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Article

Linear Diophantine Fuzzy Rough Sets on Paired Universes with Multi Stage Decision Analysis

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Abstract: Rough set (RS) and fuzzy set (FS) theories were developed to account for ambiguity in the data processing. The most persuasive and modernist abstraction of an FS is the linear Diophantine FS (LD-FS). This paper introduces a resilient hybrid linear Diophantine fuzzy RS model (LDF-RS) on paired universes based on a linear Diophantine fuzzy relation (LDF-R). This is a typical method of fuzzy RS (F-RS) and bipolar FRS (BF-RS) on two universes that are more appropriate and customizable. By using an LDF-level cut relation, the notions of lower approximation (L-A) and upper approximation (U-A) are defined. While this is going on, certain fundamental structural aspects of LD-FAs are thoroughly investigated, with some instances to back them up. This cutting-edge LDF-RS technique is crucial from both a theoretical and practical perspective in the field of medical assessment.

Keywords: fuzzy set; linear Diophantine fuzzy sets; linear Diophantine fuzzy relations; level cut relations; rough approximations on two universes; decision analysis

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1. Introduction

As one of the most effective methods for developing a set's embryonic concept, Zadeh [1] first proposed the idea of an FS in 1965. According to the attributes, FS permits grading a set's features in the range of $[0,1]$. Since the conception of the theory, FS has been developed in a variety of ways, including intuitionistic fuzzy set (IF-S) [2,3], bipolar FS (B-FS) [4], Pythagorean FS (P-FS) [5,6], q-rung orthopair FS (q-ROF-S) [7], and LD-FS [8].

In 2019, Riaz and Hashmi [8] unveiled LD-FS, one of the most exquisite and significant generalizations of FS. Using the control parameters, LD-FS eliminates the restrictions connected to the membership degree (MD) and non-membership degree (NMD) of the prevalent abstractions of IF-Ss, B-FSs, and q-ROF-S. LD-FS is the most practical mathematical model for decision making (DM), multi-attribute decision making (MADM), engineering, artificial intelligence (AI), and medicine, allowing the decision maker to freely choose the grades [8]. Today, LD-FS is the owner of a huge study (see [9–11]). Ayub et al. [12] advanced an impressive method of an LDF-R to broaden the concept of IF-R, in which they provide an in-depth analysis of its essential characteristics, algebraic structures, and application in decision analysis.

While binary relations play a significant role in several domains for the transmission of unique things. In 1971, Zadeh [13] proposed the fuzzification of binary relations and presented the idea of an F-R. Numerous significant applications of FSs and F-Rs may be found in MCDM, neural networks, databases, pattern recognition, AI, clustering, F-control, and uncertainty reasoning. A thorough analysis of FSs and F-Rs is offered in [14].

The necessity to expand F-R was similar to that of FS. In 1984, Atanassov [15] proposed the concept of IF-R. An IF-R, per Atanassov's definition [15], is a pair of F-Rs where the total of the coalition and alienation grades is less than or equal to 1. A soft set [16], being a parameterized collection of the universe objects, has robust applications in decision making. m-Polar neutrosophic topology provides a generalized topological structure for data analysis [17].

Pawlak [18,19] suggested an approach of RS to deal with uncertainty in intelligent systems as another abstraction of classical set theory. The L-A and U-A, which are used to define the M of objects in RS theory, are two sharp approximation (A) sets. The fundamental ideas of the RS theory, which reveals the hidden knowledge in information systems, are these approximations. AI, machine learning, conflict analysis, and data analysis are just a few fields where RS theory has been successfully applied.

Due to the equivalence relation (E-R) that underlies the RS theory, its application in practical situations is constrained. Numerous abstractions have been constructed to overcome the constraint of an E-R. For instance, RS based on a binary relation [20,21], a set-valued map [22], a tolerance relation [23], a similarity relation [24], a reflexive relation (R-R) and transitive relation (T-R) [25], a soft binary relation [26,27], a soft E-R [28], two E-Rs [29], a normal soft group [30], two soft binary relations, and two normal soft groups, demonstrates how an E-R may be adjusted with different granule interpretations. Zhan and Alcantud [31] proposed a new kind of soft rough covering by means of soft neighborhoods. Motivation of the proposed work is based on some existing methodologies such as attribute analysis [32], picture fuzzy aggregation [33], interval-valued picture fuzzy Maclaurin symmetric mean operator [34] complex interval-valued Pythagorean fuzzy aggregation [35], risk priority evaluation [36], roughness in soft-intersection groups [37], and roughness in modules of fractions [38]. Karamaşa et al. [39] proposed an extended SVN-AHP and MULTIMOORA method to for flight training organizations. Osintsev [40] suggested DEMATEL-ANP method for an evaluation of logistic flows in green supply chains.

1.1. Research Gap and Motivation

From all of the above-mentioned, the sequel summarizes the driving forces behind our research and the gaps that lie underneath it:

(1) With the conceptualizations of the rough FS (R-FS) and fuzzy RS (F-RS) models (see [41–44]), Dubios and Prade [45] started the unification of RS and FS. Several authors have researched this idea (see [46–48]).
(2) Incorporating two universes, Li and Wang [49] created the R-FSA imagination.
(3) Yang [50] provided some of the applications for the notion of the roughness of a crisp set of two universes.
(4) Yang et al. [51] presented the BF-RS's idea on dual universe along with some of its applications.
(5) Less research has been performed on the idea of roughness in dual universes, particularly in P-FS and q-ROF-S.
(6) Ayub et al. [52] carefully thought out a method of applying RS to LD-FS with the aid of LDF-R and its applications in DM.
(7) To the best of our knowledge, no research has been performed on the idea of LDF-S roughness using the level cut relation of an LDF-R.
(8) To close this knowledge gap in the investigation of the roughness of LD-FSs, we introduce an abstraction of LDF-Rs using the level cut relations of an LDF-R on two different universal sets.

1.2. Major Contributions

This study uses level-cut relations from an LDF-R of dual universes to examine the roughness of an LD-FS. The fore set and after set of the level cut relations are used to design the underlying operations of RSs, the L- and U-As. With the use of useful examples, certain fundamental conclusions about As are demonstrated. We also defined the terms "accuracy measure" (A-M) and "roughness measure" (R-M) for LDF-RS. Finally, an LDF-RSs application to medical diagnosis is made to demonstrate its viability in real life.

1.3. Organization of the Paper

The remainder of this article is organized as follows to facilitate the study: In Section 2, some hypothetical early conceptions of RS, LD-FS, and LDF-R are provided. Using an LDF-R and a thorough examination of the essential characteristics of approximations with examples, the concept of LDF-RS on two distinct universes is introduced in the third segment. Section 4 includes the A-M and R-M cues for the LDF-RS. The application of LDF-RSs is demonstrated with the help of an example in Section 5. Section 6 concludes the paper by summarizing the final remarks.

2. Preliminaries

This subsection consists of some essential knowledge of LD-FS, LDF-R and RS. Throughout this research, $\breve{\mathscr{U}}$, $\breve{\mathscr{U}}_1$ and $\breve{\mathscr{U}}_2$ will denote the initial universes, unless otherwise specified.

Definition 1 ([19])**.** *Let ρ be an $E-R$ on $\breve{\mathscr{U}}$. Then, the pair $(\breve{\mathscr{U}},\rho)$ is known as an R approximation space (R-AS). For any subset $\mathcal{O}$ of $\breve{\mathscr{U}}$, the L-A $\underline{\mathcal{O}}_\rho$ and the U-A $\overline{\mathcal{O}}^\rho$ are defined as follows:*

$$\underline{\mathcal{O}}_\rho = \{v \in \breve{\mathscr{U}} : [v]_\rho \subseteq \mathcal{O}\} \text{ and } \overline{\mathcal{O}}^\rho = \{v \in \breve{\mathscr{U}} : [v]_\rho \cap \mathcal{O} \neq \emptyset\}$$

where $[v]_\rho$ signifies an E-class of $v \in \breve{\mathscr{U}}$ deduced by ρ. The boundary zone is indicated and described as follows:

$$BR(\mathcal{O}) = \overline{\mathcal{O}}^\rho - \underline{\mathcal{O}}_\rho$$

If $BR(\mathcal{O}) \neq \emptyset$, then $\mathcal{O}$ is known as an RS or otherwise a crisp set or a definable set. Based on these As, Pawlak characterized a crisp set $\mathcal{O} \subseteq \breve{\mathscr{U}}$ in the sequel:

- ⋆ *$\underline{\mathcal{O}}_\rho$ consists of the definite members and is known as the positive region (PR) of $\mathcal{O}$;*
- ⋆ *$\breve{\mathscr{U}} - \overline{\mathcal{O}}^\rho$ consists of the definite non-members and is known as the negative region (NR) of $\mathcal{O}$;*
- ⋆ *$BR(\mathcal{O})$ contains questionable members that may or may not be contained in $\mathcal{O}$ and is known as the boundary region (BR).*

Recently, Riaz and Hashmi [8] introduced an efficient approach to handling uncertainties that eradicate all the limitations related to affiliation and disassociation grades of the existing models (FS,B-FS,IF-S and P-FS).

Definition 2 ([8])**.** *An LD-FS on $\breve{\mathscr{U}}$ is an object defined as follows:*

$$\pounds_{\mathcal{D}} = \{(v, <\Theta^M(v), \Theta^N(v)>, <\varpi^M(v), \varpi^M(v)>) : v \in \breve{\mathscr{U}}\}$$

where

$$\Theta^M, \Theta^N : \breve{\mathscr{U}} \to [0,1]$$

are M and NM functions and $\varpi^M(v), \varpi^N(v) \in [0,1]$ are the reference parameters of $\Theta^M(v), \Theta^N(v)$ respectively, such that $0 \leq \varpi^M(v)\Theta^M(v) + \varpi^N(v)\Theta^N(v) \leq 1$ satisfying $0 \leq \varpi^M(v) + \varpi^N(v) \leq 1$ for all $u \in \breve{\mathscr{U}}$. The hesitation part is defined as $\Lambda(v)\Pi(v) = 1 - (\varpi^M(v)\Theta^M(v) + \varpi^N(v)\Theta^N(v))$, where $\Pi(v)$ expresses the degree of indeterminacy, and $\Lambda(v)$ refers to the relevant reference parameter. We use the notion $LD-FS(\breve{\mathscr{U}})$ to represent the collection of all LD-FSs on $\breve{\mathscr{U}}$.

By using control parameters that correspond to the association and disassociation grades in Riaz and Hashmi's [8] motivation, Ayub et al. [12] have expanded the idea of IF-R [15] to LDF-R.

Definition 3 ([12])**.** *An expression of the following form is an LDF-R $\ddot{\rho}$ from $\breve{\mathscr{U}}_1$ to $\breve{\mathscr{U}}_2$:*

$$\ddot{\rho} = \{((v_1, v_2), < \Theta^M(v_1, v_2), \Theta^N(v_1, v_2) >, < \varpi^M(v_1, v_2), \varpi^N(v_1, v_2) >) : v_1 \in \breve{\mathscr{U}}_1, v_2 \in \breve{\mathscr{U}}_2\}$$

where the mappings

$$\Theta^M, \Theta^N : \breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2 \to [0,1]$$

indicate the M and NM F-Rs from $\breve{\mathscr{U}}_1$ *to* $\breve{\mathscr{U}}_2$*, respectively, and* $\varpi^M(v_1, v_2), \varpi^N(v_1, v_2) \in [0,1]$ *are the relevant reference parameters to* $\Theta^M(v_1, v_2)$ *and* $\Theta^N(v_1, v_2)$*, respectively, fulfilling the requirement* $0 \leq \varpi^M(v_1, v_2)\Theta^M(v_1, v_2) + \varpi^N(v_1, v_2)\Theta^N(v_1, v_2) \leq 1$*, for all* $(v_1, v_2) \in \breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2$ *with* $0 \leq \varpi^M(v_1, v_2) + \varpi^N(v_1, v_2) \leq 1$*. The hesitation part is defined as follows:*

$$\ddot{\gamma}(v_1, v_2)\ddot{\pi}(v_1, v_2) = 1 - (\varpi^M(v_1, v_2)\Theta^M(v_1, v_2) + \varpi^N(v_1, v_2)\Theta^N(v_1, v_2))$$

where $\ddot{\pi}(v_1, v_2)$ *is the hesitation index, and* $\ddot{\gamma}(v_1, v_2)$ *is the relevant reference parameter. For the sake of simplicity, we will use* $\ddot{\rho} = (< \Theta^M(v_1, v_2), \Theta^N(v_1, v_2) >, < \varpi^M(v_1, v_2), \varpi^N(v_1, v_2) >)$ *for an LDF-R from* $\breve{\mathscr{U}}_1$ *to* $\breve{\mathscr{U}}_2$*. The collection of all LDF-Rs from* $\breve{\mathscr{U}}_1$ *to* $\breve{\mathscr{U}}_2$ *will be designated by* $LDF - R(\breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2)$*.*

With respect to finite universes $\breve{\mathscr{U}}_1$ and $\breve{\mathscr{U}}_2$, the matrix notation of an LDF-R is given in the sequel.

Definition 4 ([12])**.** *Let* $\ddot{\rho} = (< \Theta^M(u_i, v_j), \Theta^N(u_i, v_j) >, < \varpi^M(u_i, v_j), \varpi^N(u_i, v_j) >)$ *be an LDF-R from* $\breve{\mathscr{U}}_1$ *to* $\breve{\mathscr{U}}_2$*, where* $\breve{\mathscr{U}}_1 = \{u_1, u_2, ..., u_m\}$ *and* $\breve{\mathscr{U}}_2 = \{v_1, v_2, ..., v_n\}$*. Consider* $\Theta^M(u_i, v_j) = (\Theta^M_{ij})_{m\times n}$*,* $\Theta^N(u_i, v_j) = (\Theta^N_{ij})_{m\times n}$ *and* $\varpi^M(u_i, v_j) = (\varpi^M_{ij})_{m\times n}$*,* $\varpi^N(u_i, v_j) = (\varpi^N_{ij})_{m\times n}$*, with* $0 \leq \varpi^M_{ij} + \varpi^M_{ij} \leq 1$ *fulfilling* $0 \leq \varpi^M_{ij}\Theta^M_{ij} + \varpi^M_{ij}\Theta^N_{ij} \leq 1$ *for all* i, j*, where* $1 \leq i \leq m$ *and* $1 \leq j \leq n$*. Then, the following four matrices can be used to represent* $\ddot{\rho}$*:*

$$\Theta^M = (\Theta^M_{ij})_{m\times n} = \begin{pmatrix} \Theta^M_{11} & \Theta^M_{12} & \dots & \Theta^M_{1n} \\ \Theta^M_{21} & \Theta^M_{22} & \dots & \Theta^M_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \Theta^M_{m1} & \Theta^M_{m2} & \dots & \Theta^M_{mn} \end{pmatrix}, \Theta^N = (\Theta^N_{ij})_{m\times n} = \begin{pmatrix} \Theta^N_{11} & \Theta^N_{12} & \dots & \Theta^N_{1n} \\ \Theta^N_{21} & \Theta^N_{22} & \dots & \Theta^N_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \Theta^N_{m1} & \Theta^N_{m2} & \dots & \Theta^N_{mn} \end{pmatrix},$$

$$\varpi^M = (\varpi^M_{ij})_{m\times n} = \begin{pmatrix} \varpi^M_{11} & \varpi^M_{12} & \dots & \varpi^M_{1n} \\ \varpi^M_{21} & \varpi^M_{22} & \dots & \varpi^M_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \varpi^M_{m1} & \varpi^M_{m2} & \dots & \varpi^M_{mn} \end{pmatrix}, \varpi^N = (\varpi^N_{ij})_{m\times n} = \begin{pmatrix} \varpi^N_{11} & \varpi^N_{12} & \dots & \varpi^N_{1n} \\ \varpi^N_{21} & \varpi^N_{22} & \dots & \varpi^N_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \varpi^N_{m1} & \varpi^N_{m2} & \dots & \varpi^N_{mn} \end{pmatrix}$$

The following definitions describe some basic operations on LDF-Rs.

Definition 5 ([12])**.** *Let* $\ddot{\rho}_1 = (< \Theta^M_1(v_1, v_2), \Theta^N_1(v_1, v_2) >, < \varpi^M_1(v_1, v_2), \varpi^N_1(v_1, v_2) >)$ *and* $\ddot{\rho}_2 = (< \Theta^M_2(v_1, v_2), \Theta^N_2(v_1, v_2) >, < \varpi^M_2(v_1, v_2), \varpi^N_2(v_1, v_2) >)$ *be two LDF-Rs from* $\breve{\mathscr{U}}_1$ *to* $\breve{\mathscr{U}}_2$*. Then,*

(1) $\ddot{\rho}_1 \subseteq \ddot{\rho}_2$ *if and only if*

$$\Theta^M_1(v_1, v_2) \leq \Theta^M_2(v_1, v_2) \text{ and } \Theta^N_1(v_1, v_2) \geq \Theta^N_2(v_1, v_2),$$

$$\varpi^M_1(v_1, v_2) \leq \varpi^M_2(v_1, v_2) \text{ and } \varpi^N_1(v_1, v_2) \geq \varpi^N_2(v_1, v_2)$$

(2) $\ddot{\rho}_1 \cup \ddot{\rho}_2 = (< (\Theta^M_1 \cup \Theta^M_2)(v_1, v_2), (\Theta^N_1 \cap \Theta^N_2)(v_1, v_2) >, < \varpi^M_1(v_1, v_2) \vee \varpi^M_2(v_1, v_2), \varpi^M_1(v_1, v_2) \wedge \varpi^N_2(v_1, v_2) >)$*, where*

$$(\Theta^M_1 \cup \Theta^M_2)(v_1, v_2) = \Theta^M_1(v_1, v_2) \vee \Theta^M_2(v_1, v_2) \text{ and}$$

$$(\Theta^N_1 \cap \Theta^N_2)(v_1, v_2) = \Theta^N_1(v_1, v_2) \wedge \Theta^N_2(v_1, v_2)$$

(3) $\ddot{\rho}_1 \cap \ddot{\rho}_2 = (< (\Theta_1^M \cap \Theta_2^M)(v_1, v_2), (\Theta_1^N \cup \Theta_2^N)(v_1, v_2) >, < \varpi_1^M(v_1, v_2) \wedge \varpi_2^M(v_1, v_2), \varpi_1^N(v_1, v_2) \vee \varpi_2^N(v_1, v_2) >)$, *where*

$$(\Theta_1^M \cap \Theta_2^M)(v_1, v_2) = \Theta_1^M(v_1, v_2) \wedge \Theta_2^M(v_1, v_2) \text{ and}$$

$$(\Theta_1^N \cup \Theta_2^N)(v_1, v_2) = \Theta_1^N(v_1, v_2) \vee \Theta_2^N(v_1, v_2)$$

(4) $\ddot{\rho}_1^c = (< \Theta_1^N(v_1, v_2), \Theta_1^M(v_1, v_2) >, < \varpi_1^N(v_1, v_2), \varpi_1^M(v_1, v_2) >)$.

for all $(v_1, v_2) \in \breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2$.

Definition 6 ([12]). *Let* $\ddot{\rho}_1 = (< \Theta_1^M(v_1, v_2), \Theta_1^N(v_1, v_2) >, < \varpi_1^M(v_1, v_2), \varpi_1^N(v_1, v_2) >)$ *be an LDF-R over* $\breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2$ *and* $\ddot{\rho}_2 = (< \Theta_2^M(v_1, v_2), \Theta_2^N(v_1, v_2) >, < \varpi_2^M(v_1, v_2), \varpi_2^N(v_1, v_2) >)$ *be an LDF-R over* $\breve{\mathscr{U}}_2 \times \breve{\mathscr{U}}_3$. *Then, their composition is denoted by* $\hat{\circ}$ *and is determined accordingly:*

$$\ddot{\rho}_1 \hat{\circ} \ddot{\rho}_2 = (< (\Theta_1^M \hat{\circ} \Theta_2^M)(v_1, v_3), (\Theta_1^N \hat{\circ} \Theta_2^N)(v_1, v_3) >, < (\varpi_1^M \hat{\circ} \varpi_2^M)(v_1, v_3), (\varpi_1^N \hat{\circ} \varpi_2^N)(v_1, v_3) >)$$

where

$$(\Theta_1^M \hat{\circ} \Theta_2^M)(v_1, v_3) = \vee_{x_2 \in \breve{\mathscr{U}}_2}(\Theta_1^M(v_1, v_2) \wedge \Theta_2^M(v_2, v_3))$$

$$(\Theta_1^N \hat{\circ} \Theta_2^N)(v_1, v_3) = \wedge_{x_2 \in \breve{\mathscr{U}}_2}(\Theta_1^N(v_1, v_2) \vee \Theta_2^N(v_2, v_3))$$

and

$$(\varpi_1^M \hat{\circ} \varpi_2^M)(v_1, v_3) = \vee_{u_2 \in \breve{\mathscr{U}}_2}(\varpi_1^M(v_1, v_2) \wedge \varpi_2^M(v_2, v_3))$$

$$(\varpi_1^N \hat{\circ} \varpi_2^N)(v_1, v_3) = \wedge_{u_2 \in \breve{\mathscr{U}}_2}(\varpi_1^N(v_1, v_2) \vee \varpi_2^N(v_2, v_3))$$

for all $(v_1, v_3) \in \breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_3$.

Definition 7 ([12]). *Let* $\ddot{\rho}$ *be an LDF-R on* $\breve{\mathscr{U}}$. *Then,* $\ddot{\rho}$ *is classified as:*

(1) *a reflexive LDF-R (R-LDF-R), if:*

$$\Theta^M(v, v) = 1, \Theta^N(v, v) = 0 \text{ and } \varpi^M(v, v) = 1, \varpi^N(v, v) = 0$$

for all $u \in \breve{\mathscr{U}}$.

(2) *a symmetric LDF-R (S-LDF-R), if*

$$\Theta^M(v_1, v_2) = \Theta^M(v_2, v_1), \Theta^N(v_1, v_2) = \Theta^N(v_2, v_1) \text{ and } \ddot{\alpha}(v_1, v_2) = \ddot{\alpha}(v_2, v_1), \ddot{\beta}(v_1, v_2) = \ddot{\beta}(v_2, v_1)$$

(3) *a transitive LDF-R (T-LDF-R), if*

$$\Theta^M \hat{\circ} \Theta^M \subseteq \Theta^M, \Theta^N \hat{\circ} \Theta^N \supseteq \Theta^N \text{ and } \varpi^M \hat{\circ} \varpi^M \subseteq \varpi^M, \varpi^N \hat{\circ} \varpi^N \supseteq \varpi^N$$

(4) *an equivalence LDF-R (E-LDF-R), if* $\ddot{\rho}$ *is a R-, S-, and T-LDF-R over* $\breve{\mathscr{U}}$.

If $|\breve{\mathcal{U}}| = n$, where $|.|$ indicates the quantity of items in $\breve{\mathscr{U}}$, $\ddot{\rho} = (< (\Theta_{ij}^M)_{n\times n}, (\Theta_{ij}^N)_{n\times n} >, < (\varpi_{ij}^M)_{n\times n}, (\varpi_{ij}^N)_{n\times n} >)$. Let $\Theta^M = (\Theta_{ij}^M)_{n\times n}$, $\Theta^N = (\Theta_{ij}^N)_{n\times n}$ and $\varpi^M = (\varpi_{ij}^M)_{n\times n}$, $\varpi^N = (\varpi_{ij}^N)_{n\times n}$. Then,

(1) $\ddot{\rho}$ is R, if $\Theta_{ii}^M = \varpi_{ii}^M = 1$, and $\Theta_{ii}^N = \varpi_{ii}^N = 0$, where $i, j = 1, 2, ..., n$.
(2) $\ddot{\rho}$ is S, if $(\Theta^M)^T = \Theta^M$, $(\Theta^N)^T = \Theta^N$ and $(\varpi^M)^T = \varpi^M$, $(ss\varpi^N)^T = \varpi^N$,
(3) $\ddot{\rho}$ is T, if $\Theta^M \hat{\circ} \Theta^M \subseteq \Theta^M$, $\Theta^N \hat{\circ} \Theta^N \supseteq \Theta^N$ and $\varpi^M \hat{\circ} \varpi^M \subseteq \varpi^M$, $\varpi^N \hat{\circ} \varpi^N \supseteq \varpi^N$.
(4) $\ddot{\rho}$ is E, if $\ddot{\rho}$ is R, S and T as well,

3. Some Properties of Linear Diophantine Fuzzy Relation

Ayub et al. [12] proposed the idea of LDF-R from $\breve{\mathscr{U}}_1$ to $\breve{\mathscr{U}}_2$. The purpose of this section is to introduce the idea of a level cut relation of an LDF-R. Additionally, we investigate a few of its crucial characteristics, including the R-, S-, and T-LDF-R in terms of its level cut relations.

Definition 8. *Let* $\ddot{\rho} = (<\Theta^M(v_1,v_2),\Theta^N(v_1,v_2)>,<\varpi^M(v_1,v_2),\varpi^N(v_1,v_2)>)$ *be an LDF-R from* $\breve{\mathscr{U}}_1$ *to* $\breve{\mathscr{U}}_2$*. Let* $\ddot{s},\ddot{t},\ddot{u},\ddot{v} \in [0,1]$ *be such that* $0 \le \ddot{s}\ddot{u} + \ddot{t}\ddot{v} \le 1$ *with* $0 \le \ddot{u} + \ddot{v} \le 1$*, and define the* $(<\ddot{s},\ddot{u}>,<\ddot{t},\ddot{v}>)-$*level cut relation of* $\ddot{\rho}$ *as follows:*

$$(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>} = \{(v_1,v_2) \in \breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2 : \Theta^M(v_1,v_2) \ge \ddot{s}, \varpi^M(v_1,v_2) \ge \ddot{u} \text{ and } \Theta^N(v_1,v_2) \le \ddot{t}, \varpi^N(v_1,v_2) \le \ddot{v}\}$$

where

$$(\ddot{\rho})_{<\ddot{s},\ddot{u}>} = \{(v_1,v_2) \in \breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2 : \Theta^M(v_1,v_2) \ge \ddot{s}, \varpi^M(v_1,v_2) \ge \ddot{u}\}$$

is said to be $<\ddot{s},\ddot{u}>-$*level cut relation of* $\ddot{\rho}$*, and*

$$(\ddot{\rho})^{<\ddot{t},\ddot{v}>} = \{(v_1,v_2) \in \breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2 : \Theta^N(v_1,v_2) \le \ddot{t}, \varpi^N(v_1,v_2) \le \ddot{v}\}$$

is called $<\ddot{t},\ddot{v}>-$*level cut relation of* $\ddot{\rho}$.

Theorem 1. $\ddot{\rho}$ *is R-LDF-R if and only if* $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ *is R-R on* $\breve{\mathscr{U}}$*, for all* $\ddot{s},\ddot{u},\ddot{t},\ddot{v} \in [0,1]$.

Proof. Suppose that $\ddot{\rho}$ is R-LDF-R. By Definition 7 (1), $\Theta^M(v,v) = 1 \ge \ddot{s}, \Theta^N(v,v) = 0 \le \ddot{t}$ and $\varpi^M(v,v) = 1 \ge \ddot{u}, \varpi^N(v,v) = 0 \le \ddot{v}$, for all $\ddot{s},\ddot{t},\ddot{u},\ddot{v} \in [0,1]$ such that $0 \le \ddot{s}\ddot{u} + \ddot{t}\ddot{v} \le 1$ with $0 \le \ddot{u} + \ddot{v} \le 1$. Hence, $(x,x) \in (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ for all $u \in \breve{\mathscr{U}}$.

Conversely, assume that $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ is R-R. If $\ddot{\rho}$ is not R-LDF-R, then for some $v \in \breve{\mathscr{U}}$ either $\Theta^M(v,v) \ne 1$, or $\Theta^N(v,v) \ne 0$ or $\varpi^M(v,v) \ne 1$ or $\varpi^N(v,v) \ne 0$, for some $\ddot{s},\ddot{t},\ddot{u},\ddot{v} \in [0,1]$. If $\Theta^M(v,v) \ne 1$. Taking $\ddot{s} = 1$, we have $(x,x) \notin (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$, which is a contradiction. The other three cases are similar. Hence, $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ is a R-R. □

Theorem 2. $\ddot{\rho}$ *is S-LDF-R if and only if* $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ *is S-R on* $\breve{\mathscr{U}}$*, for all* $\ddot{s},\ddot{u},\ddot{t},\ddot{v} \in [0,1]$.

Proof. Suppose that $\ddot{\rho}$ is S-LDF-R. Let $(v_1,v_2) \in (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$. By Definition 8, $\Theta^M(v_1,v_2) \ge \ddot{s}, \varpi^M(v_1,v_2) \ge \ddot{u}$ and $\Theta^N(v_1,v_2) \le \ddot{t}, \varpi^N(v_1,v_2) \le \ddot{v}$. Since $\ddot{\rho}$ is symmetric, so we have $\Theta^M(v_2,v_1) \ge \ddot{s}, \varpi^M(v_2,v_1) \ge \ddot{u}$ and $\Theta^N(v_2,v_1) \le \ddot{t}, \varpi^N(v_2,v_1) \le \ddot{v}$ (see Definition 7 (2)). Thus, $(v_2,v_1) \in (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$.

Conversely, assume that $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ is S-R on $\breve{\mathscr{U}}$. Letting $\Theta^M(v_1,v_2) = \ddot{s}, \varpi^M(v_1,v_2) = \ddot{u}$ and $\Theta^N(v_1,v_2) = \ddot{t}, \varpi^N(v_1,v_2) = \ddot{v}$, for some $\ddot{s},\ddot{t},\ddot{u},\ddot{v} \in [0,1]$ such that $0 \le \ddot{s}\ddot{u} + \ddot{t}\ddot{v} \le 1$ with $0 \le \ddot{u} + \ddot{v} \le 1$. It follows that $(v_1,v_2) \in (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$. By assumption on $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$, we have $(v_2,v_1) \in (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$. Thus, $\Theta^M(v_2,v_1) \ge \ddot{s} = \Theta^M(v_2,v_1), \varpi^M(v_2,v_1) \ge \ddot{u} = \varpi^M(v_1,v_2)$ and $\Theta^N(v_2,v_1) \le \ddot{t} = \Theta^N(v_1,v_2), \beta(v_2,v_1) \le \ddot{v} = \varpi^N(v_1,v_2)$. By using similar arguments, other inequalities can be shown. Thus, $\ddot{\rho}$ is S-LDF-R on $\breve{\mathscr{U}}$. This completes the proof. □

Proposition 1. $\ddot{\rho}$ *is T-LDF-R if and only if*

$$\Theta^M(v_1,v_2) \wedge \Theta^M(v_2,v_3) \le \Theta^M(v_1,v_3), \Theta^N(v_1,v_2) \wedge \Theta^N(v_2,v_3) \ge \Theta^N(v_1,v_3)$$

$$\text{and } \varpi^M(v_1,v_2) \wedge \varpi^M(v_2,v_3) \le \varpi^M(v_1,v_3), \varpi^N(v_1,v_2) \wedge \varpi^N(v_2,v_3) \ge \varpi^N(v_1,v_3),$$

for all $v_1,v_2,v_3 \in \breve{\mathscr{U}}$.

Proof. Suppose that $\ddot{\rho}$ is T-LDF-R on $\breve{\mathscr{U}}$. By Definition 7 (3), $(\Theta^M \hat{\circ} \Theta^M)(v_1,v_3) \subseteq \Theta^M(v_1,v_3)$, $(\Theta^N \hat{\circ} \Theta^N)(v_1,v_3) \supseteq \Theta^N(v_1,v_3)$ and $(\varpi^M \hat{\circ} \varpi^M)(v_1,v_3) \subseteq \varpi^M(v_1,v_3), (\varpi^N \hat{\circ} \varpi^N)(v_1,v_3) \supseteq \varpi^N(v_1,v_3)$, for all $v_1,v_3 \in \breve{\mathscr{U}}$. Thus, $\Theta^M(v_1,v_2) \wedge \Theta^M(v_2,v_3) \le \Theta^M(v_1,v_3), \Theta^N(v_1,v_2) \wedge \Theta^N(v_2,v_3) \ge \Theta^N(v_1,v_3)$ and $\varpi^M(v_1,v_2) \wedge \varpi^M(v_2,v_3) \le \varpi^M(v_1,v_3)$, $\varpi^N(v_1,v_2) \wedge \varpi^N(v_2,$

$v_3) \geq \varpi^N(v_1, v_3)$, for all $v_1, v_2, v_3 \in \breve{\mathscr{U}}$ (see Definition 6). The converse can be proven, similarly. $\square$

Theorem 3. *$\ddot{\rho}$ is a T-LDF-R if and only if $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ is T-R on $\breve{\mathscr{U}}$, for all $\ddot{s}, \ddot{u}, \ddot{t}, \ddot{v} \in [0,1]$.*

Proof. Suppose that $\ddot{\rho}$ is T-LDF-R. Let $(v_1, v_2), (v_2, v_3) \in (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$. Then, $\Theta^M(v_1, v_2) \wedge \Theta^M(v_2, v_3) \geq \ddot{s}, \alpha(v_1, v_2) \wedge \alpha(v_2, v_3) \geq \ddot{u}$ and $\Theta^N(v_1, v_2) \vee \Theta^N(v_2, v_3) \leq \ddot{t}, \beta(v_1, v_2) \vee \beta(v_2, v_3) \leq \ddot{v}$ (see Definition 8). Using above Proposition 1, we obtain: $\Theta^M(v_1, v_3) \geq \ddot{s}, \varpi^M(v_1, v_3) \geq \ddot{u}, \Theta^N(v_1, v_3) \leq \ddot{t}, \varpi^N(v_1, v_3) \leq \ddot{v}$. Thus, $(v_1, v_3) \in (\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$. $\square$

Theorem 4. *$\ddot{\rho}$ is an E-LDF-R if and only if $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ is an E-R on $\breve{\mathscr{U}}$, for all $\ddot{s}, \ddot{t}, \ddot{u}, \ddot{v} \in [0,1]$.*

Proof. Theorems 1–3 have a direct impact on the proof. $\square$

Now, to measure the 'resemblance', 'comparability' or 'closeness' of the objects in $\breve{\mathscr{U}}$, we define the following concept.

Definition 9. *$\ddot{\rho}$ is said to be a tolerance LDF-R (or compatible LDF-R), if it is R-LDF-R and S-LDF-R.*

To illustrate our above notions, we provide Example 2 below.

Example 1. *Let $\breve{\mathscr{U}} = \{v_1, v_2, v_3, v_4\}$. Construct an LDF-R $\ddot{\rho}$ on $\breve{\mathscr{U}}$ in matrix notation form as follows:*

$$\Theta^M = \begin{pmatrix} 1 & 0.725 & 0.862 & 0.921 \\ 0.725 & 1 & 0.815 & 0.132 \\ 0.862 & 0.815 & 1 & 0.325 \\ 0.921 & 0.132 & 0.325 & 1 \end{pmatrix}, \Theta^N = \begin{pmatrix} 0 & 0.218 & 0.125 & 0.215 \\ 0.218 & 0 & 0.651 & 0.334 \\ 0.125 & 0.651 & 0 & 0.728 \\ 0.215 & 0.334 & 0.728 & 0 \end{pmatrix},$$
$$\varpi^M = \begin{pmatrix} 1 & 0.71 & 0.81 & 0.89 \\ 0.71 & 1 & 0.75 & 0.11 \\ 0.81 & 0.75 & 1 & 0.21 \\ 0.89 & 0.11 & 0.21 & 1 \end{pmatrix}, \varpi^N = \begin{pmatrix} 0 & 0.16 & 0.10 & 0.11 \\ 0.16 & 0 & 0.25 & 0.34 \\ 0.10 & 0.25 & 0 & 0.64 \\ 0.11 & 0.34 & 0.64 & 0 \end{pmatrix}.$$

Using Definition 8 of $(<\ddot{s}, \ddot{u}>, <\ddot{t}, \ddot{v}>)$-level cut relation, we are able to obtain the following: For $\ddot{s} = \ddot{u} = 1, \ddot{t} = \ddot{v} = 0$,

$$(\ddot{\rho})^{<0,0>}_{<1,1>} = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4)\}$$

For $\ddot{s} = 0.725$, $\ddot{u} = 0.71$ and $\ddot{t} = 0.218$, $\ddot{v} = 0.16$,

$$(\ddot{\rho})^{<0.218,0.16>}_{<0.725,0.71>} = \{(v_1, v_1), (v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_2), (v_3, v_1), (v_3, v_3), (v_4, v_1), (v_4, v_4)\}$$

For $\ddot{s} = 0.862$, $\ddot{u} = 0.81$ and $\ddot{t} = 0.125$, $\ddot{v} = 0.10$,

$$(\ddot{\rho})^{<0.125,0.10>}_{<0.862,0.81>} = \{(v_1, v_1), (v_1, v_3)(v_2, v_2), (v_3, v_1), (v_3, v_3), (v_4, v_4)\}$$

For $\ddot{s} = 0.921$, $\ddot{u} = 0.89$ and $\ddot{t} = 0.215$, $\ddot{v} = 0.11$,

$$(\ddot{\rho})^{<0.215,0.11>}_{<0.921,0.89>} = \{(v_1, v_1), (v_1, v_4), (v_2, v_2), (v_3, v_3), (v_4, v_1), (v_4, v_4)\}$$

For $\ddot{s} = 0.815$, $\ddot{u} = 0.75$ and $\ddot{t} = 0.651$, $\ddot{v} = 0.25$,

$$(\ddot{\rho})^{<0.651,0.25>}_{<0.815,0.75>} = \{(v_1, v_1), (v_1, v_3), (v_1, v_4), (v_2, v_2), (v_2, v_3), (v_3, v_1), (v_3, v_2), (v_3, v_3), (v_4, v_1), (v_4, v_4)\}$$

For $\ddot{s} = 0.132$, $\ddot{u} = 0.11$ and $\ddot{t} = 0.334$, $\ddot{v} = 0.34$,

$$(\ddot{\rho})^{<0.334,0.34>}_{<0.132,0.11>} = \{(v_1,v_1),(v_1,v_2),(v_1,v_3),(v_1,v_4),(v_2,v_1),(v_2,v_2),(v_2,v_4),(v_3,v_1),(v_3,v_3),(v_4,v_1),(v_4,v_2),(v_4,v_4)\}$$

For $\ddot{s} = 0.325$, $\ddot{u} = 0.21$ and $\ddot{t} = 0.728$, $\ddot{v} = 0.64$,

$$(\ddot{\rho})^{<0.728,0.64>}_{<0.325,0.21>} = (\breve{\mathscr{U}} \times \breve{\mathscr{U}}) \setminus \{(v_2,v_4),(v_4,v_2)\}$$

It is simple to observe that $(\ddot{\rho})^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$ is an E-R on $\breve{\mathscr{U}}$, for each $\ddot{s},\ddot{u},\ddot{t},\ddot{v}$. Hence, by using Theorem 4, $\ddot{\rho}$ is an E-LDF-R on $\breve{\mathscr{U}}$.

4. Linear Diophantine Fuzzy Rough Sets on Two Universes

In literature, R-As on two different universes using F-R are initiated by Sun and Ma [48]. Since the NM part is not discussed in F-R, Yang et al. [51] extended the concept of [48] to fuzzy bipolar relation (FB-R). In this segment, we generalize this concept to LDF-R and introduce a new concept of roughness called LDF-RS on two universes based on the after sets and fore sets of the level cut relation of an LDF-R (a crisp relation).

If $\ddot{\rho} \in LDF - R(\breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2)$, then the triplet $\ddot{\mathbb{P}} = (\breve{\mathscr{U}}_1, \breve{\mathscr{U}}_2, \ddot{\rho})$ is called an LDF rough approximation space (LDF-RAS).

Definition 10. *Let $\ddot{\mathbb{P}} = (\breve{\mathscr{U}}_1, \breve{\mathscr{U}}_2, \ddot{\rho})$ be an LDF-RAS and $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$. Describe the L-A $\underline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})}$ of $\mathcal{Y}$ and the U-A $\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})}$ of $\mathcal{Y}$ as follows:*

$$\underline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})} = \{v_1 \in \breve{\mathscr{U}}_1 : \emptyset \neq v_1\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>} \subseteq \mathcal{Y}\};$$

$$\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})} = \{v_1 \in \breve{\mathscr{U}}_1 : \emptyset \neq v_1\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}, v_1\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>} \cap \mathcal{Y} \neq \emptyset\}$$

Similarly, we can define the L-A $\underline{(\mathcal{X})appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}}$ and U-A $\overline{(\mathcal{X})appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}}$ for any subset $\mathcal{X} \subseteq \breve{\mathscr{U}}_1$ as follows:

$$\underline{(\mathcal{X})appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}} = \{v_2 \in \breve{\mathscr{U}}_2 : \emptyset \neq \ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}v_2 \subseteq \mathcal{X}\}$$

$$\overline{(\mathcal{X})appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}} = \{v_2 \in \breve{\mathscr{U}}_2 : \emptyset \neq \ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}v_2, \ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}v_2 \cap \mathcal{X} \neq \emptyset\}$$

where $v_2\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>} = \{v_2 \in \breve{\mathscr{U}}_2 : (v_1,v_2) \in \ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}\}$ and $\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}v_2 = \{v_1 \in \breve{\mathscr{U}}_1 : (v_1,v_2) \in \ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}\}$.

Remark 1.

(1) *If $\breve{\mathscr{U}}_1 = \breve{\mathscr{U}}_2$, then the L-A and U-A for any $\mathcal{X} \subseteq \breve{\mathscr{U}}_1$ can also be defined as in the above Definition 10.*

(2) *All the notions and results for any subset $\mathcal{Y}$ of $\breve{\mathscr{U}}_2$ from Definition 11 to Theorem 5 can be proved in similar manners for any subset $\mathcal{X} \subseteq \breve{\mathscr{U}}_1$.*

Definition 11. *Let $\ddot{\mathbb{P}} = (\breve{\mathscr{U}}_1, \breve{\mathscr{U}}_2, \ddot{\rho})$ be an LDF-RAS and $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$. Then, the following sets are defined as follows:*

(1) $LDF - POS_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \underline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})}$;

(2) $LDF - BND_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})} - \underline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})}$;

(3) $LDF - NEG_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \breve{\mathscr{U}}_2 - \overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})} = (\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})})^c$.

are called the PR, BR and NR of $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$, respectively.

In the sequel of this manuscript, we mean $\ddot{\mathbb{P}} = (\breve{\mathscr{U}}_1, \breve{\mathscr{U}}_2, \ddot{\rho})$ as a LDF-RAS and $\ddot{s}, \ddot{u} \in (0,1]$, $\ddot{t}, \ddot{v} \in [0,1)$.

Proposition 2. *Let $\mathcal{Y}_1, \mathcal{Y}_2 \subseteq \breve{\mathscr{U}}_2$. Then,*

(1) $\underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1) \subseteq \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1)$;

(2) $\underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\emptyset) = \emptyset = \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\emptyset)$;

(3) *If* $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$, *then* $\underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1) \subseteq \underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)$;

(4) $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$, *then* $\overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1) \subseteq \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)$;

(5) $\underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1 \cap \mathcal{Y}_2) = \underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1) \cap \underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)$;

(6) $\overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1 \cap \mathcal{Y}_2) \subseteq \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1) \cap \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)$;

(7) $\underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1 \cup \mathcal{Y}_2) \supseteq \underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1) \cup \underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)$;

(8) $\overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1 \cup \mathcal{Y}_2) = \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1) \cup \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)$

Proof. All the assertions can be easily proved by using Definition 10. □

Note that: if $x\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>} \neq \emptyset$, then the assertions (1) and (2) may not hold (see Example 2).

Example 2. *Let $\breve{\mathscr{U}}_1 = \{u_1, u_2, u_3\}$ and $\breve{\mathscr{U}}_2 = \{v_1, v_2, v_3\}$ be the universal sets. Then, we define an LDF-R $\ddot{\rho}$ from $\breve{\mathscr{U}}_1$ to $\breve{\mathscr{U}}_2$ in the matrix notations given as below:*

$$\Theta^M = \begin{pmatrix} 0.77 & 0.57 & 0.67 \\ 0.55 & 0.48 & 0.50 \\ 0.68 & 0.45 & 0.43 \end{pmatrix}, \Theta^N = \begin{pmatrix} 0.71 & 0.41 & 0.56 \\ 0.80 & 0.72 & 0.46 \\ 0.54 & 0.40 & 0.22 \end{pmatrix},$$
$$\varpi^M = \begin{pmatrix} 0.51 & 0.50 & 0.61 \\ 0.46 & 0.40 & 0.37 \\ 0.54 & 0.39 & 0.35 \end{pmatrix}, \varpi^N = \begin{pmatrix} 0.49 & 0.46 & 0.38 \\ 0.52 & 0.58 & 0.58 \\ 0.45 & 0.56 & 0.61 \end{pmatrix}.$$

Using Definition 8 of $(<\ddot{s},\ddot{u}>,<\ddot{t},\ddot{v}>)$-level cut relation, for $\ddot{s} = 0.77, \ddot{u} = 0.51$, $\ddot{t} = 0.71, \ddot{v} = 0.49$, we can obtain:

$$u_1\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>} = \{v_1\}, u_2\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>} = u_3\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>} = \emptyset$$

Suppose $\mathcal{Y} = \{v_1, v_2\}$. Then by Definition 10,

$$\underline{(\mathcal{Y})appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \mathscr{U}_1, \overline{(\mathcal{Y})appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \{u_1\}$$

$$\underline{(\emptyset)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \{u_2, u_3\}, \overline{(\emptyset)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \emptyset$$

$$\underline{(\mathscr{U}_2)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \mathscr{U}_1, \overline{(\mathscr{U}_2)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \{u_1\}$$

Thus, we obtain that $\underline{(\emptyset)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} \neq \emptyset$ and $\overline{(\mathscr{U}_2)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} \neq \mathscr{U}_1$. However, if $u\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>} \neq \emptyset$, then:

$$\underline{(\mathscr{U}_2)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \overline{(\mathscr{U}_2)appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}} = \{u_1\} \neq \mathscr{U}_1$$

(see Proposition 3).

Proposition 3. *Let $\ddot{\rho}$ be a R-LDF-R on $\breve{\mathscr{U}}_1$ and $\ddot{s}, \ddot{u} \in (0,1]$, $\ddot{t}, \ddot{v} \in [0,1)$. For any subset $\mathcal{Y} \subseteq \ddot{\mathscr{U}}_1$, the following properties hold:*

(1) $\underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}) \subseteq \mathcal{Y} \subseteq \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})$;

(2) $\underline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\breve{\mathscr{U}}_1) = \breve{\mathscr{U}}_1 = \overline{appr}_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\breve{\mathscr{U}}_1)$.

Proof. The proof is straightforward. □

Lemma 1. *Suppose that $\ddot{s}_1, \ddot{s}_2, \ddot{u}_1, \ddot{u}_2 \in (0,1]$ and $\ddot{t}_1, \ddot{t}_2, \ddot{v}_1, \ddot{v}_2 \in [0,1)$ such that $\ddot{s}_1 \leq \ddot{s}_2$, $\ddot{u}_1 \leq \ddot{u}_2$ and $\ddot{t}_2 \leq \ddot{t}_1$, $\ddot{v}_2 \leq \ddot{v}_1$. Then,*

$$\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>} \subseteq \ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>}.$$

Proof. Let $(v_1, v_2) \in \ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>}$. Using Definition 8, $\Theta^M(v_1, v_2) \geq \ddot{s}_2$, $\varpi^M(v_1, v_2) \geq \ddot{u}_2$ and $\Theta^N(v_1, v_2) \leq \ddot{t}_2$, $\varpi^N(v_1, v_2) \leq \ddot{v}_2$. Since $\ddot{s}_1 \leq \ddot{s}_2$, $\ddot{u}_1 \leq \ddot{u}_2$ and $\ddot{t}_2 \leq \ddot{t}_1$, $\ddot{v}_2 \leq \ddot{v}_1$, so

$$\Theta^M(v_1, v_2) \geq \ddot{s}_2 \geq \ddot{s}_1, \varpi^M(v_1, v_2) \geq \ddot{u}_2 \geq \ddot{u}_1 \text{ and } \Theta^N(v_1, v_2) \leq \ddot{t}_2 \leq \ddot{t}_1, \varpi^N(v_1, v_2) \leq \ddot{v}_2 \leq \ddot{v}_1$$

Hence, $\Theta^M(v_1, v_2) \geq \ddot{s}_1$, $\varpi^M(v_1, v_2) \geq \ddot{u}_1$ and $\Theta^N(v_1, v_2) \leq \ddot{t}_1$, $\varpi^N(v_1, v_2) \leq \ddot{v}_1$. Thus $(v_1, v_2) \in \ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>}$. □

Proposition 4. *With the same assumptions as in the above Lemma 1, suppose that $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$. Then, the following assertions are true:*

(1) $\overline{appr}_{\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>}}(\mathcal{Y}) \subseteq \overline{appr}_{\ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>}}(\mathcal{Y})$,

(2) $\underline{appr}_{\ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>}}(\mathcal{Y}) \subseteq \underline{appr}_{\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>}}(\mathcal{Y})$.

Proof. (1) Let $v_1 \in \overline{appr}_{\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>}}(\mathcal{Y})$. From Definition 10, $v_2 \in v_1\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>} \cap \mathcal{Y}$ for some $v_2 \in \mathscr{U}_1$. Since $v_1\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>} \subseteq v_1\ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>}$, therefore $v_2 \in v_1\ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>} \cap \mathcal{Y}$ (using Lemma 1). Hence, $v_1 \in \overline{appr}_{\ddot{\rho}^{<t_1,v_1>}_{<\ddot{s}_1,\ddot{u}_1>}}(\mathcal{Y})$.

(2) Let $v_1 \in \underline{appr}_{\ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>}}(\mathcal{Y})$. By Definition 10, $v_1\ddot{\rho}^{<\ddot{t}_1,\ddot{v}_1>}_{<\ddot{s}_1,\ddot{u}_1>} \subseteq \mathcal{Y}$. From Lemma 1, $v_1\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>} \subseteq \mathcal{Y}$. This proves that $v_1 \in \underline{appr}_{\ddot{\rho}^{<\ddot{t}_2,\ddot{v}_2>}_{<\ddot{s}_2,\ddot{u}_2>}}(\mathcal{Y})$. □

The inclusions in Proposition 4 may not hold, as is demonstrated in the sequel.

Example 3. *Let us revisit Example 2, assume $\ddot{s}_1 = 0.55, \ddot{u}_1 = 0.46, \ddot{t}_1 = 0.80, \ddot{v}_1 = 0.52$ and $\ddot{s}_2 = 0.77, \ddot{u}_2 = 0.51, \ddot{t}_2 = 0.71, \ddot{v}_2 = 0.49$. Then by Definition 8,*

$$u_1\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>} = \mathscr{U}_2, u_2\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>} = u_3\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>} = \{v_1\}$$

$$u_1\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>} = \{v_1\}, u_2\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>} = u_3\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>} = \emptyset$$

Take $\mathcal{Y} = \{v_1\}$, then by Definition 10, we have:

$$\underline{appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}}(\mathcal{Y}) = \overline{appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}}(\mathcal{Y}) = \{u_1\}$$

$$\underline{appr}_{\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>}}(\mathcal{Y}) = \{u_2, u_3\}, \overline{appr}_{\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>}}(\mathcal{Y}) = \mathscr{U}_1$$

Since $\ddot{s}_1 < \ddot{s}_2$, $\ddot{u}_1 < \ddot{u}_2$ and $\ddot{t}_1 > \ddot{t}_2$, $\ddot{v}_1 > \ddot{v}_2$, but $\overline{appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}}(\mathcal{Y})$ and $\underline{appr}_{\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>}}(\mathcal{Y}) \nsubseteq \underline{appr}_{\ddot{\rho}^{<0.71,0.49>}_{<0.77,0.51>}}(\mathcal{Y})$.

Lemma 2. *Let $\ddot{\rho}_1, \ddot{\rho}_2 \in LDF-R(\breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2)$ be such that $\ddot{\rho}_1 \subseteq \ddot{\rho}_2$. Then,*

$$\ddot{\rho}_1{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>} \subseteq \ddot{\rho}_2{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}$$

Proof. Let $(v_1, v_2) \in \ddot{\rho}_1{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}$. By Definition 8, $\Theta_1^M(v_1, v_2) \geq \ddot{s}$, $\varpi_1^M(v_1, v_2) \geq \ddot{u}$ and $\Theta_1^N(v_1, v_2) \leq \ddot{t}$, $\varpi_1^N(v_1, v_2) \leq \ddot{v}$. Since $\ddot{\rho}_1 \subseteq \ddot{\rho}_2$, therefore $\ddot{s} \leq \Theta_1^M(v_1, v_2) \leq \Theta_2^M(v_1, v_2), \ddot{u} \leq \varpi_1^M(v_1, v_2) \leq \varpi_2^M(v_1, v_2)$ and $\ddot{t} \geq \Theta_1^N(v_1, v_2) \geq \Theta_2^N(v_1, v_2), \ddot{v} \geq \varpi_1^N(v_1, v_2) \geq \varpi_2^N(v_1, v_2)$. Hence, $\Theta_2^M(v_1, v_2) \geq \ddot{s}$, $\varpi_2^M(v_1, v_2) \geq \ddot{u}$ and $\Theta_2^N(v_1, v_2) \leq \ddot{t}$, $\varpi_2^N(v_1, v_2) \leq \ddot{v}$. Thus, $(v_1, v_2) \in \ddot{\rho}_2{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}$. □

Proposition 5. *With the same notations as in Lemma 2, assume that $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$. Then,*

(1) $\underline{appr}_{\ddot{\rho}_2{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y}) \subseteq \underline{appr}_{\ddot{\rho}_1{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y})$,

(2) $\overline{appr}_{\ddot{\rho}_1{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y}) \subseteq \overline{appr}_{\ddot{\rho}_2{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y})$.

Proof. (1) Let $v \in \underline{appr}_{\ddot{\rho}_2{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y})$. Then, $v\ddot{\rho}_2{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>} \subseteq \mathcal{Y}$. By Lemma 2, $v\ddot{\rho}_1{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>} \subseteq v\ddot{\rho}_2{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>} \subseteq \mathcal{Y}$. Hence, $x\ddot{\rho}_1{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>} \subseteq \mathcal{Y}$. This proves that $v \in \underline{appr}_{\ddot{\rho}_1{}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y})$. Similar to the proof of (1), proof of (2). □

5. Accuracy Measure and Roughness Measure for LDF-RSs on Two Universes

The concept of A-M and R-M was first invented by Pawlak in 1982 in order to define the imprecision of R-As. Our perception of the accuracy of the data relating to an E-R for a given classification is based on these numerical measures. In [51], Yang et al. gave the idea of A-M and R-M for BF-RSs on dual universes. In this passage, we extend this concept to LDF-RSs on two universes.

With respect to a Pawlak A-S $P = (\breve{\mathscr{U}}, \rho)$, where ρ is an E-R on $\breve{\mathscr{U}}$. Then the A-M and R-M of $\mathcal{O}$ of $\breve{\mathscr{U}}$ are defined as follows, respectively:

$$AM(\mathcal{O}) = \frac{\underline{\rho(\mathcal{O})}}{\overline{\rho(\mathcal{O})}} \text{ and } RM(\mathcal{O}) = 1 - AM(\mathcal{O}).$$

We define the subsequent ideas by using the same pattern.

Definition 12. *Let $\ddot{\mathbb{P}} = (\breve{\mathscr{U}}_1, \breve{\mathscr{U}}_2, \ddot{\rho})$ be an LDF-RAS and $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$, define the AM of $\mathcal{Y}$ with respect to $\ddot{\rho}$ as follows:*

$$\mathbb{AM}(\mathcal{Y}) = \frac{\underline{|appr_{\ddot{\rho}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y})|}}{\overline{|appr_{\ddot{\rho}_{<\ddot{s},\ddot{u}>}^{<\ddot{t},\ddot{v}>}}(\mathcal{Y})|}}$$

where $|.|$ indicates the number of elements in the sets. After that, we define the RM of $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$ with respect to $\ddot{\rho}$ as follows:

$$\mathbb{RM}(\mathcal{Y}) = 1 - \mathbb{AM}(\mathcal{Y})$$

Remark 2. *The following points can be deduced from definition 12 given above:*

(1) $\mathbb{AM}(\mathcal{Y}), \mathbb{RM}(\mathcal{Y}) \in [0, 1]$.

(2) *If $\ddot{s} = \ddot{u} = 1$ and $\ddot{t} = \ddot{v} = 0$, then $\mathbb{AM}(\mathcal{Y}) = 1$ and $\mathbb{RM}(\mathcal{Y}) = 0$.*

In the following, we construct an example for the clarification of the above Definition 12.

Example 4. *In Example 3, for $\ddot{s}_1 = 0.55, \ddot{u}_1 = 0.46, \ddot{t}_1 = 0.80, \ddot{v}_1 = 0.52$ and $\mathcal{Y} = \{y_1\}$, we have:*

$$\underline{appr}_{\ddot{\rho}_{<0.77,0.51>}^{<0.71,0.49>}}(\mathcal{Y}) = \overline{appr}_{\ddot{\rho}_{<0.77,0.51>}^{<0.71,0.49>}}(\mathcal{Y}) = \{x_1\}$$

Thus, by Definition 12, $\mathbb{MA}(\mathcal{Y}) = 1$ and $\mathbb{MR}(\mathcal{Y}) = 0$. Hence, our information related to $\ddot{\rho}$ is accurate up to grade 1, which means that $\ddot{\rho}$ describes the objects of $\mathcal{Y}$ absolutely accurately. On the other hand, for $\ddot{s}_2 = 0.77, \ddot{u}_2 = 0.51, \ddot{t}_2 = 0.71, \ddot{v}_2 = 0.49$ and $\mathcal{Y} = \{y_1\}$, we have:

$$\underline{appr_{\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>}}}(\mathcal{Y}) = \{x_2, x_3\}, \overline{appr_{\ddot{\rho}^{<0.80,0.52>}_{<0.55,0.46>}}(\mathcal{Y})} = \breve{\mathscr{U}}_1$$

Then, $\mathbb{MA}(\mathcal{Y}) = \frac{2}{3}$ and $\mathbb{MR}(\mathcal{Y}) = \frac{1}{3}$. Hence, our information related to $\ddot{\rho}$ is accurate up to grade 0.6666, which means that $\ddot{\rho}$ describes the items of $\breve{\mathscr{U}}_2$ accurately up to grade 0.6666.

In the following result, we describe a connection of the A-M $\mathbb{AM}(\mathcal{Y})$ and R-M $\mathbb{RM}(\mathcal{Y})$ about the union and intersection of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ on the universe $\breve{\mathscr{U}}_2$.

Theorem 5. *Let $\ddot{\mathbb{P}} = (\breve{\mathscr{U}}_1, \breve{\mathscr{U}}_2, \ddot{\rho})$ be a LDF-RAS and $\mathcal{Y}_1, \mathcal{Y}_2$ are any non-empty subsets of $\breve{\mathscr{U}}_2$. Then, A-M and R-M of $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_1 \cup \mathcal{Y}_2$ and $\mathcal{Y}_1 \cap \mathcal{Y}_2$ the following relations;*

(1) $\mathbb{MR}(\mathcal{Y}_1 \cup \mathcal{Y}_2)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1)} \cup \overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)}| \leq \mathbb{MR}(\mathcal{Y}_1)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1)}| + \mathbb{MR}(\mathcal{Y}_2)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)}| - \mathbb{MR}(\mathcal{Y}_1 \cap \mathcal{Y}_2)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1)} \cap \overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)}|$;

(2) $\mathbb{MA}(\mathcal{Y}_1 \cup \mathcal{Y}_2)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1)} \cup \overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)}| \geq \mathbb{MA}(\mathcal{Y}_1)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1)}| + \mathbb{MA}(\mathcal{Y}_2)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)}| - \mathbb{MA}(\mathcal{Y}_1 \cap \mathcal{Y}_2)|\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_1)} \cap \overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y}_2)}|$

Proof. The proof resembles that of Theorem 3.3 in [51]. □

6. An Application of LDF-RSs on Two Different Universes

In the literature, a number of scientists have developed various techniques for medical diagnosis. Sun and Ma [48] presented an application of the F-RS model on two distinct domains in clinical diagnosis systems. Since the information is insufficient in the case of F-RS, Yang et al. [51] expanded the idea of Sun and Ma [48] to BF-RS model on two distinct cosmologies. LD-FSs are more efficient in decision analysis than the prevailing concepts of FS, IF-S, B-FS and q-ROF-S. Therefore, we need to extend the existing technique of BF-RS to a more general and robust model, namely LDF-RS on two contrasting universes and utilize this notion in clinical diagnosis.

Suppose that $\breve{\mathscr{U}}_1$ refers to the collection of afflicted people and $\breve{\mathscr{U}}_2$ indicates the group of symptoms. Let $\ddot{\mathbb{P}} = (\breve{\mathscr{U}}_1, \breve{\mathscr{U}}_2, \ddot{\rho})$ be LDF-RAS. If $(v_1, v_2) \in \ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}$, for all $v_1 \in \breve{\mathscr{U}}_1$ and $v_2 \in \breve{\mathscr{U}}_2$, then we say that the sufferer x has the symptom y and the percentage of the patient who exhibits symptom y is at least $\ddot{s}$ and the degree of its corresponding parameter is not less than $\ddot{u}$, the sufferer's degree of symptom y non-existence is not greater than $\ddot{t}$, and the degree of its corresponding parameter is not greater than $\ddot{v}$.

We are aware that a certain illness has a number of common symptoms. We denote a certain disease by $\mathcal{Y} = \{\mathfrak{y}_i \in \breve{\mathscr{U}}_2 : i \in I\}$ for any $\mathcal{Y} \subseteq \breve{\mathscr{U}}_2$ and make the following inferences using the PR, NR, and BR described in Definition 11:

Let $v \in \breve{\mathscr{U}}_1$ be a given certain sufferer. Then,

(1) If $v \in LDF - POS_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \underline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}}(\mathcal{Y})$ and $v\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>} \neq \emptyset$, that is, he must have illness $\mathcal{Y}$, at which point the patient urgently requires treatment.

(2) If $v \in LDF - BND_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})} - \underline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}}(\mathcal{Y})$, consequently, he will be the doctor's second choice because he is not diagnosed based on these symptoms, even though he may have the disease $\mathcal{Y}$.

(3) If $v \in \mathcal{LDFNEG}_{\ddot{\mathbb{P}}}(\mathcal{Y})$, that is, $v \in (\overline{appr_{\ddot{\rho}^{<\ddot{t},\ddot{v}>}_{<\ddot{s},\ddot{u}>}}(\mathcal{Y})})^c$, consequently, he does not have illness $\mathcal{Y}$ and does not require treatment.

Let us use a specific case to demonstrate this.

Example 5. *Let $\breve{\mathscr{U}}_1 = \{p_1, p_2, p_3, p_4\}$ be the group of certain victims and $\breve{\mathscr{U}}_2 = \{q_1, q_2, q_3\}$ be the set of some symptoms. Consider an LDF-R $\ddot{\rho}$ from $\breve{\mathscr{U}}_1$ to $\breve{\mathscr{U}}_2$. It describes the M and NM grades, together with the grades of their parameters, for each patient pi in relation to the symptom qj in the following matrices:*

$$\Theta^M = \begin{pmatrix} 0.80 & 0.54 & 0.68 \\ 0.71 & 0.45 & 0.40 \\ 0.57 & 0.36 & 0.75 \\ 0.85 & 0.81 & 0.62 \end{pmatrix}, \Theta^N = \begin{pmatrix} 0.35 & 0.46 & 0.38 \\ 0.36 & 0.72 & 0.43 \\ 0.46 & 0.56 & 0.47 \\ 0.21 & 0.32 & 0.25 \end{pmatrix},$$

$$\varpi^M = \begin{pmatrix} 0.71 & 0.50 & 0.62 \\ 0.62 & 0.38 & 0.30 \\ 0.46 & 0.26 & 0.60 \\ 0.80 & 0.78 & 0.59 \end{pmatrix}, \beta = \begin{pmatrix} 0.24 & 0.48 & 0.38 \\ 0.38 & 0.52 & 0.70 \\ 0.54 & 0.66 & 0.40 \\ 0.20 & 0.18 & 0.28 \end{pmatrix}.$$

Let $\mathcal{Y} = \{q_1, q_2\}$ symbolize a specific sickness, and there are two signs of this condition in clinic.

Case-1: *For $\ddot{s} = 0.45$, $\ddot{u} = 0.38$ and $\ddot{t} = 0.72$, $\ddot{v} = 0.52$, we have:*

$$p_1\ddot{\rho}^{<0.72,0.52>}_{<0.45,0.38>} = p_4\ddot{\rho}^{<0.72,0.52>}_{<0.45,0.38>} = \breve{\mathscr{U}}_2, p_2\ddot{\rho}^{<0.72,0.52>}_{<0.45,0.38>} = \{q_1, q_2\}, p_3\ddot{\rho}^{<0.72,0.52>}_{<0.45,0.38>} = \{q_3\}$$

(see Definition 8). By simple computations, the L-A and U-A of $\mathcal{Y}$ are given below:

$$\underline{appr_{\ddot{\rho}^{<0.72,0.52>}_{<0.45,0.38>}}(\mathcal{Y})} = \{p_2\}, \overline{appr_{\ddot{\rho}^{<0.72,0.52>}_{<0.45,0.38>}}(\mathcal{Y})} = \{p_1, p_2, p_4\}$$

Using Definition 10, $LDF - POS_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \{p_2\}$, $LDF - BND_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \{p_1, p_4\}$ and $LDF - NEG_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \{p_3\}$. Furthermore, by Definition 12, the A-M and R-M are calculated as:

$$\mathbb{MA}(\mathcal{Y}) = \frac{1}{3}, \mathbb{MR}(\mathcal{Y}) = \frac{2}{3}$$

Thus, we interpret the subsequent results:

(1) *Patient $p2$ must be afflicted with illness $\mathcal{Y}$ and requires emergency medical attention.*
(2) *We cannot guarantee that patients $p1$ and $p4$ are suffering from illness $\mathcal{Y}$ based on these symptoms. The doctor will therefore choose the second option.*
(3) *The sickness $\mathcal{Y}$ does not affect patient $p3$.*

Case-2: *For $\ddot{s} = 0.57$, $\ddot{u} = 0.46$ and $\ddot{t} = 0.46$, $\ddot{v} = 0.54$, we have:*

$$p_1\ddot{\rho}^{<0.46,0.54>}_{<0.57,0.46>} = \{q_1, q_3\}, p_2\ddot{\rho}^{<0.46,0.54>}_{<0.57,0.46>} = \{q_1\} = p_3\ddot{\rho}^{<0.46,0.54>}_{<0.57,0.46>}, p_4\ddot{\rho}^{<0.46,0.54>}_{<0.57,0.46>} = \breve{\mathscr{U}}_2.$$

(using Definition 8). By simple calculations, the L- and U-As of $\mathcal{Y}$ are as follows:

$$\underline{appr_{\ddot{\rho}^{<0.46,0.54>}_{<0.57,0.46>}}(\mathcal{Y})} = \{p_2, p_3\}, \overline{appr_{\ddot{\rho}^{<0.46,0.54>}_{<0.57,0.46>}}(\mathcal{Y})} = \breve{\mathscr{U}}_2$$

Using Definition 10, $\mathcal{LDF}POS_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \{p_2, p_3\}$, $\mathcal{LDF}BND_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \{p_1, p_4\}$ and $\mathcal{LDF}NEG_{\ddot{\mathbb{P}}}(\mathcal{Y}) = \emptyset$. Further, using Definition 12, the A-M and R-M are computed as follows:

$$\mathbb{MA}(\mathcal{Y}) = \frac{1}{2}, \mathbb{MR}(\mathcal{Y}) = \frac{1}{2}$$

Thus, we conclude that:

(1) *Patients $p2$ and $p3$ must endure illness $\mathcal{Y}$, and he requires prompt medical attention.*
(2) *Regarding patients $p1$ and $p4$, we cannot guarantee whether or not they are experiencing the symptoms of illness $\mathcal{Y}$. The doctor will therefore choose the second option.*
(3) *No one who suffers has a healthy diagnosis.*

Remark 3.

($\diamond$) *Based on the analysis discussed earlier, we may infer that decision precision rises with approximation precision, as in [51]. Thus, a precise decision can be made by a doctor using the proposed method of LDF-RSs.*

($\diamond$) *Furthermore, our proposed technique of LDF-RSs allows reducing the likelihood of a surgical misconception.*

($\diamond$) *Additionally, the LDF-RS model, and because of the application of control or reference factors found in LD-FSs, the applied approach may help decision-makers arrive at a precise and scientific conclusion in circumstances where they frequently encounter one another.*

Comparative Analysis

In this section, we contrast our findings with a few of Yang et al. [51], Sun and Ma [48] and Ayub et al.'s [52] previously used methods.

Example 6. *For [48], consider our previous example 5, where $\breve{\mathscr{U}}_1 = \{p_1, p_2, p_3, p_4\}$ and $\breve{\mathscr{U}}_2 = \{q_1, q_2, q_3\}$. The following describes the M grades for each patient pi in connection to the symptom qj and F-R Θ^M on $\breve{\mathscr{U}}_1 \times \breve{\mathscr{U}}_2$:*

$$\Theta^M = \begin{pmatrix} 0.80 & 0.54 & 0.68 \\ 0.71 & 0.45 & 0.40 \\ 0.57 & 0.36 & 0.75 \\ 0.85 & 0.81 & 0.62 \end{pmatrix},$$

Using Definition 3.3 of [48] for level cuts, we obtain the following for $\ddot{s} = 0.45$:

$$p_1\Theta^M_{0.45} = p_4\Theta^M_{0.45} = \breve{\mathscr{U}}_2, p_3\Theta^M_{0.45} = \{q_1, q_3\}, p_2\Theta^M_{0.45} = \{q_1, q_2\}$$

For $\mathcal{Y} = \{q_1, q_2\}$, the L- and U-As are obtained by using Definition 3.3 of [48] below:

$$\underline{appr_{\Theta^M_{0.45}}}(\mathcal{Y}) = \{p_2\}, \overline{appr_{\Theta^M_{0.45}}}(\mathcal{Y}) = \breve{\mathscr{U}}_2$$

Therefore, $P - R(\mathcal{Y}) = \{p_2\}$, $B - R(\mathcal{Y}) = \{p_1, p_3, p_4\}$ and $N - R(\mathcal{Y}) = \emptyset$. As a result, the following conclusions may be made from this information:

(1) *Patient p2 needs immediate medical care as he must deal with the sickness $\mathcal{Y}$.*
(2) *We are unable to confirm if patients p1, p3, and p4 are displaying the signs of sickness $\mathcal{Y}$. Therefore, the doctor will select choice number two.*
(3) *Nobody who is ill has a clear diagnosis.*

For $\ddot{s} = 0.57$, we have:

$$p_1\Theta^M_{0.57} = p_3\Theta^M_{0.57} = \{q_1, q_3\}, p_2\Theta^M_{0.57} = \{q_1\}, p_4\Theta^M_{0.57} = \breve{\mathscr{U}}_2$$

The L- and U-As for $\mathcal{Y}$ are found by applying Definition 3.3 of [48] below:

$$\underline{appr_{\Theta^M_{0.57}}}(\mathcal{Y}) = \{p_3\}, \overline{appr_{\Theta^M_{0.57}}}(\mathcal{Y}) = \breve{\mathscr{U}}_2$$

Therefore, $P - R(\mathcal{Y}) = \{p_3\}$, $B - R(\mathcal{Y}) = \{p_1, p_2, p_4\}$ and $N - R(\mathcal{Y}) = \emptyset$. Thus, it follows that:

(1) *Patient p3 is suffering from illness $\mathcal{Y}$ and needs immediate medical care.*
(2) *We are unable to confirm if patients p1, p2, and p4 are displaying the signs of sickness $\mathcal{Y}$. Therefore, the doctor will select choice number two.*
(3) *There is no healthy diagnosis for someone who is suffering.*

Example 7. *We use the same Example 5 with BF-R which is expressed in the Table 1 for [51]:*

Table 1. ρ_B.

$\mathcal{U}_1 \backslash \mathcal{U}_2$	q_1	q_2	q_3
p_1	$< 0.80, 0.20 >$	$< 0.54, 0.46 >$	$< 0.68, 0.30 >$
p_2	$< 0.71, 0.25 >$	$< 0.45, 0.45 >$	$< 0.40, 0.43 >$
p_3	$< 0.57, 0.40 >$	$< 0.36, 0.56 >$	$< 0.75, 0.25 >$
p_4	$< 0.85, 0.12 >$	$< 0.81, 0.15 >$	$< 0.62, 0.25 >$

Using Definition 3.1 of [51], the $< \ddot{s}, \ddot{t} >$ *−level cuts for* $\ddot{s} = 0.45$ *and* $\ddot{t} = 0.54$*, we have the sequel:*

$$p_1(\rho_B)^{<0.45,0.54>} = p_4(\rho_B)^{<0.45,0.54>} = \mathcal{U}_2, p_2(\rho_B)^{<0.45,0.54>} = \{q_1, q_2\}, p_3(\rho_B)^{<0.45,0.54>} = \{q_3\}$$

From Definition 3.2 of [51], the L-, and U-As of $\mathcal{Y}$ *are given below:*

$$\underline{appr_{(\rho_B)^{<0.45,0.54>}}}(\mathcal{Y}) = \{p_2\}, \overline{appr_{(\rho_B)^{<0.45,0.54>}}}(\mathcal{Y}) = \{p_1, p_2, p_4\}$$

Therefore, $P - R(\mathcal{Y}) = \{p_2\}$, $B - R(\mathcal{Y}) = \{p_1, p_4\}$ *and* $N - R(\mathcal{Y}) = \{p_3\}$*. Thus, based on these findings, the following inferences can be made:*

(1) Patient p2 must suffer from disease $\mathcal{Y}$*, so he requires urgent medical attention.*

(2) We are uncertain as to whether patients p1 and p4 are exhibiting the signs of sickness $\mathcal{Y}$*. Therefore, the doctor will select choice number two.*

(3) Patient p_3 *was declared to be in good health and does not require any additional care.*

Now, for $\ddot{s} = 0.57$ *and* $\ddot{t} = 0.40$*, using Definition 3.1 of [51] for* $< \ddot{s}, \ddot{t} >$ *−level cuts, we obtain the following:*

$$p_1(\rho_B)^{<0.45,0.54>} = p_3(\rho_B)^{<0.45,0.54>} = \{q_1, q_3\}, p_2(\rho_B)^{<0.45,0.54>} = \{q_1\}, p_4(\rho_B)^{<0.45,0.54>} = \mathcal{U}_2$$

By using Definition 3.2 of [51] and simple calculations, we obtain the L-A and U-A of $\mathcal{Y}$ *in the sequel:*

$$\underline{appr_{(\rho_B)^{<0.57,0.40>}}}(\mathcal{Y}) = \{p_2\}, \overline{appr_{(\rho_B)^{<0.57,0.40>}}}(\mathcal{Y}) = \mathcal{U}_2$$

Therefore, $P - R(\mathcal{Y}) = \{p_2\}$, $B - R(\mathcal{Y}) = \{p_1, p_3, p_4\}$ *and* $N - R(\mathcal{Y}) = \emptyset$*. Based on these results, we conclude that:*

(1) Patient p2 has to have illness $\mathcal{Y}$*, so he needs to get medical help right away.*

(2) We cannot guarantee that patients p1, p3, and p4 are displaying the signs of sickness $\mathcal{Y}$ *or not. Therefore, the doctor will select choice number two.*

(3) Nobody who is in pain has a good diagnosis.

Example 8. *For [52], consider the same LDF-R as in Example 5. By using Definition 9 of [52], we obtain the L-, and U-As for* $\mathcal{Y} = \{q_1, q_2\}$ *and* $\ddot{s} = 0.45$, $\ddot{u} = 0.38$ *as follows:*

$$\underline{\ddot{\rho}(\mathcal{Y})}_{<0.45,0.38>} = \{p_2\}, \overline{\ddot{\rho}(\mathcal{Y})}^{<0.45,0.38>} = \mathcal{U}_1$$

For $\ddot{t} = 0.72$ *and* $\ddot{v} = 0.52$*, the L-A and U-A are as follows:*

$$\underline{\ddot{\rho}(\mathcal{Y})}_{<0.72,0.52>} = \{p_1, p_2, p_4\}, \overline{\ddot{\rho}(\mathcal{Y})}^{<0.72,0.52>} = \emptyset$$

Thus, $P - R(\mathcal{Y}) = (\{p_2\}, \emptyset)$, $B - R(\mathcal{Y}) = (\{p_1, p_3, p_4\}, \{p_1, p_2, p_4\})$ *and* $N - R(\mathcal{Y}) = (\emptyset, \{p_3\})$*. These findings allow for the following inferences:*

(1) Patient p2 must deal with the ailment $\mathcal{Y}$*, necessitating immediate medical attention. Since there is no other patient in the area, we can declare with certainty that this patient does not have illness* $\mathcal{Y}$*.*

(2) We cannot ensure that patients p1, p3, and p4 are exhibiting the symptoms of sickness $\mathcal{Y}$*. Consequently, the doctor will pick option number two.*

(3) Nobody who is in pain has a good diagnosis.
Now, for $\ddot{s} = 0.57$, $\ddot{u} = 0.46$ the L-, and U-As are as follows:

$$\underline{\ddot{\rho}(\mathcal{Y})}_{<0.57,0.46>} = \{p_2\}, \overline{\ddot{\rho}(\mathcal{Y})}^{<0.45,0.38>} = \mathcal{U}_1$$

For $\ddot{t} = 0.46$ and $\ddot{v} = 0.54$, the L-A and U-As of $\mathcal{Y}$ are as follows:

$$\underline{\ddot{\rho}(\mathcal{Y})}_{<0.46,0.54>} = \mathcal{U}_1, \overline{\ddot{\rho}(\mathcal{Y})}^{<0.46,0.54>} = \emptyset$$

Thus, $P - R(\mathcal{Y}) = (\{p_2\}, \emptyset)$, $B - R(\mathcal{Y}) = (\{p_1, p_3, p_4\}, \mathcal{U}_1)$ and $N - R(\mathcal{Y}) = (\emptyset, \emptyset)$. These lead us to conclude that:

(1) Patient $p2$ must deal with ailment $\mathcal{Y}$, necessitating immediate medical attention. Since there is no other patient in the area, we can declare with certainty that this patient does not have illness $\mathcal{Y}$.
(2) We cannot confirm whether patients $p1$, $p3$, and $p4$ are exhibiting the signs of sickness $\mathcal{Y}$. As a result, the doctor will go with option number two.
(3) No one with a diagnosis of illness is healthy.

7. Conclusions

The concept of LD-FS is a very powerful and convenient tool to describe the uncertainties in many practical problems, which involves decisions. The decision makers can freely choose the degree of truthness and the degree of falsity by making the use of reference or control parameters. Thus, LD-FS enhanced the space of truthness degree and falsity degree and removed the limitations of these degrees as in the existing concepts of FS, IF-S, B-FS, P-FS and q-ROF-S. In this paper, the existing notions of the F-RS model and BF-RS model on two universes have been generalized into the LDF-RS model on two universes as a more convenient and a robust model. The basic notions of lower and upper LDF-RAS have been defined by employing the after sets and fore sets of the $(<\ddot{s}, \ddot{u}>, <\ddot{t}, \ddot{v}>)$-level cut relation of an LDF-Rs. Some important results related to the L- and U-As have been proved with illustrative examples. Furthermore, to illustrate the application of LDF-RSs, an example has been employed. Further research on the proposed ideas of this research paper applied to other practical applications is needed, which may lead to many fruitful outcomes.

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