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Existence Results for Coupled Implicit ψ-Riemann–Liouville Fractional Differential Equations with Nonlocal Conditions

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Abstract: In this paper, we study the existence and uniqueness of solutions for a coupled implicit system involving ψ-Riemann–Liouville fractional derivative with nonlocal conditions. We first transformed the coupled implicit problem into an integral system and then analyzed the uniqueness and existence of this integral system by means of Banach fixed-point theorem and Krasnoselskiis fixed-point theorem. Some known results in the literature are extended. Finally, an example is given to illustrate our theoretical result.

Keywords: coupled fractional differential equations; ψ-Riemann–Liouville fractional derivative; fixed-point theorems; existence and uniqueness

MSC: 34A08; 26A33; 34A12

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1. Introduction

The fractional calculus is an important branch of mathematics and its wide applications to many fields, such engineering, economics, physics, chemistry, finance, control of dynamical systems, and so on—see [1–7], and the references cited therein. One of the proposed generalizations of the fractional calculus operators is the ψ-fractional operator—see [8–10] and references therein for its wide applications. Some properties of this operator could be found in [11–13].

As we all know, the coupled system of fractional differential equations is becoming a more popular research field due to its vast applications in real-time problems, namely anomalous diffusion, chaotic systems, disease models, and ecological models [14–16]. Recently, the coupled system of fractional differential equations has been considered extensively in the literature. Alsaedi et al. [17] researched the uniqueness and existence of solutions for a nonlinear system of Riemann–Liouville fractional differential equations equipped with nonseparated semi-coupled integro-multipoint boundary conditions. Baleanu et al. [18] studied the uniqueness existence and Ulam stability for a coupled system involving generalized Sturm–Liouville problems and Langevin fractional differential equations described by Atangana–Baleanu–Caputo derivatives by virtue of the notable Mittag–Leffler kernel. Muthaiah et al. [19] presented the existence, uniqueness, and Hyers–Ulam stability of the coupled system of Caputo–Hadamard-type fractional differential equations with multipoint and nonlocal integral boundary conditions. Based on the features of the Hadamard fractional derivative, the implementation of fixed-point theorems, the employment of Urs's stability approach, and the existence, uniqueness, and stability of the coupled system of nonlinear Langevin equations involving Caputo–Hadamard fractional derivative—subject to nonperiodic boundary conditions—are established by Matar et al. [20]. In [21], by using the coincidence degree theory, Zhang et al. established the existence and uniqueness theorems for the coupled systems of implicit fractional differential equations with periodic boundary conditions.

In recent years, the study of basic theories of initial and boundary value problems for implicit fractional differential equations and integral equations with Caputo fractional derivative and Riemann–Liouville fractional derivative has been paid to much attention. In [22], Benchohra and Souid obtained integrable solutions for initial value problem of implicit fractional differential equations. Nieto et al. [23] studied initial value problem for an implicit fractional differential equation using a fixed-point theory and approximation method. Furthermore, in [24] Benchohra and Bouriah established existence and various stability results for a class of boundary value problem for implicit fractional differential equation with Caputo fractional derivative. Implicit fractional differential equations play a key role in different problems, the readers are referred to see [25–28].

In [29], Benchohra et al. discussed the existence and Ulam stability analysis of the following nonlinear implicit fractional differential equation with initial value condition:

$$\begin{cases} D^{\alpha}_{0^+}x(t) = f(t, x(t), D^{\alpha}_{0^+}x(t)), & t \in (0, T], \\ t^{1-\alpha}x(t)|_{t=0} = x_0, & x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $D^{\alpha}_{0^+}$ is the standard Riemann–Liouville fractional derivative, $f : (0; T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $0 < \alpha < 1$.

Very recently, in [30], Lachouri et al. studied the existence and uniqueness of solutions for the following nonlinear implicit Riemann–Liouville fractional differential equation with nonlocal condition:

$$\begin{cases} D^{\alpha}_{0^+}x(t) = f(t, x(t), D^{\alpha}_{0^+}x(t)), & t \in (0, T], \\ t^{1-\alpha}x(t)|_{t=0} = x_0 - g(x), & x_0 \in \mathbb{R}, \end{cases} \tag{2}$$

where $D^{\alpha}_{0^+}$ and f are as in (1.1), $g : C((0, T], \mathbb{R}) \to \mathbb{R}$ is a continuous nonlinear function.

Motivated by the above works, we consider the following coupled implicit ψ-Riemann–Liouville fractional differential equations with nonlocal conditions:

$$\begin{cases} D^{\alpha,\psi}_{0^+}x(t) = f(t, y(t), D^{\alpha,\psi}_{0^+}x(t)), & t \in (0, T], \\ D^{\alpha,\psi}_{0^+}y(t) = g(t, x(t), D^{\alpha,\psi}_{0^+}y(t)), & t \in (0, T], \\ (\psi(t) - \psi(0))^{1-\alpha}x(t)|_{t=0} = y_0 - r(y), & y_0 \in \mathbb{R}, \\ (\psi(t) - \psi(0))^{1-\alpha}y(t)|_{t=0} = x_0 - h(x), & x_0 \in \mathbb{R}, \end{cases} \tag{3}$$

where $D^{\alpha,\psi}_{0^+}x(t)$ is the Riemann–Liouville fractional derivative of a function x with respect to another function ψ, which is increasing, and $\psi'(t) \neq 0$ for all $t \in [0, T]$, $f, g : (0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions, and $0 < \alpha < 1$, $h, r : C((0, T], \mathbb{R}) \to \mathbb{R}$ are two continuous nonlinear functions.

To the best of our knowledge, there are no papers on coupled implicit fractional differential equations including fractional derivative of a function with respect to another function. We cover this gap in this paper.

In this paper, our aim is to present the sufficient conditions for the existence and uniqueness of solutions for coupled implicit system (3). First of all, we transform (3) into an integral system and then we study the existence and uniqueness of solutions by the Banach and Krasnoselskii fixed-point theorems. Finally, an example is given to illustrate our main results. Our results extend the main results of [30].

This paper will be organized as follows. In Section 2, we will briefly recall some notations, definitions and preliminaries. Section 3 is devoted to proving the existence and uniqueness of the solution for system (3). In Section 4, an example is given to illustrate our theoretical result. Finally, we present some conclusions in Section 5.

2. Preliminaries

In this section, we provided some basic definitions and lemmas which are used in the sequel.

Definition 1 ([13])**.** *Let $\alpha > 0$, f be an integrable function defined on $[a,b]$ and $\psi \in C^1([a,b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$. The left ψ-Riemann–Liouville fractional integral operator of order α of a function f is defined by:*

$$I_{a^+}^{\alpha,\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds.$$

Definition 2 ([13])**.** *Let $n-1 < \alpha < n$, $f \in C^n([a,b])$ and $\psi \in C^n([a,b])$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$. The left ψ-Riemann–Liouville fractional derivative of order α of a function f is defined by:*

$$\begin{aligned} D_{a^+}^{\alpha,\psi} f(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\psi} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} f(s) ds, \end{aligned}$$

where $n = [\alpha] + 1$.

Lemma 1 ([13])**.** *Let $\alpha > 0$ and $\beta > 0$, then*

(i) $I_{a^+}^{\alpha,\psi} (\psi(s) - \psi(a))^{\beta-1}(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(a))^{\beta+\alpha-1}$,

(ii) $D_{a^+}^{\alpha,\psi} (\psi(s) - \psi(a))^{\beta-1}(t) = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(a))^{\beta-\alpha-1}$.

In the following, we will give the combinations of the fractional integral and the fractional derivatives of a function with respect to another function.

Lemma 2 ([11])**.** *Let $f \in C^n([a,b])$ and $n-1 < \alpha < n$. Then we have*

(1) $D_{a^+}^{\alpha,\psi} I_{a^+}^{\alpha,\psi} f(t) = f(t)$;

(2) $I_{a^+}^{\alpha,\psi} D_{a^+}^{\alpha,\psi} f(t) = f(t) - \displaystyle\sum_{k=1}^{n} \frac{f^{[k-1]}(a^+)}{\Gamma(k-\alpha)} (\psi(t) - \psi(a))^{k-\alpha}$,

where $f^{[k]}(t) := \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^k f(t)$ on $[a,b]$. In particular, given $\alpha \in (0,1)$, one has

$$I_{a^+}^{\alpha,\psi} D_{a^+}^{\alpha,\psi} f(t) = f(t) - c(t-a)^{\alpha-1},$$

where c is a constant.

Lemma 3. *(x,y) solves (3) if—and only if—it is a solution of integral system.*

$$\begin{cases} x(t) = (\psi(t) - \psi(0))^{\alpha-1}(y_0 - r(y)) + \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} f(s, y(s), D_{0^+}^{\alpha,\psi} x(s)) \psi'(s) ds, \\ y(t) = (\psi(t) - \psi(0))^{\alpha-1}(x_0 - h(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} g(s, x(s), D_{0^+}^{\alpha,\psi} y(s)) \psi'(s) ds. \end{cases} \tag{4}$$

Proof. If (x,y) satisfies the problem (3), then applying $I_{0^+}^{\alpha,\psi}$ to both sides of the first equation and second equation of (3), respectively, we have

$$I_{0^+}^{\alpha,\psi} D_{0^+}^{\alpha,\psi} x(t) = I_{0^+}^{\alpha,\psi} f(t, y(t), D_{0^+}^{\alpha,\psi} x(t)),$$

and

$$I_{0^+}^{\alpha,\phi} D_{0^+}^{\alpha,\psi} y(t) = I_{0^+}^{\alpha,\psi} g(t, x(t), D_{0^+}^{\alpha,\psi} y(t)).$$

By Lemma 2, we obtain

$$\begin{cases} x(t) = c_1(\psi(t) - \psi(0))^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} f(s, y(s), D_{0^+}^{\alpha,\psi} x(s)) \psi'(s) ds, \\ y(t) = c_2(\psi(t) - \psi(0))^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} g(s, x(s), D_{0^+}^{\alpha,\psi} y(s)) \psi'(s) ds, \end{cases} \tag{5}$$

where $t \in (0, T]$. In view of the following conditions:

$$(\psi(t) - \psi(0))^{1-\alpha} x(t)|_{t=0} = y_0 - r(y), \quad (\psi(t) - \psi(0))^{1-\alpha} y(t)|_{t=0} = x_0 - h(x),$$

we obtain

$$c_1 = y_0 - r(y), \quad c_2 = x_0 - h(x). \tag{6}$$

Substituting (6) into (5) we obtain the integral system (4). □

Theorem 1 ((Krasnoselskll's fixed point theorem) [31]). *Let Ω be a non-empty closed bounded convex subset of a Banach space E. Suppose that F_1 and F_2 map Ω into E, such that*

(i) $F_1 x + F_2 y \in \Omega$ for all $x, y \in \Omega$;
(ii) F_1 is continuous and compact;
(iii) F_2 is a contraction with constant $k < 1$.

Then, there is a $z \in \Omega$, with $F_1 z + F_2 z = z$.

3. Main Results

Let $\gamma > 0$, $T > 0$ and $J = [0, T]$, we denote the weighted space of the following continuous functions:

$$C_{\gamma,\psi}(J, \mathbb{R}) = \{x : (0, T] \to \mathbb{R} | (\psi(t) - \psi(0))^{\gamma} x \in C(J, \mathbb{R})\},$$

with the norm

$$\|x\|_{C_{\gamma,\psi}} = \sup_{t \in J} |(\psi(t) - \psi(0))^{\gamma} x(t)|.$$

In fact, we have (i) $\|x\|_{C_{\gamma,\psi}} \geq 0$, (ii) $\|kx\|_{C_{\gamma,\psi}} = |k| \|x\|_{C_{\gamma,\psi}}$, and
(iii) $\|x + y\|_{C_{\gamma,\psi}} = \sup_{t \in J} |(\psi(t) - \psi(0))^{\gamma} (x(t) + y(t))|$
$\leq \sup_{t \in J} |(\psi(t) - \psi(0))^{\gamma} x(t)| + \sup_{t \in J} |(\psi(t) - \psi(0))^{\gamma} y(t)| = \|x\|_{C_{\gamma,\psi}} + \|y\|_{C_{\gamma,\psi}}$.
Thus, $C_{\gamma,\psi}(J, \mathbb{R})$ is a Banach space.
Define the operators $A : C_{1-\alpha,\psi}(J, \mathbb{R}) \times C_{1-\alpha,\psi}(J, \mathbb{R}) \to C_{1-\alpha,\psi}(J, \mathbb{R}) \times C_{1-\alpha,\psi}(J, \mathbb{R})$ by

$$A(x, y)(t) = (A_1(x, y)(t), A_2(x, y)(t)), \quad t \in (0, T], \tag{7}$$

where

$$A_1(x, y)(t) = (\psi(t) - \psi(0))^{\alpha-1} (y_0 - r(y)) + \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} p(s) \psi'(s) ds,$$

and

$$A_2(x, y)(t) = (\psi(t) - \psi(0))^{\alpha-1} (x_0 - h(x)) + \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} q(s) \psi'(s) ds,$$

here, $p, q : (0, T] \to \mathbb{R}$ are two functions satisfying the functional equations

$$p(t) = f(t, y(t), p(t)), \quad q(t) = g(t, x(t), q(t)). \tag{8}$$

The operator A is well-defined, i.e., for every $(x, y) \in C_{1-\alpha,\psi}(J, \mathbb{R}) \times C_{1-\alpha,\psi}(J, \mathbb{R})$ and $t > 0$, the following integrals

$$\frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) p(s) ds,$$

and

$$\frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} \psi'(s) q(s) ds$$

belong to $C_{1-\alpha,\psi}(J,\mathbb{R})$.

For convenience, we allow the following hypothesis.

Hypothesis 1 (H1). *There exist constants $L_1, l_1 > 0$ and $L_2, l_2 \in (0,1)$ such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq L_1|u_1 - u_2| + L_2|v_1 - v_2|,$$

$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \leq l_1|u_1 - u_2| + l_2|v_1 - v_2|,$$

for $t \in (0,T]$, $u_1, u_2, v_1, v_2 \in \mathbb{R}$ and $f(\cdot,0,0), g(\cdot,0,0) \in C_{1-\alpha,\psi}(J,\mathbb{R})$.

Hypothesis 2 (H2). *There exist two constant $b_1, b_2 \in (0,1)$, such that*

$$|h(x) - h(y)| \leq b_1\|x - y\|_{C_{1-\alpha,\psi}}, \quad |r(x) - r(y)| \leq b_2\|x - y\|_{C_{1-\alpha,\psi}},$$

for $x, y \in C_{1-\alpha,\psi}(J,\mathbb{R})$.

Hypothesis 3 (H3). *There exist $m_1, n_1 \in C_{1-\alpha,\psi}(J,\mathbb{R}^+)$, $m_2, n_2, m_3, n_3 \in C(J,\mathbb{R}^+)$ with $m_3^* = \sup_{t\in J} m_3(t) < 1$ and $n_3^* = \sup_{t\in J} n_3(t) < 1$, such that*

$$|f(t,u,v)| \leq m_1(t) + m_2(t)|u| + m_3(t)|v|,$$

$$|g(t,u,v)| \leq n_1(t) + n_2(t)|u| + n_3(t)|v|,$$

for $t \in (0,T]$ and each $u, vs. \in \mathbb{R}$.

Theorem 2. *Assume that (H1)–(H2) hold. If the following is true:*

$$k = \max\left\{ b_2 + \frac{L_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-L_2)}(\psi(T) - \psi(0))^{\alpha}, b_1 + \frac{l_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-l_2)}(\psi(T) - \psi(0))^{\alpha} \right\} < 1, \tag{9}$$

then there exists a unique solution for the BVP (3) in the space $C_{1-\alpha,\psi}(J,\mathbb{R}) \times C_{1-\alpha,\psi}(J,\mathbb{R})$.

Proof. In the following, we will prove that the operator A has unique fixed point. From (8), one has by condition (H1) that

$$|p(t)| \leq |f(t,y(t),p(t)) - f(t,0,0)| + |f(t,0,0)| \leq L_1|y(t)| + L_2|p(t)| + |f(t,0,0)|,$$

which implies that

$$|p(t)| \leq \frac{L_1}{1-L_2}|y(t)| + e_1(\psi(t) - \psi(0))^{\alpha-1}, \quad t \in (0,T], \tag{10}$$

where $e_1 = \frac{\sup_{t\in J}|(\psi(t)-\psi(0))^{1-\alpha}f(t,0,0)|}{1-L_2} < +\infty$. Similarly, we have

$$|q(t)| \leq \frac{l_1}{1-l_2}|x(t)| + e_2(\psi(t) - \psi(0))^{\alpha-1}, \quad t \in (0,T], \tag{11}$$

where $e_2 = \frac{\sup_{t\in J}|(\psi(t)-\psi(0))^{1-\alpha}g(t,0,0)|}{1-l_2} < +\infty$. For each $x, y \in C_{1-\alpha,\psi}(J,\mathbb{R})$, by Lemma 1, we obtain

$$\left|\tfrac{(\psi(t)-\psi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)p(s)ds\right|$$
$$\leq \tfrac{(\psi(t)-\psi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)|p(s)|ds$$
$$\leq \tfrac{(\psi(t)-\psi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\left(\tfrac{L_1}{1-L_2}|y(s)|+e_1(\psi(s)-\psi(0))^{\alpha-1}\right)ds$$
$$\leq \tfrac{(\psi(t)-\psi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)(\psi(s)-\psi(0))^{\alpha-1}$$
$$\cdot\left(\tfrac{L_1}{1-L_2}|(\psi(s)-\psi(0))^{1-\alpha}y(s)|+e_1\right)ds$$
$$\leq \left(\tfrac{L_1}{1-L_2}\|y\|_{C_{1-\alpha,\psi}}+e_1\right)(\psi(t)-\psi(0))^{1-\alpha}I^{\alpha,\psi}((\psi(t)-\psi(0))^{\alpha-1})$$
$$\leq \left(\tfrac{L_1}{1-L_2}\|y\|_{C_{1-\alpha,\psi}}+e_1\right)\tfrac{\Gamma(\alpha)(\psi(t)-\psi(0))^{\alpha}}{\Gamma(2\alpha)}$$
$$\leq \left(\tfrac{L_1}{1-L_2}\|y\|_{C_{1-\alpha,\psi}}+e_1\right)\tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(T)-\psi(0))^{\alpha}.$$

Similarly, we have

$$\left|\frac{(\psi(t)-\psi(0))^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)q(s)ds\right|$$
$$\leq \left(\frac{l_1}{1-l_2}\|x\|_{C_{1-\alpha,\psi}}+e_2\right)\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(T)-\psi(0))^{\alpha}.$$

Thus, the integrals exist and belong to $C_{1-\alpha,\psi}(J,\mathbb{R})$. Let $(x,y),(u,v)\in C_{1-\alpha,\psi}(J,\mathbb{R})\times C_{1-\alpha,\psi}(J,\mathbb{R})$. Then, for each $t\in(0,T]$, we obtain

$$\begin{aligned}|A_1(x,y)(t)-A_1(u,v))(t)| &\leq (\psi(t)-\psi(0))^{\alpha-1}|r(y)-r(v)|\\ &+\tfrac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)|p_y(s)-p_v(s)|ds,\end{aligned}$$

and

$$\begin{aligned}|A_2(x,y)(t)-A_2(u,v)(t)| &\leq (\psi(t)-\psi(0))^{\alpha-1}|h(x)-h(u)|\\ &+\tfrac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)|q_x(s)-q_u(s)|ds,\end{aligned}$$

where $p_y,p_v,q_x,q_u\in C_{1-\alpha,\psi}(J,\mathbb{R})$, such that

$$p_y=f(t,y(t),p_y(t)),\quad p_v=f(t,v(t),p_v(t)),$$
$$q_x=g(t,x(t),q_x(t)),\quad q_u=g(t,u(t),q_u(t)).$$

In view of (H1), one has

$$\begin{aligned}|p_y(t)-p_v(t)| &= |f(t,y(t),p_y(t))-f(t,v(t),p_v(t))|\\ &\leq L_1|y(t)-v(t)|+L_2|p_y(t)-p_v(t)|,\end{aligned}$$

and

$$\begin{aligned}|q_x(t)-q_u(t)| &= |g(t,x(t),q_x(t))-g(t,u(t),q_u(t))|\\ &\leq l_1|x(t)-u(t)|+l_2|q_x(t)-q_u(t)|,\end{aligned}$$

which implies that

$$|p_y(t)-p_v(t)|\leq\frac{L_1}{1-L_2}|y(t)-v(t)|,\quad |q_x(t)-q_u(t)|\leq\frac{l_1}{1-l_2}|x(t)-u(t)|.$$

Hence, for every $t \in (0, T]$

$$|A_1(x,y)(t) - A_1(u,v)(t)| \leq (\psi(t)-\psi(0))^{\alpha-1}|r(y)-r(v)|$$
$$+\tfrac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)|p_y(s)-p_v(s)|ds$$
$$\leq b_2(\psi(t)-\psi(0))^{\alpha-1}\|y-v\|_{C_{1-\alpha,\psi}} + \tfrac{L_1}{\Gamma(\alpha)(1-L_2)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)|y(s)-v(s)|ds$$
$$\leq b_2(\psi(t)-\psi(0))^{\alpha-1}\|y-v\|_{C_{1-\alpha,\psi}}$$
$$+\tfrac{L_1}{\Gamma(\alpha)(1-L_2)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}|(\psi(s)-\psi(0))^{1-\alpha}(y(s)-v(s))|\psi'(s)ds$$
$$\leq b_2(\psi(t)-\psi(0))^{\alpha-1}\|y-v\|_{C_{1-\alpha,\psi}}$$
$$+\tfrac{L_1}{1-L_2}I^{\alpha,\psi}((\psi(t)-\psi(0))^{\alpha-1})\|y-v\|_{C_{1-\alpha,\psi}}$$
$$\leq b_2(\psi(t)-\psi(0))^{\alpha-1}\|y-v\|_{C_{1-\alpha,\psi}}$$
$$+\tfrac{L_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-L_2)}(\psi(t)-\psi(0))^{2\alpha-1}\|y-v\|_{C_{1-\alpha,\psi}}.$$

So, we obtain the following:

$$(\psi(t)-\psi(0))^{1-\alpha}|A_1(x,y)(t)-A_1(u,v)(t)|$$
$$\leq \left(b_2 + \tfrac{L_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-L_2)}(\psi(t)-\psi(0))^{\alpha}\right)\|y-v\|_{C_{1-\alpha,\psi}}$$

That is, as follows:

$$\|A_1(x,y)-A_1(u,v)\|_{C_{1-\alpha,\psi}} \leq \left(b_2 + \frac{L_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-L_2)}(\psi(T)-\psi(0))^{\alpha}\right)\|y-v\|_{C_{1-\alpha,\psi}}.$$

Similarly, we can obtain the following:

$$\|A_2(x,y)-A_2(u,v)\|_{C_{1-\alpha,\psi}} \leq \left(b_1 + \frac{l_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-l_2)}(\psi(T)-\psi(0))^{\alpha}\right)\|x-u\|_{C_{1-\alpha,\psi}}.$$

Thus, we have

$$\|A(x,y)-A(u,v)\|_{C_{1-\alpha,\psi}\times C_{1-\alpha,\psi}} = \|A_1(x,y)-A_1(u,v)\|_{C_{1-\alpha,\psi}} + \|A_2(x,y)-A_2(u,v)\|_{C_{1-\alpha,\psi}}$$
$$\leq \max\left\{b_2 + \tfrac{L_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-L_2)}(\psi(T)-\psi(0))^{\alpha}, b_1 + \tfrac{l_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-l_2)}(\psi(T)-\psi(0))^{\alpha}\right\}$$
$$\cdot(\|x-u\|_{C_{1-\alpha,\psi}} + \|y-v\|_{C_{1-\alpha,\psi}})$$
$$= k\|(x,y)-(u,v)\|_{C_{1-\alpha,\psi}\times C_{1-\alpha,\psi}}.$$

where $k = \max\left\{b_2 + \tfrac{L_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-L_2)}(\psi(T)-\psi(0))^{\alpha}, b_1 + \tfrac{l_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-l_2)}(\psi(T)-\psi(0))^{\alpha}\right\}$. From (7), we know that A is a contraction operator. By using of Banach's fixed-point theorem, we obtain that A has a unique fixed point which is a unique solution of the problem (3). □

Theorem 3. *Suppose that (H1)–(H3) hold. If*

$$\mu = \max\left\{b_2 + \frac{m_2^*\Gamma(\alpha)(\psi(T)-\psi(0))^{\alpha}}{(1-m_3^*)\Gamma(2\alpha)}, b_1 + \frac{n_2^*\Gamma(\alpha)(\psi(T)-\psi(0))^{\alpha}}{(1-n_3^*)\Gamma(2\alpha)}\right\} < 1, \qquad (12)$$

where $m_2^* = \sup_{t\in J} m_2(t)$ *and* $n_2^* = \sup_{t\in J} n_2(t)$*. Then, the BVP (3) has at least one solution in* Ω*.*

Proof. Let

$$R = \frac{1}{1-\mu},$$
$$\Delta = |x_0| + |y_0| + Q_1 + Q_2 + \frac{m_1^\diamond \Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-m_3^*)\Gamma(2\alpha)} + \frac{n_1^\diamond \Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-n_3^*)\Gamma(2\alpha)},$$

where $m_1^\diamond = \sup_{t\in J}\{(\psi(t)-\psi(0))^{1-\alpha} m_1(t)\}$, $n_1^\diamond = \sup_{t\in J}\{(\psi(t)-\psi(0))^{1-\alpha} n_1(t)\}$, $Q_1 = |h(0)|$ and $Q_2 = |r(0)|$.

Set the non-empty closed bounded convex subset as follows:

$$\Omega = \{(x,y) \in C_{1-\alpha,\psi}(J,\mathbb{R}) \times C_{1-\alpha,\psi}(J,\mathbb{R}) : \|(x,y)\|_{C_{1-\alpha,\psi}\times C_{1-\alpha,\psi}} \le M\},$$

where $M \ge R\Delta$ is fixed. Define two operators F_1, F_2 on Ω as follows:

$$F_1(x,y)(t) = ((\psi(t)-\psi(0))^{\alpha-1}(y_0 - r(y)), (\psi(t)-\psi(0))^{\alpha-1}(x_0 - h(x))),$$
$$F_2(x,y)(t) = \left(\frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1} p_x(s)\psi'(s)ds, \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1} q_y(s)\psi'(s)ds\right),$$

where $p_x, q_y : (0,T] \to \mathbb{R}$ are two functions satisfying the following functional equations:

$$p_x(t) = f(t, y(t), p_x(t)), \quad q_y(t) = g(t, x(t), q_y(t)). \tag{13}$$

In the following, we will prove that the operator $F_1 + F_2$ in Ω has at least one fixed point by using Krasnoselskii's fixed-point theorem. The proof will be given in four steps.

Step 1. We prove that $F_1(x,y) + F_2(u,v) \in \Omega$ for all $(x,y),(u,v) \in \Omega$.

For any $(x,y),(u,v) \in \Omega$ and $t \in (0,T]$, one has

$$F_1(x,y)(t) + F_2(u,v)(t) = (H_1(x,y,u,v), H_2(x,y,u,v)),$$

where

$$H_1(x,y,u,v) = (\psi(t)-\psi(0))^{\alpha-1}(y_0 - r(y)) + \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1} p_u(s)\psi'(s)ds,$$
$$H_2(x,y,u,v) = (\psi(t)-\psi(0))^{\alpha-1}(x_0 - h(x)) + \frac{1}{\Gamma(\alpha)}\int_0^t (\psi(t)-\psi(s))^{\alpha-1} q_v(s)\psi'(s)ds.$$

Here, $p_u, q_v : (0,T] \to \mathbb{R}$ are two functions satisfying the the functional equations:

$$p_u(t) = f(t, v(t), p_u(t)), \quad q_v(t) = g(t, u(t), q_v(t)).$$

From (H3), for any $t \in (0,T]$, we obtain

$$\begin{aligned}
&|(\psi(t)-\psi(0))^{1-\alpha} p_u(t)| = |(\psi(t)-\psi(0))^{1-\alpha} f(t, v(t), p_u(t))| \\
&\le (\psi(t)-\psi(0))^{1-\alpha} m_1(t) + m_2(t)|(\psi(t)-\psi(0))^{1-\alpha} v(t)| + m_3(t)|(\psi(t)-\psi(0))^{1-\alpha} p_u(t)| \\
&\le m_1^\diamond + m_2^*\|v\|_{C_{1-\alpha,\psi}} + m_3^*|(\psi(t)-\psi(0))^{1-\alpha} p_u(t)|,
\end{aligned}$$

which implies that

$$|(\psi(t)-\psi(0))^{1-\alpha} p_u(t)| \le \frac{m_1^\diamond + m_2^*\|v\|_{C_{1-\alpha,\psi}}}{1-m_3^*}. \tag{14}$$

Similarly, we can obtain

$$|(\psi(t)-\psi(0))^{1-\alpha} q_v(t)| \le \frac{n_1^\diamond + n_2^*\|u\|_{C_{1-\alpha,\psi}}}{1-n_3^*}. \tag{15}$$

For any $(x,y),(u,v) \in \Omega$ and $t \in (0,T]$, by (14), we obtain the following:

$$\begin{aligned}
&|H_1(x,y,u,v)| \le (\psi(t)-\psi(0))^{\alpha-1}|y_0| + (\psi(t)-\psi(0))^{\alpha-1}|r(y)-r(0)| + (\psi(t)-\psi(0))^{\alpha-1}|r(0)| \\
&+\tfrac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}|(\psi(s)-\psi(0))^{1-\alpha}p_u(s)|\psi'(s)ds \\
&\le (\psi(t)-\psi(0))^{\alpha-1}|y_0| + (\psi(t)-\psi(0))^{\alpha-1}b_2\|y\|_{C_{1-\alpha,\psi}} + (\psi(t)-\psi(0))^{\alpha-1}Q_2 \\
&+\tfrac{m_1^\diamond+m_2^*\|v\|_{C_{1-\alpha,\psi}}}{1-m_3^*}\tfrac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}\psi'(s)ds \\
&= (\psi(t)-\psi(0))^{\alpha-1}|y_0| + (\psi(t)-\psi(0))^{\alpha-1}b_2\|y\|_{C_{1-\alpha,\psi}} + (\psi(t)-\psi(0))^{\alpha-1}Q_2 \\
&+\tfrac{m_1^\diamond+m_2^*\|v\|_{C_{1-\alpha,\psi}}}{1-m_3^*}\tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(t)-\psi(0))^{2\alpha-1}.
\end{aligned} \tag{16}$$

Thus,

$$\begin{aligned}
&(\psi(t)-\psi(0))^{1-\alpha}|H_1(x,y,u,v)| \\
&\le |y_0| + Q_2 + \tfrac{m_1^\diamond\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-m_3^*)\Gamma(2\alpha)} + b_2\|y\|_{C_{1-\alpha,\psi}} + \tfrac{m_2^*\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-m_3^*)\Gamma(2\alpha)}\|v\|_{C_{1-\alpha,\psi}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&(\psi(t)-\psi(0))^{1-\alpha}|H_2(x,y,u,v)| \\
&\le |x_0| + Q_1 + \tfrac{n_1^\diamond\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-n_3^*)\Gamma(2\alpha)} + b_1\|x\|_{C_{1-\alpha,\psi}} + \tfrac{n_2^*\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-n_3^*)\Gamma(2\alpha)}\|u\|_{C_{1-\alpha,\psi}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|F_1(x,y)+F_2(u,v)\|_{C_{1-\alpha,\psi}} = \|H_1(x,y,u,v\|_{C_{1-\alpha,\psi}} + \|H_2(x,y,u,v\|_{C_{1-\alpha,\psi}} \\
&\le |x_0| + |y_0| + Q_1 + Q_2 + \tfrac{m_1^\diamond\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-m_3^*)\Gamma(2\alpha)} + \tfrac{n_1^\diamond\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-n_3^*)\Gamma(2\alpha)} \\
&+\Big(\max\Big\{b_2 + \tfrac{m_2^*\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-m_3^*)\Gamma(2\alpha)}, b_1 + \tfrac{n_2^*\Gamma(\alpha)(\psi(T)-\psi(0))^\alpha}{(1-n_3^*)\Gamma(2\alpha)}\Big\}\Big)M \\
&= \Delta + \mu M \le \tfrac{M}{R} + \Big(1-\tfrac{1}{R}\Big)M = M.
\end{aligned}$$

Thus, $F_1(x,y) + F_2(u,v) \in \Omega$ for all $x, y \in \Omega$.
Step 2. We show that F_1 is a contraction mapping.
For any $(x,y),(u,v) \in \Omega\times\Omega$, we obtain the following by (H2):

$$F_1(x,y)(t) - F_1(u,v)(t) = (F_{11}, F_{12}),$$

where $F_{11} = (\psi(t)-\psi(0))^{\alpha-1}(r(v)-r(y))$ and $F_{12} = (\psi(t)-\psi(0))^{\alpha-1}(h(u)-h(x))$.
So,

$$\begin{aligned}
&|F_{11}| \le (\psi(t)-\psi(0))^{\alpha-1}b_2\|y-v\|_{C_{1-\alpha,\psi}}, \quad |F_{12}| \le (\psi(t)-\psi(0))^{\alpha-1}b_1\|x-u\|_{C_{1-\alpha,\psi}}. \\
&\|F_1(x,y)-F_1(u,v)\|_{C_{1-\alpha,\psi}\times C_{1-\alpha,\psi}} \\
&= \sup_{t\in J}|(\psi(t)-\psi(0))^{1-\alpha}F_{11}| + \sup_{t\in J}|(\psi(t)-\psi(0))^{1-\alpha}F_{12}| \\
&\le b_1\|x-u\|_{C_{1-\alpha,\psi}} + b_2\|y-v\|_{C_{1-\alpha,\psi}} \\
&\le \max\{b_1,b_2\}(\|x-u\|_{C_{1-\alpha,\psi}} + \|y-v\|_{C_{1-\alpha,\psi}}) \\
&= \max\{b_1,b_2\}(\|(x,y)-(u,v)\|_{C_{1-\alpha,\psi}\times C_{1-\alpha,\psi}}.
\end{aligned}$$

Hence, the operator F_1 is a contraction.
Step 3. We show that F_2 is continuous.

Let $\{(x_n, y_n)\}$ be a sequence such that $(x_n, y_n) \to (x, y)$ in $C_{1-\alpha,\psi}(J, \mathbb{R}) \times C_{1-\alpha,\psi}(J, \mathbb{R})$, then, for each $t \in (0, T]$, we have

$$F_2(x_n, y_n)(t) - F_2(x, y)(t) = (F_{21}, F_{22}),$$

where

$$F_{21} = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} (p_{x_n}(s) - p_x(s))\psi'(s)ds, \tag{17}$$

$$F_{22} = \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t) - \psi(s))^{\alpha-1} (q_{y_n}(s) - q_y(s))\psi'(s)ds. \tag{18}$$

where $p_{x_n}, p_x, q_{y_n}, q_y \in C_{1-\alpha,\psi}(J, \mathbb{R})$ be such that

$$\begin{aligned} p_{x_n}(t) &= f(t, y_n(t), p_{x_n}(t)), \quad p_x(t) = f(t, y(t), p_x(t)), \\ q_{y_n}(t) &= g(t, x_n(t), q_{y_n}(t)), \quad q_y(t) = g(t, x(t), q_y(t)). \end{aligned}$$

By (H1), one has

$$\begin{aligned} |p_{x_n}(t) - p_x(t)| &= |f(t, y_n(t), p_{x_n}(t)) - f(t, y(t), p_x(t))| \\ &\le L_1|y_n(t) - y(t)| + L_2|p_{x_n}(t) - p_x(t)|, \\ |q_{y_n}(t) - q_y(t)| &= |g(t, x_n(t), q_{y_n}(t)) - g(t, x(t), q_y(t))| \\ &\le l_1|x_n(t) - x(t)| + l_2|q_{y_n}(t) - q_y(t)|. \end{aligned}$$

Then,

$$|p_{x_n}(t) - p_x(t)| \le \frac{L_1}{1 - L_2}|y_n(t) - y(t)|, \tag{19}$$

and

$$|q_{y_n}(t) - q_y(t)| \le \frac{l_1}{1 - l_2}|x_n(t) - x(t)|. \tag{20}$$

By replacing (19) in Equation (17), we obtain the following:

$$\begin{aligned} &|F_{21}| \le \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} |p_{x_n}(s) - p_x(s)|\psi'(s)ds \\ &\le \tfrac{L_1}{(1-L_2)\Gamma(\alpha)} \textstyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} |y_n(s) - y(s)|\psi'(s)ds \\ &= \tfrac{L_1}{(1-L_2)\Gamma(\alpha)} \textstyle\int_0^t (\psi(t) - \psi(s))^{\alpha-1} (\psi(s) - \psi(0))^{\alpha-1} |(\psi(s) - \psi(0))^{1-\alpha}(y_n(s) - y(s))|\psi'(s)ds \\ &= \tfrac{L_1}{1-L_2} I_{\alpha,\psi}((\psi(t) - \psi(0))^{\alpha-1}) \|y_n - y\|_{C_{1-\alpha,\psi}} \\ &= \tfrac{L_1\Gamma(\alpha)}{(1-L_2)\Gamma(2\alpha)} (\psi(t) - \psi(0))^{2\alpha-1} \|y_n - y\|_{C_{1-\alpha,\psi}}. \end{aligned} \tag{21}$$

Similarly, we can obtain

$$|F_{22}| \le \frac{l_1\Gamma(\alpha)}{(1 - l_2)\Gamma(2\alpha)} (\psi(t) - \psi(0))^{2\alpha-1} \|x_n - x\|_{C_{1-\alpha,\psi}}. \tag{22}$$

From (21), and (22), one has

$$\|F_2(x_n,y_n)-F_2(x,y)\|_{C_{1-\alpha,\psi}} = \|(\psi(t)-\psi(0))^{1-\alpha}F_{21}\|_{C_{1-\alpha,\psi}} + \|(\psi(t)-\psi(0))^{1-\alpha}F_{22}\|_{C_{1-\alpha,\psi}}$$
$$\le \tfrac{L_1\Gamma(\alpha)}{(1-L_2)\Gamma(2\alpha)}(\psi(t)-\psi(0))^{\alpha}\|y_n-y\|_{C_{1-\alpha,\psi}}$$
$$+\tfrac{l_1\Gamma(\alpha)}{(1-l_2)\Gamma(2\alpha)}(\psi(t)-\psi(0))^{\alpha}\|x_n-x\|_{C_{1-\alpha,\psi}}$$
$$\le \max\Big\{\tfrac{L_1}{1-L_2},\tfrac{l_1}{1-l_2}\Big\}\tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(T)-\psi(0))^{\alpha}(\|x_n-x\|_{C_{1-\alpha,\psi}}+\|y_n-y\|_{C_{1-\alpha,\psi}})$$
$$= \max\Big\{\tfrac{L_1}{1-L_2},\tfrac{l_1}{1-l_2}\Big\}\tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(T)-\psi(0))^{\alpha}\|(x_n,y_n)-(x,y)\|_{C_{1-\alpha,\psi}},$$

which implies that

$$\|F_2(x_n,y_n)-F_2(x,y)\|_{C_{1-\alpha,\psi}} \to 0 \quad \text{as } n\to\infty.$$

That is, F_2 is continuous.

Step 4. We prove that F_2 is compact.

For each $(x,y)\in\Omega\times\Omega$ and $t\in(0,T]$, one has

$$F_2(x,y)(t) = \left(\frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}p_x(s)\psi'(s)ds, \frac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}q_y(s)\psi'(s)ds\right),$$

where p_x, q_y are as in (13). Similar to the proof of (16), we obtain

$$\left|\tfrac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}p_x(s)\psi'(s)ds\right|$$
$$\le \tfrac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}|(\psi(s)-\psi(0))^{1-\alpha}p_x(s)|\psi'(s)ds$$
$$\le \left(\tfrac{m_1^{\diamond}+m_2^{*}M}{1-m_3^{*}}\right)\tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(t)-\psi(0))^{2\alpha-1},$$

and

$$\left|\tfrac{1}{\Gamma(\alpha)}\int_0^t(\psi(t)-\psi(s))^{\alpha-1}q_y(s)\psi'(s)ds\right|$$
$$\le \left(\tfrac{n_1^{\diamond}+n_2^{*}M}{1-n_3^{*}}\right)\tfrac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(t)-\psi(0))^{2\alpha-1}.$$

Hence,

$$\|F_2(x,y)\|_{C_{1-\alpha,\psi}} \le \max\left\{\frac{m_1^{\diamond}+m_2^{*}M}{1-m_3^{*}},\frac{n_1^{\diamond}+n_2^{*}M}{1-n_3^{*}}\right\}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}(\psi(T)-\psi(0))^{\alpha}.$$

Thus, $F_2(\Omega\times\Omega)$ is uniformly bounded.

Finally, we show that $F_2(\Omega\times\Omega)$ is equicontinuous, let $0<t_1<t_2\le T$ and $(x,y)\in\Omega\times\Omega$. Then,

$$(\psi(t_2)-\psi(0))^{1-\alpha}F_2(x,y)(t_2)-(\psi(t_1)-\psi(0))^{1-\alpha}F_2(x,y)(t_1) = (K_{21},K_{22}),$$

where

$$K_{21} = \tfrac{1}{\Gamma(\alpha)}\Big[\int_0^{t_1}+\int_{t_1}^{t_2}(\psi(t_2)-\psi(0))^{1-\alpha}(\psi(t_2)-\psi(s))^{\alpha-1}p_x(s)\psi'(s)ds$$
$$-\int_0^{t_1}(\psi(t_1)-\psi(0))^{1-\alpha}(\psi(t_1)-\psi(s))^{\alpha-1}p_x(s)\psi'(s)ds\Big],$$
$$K_{22} = \tfrac{1}{\Gamma(\alpha)}\Big[\int_0^{t_1}+\int_{t_1}^{t_2}(\psi(t_2)-\psi(0))^{1-\alpha}(\psi(t_2)-\psi(s))^{\alpha-1}q_y(s)\psi'(s)ds$$
$$-\int_0^{t_1}(\psi(t_1)-\psi(0))^{1-\alpha}(\psi(t_1)-\psi(s))^{\alpha-1}q_y(s)\psi'(s)ds\Big].$$

Then, by (14), we obtain the following:

$$
\begin{aligned}
|K_{21}| &\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |(\psi(t_2)-\psi(0))^{1-\alpha}(\psi(t_2)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}\\
&-(\psi(t_1)-\psi(0))^{1-\alpha}(\psi(t_1)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}||(\psi(s)-\psi(0))^{1-\alpha}p_x(s)|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(\psi(t_2)-\psi(0))^{1-\alpha}(\psi(t_2)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}|(\psi(s)-\psi(0))^{1-\alpha}p_x(s)|ds\\
&\leq \frac{m_1^\diamond+m_2^*M}{(1-m_3^*)\Gamma(\alpha)}\int_0^{t_1} |(\psi(t_2)-\psi(0))^{1-\alpha}(\psi(t_2)-\psi(s))^{\alpha-1}\\
&-(\psi(t_1)-\psi(0))^{1-\alpha}(\psi(t_1)-\psi(s))^{\alpha-1}|(\psi(s)-\psi(0))^{\alpha-1}ds\\
&+\frac{m_1^\diamond+m_2^*M}{(1-m_3^*)\Gamma(\alpha)}\int_{t_1}^{t_2}(\psi(t_2)-\psi(0))^{1-\alpha}(\psi(t_2)-\psi(s))^{\alpha-1}(\psi(s)-\psi(0))^{\alpha-1}ds \to 0,
\end{aligned}
$$

as $t_2 \to t_1$. Similarly, we obtain that $K_{22} \to 0$ as $t_2 \to t_1$. Which yields that $F_2(\Omega\times\Omega)$ is equicontinuous, Then, by the Ascoli–Arzela theorem, the operator F_2 is compact.

All the assume of the Theorem 1 are satisfied. Therefore, there exists a fixed point, $(x,y) \in \Omega\times\Omega$, such that $(x,y) = F_1(x,y)+F_2(x,y)$, which is a solution of the problem (3). $\square$

4. Example

Consider the following coupled implicit ψ-Riemann–Liouville fractional differential equations with nonlocal conditions

$$
\begin{cases}
D_{0^+}^{\frac{4}{7},\ln(1+t)}x(t) = f(t,y(t),D_{0^+}^{\frac{4}{7},\ln(1+t)}x(t)), & t\in(0,1],\\
D_{0^+}^{\frac{4}{7},\ln(1+t)}y(t) = g(t,x(t),D_{0^+}^{\frac{4}{7},\ln(1+t)}y(t)), & t\in(0,1],\\
\ln^{\frac{3}{7}}(1+t)x(t)|_{t=0} = \frac{1}{3}-\sum\limits_{i=1}^{n} c_i\ln^{\frac{3}{7}}(1+t)y(t_i),\\
\ln^{\frac{3}{7}}(1+t)y(t)|_{t=0} = \frac{1}{4}-\sum\limits_{i=1}^{n} d_i\ln^{\frac{3}{7}}(1+t)x(t_i),
\end{cases}
\tag{23}
$$

where $0<t_1<\cdots<t_n<1$, c_i and d_i are positive constants with $\sum\limits_{i=1}^{n} c_i \leq \frac{1}{3}$ and $\sum\limits_{i=1}^{n} d_i \leq \frac{2}{5}$. Let

$$
f(t,u,v) = \frac{1}{(2+t)^2(2+|u|+|v|)} + \frac{2\sin t}{\ln^{\frac{3}{7}}(1+t)},\quad t\in(0,1],\ u,vs.\in\mathbb{R},
$$
$$
g(t,u,v) = \frac{1}{3\exp(2-t)}\arctan(1+3|u|+|v|) + \frac{\cos t}{\ln^{\frac{3}{7}}(1+t)},\quad t\in(0,1],\ u,vs.\in\mathbb{R}.
$$

Thus, we have

$$
C_{1-\alpha,\psi}([0,1],\mathbb{R}) = C_{\frac{3}{7},\ln(1+t)}([0,1],\mathbb{R}) = \{h:(0,1]\to\mathbb{R} : \ln^{\frac{3}{7}}(1+t)h\in C([0,1],\mathbb{R})\},
$$

where $\alpha = \frac{4}{7}$. Obviously, the functions f and g are continuous, $f(\cdot,0,0), g(\cdot,0,0) \in C_{\frac{3}{7},\ln(1+t)}([0,1],\mathbb{R})$. For any $u_1,u_2,v_1,v_2\in\mathbb{R}$ and $t\in(0,1]$, one has

$$
\begin{aligned}
&|f(t,u_1,v_1)-f(t,u_2,v_2)| = \frac{1}{(2+t)^2}\left|\frac{1}{2+|u_1|+|v_1|}-\frac{1}{2+|u_2|+|v_2|}\right|\\
&\leq \frac{|u_1-u_2|+|v_1-v_2|}{(2+t)^2(2+|u_1|+|v_1|)(2+|u_2|+|v_2|)}\\
&\leq \frac{1}{4}(|u_1-u_2|+|v_1-v_2|),\\
&|g(t,u_1,v_1)-g(t,u_2,v_2)| = \frac{1}{3\exp(2-t)}|\arctan(1+3|u_1|+|v_1|)-\arctan(1+3|u_2|+|v_2|)|\\
&\leq \frac{1}{3e}(3|u_1-u_2|+|v_1-v_2|).
\end{aligned}
$$

Moreover, set the following:

$$r(y) = \sum_{i=1}^{n} c_i \ln^{\frac{3}{7}}(1+t)y(t_i), \quad h(x) = \sum_{i=1}^{n} d_i \ln^{\frac{3}{7}}(1+t)x(t_i),$$

For each $x_1, x_2, y_1, y_2 \in \mathbb{R}$, we have

$$|h(x_1) - h(x_2)| \leq \sum_{i=1}^{n} d_i \ln^{\frac{3}{7}}(1+t)|x_1(t_i) - x_2(t_i)|$$

$$\leq \sum_{i=1}^{n} d_i \|x_1 - x_2\|_{C_{\frac{3}{7},\ln(1+t)}} \leq \frac{2}{5}\|x_1 - x_2\|_{C_{\frac{3}{7},\ln(1+t)}},$$

and

$$|r(y_1) - r(y_2)| \leq \sum_{i=1}^{n} c_i \ln^{\frac{3}{7}}(1+t)|y_1(t_i) - y_2(t_i)|$$

$$\leq \sum_{i=1}^{n} c_i \|y_1 - y_2\|_{C_{\frac{3}{7},\ln(1+t)}} \leq \frac{1}{3}\|y_1 - y_2\|_{C_{\frac{3}{7},\ln(1+t)}}.$$

So, conditions (H1) and (H2) are satisfied with $L_1 = \frac{1}{4}, L_2 = \frac{1}{4}, l_1 = \frac{1}{e}, l_2 = \frac{1}{3e}, b_1 = \frac{2}{5}$ and $b_2 = \frac{1}{3}$. Moreover, the following condition:

$$\max\left\{b_2 + \frac{L_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-L_2)}(\psi(T) - \psi(0))^\alpha, b_1 + \frac{l_1\Gamma(\alpha)}{\Gamma(2\alpha)(1-l_2)}(\psi(T) - \psi(0))^\alpha\right\}$$
$$= \max\left\{\frac{1}{3} + 0.4504, \frac{2}{5} + 0.5666\right\} = 0.9666 < 1,$$

is satisfied with $T = 1$. By Theorem 2, we have that the problem (23) has a unique coupled solution in the space $C_{\frac{3}{7},\ln(1+t)}([0,1], \mathbb{R}) \times C_{\frac{3}{7},\ln(1+t)}([0,1], \mathbb{R})$.

5. Conclusions

In this paper, we investigated a coupled implicit system that has ψ-Riemann–Liouville fractional derivative and nonlocal conditions. The interesting point is that two fractional implicit equations are coupled. By Banach fixed-point theorem and Krasnoselskii's fixed-point theorem, the uniqueness and the existence results are proved. Our results obtained in this paper is new and complements the existing literature on this topic. We will study the corresponding problem in future research, and we hope to be able to make some progress.

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