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A Class of Quasilinear Equations with Riemann–Liouville Derivatives and Bounded Operators

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Abstract: The existence and uniqueness of a local solution is proved for the incomplete Cauchy type problem to multi-term quasilinear fractional differential equations in Banach spaces with Riemann–Liouville derivatives and bounded operators at them. Nonlinearity in the equation is assumed to be Lipschitz continuous and dependent on lower order fractional derivatives, which orders have the same fractional part as the order of the highest fractional derivative. The obtained abstract result is applied to study a class of initial-boundary value problems to time-fractional order equations with polynomials of an elliptic self-adjoint differential operator with respect to spatial variables as linear operators at the time-fractional derivatives. The nonlinear operator in the considered partial differential equations is assumed to be smooth with respect to phase variables.

Keywords: multi-term fractional differential equation; quasilinear equation; Riemann–Liouville fractional derivative; defect of Cauchy type problem; fixed point theorem; initial-boundary value problem

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1. Introduction

In recent decades, problems with fractional derivatives have been studied by many authors [1–5]. Now fractional integro-differential calculus is an important tool in modeling various phenomena that arise in physics, chemistry, mathematical biology, engineering, etc. (see e.g., [6,7]).

The purpose of this paper is to study the local unique solvability of initial value problems for multi-term equations in Banach spaces with fractional Riemann–Liouville derivatives $D_t^\beta z$, $\beta > 0$, fractional Riemann–Liouville integrals $J_t^\beta z$, $\beta \geq 0$, and with nonlinearity, which depends on fractional derivatives of lower orders

$$D_t^\alpha z(t) = \sum_{j=1}^{m-1} A_j D_t^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D_t^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J_t^{\beta_s} z(t)$$
$$+ F(t, D_t^{\alpha-m} z(t), D_t^{\alpha-m+1} z(t), \dots, D_t^{\alpha-1} z(t)). \tag{1}$$

Operators A_j, $j = 1, 2, \dots, m-1$, B_l, $l = 1, 2, \dots, n$, C_s, $s = 1, 2, \dots, r$ are supposed to be bounded on a Banach space $\mathcal{Z}$, a nonlinear map $F \in C(Z; \mathcal{Z})$, where Z is an open set in $\mathbb{R} \times \mathcal{Z}^m$.

Note that unique solvability issues for the Cauchy problem to multi-term linear equation of form (1) with Gerasimov—Caputo derivatives and bounded operators at them were studied in [8], various classes of nonlinear equations with Gerasimov—Caputo derivatives [9–11], or with a unique Riemann–Liouville derivative in a linear part of an equation [12,13] have been studied before.

Linear equations of form (1) with Riemann–Liouville derivatives were studied in the work [14] in the case of bounded operators in the equation, and in [15] in the case of closed operators. In [14] it was shown, that the Cauchy type problem for an equation with several Riemann–Liouville derivatives has the so-called defect m^*, when several initial data must be zero in the lower order initial conditions for the solvability of the problem. So, a natural initial value problem for a multi-term equation of such type is, generally speaking, the incomplete Cauchy problem

$$D_t^{\alpha-m+k}z(t_0) = z_k,\ k = m^*, m^*+1, \dots, m-1. \tag{2}$$

Section 2 of this work contains the unique solvability theorem for linear ($F \equiv f(t)$) problem (1), (2) from the work [14].

In Section 3, firstly problem (1), (2) is reduced to the integro-differential equation

$$z(t) = \sum_{p=m^*}^{m-1} Z_p(t-t_0)z_p + \int_{t_0}^{t} Z_{m-1}(t-s)F(s, D_s^{\alpha-m}z(s), \dots, D_s^{\alpha-1}z(s))ds, \tag{3}$$

where $\{Z_p(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$, $p = m^*, m^*+1, \dots, m-1$ are the p-resolving families of operators for linear Equation (1). Next, under the condition of Lipschitzian continuity of the nonlinear operator F, using the theorem of contraction mapping for Equation (3), we prove the unique solvability of problem (1), (2) on a small enough interval.

Finally, in the last section a theorem of a local in time unique solution existence is obtained for initial-boundary value problems to a class of quasilinear equations with time-fractional derivatives, where linear operators are polynomials of an elliptic self-adjoint operator, which is differential with respect to spatial variables.

2. Preliminary Results

Let us consider the fractional integral and fractional derivative of Riemann–Liouville with the initial point at $t_0 \in \mathbb{R}$:

$$J_t^\alpha h(t) := \int_{t_0}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds, \quad D_t^\alpha h(t) = D_t^m J_t^{m-\alpha} h(t), \quad t > t_0,$$

where $m-1 < \alpha \le m \in \mathbb{N}$, i.e., $m := \lceil \alpha \rceil$.

By $\mathfrak{L}[h]$ denote the Laplace transform of a function $h : \mathbb{R}_+ \to \mathcal{Z}$. For the fractional integral and the fractional derivative of Riemann–Liouville we have the equalities [2]

$$\mathfrak{L}[J_t^\alpha h](\lambda) = \lambda^{-\alpha}\mathfrak{L}[h](\lambda), \quad \mathfrak{L}[D_t^\alpha h](\lambda) = \lambda^\alpha \mathfrak{L}[h](\lambda) - \sum_{k=0}^{m-1} \lambda^{m-1-k} D_t^{\alpha-m+k} h(0),$$

Hereafter $D_t^{\alpha-m+k}h(0) := \lim\limits_{t\to 0+} D_t^{\alpha-m+k}h(t)$.

Let $\mathcal{Z}$ be a Banach space, $\mathcal{L}(\mathcal{Z})$ be the Banach space of bounded linear operators on $\mathcal{Z}$, $T > t_0$. Consider the inhomogeneous equation

$$D_t^\alpha z(t) = \sum_{j=1}^{m-1} A_j D_t^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D_t^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J_t^{\beta_s} z(t) + f(t),\ t \in (t_0, T). \tag{4}$$

Here $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l := \lceil \alpha_l \rceil$, $m := \lceil \alpha \rceil$, $\alpha_l - m_l \ne \alpha - m$, $l = 1, 2, \dots, n$, $\beta_1 > \beta_2 > \cdots > \beta_r \ge 0$, operators A_j, $j = 1, 2, \dots, m-1$, B_l, $l = 1, 2, \dots, n$, C_s, $s = 1, 2, \dots, r$, are linear and bounded in $\mathcal{Z}$. Let

$$\underline{\alpha} := \max\{\alpha_l : l \in \{1,2,\dots,n\},\, \alpha_l - m_l < \alpha - m\}, \quad \underline{m} = \lceil \underline{\alpha} \rceil,$$

$$\overline{\alpha} := \max\{\alpha_l : l \in \{1,2,\dots,n\},\, \alpha_l - m_l > \alpha - m\}, \quad \overline{m} = \lceil \overline{\alpha} \rceil.$$

Denote by $m^* := \max\{\underline{m} - 1, \overline{m}\}$ the defect of the Cauchy type problem for Equation (4) [14].

A solution of the incomplete Cauchy type problem

$$D_t^{\alpha-m+k} z(t_0) = z_k, \; k = m^*, m^*+1, \dots, m-1, \tag{5}$$

for (4) is a function $z : (t_0, T] \to \mathcal{Z}$ such that $J_t^{m-\alpha} z \in C^m((t_0,T];\mathcal{Z}) \cap C^{m-1}([t_0,T];\mathcal{Z})$, $J_t^{m_l-\alpha_l} z \in C^{m_l}((t_0,T];\mathcal{Z})$, $l = 1,2,\dots,n$, $J_t^{\beta_s} z \in C((t_0,T];\mathcal{Z})$, $s = 1,2,\dots,r$, while equality (4) for $t \in (t_0,T]$ and (5) hold.

Put $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_0 = \{\lambda \in \mathbb{C} : |\lambda| = r_0, \arg\lambda \in (-\pi,\pi)\}$, $\Gamma_\pm = \{\lambda \in \mathbb{C} : \arg\lambda = \pm\pi, |\lambda| \in [r_0, \infty)\}$,

$$R_\lambda := \left(I - \sum_{j=1}^{m-1} \lambda^{j-m} A_j - \sum_{l=1}^{n} \lambda^{\alpha_l - \alpha} B_l - \sum_{s=1}^{r} \lambda^{-\beta_s - \alpha} C_s \right)^{-1},$$

$$Z_p(t) = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha} R_\lambda \cdot \left(\lambda^{m-1-p} I - \sum_{j=p+1}^{m-1} \lambda^{j-1-p} A_j \right) e^{\lambda t} d\lambda, \; p = 0,1,\dots,m-1, \; t > 0.$$

Substitute in ([14], Theorem 2) $t - t_0$ instead of t and obtain the next result.

Theorem 1 ([14]). *Let $m-1 < \alpha \le m \in \mathbb{N}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \le m_l \in \mathbb{N}$, $\alpha_l - m_l \ne \alpha - m$, $l = 1,2,\dots,n$, $\beta_1 > \beta_2 > \cdots > \beta_r \ge 0$, $A_j \in \mathcal{L}(\mathcal{Z})$, $j = 1,2,\dots,m-1$, $B_l \in \mathcal{L}(\mathcal{Z})$, $l = 1,2,\dots,n$, $C_s \in \mathcal{L}(\mathcal{Z})$, $s = 1,2,\dots,r$, $z_k \in \mathcal{Z}$, $k = m^*, m^*+1,\dots,m-1$, $f \in C((t_0,T);\mathcal{Z}) \cap L_1(t_0,T;\mathcal{Z})$. Then there exists an unique solution to (4), (5). It has the form*

$$z(t) = \sum_{p=m^*}^{m-1} Z_p(t-t_0) z_p + \int_{t_0}^{t} Z_{m-1}(t-s) f(s) ds.$$

3. Quasilinear Equation

Let Z be an open set in $\mathbb{R} \times \mathcal{Z}^m$, $F : Z \to \mathcal{Z}$, consider the quasilinear equation

$$D_t^\alpha z(t) = \sum_{j=1}^{m-1} A_j D_t^{\alpha-m+j} z(t) + \sum_{l=1}^{n} B_l D_t^{\alpha_l} z(t) + \sum_{s=1}^{r} C_s J_t^{\beta_s} z(t)$$
$$+ F(t, D_t^{\alpha-m} z(t), D_t^{\alpha-m+1} z(t), \dots, D_t^{\alpha-1} z(t)). \tag{6}$$

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$$D_t^{\alpha-m+k} z(t_0) = z_k, \; k = m^*, m^*+1, \dots, m-1, \tag{7}$$

for Equation (6) on $(t_0, t_1]$ will be called such function $z \in C((t_0,t_1];\mathcal{Z})$, that $J_t^{m-\alpha} z \in C^m((t_0,t_1];\mathcal{Z}) \cap C^{m-1}([t_0,t_1];\mathcal{Z})$, $J_t^{m_l-\alpha_l} z \in C^{m_l}((t_0,t_1];\mathcal{Z})$, $l = 1,2,\dots,n$, and $J_t^{\beta_s} z \in C((t_0,t_1];\mathcal{Z})$, $s = 1,2,\dots,r$, the inclusion $(t, D_t^{\alpha-m} z(t), D_t^{\alpha-m+1} z(t), \dots, D_t^{\alpha-1} z(t)) \in Z$ and equality (6) are valid for all $t \in (t_0,t_1]$, conditions (7) are fulfilled.

Let us introduce the notations $\overline{x} := (x_0, x_1, \dots, x_{m-1}) \in \mathcal{Z}^m$, $S_\delta(\overline{x}) = \{\overline{y} \in \mathcal{Z}^{m-1} : \|y_k - x_k\|_{\mathcal{Z}} \le \delta, \, k = 0,1,\dots,m-1\}$.

A mapping $F : Z \to \mathcal{Z}$ is called locally Lipschitzian in $\overline{x}$, if for every $(t,\overline{x}) \in Z$ there exist such $\delta > 0$, $l > 0$, that $[t-\delta, t+\delta] \times S_\delta(\overline{x}) \subset Z$, and for all $(s,\overline{y}),(s,\overline{v}) \in [t-\delta,t+\delta] \times S_\delta(\overline{x})$ the inequality

$$\|F(s,\overline{y}) - F(s,\overline{v})\|_{\mathcal{Z}} \le l \sum_{k=0}^{m-1} \|y_k - v_k\|_{\mathcal{Z}}$$

is satisfied.

Lemma 1. *Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m = \lceil \alpha \rceil$, $m_l = \lceil \alpha_l \rceil$, $\alpha_l - m_l \neq \alpha - m$, $l = 1,2,\ldots,n$, $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$, $A_j \in \mathcal{L}(\mathcal{Z})$, $j = 1,2,\ldots,m-1$, $B_l \in \mathcal{L}(\mathcal{Z})$, $l = 1,2,\ldots,n$, $C_s \in \mathcal{L}(\mathcal{Z})$, $s = 1,2,\ldots,r$, $z_k \in \mathcal{Z}$, $k = m^*, m^*+1,\ldots,m-1$, Z be an open set in $\mathbb{R} \times \mathcal{Z}^m$, $(t_0,0,0,\ldots,0,z_{m^*},z_{m^*+1},\ldots,z_{m-1}) \in Z$, $F \in C(Z;\mathcal{Z})$. Then a function $z : (t_0,t_1] \to \mathcal{Z}$ is a solution of problem (6), (7) on $(t_0,t_1]$, if and only if $J_t^{m-\alpha} z \in C^{m-1}([t_0,t_1];\mathcal{Z})$ and for all $t \in (t_0,t_1]$*

$$z(t) = \sum_{p=m^*}^{m-1} Z_p(t-t_0)z_p + \int_{t_0}^{t} Z_{m-1}(t-s)F(s, D_s^{\alpha-m}z(s),\ldots,D_s^{\alpha-1}z(s))ds. \tag{8}$$

Proof. If z is a solution of problem (6), (7), then the mapping

$$t \to F(t, D_t^{\alpha-m}z(t), D_t^{\alpha-m+1}z(t),\ldots,D_t^{\alpha-1}z(t))$$

acts continuously from $[t_0,t_1]$ into $\mathcal{Z}$ due to the definition of the solution at small enough $t_1 - t_0$. By Theorem 2 (see [14]) a solution satisfies Equation (8).

Let z satisfy Equation (8), then one can verify that z is a solution to problem (6), (7) due to Theorem 1 [14] and by repeating word to word the proof of Lemma 3 in [14]. $\square$

Theorem 2. *Let $m-1 < \alpha \leq m \in \mathbb{N}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \leq m_l \in \mathbb{N}$, $\alpha_l - m_l \neq \alpha - m$, $l = 1,2,\ldots,n$, $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$, $A_j \in \mathcal{L}(\mathcal{Z})$, $j = 1,2,\ldots,m-1$, $B_l \in \mathcal{L}(\mathcal{Z})$, $l = 1,2,\ldots,n$, $C_s \in \mathcal{L}(\mathcal{Z})$, $s = 1,2,\ldots,r$, $z_k \in \mathcal{Z}$, $k = m^*,m^*+1,\ldots,m-1$, Z be an open set in $\mathbb{R} \times \mathcal{Z}^m$, $(t_0,0,0,\ldots,0,z_{m^*},z_{m^*+1},\ldots,z_{m-1}) \in Z$, a mapping $F \in C(Z;\mathcal{Z})$ is locally Lipschitzian in $\overline{x}$. Then there exists such $t_1 > t_0$, that problem (6), (7) has an unique solution on $(t_0,t_1]$.*

Proof. Take $y := J_t^{m-\alpha}z \in C^{m-1}([t_0,t_1],\mathcal{Z})$, then $y^{(k)} = D_t^{\alpha-m+k}z$, $k = 1,2,\ldots,m-1$. Then the mapping $t \to F(t,y(t),y^{(1)}(t),\ldots,y^{(m-1)}(t))$ acts continuously from $[t_0,t_1]$ into $\mathcal{Z}$. By Lemma 1 it suffices to show that the equation

$$y(t) = \sum_{p=m^*}^{m-1} J_t^{m-\alpha} Z_p(t-t_0)z_p + J_t^{m-\alpha} \int_{t_0}^{t} Z_{m-1}(t-s)F(s,y(s),y^{(1)}(s),\ldots,y^{(m-1)}(s))ds \tag{9}$$

has an unique solution $y \in C^{m-1}([t_0,t_1],\mathcal{Z})$ for some $t_1 > t_0$.

It was proved in Theorem 1 [14] that $D_t^{\alpha-m+n}Z_{m-1}(0) = 0$, $n = 0,1,\ldots,m-2$. Since for all $p = 0,1,\ldots,m-1$

$$\left\| \frac{\lambda^{-\alpha}R_\lambda}{\mu - \lambda} \left(\lambda^{m-1-p}I - \sum_{j=p+1}^{m-1} \lambda^{j-1-p}A_j \right) \right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{C_1}{|\lambda|^{\alpha-m+2}},$$

$\alpha - m + 2 > 1$, so, $\mathfrak{L}[D_t^{\alpha-m+n}Z_{m-1}](\mu) = \mu^{n-m}R_\mu$, at $t \in [t_0,t_1]$, $n = 0,1,\ldots,m-2$,

$$\|D_t^{\alpha-m+n}Z_{m-1}(t)\|_{\mathcal{L}(\mathcal{Z})} \le \frac{1}{2\pi}\int_{\Gamma} \|\lambda^{n-m}R_\lambda\|_{\mathcal{L}(\mathcal{Z})}|e^{\lambda t}|ds \le C_2\int_{\delta}^{\infty} r^{n-m}dr + C_3 \le C_4. \quad (10)$$

At $n = m-1$ we have

$$D_t^{\alpha-1}Z_{m-1}(t) = \frac{1}{2\pi i}\int_{\Gamma}\frac{R_\lambda}{\lambda}e^{\lambda t}d\lambda$$

$$= I + \frac{1}{2\pi i}\int_{\Gamma}\lambda^{-1}\left(\sum_{j=1}^{m-1}\lambda^{j-m}A_j + \sum_{l=1}^{n}\lambda^{\alpha_l-\alpha}B_l + \sum_{s=1}^{r}\lambda^{-\beta_s-\alpha}C_s\right)R_\lambda e^{\lambda t}d\lambda,$$

for $\lambda \in \Gamma$

$$\left\|\lambda^{-1}\left(\sum_{j=1}^{m-1}\lambda^{j-m}A_j + \sum_{l=1}^{n}\lambda^{\alpha_l-\alpha}B_l + \sum_{s=1}^{r}\lambda^{-\beta_s-\alpha}C_s\right)R_\lambda\right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_5}{|\lambda|^{1+\delta}},$$

where $\delta = \min\{1, \alpha-\alpha_l : l = 1,2,\dots,n\}$. Consequently, at $t \in [t_0, t_1]$

$$\|D_t^{\alpha-1}Z_{m-1}(t)\|_{\mathcal{L}(\mathcal{Z})} \le C_6. \quad (11)$$

Let $\tau > 0$ and $\delta > 0$ be such that $[t_0, t_0+\tau] \times S_\delta(\bar{z}) \subset Z$, where $\bar{z} = (0, 0, \dots, 0, z_{m^*}, z_{m^*+1}, \dots, z_{m-1})$ is constructed using initial data (7). Denote by $\mathcal{S}$ the set of functions $y \in C^{m-1}([t_0, t_0+\tau]; \mathcal{Z})$ such that $\|y^{(q)}(t)\| \le \delta$, $q = 0,1,\dots,m^*-1$, $\|y^{(k)}(t) - z_k\| \le \delta$, $k = m^*, m^*+1, \dots, m-1$ for $t_0 \le t \le t_0+\tau$. We define a metric on $\mathcal{S}$

$$d(y,v) := \sum_{k=0}^{m-1}\sup_{t\in[t_0,t_0+\tau]}\|y^{(k)}(t) - v^{(k)}(t)\|_{\mathcal{Z}},$$

then $\mathcal{S}$ is a complete metric space.

Note that

$$J_t^{m-\alpha}\int_{t_0}^{t} Z_{m-1}(t-s)F(s, y(s), y^{(1)}(s), \dots, y^{(m-1)}(s))\,ds$$

$$= \int_{t_0}^{t} J_t^{m-\alpha}Z_{m-1}(t-s)F(s, y(s), y^{(1)}(s), \dots, y^{(m-1)}(s))\,ds.$$

This equality can be proved by changing the order of integration in its left part.

Define for $y \in \mathcal{S}$

$$G(y)(t) := \sum_{p=m^*}^{m-1} J_t^{m-\alpha}Z_p(t-t_0)z_p + \int_{t_0}^{t} J_t^{m-\alpha}Z_{m-1}(t-s)F(s, y(s), y^{(1)}(s), \dots, y^{(m-1)}(s))\,ds$$

for $t \in [t_0, t_0+\tau]$. Let us prove that G maps the metric space $\mathcal{S}$ into itself and it is a contraction operator, if $\tau > 0$ is sufficiently small. Indeed, for $n = 0,1,\dots,m-1$

$$[G(y)]^{(n)}(t) = \sum_{p=m^*}^{m-1} D_t^{\alpha-m+n}Z_p(t-t_0)z_p$$

$$+ \int_{t_0}^{t} D_t^{\alpha-m+n}Z_{m-1}(t-s)F(s, y(s), y^{(1)}(s), \dots, y^{(m-1)}(s))ds,$$

since $D_t^{\alpha-m+n}Z_{m-1}(0) = 0$, $n = 0,1,\dots,m-2$. By Theorem 2 [14] we have $G(y) \in C^{m-1}([t_0,t_0+\tau];\mathcal{Z})$, $[G(y)]^{(q)}(t_0) = 0$, $q = 0,1,\dots,m^*-1$, $[G(y)]^{(k)}(t_0) = z_k$, $k = m^*,m^*+1,\dots,m-1$. Therefore, for $t \in [t_0,t_0+\tau]$ $\|[G(y)]^{(q)}(t)\|_{\mathcal{Z}} \le \delta$, $q = 0,1,\dots,m^*-1$, $\|[G(y)]^{(k)}(t) - z_k\|_{\mathcal{Z}} \le \delta$, $k = m^*,m^*+1,\dots,m-1$, for a small enough $\tau > 0$. So, $G : \mathcal{S} \to \mathcal{S}$.

Denote $F(t,\overline{D}y(t)) := F(t,y(t),y^{(1)}(t),\dots,y^{(m-1)}(t))$ for brevity. We have at $n = 0,1,\dots,m-1$, $t \in [t_0,t_0+\tau]$ due to (10), (11)

$$\|[G(y)]^{(n)}(t) - [G(v)]^{(n)}(t)\|_{\mathcal{Z}} = \left\| \int_{t_0}^{t} D_t^{\alpha-m+n}Z_{m-1}(t-s)\big(F(s,\overline{D}y(s)) - F(s,\overline{D}v(s))\big)ds \right\|$$

$$\le \tau \sup_{t\in[t_0,t_0+\tau]} \|D_t^{\alpha-m+n}Z_{m-1}(t)\|_{\mathcal{L}(\mathcal{Z})} l \sum_{k=0}^{m-1} \sup_{t\in[t_0,t_0+\tau]} \|y^{(k)}(t) - v^{(k)}(t)\|_{\mathcal{Z}} ds$$

$$\le C_7 \tau d(y,v) \le \frac{d(y,v)}{2m}$$

for small enough τ. Therefore, $d(G(y),G(v)) \le \frac{1}{2}d(y,v)$, the operator G has a unique fixed point $y_0 \in \mathcal{S}$, it is an unique local solution of integro-differential Equation (9). Thus, there exists a unique solution to problem (6), (7) on the segment $[t_0,t_0+\tau]$, it is uniquely defined by the equality $z = D_t^{m-\alpha}y_0$. $\square$

4. A Class of Initial-Boundary Value Problems

Assume given the polynomials

$$P_1(\lambda) = \sum_{p=0}^{\nu} a_p\lambda^p,\ P_2^j(\lambda) = \sum_{p=0}^{\nu} b_p^j\lambda^p,\ P_3^l(\lambda) = \sum_{p=0}^{\nu} c_p^l\lambda^p,\ P_4^s(\lambda) = \sum_{p=0}^{\nu} d_p^s\lambda^p,$$

$a_p, b_p^j, c_p^l, d_p^s \in \mathbb{C}$, $p = 0,1,\dots,\nu \in \mathbb{N}$, $j = 1,2,\dots,m-1$, $l = 1,2,\dots,n$, $s = 1,2,\dots,r$, $a_\nu \ne 0$, $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary $\partial\Omega$,

$$(\mathcal{A}u)(\xi) = \sum_{|q|\le 2\rho} a_q(\xi) \frac{\partial^{|q|}u(\xi)}{\partial\xi_1^{q_1}\partial\xi_2^{q_2}\dots\partial\xi_d^{q_d}},\ \ a_q \in C^\infty(\overline{\Omega}),$$

$$(\mathcal{B}_l u)(\xi) = \sum_{|q|\le \rho_l} b_{lq}(\xi) \frac{\partial^{|q|}u(\xi)}{\partial\xi_1^{q_1}\partial\xi_2^{q_2}\dots\partial\xi_d^{q_d}},\ \ b_{lq} \in C^\infty(\partial\Omega),\ l = 1,2,\dots,\rho,$$

$q = (q_1,q_2,\dots,q_d) \in \mathbb{N}_0^d$, $|q| = q_1+\dots+q_d$, and the operator pencil $\mathcal{A},\mathcal{B}_1,\mathcal{B}_2,\dots,\mathcal{B}_\rho$ is regularly elliptic [16]. Define the operator $\mathcal{A}_1 \in \mathcal{C}l(L_2(\Omega))$ with the domain

$$D_{\mathcal{A}_1} = H^{2\rho}_{\{\mathcal{B}_l\}}(\Omega) := \{v \in H^{2\rho}(\Omega) : \mathcal{B}_l v(\xi) = 0,\ l = 1,2,\dots,\rho,\ \xi \in \partial\Omega\}$$

by the rule $\mathcal{A}_1 u := \mathcal{A}u$. Suppose that $\mathcal{A}_1$ is a selfadjoint operator; then its spectrum $\sigma(\mathcal{A}_1)$ is real and discrete [16]. Moreover, assume that the spectrum $\sigma(\mathcal{A}_1)$ is bounded from the right and does not contain zero, $\{\varphi_k : k \in \mathbb{N}\}$ is an orthonormal in $L_2(\Omega)$ system of eigenfunctions of $\mathcal{A}_1$ in $L_2(\Omega)$ which is enumerated in nonincreasing order of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ with their multiplicities counted.

Take $m-1 < \alpha \le m \in \mathbb{N}$, $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha$, $m_l - 1 < \alpha_l \le m_l \in \mathbb{N}$, $\alpha_l - m_l \ne \alpha - m$, $l = 1,2,\dots,n$, $\beta_1 > \beta_2 > \dots > \beta_r \ge 0$, $H : \mathbb{R}^{m+1} \to \mathbb{R}$. Denote by m^* the defect of the Cauchy type problem, which is defined by the set of numbers $\alpha_1,\alpha_2,\dots,\alpha_n,\alpha$ (see the second section), and consider the initial-boundary value problem

$$D_t^{\alpha-m+k}u(\xi,0) = u_k(\xi),\ k = m^*,m^*+1,\dots,m-1,\quad \xi \in \Omega, \tag{12}$$

$$\mathcal{B}_l\mathcal{A}^k u(\xi,t)=0,\ \ k=0,1,\dots,\nu-1,\ \ l=1,2,\dots,\rho,\ \ (\xi,t)\in\partial\Omega\times(t_0,t_1], \tag{13}$$

$$\begin{aligned} P_1(\mathcal{A})D_t^{\alpha}u(\xi,t) &= \sum_{j=1}^{m-1} P_2^j(\mathcal{A})D_t^{\alpha-m+j}u(\xi,t) \\ &+\sum_{l=1}^{n} P_3^l(\mathcal{A})D_t^{\alpha_l}u(\xi,t)+\sum_{s=1}^{r} P_4^s(\mathcal{A})J_t^{\beta_s}u(\xi,t) \\ +H(\xi,D_t^{\alpha-m}(\xi,t),D_t^{\alpha-m+1}&(\xi,t),\dots,D_t^{\alpha-1}(\xi,t)),\ \ (\xi,t)\in\Omega\times(t_0,t_1]. \end{aligned} \tag{14}$$

Put $\rho_0\geq 0$, $\mathcal{X}:=\{v\in H^{2\rho\nu+\rho_0}(\Omega):\mathcal{B}_l\mathcal{A}^k v(\xi)=0, k=0,1,\dots,\nu-1, l=1,2,\dots,\rho,\ \xi\in\partial\Omega\}$, $\mathcal{Y}:=H^{\rho_0}(\Omega)$ is a Sobolev space $W_2^{\rho_0}(\Omega)$ for $\rho_0>0$, or the Lebesgue space $L^{\rho_0}(\Omega)$, if $\rho_0=0$; $L:=P_1(\mathcal{A})\in\mathcal{L}(\mathcal{X};\mathcal{Y})$, $M_j:=P_2^j(\mathcal{A})\in\mathcal{L}(\mathcal{X};\mathcal{Y})$, $j=1,2,\dots,m-1$, $N_l:=P_3^l(\mathcal{A})\in\mathcal{L}(\mathcal{X};\mathcal{Y})$, $l=1,2,\dots,n$, $S_s:=P_4^s(\mathcal{A})\in\mathcal{L}(\mathcal{X};\mathcal{Y})$, $s=1,2,\dots,r$.

If $P_1(\lambda_k)\neq 0$ for all $k\in\mathbb{N}$, then there exists the inverse operator $L^{-1}\in\mathcal{L}(\mathcal{Y};\mathcal{X})$ and (12)–(14) is representable in form (6), (7), where $\mathcal{Z}=\mathcal{X}$, $A_j=L^{-1}M_j\in\mathcal{L}(\mathcal{Z})$, $j=1,2,\dots,m-1$, $B_l=L^{-1}N_l\in\mathcal{L}(\mathcal{Z})$, $l=1,2,\dots,n$, $C_s=L^{-1}S_s\in\mathcal{L}(\mathcal{Z})$, $s=1,2,\dots,r$, $z_k=u_k(\cdot)$, $k=m^*,m^*+1,\dots,m-1$, $F(x_0,x_1,\dots,x_{m-1})=L^{-1}H(\cdot,x_0,x_1,\dots,x_{m-1})$.

Theorem 3. *Let $m-1<\alpha\leq m\in\mathbb{N}$, $0<\alpha_1<\alpha_2<\dots<\alpha_n<\alpha$, $m_l-1<\alpha_l\leq m_l\in\mathbb{N}$, $\alpha_l-m_l\neq\alpha-m$, $l=1,2,\dots,n$, $\beta_1>\beta_2>\dots>\beta_r\geq 0$, the spectrum $\sigma(\mathcal{A}_1)$ do not contain the origin and zeros of the polynomial P_1, $4\rho\nu+2\rho_0>d$, $u_k\in\mathcal{X}$, $k=m^*,m^*+1,\dots,m-1$, $H\in C^\infty(\Omega\times\mathbb{R}^n;\mathbb{R})$. Then at some $t_1>t_0$ there exists an unique solution of problem (12)–(14).*

Proof. In this problem the domain of nonlinear operator is $Z=\mathbb{R}\times\mathcal{X}^m$ and due to the inequality $4\rho\nu+2\rho_0>d$ by Proposition 1 ([17], Appendix B) we have

$$H(\cdot,x_0(\cdot),x_1(\cdot),\dots,x_{n-1}(\cdot))\in C^\infty(\mathcal{X}^m;H^{2\rho\nu+\rho_0}(\Omega)),$$

hence, $F(x_0(\cdot),x_1(\cdot),\dots,x_{m-1}(\cdot)):=L^{-1}H(\cdot,x_0(\cdot),x_1(\cdot),\dots,x_{m-1}(\cdot))\in C^\infty(\mathcal{X}^m;\mathcal{X})$. Then by Theorem 2 we obtain the statement of this theorem. □

Example 1. *Take $\alpha=5/2$, $m=3$, $n=1$, $r=1$, $\alpha_1=2/3$, $\beta_1=1/2$, $\nu=2$, $P_1(\lambda)=\lambda^2$, $P_2^1(\lambda)=b_0+b_1\lambda+b_2\lambda^2$, $P_2^2(\lambda)\equiv 0$, $P_3^1(\lambda)=c_0+c_1\lambda+c_2\lambda^2$, $P_4^1(\lambda)=d_0+d_1\lambda+d_2\lambda^2$, $d=1$, $\Omega=(0,\pi)$, $\rho=1$, $\mathcal{A}u=\frac{\partial^2 u}{\partial\xi^2}$, $\mathcal{B}_1=I$. Then $\underline{\alpha}:=\max\varnothing:=0$, $\underline{m}:=\lceil 0\rceil=0$, $\overline{\alpha}:=\max\{2/3\}=2/3$, $\overline{m}:=\lceil 2/3\rceil=1$, $m^*=1$, problem (12)–(14) has the form*

$$\begin{aligned} D_t^{5/2}\frac{\partial^4 u}{\partial\xi^4}(\xi,t) &= \left(b_0+b_1\frac{\partial^2}{\partial\xi^2}+b_2\frac{\partial^4}{\partial\xi^4}\right)D_t^{1/2}u(\xi,t) \\ +\left(c_0+c_1\frac{\partial^2}{\partial\xi^2}+c_2\frac{\partial^4}{\partial\xi^4}\right)&D_t^{2/3}u(\xi,t)+\left(d_0+d_1\frac{\partial^2}{\partial\xi^2}+d_2\frac{\partial^4}{\partial\xi^4}\right)J_t^{1/2}u(\xi,t) \\ +F(\xi,J_t^{1/2}u(\xi,t),D_t^{1/2}&u(\xi,t),D_t^{3/2}u(\xi,t)),\ \ (\xi,t)\in(0,\pi)\times(t_0,t_1], \end{aligned}$$

$$u(0,t)=u(\pi,t)=\frac{\partial^2 u}{\partial\xi^2}(0,t)=\frac{\partial^2 u}{\partial\xi^2}(\pi,t)=0,\ \ t\in(t_0,t_1],$$

$$D^{1/2}u(\xi,0)=u_1(\xi),\quad D^{3/2}u(\xi,0)=u_2(\xi)\quad \xi\in(0,\pi).$$

5. Conclusions

The local solvability is shown for the incomplete Cauchy type problem to a solved with respect to a highest derivative multi-term fractional differential equation with bounded operators at Riemann–Liouville derivatives in a Banach space with locally Lipschitzian nonlinear part. The results of the work [14] on inhomogeneous linear multi-term equation are used here for the research of the quasilinear equation, depending on lower order fractional derivatives with orders, which fractional part is equal to the fractional part of the highest fractional derivative. The abstract result was applied to the investigation of initial-boundary value problems to partial differential equations containing polynomials with respect to self-adjoint elliptic differential in spatial variables operator at time-fractional derivatives. Here the highest time-fractional partial derivative acts on the highest spatial derivative.

Our next step is to abandon this condition by allowing unlimited operators in an abstract equation. The linear case of this type has been investigated in [15], the nonlinear one has not yet been studied. Another significant step planned by the authors in the coming papers will be the rejection of conditions for the fractional part of the orders of derivatives on which the nonlinear operator depends (see above).

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