpi), S(\varpi), T(\varpi)), D^*(T(\kappa), S(\kappa), T(\varpi)) \end{matrix}\right\}\right). \end{aligned}$$

Hence, all conditions of Theorem 4 are satisfied and $u_o = 0$ *is the unique common fixed point of S and T.*

Example 15. *Let* $E = [2, 20]$ *and* $D^*(\lambda, \kappa, \varpi) = \max\{|\lambda - \kappa|, |\kappa - \varpi|, |\lambda - \varpi|\}$ *for all* $\lambda, \kappa, \varpi \in E$. *Define the mappings* $S, T : E \to E$ *by*

$$S(\lambda) = \begin{cases} 2 & \text{if} \quad \lambda = 2 \\ 6 & \text{if} \quad 2 < \lambda \leq 5 \\ 2 & \text{if} \quad 5 < \lambda \leq 20, \end{cases} \qquad \text{and} \quad T(\lambda) = \begin{cases} 2 & \text{if} \quad \lambda = 2 \\ 14 & \text{if} \quad 2 < \lambda \leq 5 \\ \frac{4\lambda+10}{15} & \text{if} \quad 5 < \lambda \leq 20. \end{cases}$$

Moreover, suppose that $\psi(t) = \frac{t}{2}$ *for all* $t \geq 0$. *Then, it is clear that* $T(E)$ *is a closed subspace of E and S and T are weakly compatible. If we consider the sequence* $\{\lambda_n\} = \{5 + \frac{1}{n}\}$, *then* $S\lambda_n \to 2$ *and* $T\lambda_n \to 2$ *as* $n \to +\infty$. *Thus, S and T satisfy the (E.A) property.*
On the other hand, a simple calculation gives that,

$$D^*(S\lambda, fy, fz) \leq \psi(D^*(T\lambda, gy, gz)) \quad \text{for all} \quad \lambda, \kappa, \varpi \in \lambda,$$

so in particular (28) holds. Finally, all hypotheses of Theorem 4 are satisfied and $u_o = 2$ *is the unique common fixed point of S and T.*

Corollary 2. *Theorems 3 and 4 remain true if we replace, respectively,* D^*_S*-W.C., weakly compatible and* $D^*_{R_S}$*-W.C. by any one of them (retaining the rest of hypothesis).*

Corollary 3. *Some corollaries could be derived from Theorems 1, 3 and 4 by taking* $\varpi = \kappa$ *or* $T = Id_\lambda$.

3. Application to Integral Equations

In this section, we will use Theorem 4 to show that there is a solution to the following integral equation:

$$\lambda(t) = h(\lambda(t)) + \int_0^t m(t, s)H(s, \lambda(s))ds; \quad t \in [0, 1] \tag{31}$$

where,

1. $h(t) : R \to R$ is a continuous function.
2. $m(t, s) : [0, 1] \times [0, 1] \to R$ are continuous functions.

3. $H(t,s) : [0,1] \times R \to R$ are continuous functions.

Let $E = C([0,1])$ be the set of all real continuous functions on $[0,1]$, endowed with the D^*-metric

$$D^*(u,v,w) = \begin{cases} 0, & \text{if } u = v = w; \\ \max\{\sup_{t\in[0,1]} |u|, \sup_{t\in[0,1]} |v|, \sup_{t\in[0,1]} |w|\} & \text{otherwise.} \end{cases}$$

Clearly, (E, D^*) is a complete D^*-metric space.

Theorem 5. *The integral Equation (31) has a solution u such that $u \in C([0,1])$ if the following conditions hold:*

1. $|h(t)| \le \beta|t|$
2. $\sup_{t\in[0,1]} m(t,s) \le c_0$, *where c_0 is any positive constant.*
3. $|H(s,t)| \le \alpha|t|$, *where $0 < \alpha c_0 + \beta < 1$.*

Proof. Define mappings $S, T : E \to E$ by

$$T(\lambda(t)) = \lambda(t) \tag{32}$$

$$S\lambda(t) = h(\lambda(t)) + \int_0^t m(t,s)H(s,\lambda(s))ds; \quad t \in [0,1]. \tag{33}$$

Now, we prove that S, T are weakly compatible mappings.

Suppose that $\lambda_0(t)$ is a coincidence point of S and T. That is $S(\lambda_0(t)) = T(\lambda_0(t)) = w_0(t)$, then

$S(T(\lambda_0(t))) = S(\lambda_0(t)) = T(S(\lambda_0(t)))$, so S and T are weakly compatible mappings.

Clearly the condition (2) of Theorem 4 is satisfied. Now, we prove condition (1) of Theorem 4.

Let $\lambda_n(t) = \frac{t}{n}; t \in [0,1]$, then using conditions (1) and (2) of Theorem (5) we get

$$\begin{aligned} |S\lambda_n(t)| &= |h(\frac{t}{n}) + \int_0^t m(t,s)H(s,\frac{s}{n})ds| \\ &\le |h(\frac{t}{n})| + |\int_0^t m(t,s)H(s,\frac{s}{n})ds| \\ &\le |h(\frac{t}{n})| + \int_0^t |m(t,s)||H(s,\frac{s}{n})ds| \\ &\le \beta\frac{t}{n} + \int_0^t c_0\alpha\frac{s}{n}ds \\ &= \beta\frac{t}{n} + c_0\alpha\frac{t^2}{2n}. \end{aligned}$$

Now, $0 \le \lim_{n\to+\infty} |S\lambda_n(t)| \le 0$, hence $\lim_{n\to+\infty} |S\lambda_n(t)| = 0$ and so $\lim_{n\to+\infty} S\lambda_n(t) = 0$. Moreover, $\lim_{n\to+\infty} T\lambda_n(t) = \lim_{n\to+\infty} \frac{t}{n} = 0$. Hence,

$$\lim_{n\to+\infty} D^*(S(\lambda_n(t)), S(\lambda_n(t)), 0) = \lim_{n\to+\infty} \sup_{t\in[0,1]} |S\lambda_n(t)| = 0,$$

and

$$\lim_{n\to+\infty} D^*(T(\lambda_n(t)), T(\lambda_n(t)), 0) = \lim_{n\to+\infty} \sup_{t\in[0,1]} |T\lambda_n(t)| = 0.$$

Therefore, (S, T) satisfied the (E.A) property.

Finally, we will show that condition (3) of Theorem (4) is satisfied. Let $\lambda(t), \kappa(t), \varpi(t) \in E$. Then, for all $t \in [0,1]$, we have

$$\begin{aligned}
|S(\lambda(t))| &= |h(\lambda(t)) + \int_0^t m(t,s)H(s,\lambda(s))\ ds| \\
&\leq |h(\lambda(t))| + \int_0^t |m(t,s)||H(s,\lambda(s))|\ ds \\
&\leq \beta|\lambda(t)| + \int_0^t \sup_{t\in[0,1]} |m(t,s)| \sup_{t\in[0,1]} |H(s,\lambda(s))|\ ds \\
&\leq \beta \sup_{t\in[0,1]} |\lambda(t)| + c_0 \int_0^t \sup_{t\in[0,1]} |H(s,\lambda(s))|\ ds \\
&\leq \beta \sup_{t\in[0,1]} |\lambda(t)| + c_0\alpha \sup_{t\in[0,1]} |\lambda(t)| \\
&= (c_0\alpha + \beta) \sup_{t\in[0,1]} |T(\lambda(t))|.
\end{aligned}$$

Hence,

$$\sup_{t\in[0,1]} |S(\lambda(t))| \leq \gamma \sup_{t\in[0,1]} |T(\lambda(t))|, \text{where } 0 \leq \gamma = c_0\alpha + \beta < 1. \tag{34}$$

Similarly, one can show that

$$\sup_{t\in[0,1]} |S(\kappa(t))| \leq \gamma \sup_{t\in[0,1]} |T(\kappa(t))|, \text{where } 0 \leq \gamma = c_0\alpha + \beta < 1, \tag{35}$$

$$\sup_{t\in[0,1]} |S(\varpi(t))| \leq \gamma \sup_{t\in[0,1]} |T(\varpi(t))|, \text{where } 0 \leq \gamma = c_0\alpha + \beta < 1. \tag{36}$$

Thus,

$$\begin{aligned}
D^*(S(\lambda(t)), S(\kappa(t)), S(\varpi(t))) &= \max\{\sup_{t\in[0,1]} |S(\lambda(t))|, \sup_{t\in[0,1]} |S(\kappa(t))|, \sup_{t\in[0,1]} |S(\varpi(t))|\} \\
&\leq \gamma \max\{\sup_{t\in[0,1]} |\lambda(t)|, \sup_{t\in[0,1]} |\kappa(t)|, \sup_{t\in[0,1]} |\varpi(t)|\} \\
&= \gamma \max\{\sup_{t\in[0,1]} |T(\lambda(t))|, \sup_{t\in[0,1]} |T(\kappa(t))|, \sup_{t\in[0,1]} |T(\varpi(t))|\} \\
&= \gamma D^*(T(\lambda(t)), T(\kappa(t)), T(\varpi(t))) \\
&\leq \gamma\left(\max\left\{\begin{matrix} D^*(T(\lambda), T(\kappa), T(\varpi)), D^*(T(\lambda), S(\lambda), T(\varpi)), \\ D^*(T(\varpi), S(\varpi), T(\varpi)), D^*(T(\kappa), S(\kappa), T(\varpi)) \end{matrix}\right\}\right) \\
&= \psi\left(\max\left\{\begin{matrix} D^*(T(\lambda), T(\kappa), T(\varpi)), D^*(T(\lambda), S(\lambda), T(\varpi)), \\ D^*(T(\varpi), S(\varpi), T(\varpi)), D^*(T(\kappa), S(\kappa), T(\varpi)) \end{matrix}\right\}\right).
\end{aligned}$$

Hence, all conditions of Theorem 4 hold with $\psi(t) = \gamma t$ and the mappings S, T have a common fixed point $u_o(t) \in C([0,1])$ which is a solution to the Equation (31). $\square$

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