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Article

Schur-Convexity of the Mean of Convex Functions for Two Variables

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Abstract: The results of Schur convexity established by Elezovic and Pecaric for the average of convex functions are generalized relative to the case of the means for two-variable convex functions. As an application, some binary mean inequalities are given.

Keywords: inequality; Schur-convex function; Hadamard's inequality; convex functions of two variables; mean

MSC: 26A51; 26D15; B25

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1. Introduction

Let $\mathbb{R}$ be a set of real numbers, g be a convex function defined on the interval $I \subseteq \mathbb{R} \to \mathbb{R}$ and $c, d \in I$, $c < d$. Then

$$g\left(\frac{d+c}{2}\right) \le \frac{1}{d-c}\int_c^d g(t)\,\mathrm{d}t \le \frac{g(d)+g(c)}{2}. \tag{1}$$

This is the famous Hadamard's inequality for convex functions.

In 2000, utilizing Hadamard's inequality, Elezovic and Pecaric [1] researched Schur-convexity on the lower and upper limit of the integral for the mean of the convex functions and obtained the following important and profound theorem.

Theorem 1 ([1]). *Let I be an interval with nonempty interior on $\mathbb{R}$ and g be a continuous function on I. Then,*

$$\Phi(c,d) = \begin{cases} \frac{1}{d-c}\int_c^d g(s)ds, & c,d \in I,\ d \ne c \\ g(c), & d = c \end{cases}$$

is Schur convex (Schur concave, resp.) on $I \times I$ iff g is convex (concave, resp.) on I.

In recent years, this result attracted the attention of many scholars (see references [2–12] and Chapter II of the monograph [13] and its references).

In this paper, the result of theorem 1 is generalized to the case of bivariate convex functions, and some bivariate mean inequalities are established.

Theorem 2. *Let I be an interval with non-empty interior on $\mathbb{R}$ and $g(s,t)$ be a continuous function on $I \times I$. If g is convex (or concave resp.) on $I \times I$, then*

$$G(u,v) = \begin{cases} \frac{1}{(v-u)^2}\int_u^v\int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t, & (u,v) \in I \times I,\ u \ne v \\ g(u,u), & (u,v) \in I \times I,\ u = v \end{cases} \tag{2}$$

is Schur convex (or Schur concave, resp.) on $I \times I$.

2. Definitions and Lemmas

To prove Theorem 2, we provide the following lemmas and definitions.

Definition 1. *Let (x_1, x_2) and $(y_1, y_2) \in \mathbb{R} \times \mathbb{R}$.*

(1) A set $\Omega \subset \mathbb{R} \times \mathbb{R}$ is said to be convex if $(x_1, x_2), (y_1, y_2) \in \Omega$ and $0 \le \beta \le 1$ implies

$$(\beta x_1 + (1-\beta)y_1, \beta x_2 + (1-\beta)y_2) \in \Omega.$$

(2) Let $\Omega \subset \mathbb{R} \times \mathbb{R}$ be convex set. A function ψ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if, for all $\beta \in [0,1]$ and all $(x_1, x_2), (y_1, y_2) \in \Omega$, inequality

$$\psi(\beta x_1 + (1-\beta)y_1, \beta x_2 + (1-\beta)y_2) \le \beta\psi(x_1, x_2) + (1-\beta)\psi(y_1, y_2) \tag{3}$$

holds. If, for all $\beta \in [0,1]$ and all $(x_1, x_2), (y_1, y_2) \in \Omega$, the strict inequality in (3) holds, then ψ is said to be strictly convex. ψ is called concave (or strictly concave, resp.) iff $-\psi$ is convex (or strictly convex, resp.)

Definition 2 ([14,15])**.** *Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$, (x_1, x_2) and $(y_1, y_2) \in \Omega$, and let $\varphi : \Omega \to \mathbb{R}$:*

(1) (x_1, x_2) is said to be majorized by (y_1, y_2) (in symbols $(x_1, x_2) \prec (y_1, y_2)$) if $\max\{x_1, x_2\} \le \max\{y_1, y_2\}$ and $x_1 + x_2 = y_1 + y_2$.

(2) ψ is said to be a Schur-convex function on Ω if $(x_1, x_2) \prec (y_1, y_2)$ on Ω implies $\psi(x_1, x_2) \prec \psi(y_1, y_2)$, and ψ is said to be a Schur-concave function on Ω iff $-\psi$ is a Schur-convex function.

Lemma 1 ([14] (p. 5))**.** *Let $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}$. Then*

$$\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right) \prec (x_1, x_2).$$

Lemma 2 ([14] (p. 5))**.** *Let $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ be symmetric set with a nonempty interior Ω°. $\psi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°. Then, function ψ is Schur convex (or Schur concave, resp.) iff ψ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial \psi}{\partial x_1} - \frac{\partial \psi}{\partial x_2}\right) \ge 0 (or \le 0, resp.)$$

holds for any $(x_1, x_2) \in \Omega^\circ$.

Lemma 3 ([16])**.** *Let $\varphi(x, w)$ and $\frac{\partial \varphi(x,w)}{\partial w}$ be continuous on*

$$D = \{(x, w) : a \le x \le b, c \le w \le d\}; let$$

$a(w), b(w)$ and their derivatives be continuous on $[c, d]$; $v \in [c, d]$ implies $a(w), b(w) \in [a, b]$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}w} \int_{a(w)}^{b(w)} \varphi(x, w)\,\mathrm{d}x = \int_{a(w)}^{b(w)} \frac{\partial \varphi(x, w)}{\partial w}\,\mathrm{d}x + \varphi(b(w), u)b'(w) - \varphi(a(w), w)a'(w). \tag{4}$$

Lemma 4. *Let $g(s, t)$ be continuous on rectangle $[a, p; a, q]$, $G(c, d) = \int_c^d \int_c^d g(s, t)\,\mathrm{d}s\,\mathrm{d}t$. If $c = c(b)$ and $d = d(b)$ are differentiable with b, $a \le c(b) \le p$ and $a \le d(b) \le q$, then*

$$\begin{aligned} \frac{\partial G}{\partial b} = & \int_c^d g(s, d)d'(b)\,\mathrm{d}s - \int_c^d g(s, c)c'(b)\,\mathrm{d}s \\ & + d'(b) \int_c^d g(d, t)\,\mathrm{d}t - c'(b) \int_c^d g(c, t)\,\mathrm{d}t. \end{aligned} \tag{5}$$

Proof. Let $\varphi(s,b) = \int_c^d g(s,t)\,\mathrm{d}t$. Then,

$$\frac{\partial \varphi(s,b)}{\partial b} = g(s,d)d'(b) - g(s,c)c'(b).$$

By Lemma 3, we have

$$\begin{aligned}
\frac{\partial G}{\partial b} &= \frac{\mathrm{d}}{\mathrm{d}b}\int_c^d \varphi(s,b)\,\mathrm{d}s \\
&= \int_c^d \frac{\partial \varphi(s,b)}{\partial b}\,\mathrm{d}s + \varphi(d,b)d'(b) - \varphi(c,b)c'(b) \\
&= \int_c^d g(s,d)d'(b)\,\mathrm{d}s - \int_c^d g(s,c)c'(b)\,\mathrm{d}s \\
&+ d'(b)\int_c^d g(d,s)\,\mathrm{d}s - c'(b)\int_c^d g(c,s)\,\mathrm{d}s.
\end{aligned}$$

□

Remark 1. *In passing, it is pointed out that (9) in Lemma 5 of reference [2] is incorrect and should be replaced by (4) of this paper.*

Lemma 5. *Let I be an interval with nonempty interior on $\mathbb{R}$ and $g(s,t)$ be a continuous function on $I \times I$. For $(u,v) \in I \times I, u \neq v$, let $G(u,v) = \int_u^v \int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t$. Then,*

$$\frac{\partial G}{\partial v} = \int_u^v g(s,v)\,\mathrm{d}s + \int_u^v g(v,t)\,\mathrm{d}t, \tag{6}$$

$$\frac{\partial G}{\partial u} = -\left(\int_u^v g(s,u)\,\mathrm{d}s + \int_u^v g(u,t)\,\mathrm{d}t\right). \tag{7}$$

Proof. By taking $c(b) = a$ and $d(b) = b$, we have $c'(b) = 0$ and $d'(b) = 1$. By (5) in Lemma 4, we obtain (6).

Notice that $G(u,v) = \int_v^u \int_v^u g(s,t)\,\mathrm{d}s\,\mathrm{d}t$; from (5), we have

$$\frac{\partial G}{\partial u} = \int_v^u g(s,u)\,\mathrm{d}s + \int_v^u g(u,t)\,\mathrm{d}t = -\left(\int_u^v g(s,u)\,\mathrm{d}s + \int_u^v g(u,t)\,\mathrm{d}t\right).$$

□

Lemma 6 ([14] (p. 38, Proposition 4.3) and [15] (p. 644, B.3.d))**.** *Let $\Omega \subset \mathbb{R} \times \mathbb{R}$ be an open convex set and let $\psi(x,y) : \Omega \to \mathbb{R}$ be twice differentiable. Then, ψ is convex on Ω iff the Hessian matrix*

$$H(x,y) = \begin{pmatrix} \frac{\partial^2 \psi}{\partial x \partial x} & \frac{\partial^2 \psi}{\partial x \partial y} \\ \frac{\partial^2 \psi}{\partial y \partial x} & \frac{\partial^2 \psi}{\partial y \partial y} \end{pmatrix}$$

is non-negative definite on Ω. If $H(\boldsymbol{x})$ is positive definite on Ω, then ψ is strictly convex on Ω.

3. Proofs of Main Results

Proof of Theorem 2. Let $g(s,t)$be convex on $I \times I$. $G(u,v)$ is evidently symmetric. By Lemma 5, we have

$$\frac{\partial G(u,v)}{\partial v} = \frac{-2}{(v-u)^3}\int_u^v \int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t + \frac{1}{(v-u)^2}\left(\int_u^v g(s,v)\,\mathrm{d}s + \int_u^v g(v,t)\,\mathrm{d}t\right).$$

$$\frac{\partial G(u,v)}{\partial u} = \frac{2}{(v-u)^3}\int_u^v \int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t - \frac{1}{(v-u)^2}\left(\int_u^v g(s,u)\,\mathrm{d}s + \int_u^v g(u,t)\,\mathrm{d}t\right).$$

$$\Delta := (v-u)\left(\frac{\partial G(u,v)}{\partial v} - \frac{\partial G(u,v)}{\partial u}\right) = -\frac{4}{(v-u)^2}\int_u^v\int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t$$
$$+\frac{1}{v-u}\int_u^v (g(s,v)+g(s,u))\,\mathrm{d}s + \frac{1}{v-u}\int_u^v (g(u,t)+g(v,t))\,\mathrm{d}t$$

By Hadamards inequality, we have

$$\frac{2}{(v-u)^2}\int_u^v\int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t = \frac{2}{v-u}\int_u^v\left(\frac{1}{v-u}\int_u^v g(s,t)\,\mathrm{d}s\right)\mathrm{d}t$$
$$\leq \frac{2}{v-ua}\int_u^v \frac{g(u,t)+g(v,t)}{2}\,\mathrm{d}t = \frac{1}{v-u}\int_u^v a(g(u,t)+g(v,t))\,\mathrm{d}t$$

and

$$\frac{2}{(v-u)^2}\int_u^v\int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t = \frac{2}{v-u}\int_u^v\left(\frac{1}{v-u}\int_u^v g(s,t)\,\mathrm{d}t\right)\mathrm{d}s$$
$$\leq \frac{2}{v-u}\int_u^v \frac{g(s,u)+g(s,v)}{2}\,\mathrm{d}s = \frac{1}{v-u}\int_u^v (g(s,u)+g(s,v))\,\mathrm{d}s.$$

Moreover, we have

$$\frac{4}{(v-u)^2}\int_u^v\int_u^v g(s,t)\,\mathrm{d}s\,\mathrm{d}t$$
$$\leq \frac{1}{v-u}\int_u^v (g(s,v)+g(s,u))\,\mathrm{d}s + \frac{1}{v-u}\int_u^v (g(u,t)+g(v,t))\,\mathrm{d}t.$$

Therefore, $\Delta \geq 0$, so $G(u,v)$ is Schur-convex on $I \times I$.
When $g(s,t)$ is a concave function on $I \times I$, it can be proved with similar methods. □

4. Application on Binary Mean

Theorem 3. *Let $c > 0$ and $d > 0$. If $c \neq d, 0 < s < 1$, then*

$$A(d,c) \geq S_{s+1}^{s}(d,c)S_s^{s-1}(d,c) \geq \frac{(c+d)^{2s-1}}{s(s+1)}, \tag{8}$$

where $A(d,c) = \frac{c+d}{2}$ and $S_s(d,c) = \left(\frac{d^s-c^s}{s(d-c)}\right)^{\frac{1}{s-1}}$ are the arithmetic mean and the s-order Stolarsky mean of positive numbers c and d, respectively.

Proof. Let $x > 0, y > 0$ and $0 < s < 1$. From Theorem 4 in the reference [17], we know that $g(x,y) = x^s y^{1-s}$ is concave on $(0,+\infty) \times (0,+\infty)$. For $c \neq d$, by Theorem 2, from $(\frac{d+c}{2}, \frac{d+c}{2}) \prec (c,d) \prec (d+c,0)$, it follows that

$$\begin{aligned}
G(d+c,0) =& \frac{1}{(d+c-0)^2}\int_c^d\int_0^{d+c} x^s y^{1-s}\,\mathrm{d}x\,\mathrm{d}y\\
=& \frac{1}{(d+c)^2}\int_0^{d+c} x^s\,\mathrm{d}x\int_0^{d+c} y^{1-s}\,\mathrm{d}y\\
=& \frac{1}{(d+c)^2}\frac{(c+d)^{s+1}}{s+1}\frac{(c+d)^s}{s} = \frac{(c+d)^{2s-1}}{s(s+1)}\\
\leq& G(c,d) = \frac{1}{(d-c)^2}\int_c^d\int_c^d x^s y^{1-s}\,\mathrm{d}x\,\mathrm{d}y\\
=& \frac{1}{(d-c)^2}\int_c^d x^s\,\mathrm{d}x\int_c^d y^{1-s}\,\mathrm{d}y\\
=& \frac{1}{(d-c)^2}\frac{d^{s+1}-c^{s+1}}{s+1}\frac{d^s-c^s}{s}\\
\leq& G\left(\frac{d+c}{2}, \frac{d+c}{2}\right) = \frac{d+c}{2},
\end{aligned}$$

That is, we obtain the following.

$$\frac{(c+d)^{2s-1}}{s(s+1)} \leq S_{s+1}^{s}(d,c)S_{s}^{s-1}(d,c) = \frac{d^{s+1}-c^{s+1}}{(s+1)(d-c)} \cdot \frac{d^{s}-c^{s}}{s(d-c)} \leq \frac{d+c}{2} = A(d,c).$$

□

Theorem 4. *Let $c > 0, d > 0$. Then,*

$$\log\left(\frac{A(d,c)}{B(d,c)}\right)^{2} \geq \left(\frac{c-d}{d+c}\right)^{2}, \tag{9}$$

where $B(d,c) = \sqrt{dc}$ is the geometric mean of of positive numbers c and d.

Proof. From reference [17], we know that the function $g(x,y) = \frac{1}{(x+y)^2}$ is convex on $(0,+\infty) \times (0,+\infty)$. For $c > 0, d > 0$ and $d \neq c$, by Theorem 2, from $(\frac{d+c}{2}, \frac{d+c}{2}) \prec (d,c)$, it follows that

$$\begin{aligned} G(c,d) =& \frac{1}{(d-c)^2}\int_{c}^{d}\int_{c}^{d}\frac{1}{(x+y)^2}\,\mathrm{d}x\,\mathrm{d}y \\ =& \frac{1}{(d-c)^2}\int_{c}^{d}\left(\frac{1}{c+y}-\frac{1}{d+y}\right)\mathrm{d}y \\ =& \frac{1}{(d-c)^2}[(\log(d+c)-\log(2c))-(\log(2d)-\log(d+c))] \\ \geq& G\left(\frac{d+c}{2},\frac{d+c}{2}\right) = \frac{1}{(d+c)^2}, \end{aligned}$$

That is, we obtain the following.

$$\log\left(\frac{A(d,c)}{B(d,c)}\right)^{2} = \log\frac{(d+c)^2}{4dc} \geq \left(\frac{c-d}{d+c}\right)^{2}.$$

□

Theorem 5. *Let $c > 0, d > 0$. Then,*

$$H_e(c^2,d^2) \geq A^2(c,d), \tag{10}$$

where $H_e(c,d) = \frac{c+\sqrt{cd}+d}{3}$ is the Heronian mean of positive numbers c and d.

Proof. From reference [18], we know that the function of two variables

$$\psi(x,y) = \frac{x^2}{2r^2} + \frac{y^2}{2s^2}$$

is a convex function on $(0,+\infty) \times (0,+\infty)$, where $s > 0$ and $r > 0$. For $d > 0, c > 0$, and $c \neq d$, by Theorem 2, from $(\frac{d+c}{2}, \frac{d+c}{2}) \prec (d,c)$, it follows that

$$\begin{aligned}G(c,d) =&\frac{1}{(d-c)^2}\int_c^d\int_c^d\left(\frac{x^2}{2r^2}+\frac{y^2}{2s^2}\right)\mathrm{d}x\,\mathrm{d}y\\ =&\frac{1}{(d-c)^2}\int_c^d\left(\frac{d^3-c^3}{6r^2}+\frac{y^2(d-c)}{2s^2}\right)\mathrm{d}y\\ =&\frac{1}{(d-c)^2}\left(\frac{(d^3-c^3)(d-c)}{6r^2}+\frac{(d^3-c^3)(d-c)}{6s^2}\right)\\ =&\frac{1}{(d-c)^2}\cdot\frac{(d^3-c^3)(d-c)}{6}\left(\frac{1}{r^2}+\frac{1}{s^2}\right)\\ \geq& G\left(\frac{d+c}{2},\frac{d+c}{2}\right)=\frac{(c+d)^2}{8}\left(\frac{1}{r^2}+\frac{1}{s^2}\right),\end{aligned}$$

namely

$$H_e(c^2,d^2)=\frac{c^2+cd+d^2}{3}=\frac{(d^3-c^3)}{3(d-c)}\geq\frac{(d+c)^2}{4}=A^2(d,c).$$

□

Theorem 6. *Let $c>0, d>0$. We have*

$$H_e(c^2,d^2)\geq L(d,c)A(d,c), \tag{11}$$

where $L(d,c)=\frac{d-c}{\log d-\log c}$ is the logarithmic mean of positive numbers c and d.

Proof. Let $g(x,y)=y^2x^{-1}, x>0, y>0$. Then,

$$g_{xx}=2x^{-3}y^2,\ \ g_{xy}=-2x^{-2}y=g_{yx}, g_{yy}=2x^{-1}.$$

The Hesse matrix of $g(x,y)$ is

$$H=\begin{pmatrix}2x^{-3}y^2 & -2x^{-2}y\\ -2x^{-2}y & 2x^{-1}\end{pmatrix}.$$

$$\det(H-\lambda I)=\det\begin{pmatrix}2x^{-3}y^2-\lambda & -2x^{-2}y\\ -2x^{-2}y & 2x^{-1}-\lambda\end{pmatrix}=0$$

$$\Rightarrow\lambda(\lambda-2x^{-3}y^2-2x^{-1})=0\Rightarrow\lambda_1=0,\lambda_2=2x^{-3}y^2+2x^{-1}>0.$$

Therefore, matrix H is positive semidefinite, so it is known that $g(x,y)$ is a convex function on $(0,+\infty)\times(0,+\infty)$. For $d>0, c>0$ and $d\neq c$, by Theorem 2, from $(\frac{d+c}{2},\frac{d+c}{2})\prec(d,c)$, it follows that

$$\begin{aligned}G(c,d) =&\frac{1}{(d-c)^2}\int_c^d\int_c^d y^2x^{-1}\,\mathrm{d}x\,\mathrm{d}y\\ =&\frac{\log d-\log c}{d-c}\cdot\frac{d^2+cd+c^2}{3}\geq\frac{\left(\frac{d+c}{2}\right)^2}{\frac{c+c}{2}}=\frac{d+c}{2},\end{aligned}$$

which is

$$H_e(c^2,d^2)\geq L(d,c)A(d,c).$$

□

Theorem 7. *Let $d > 0, c > 0, d \neq c$. Then*

$$\widetilde{E}(d,c) \leq A(d,c)e^{(d+c)}\left(\frac{d-c}{e^d-e^c}\right)^2 \leq A(d,c), \tag{12}$$

where

$$\widetilde{E}(d,c) = \begin{cases} \frac{ce^d - de^c}{e^d - e^c} + 1,\ d,c \in I,\ d \neq c \\ c,\ c = d \end{cases}$$

is exponent type mean of positive numbers c and d (see [13] (p. 134)).

Proof. Let $g(x,y) = xe^{-(x+y)}, y > 0, x > 0$. From reference [19], we know that function $g(x,y)$ is convex on $\mathbb{R} \times \mathbb{R}$. For $d > 0, c > 0$, and $d \neq c$ by Theorem 2 from $(\frac{d+c}{2}, \frac{d+c}{2}) \prec (d,c)$, it follows that

$$\begin{aligned} G(c,d) =& \frac{1}{(c-d)^2}\int_c^d\int_c^d xe^{-x-y}\,\mathrm{d}x\,\mathrm{d}y \\ =& \frac{1}{(c-d)^2}\int_c^d xe^{-x}\,\mathrm{d}x\int_c^d e^{-y}\,\mathrm{d}y \\ =& \frac{1}{(c-d)^2}\left(\frac{c+1}{e^c} - \frac{d+1}{e^d}\right)\cdot\left(\frac{1}{e^c} - \frac{1}{e^d}\right) \\ =& \frac{1}{(d-c)^2}\frac{(ce^d - de^c) + (e^d - e^c)}{e^{(c+d)}}\cdot\frac{e^d - e^c}{e^{(c+d)}} \\ \leq& G\left(\frac{d+c}{2}, \frac{d+c}{2}\right) = \frac{c+d}{2}\frac{1}{e^{(d+c)}}, \end{aligned}$$

which is

$$\frac{ce^d - de^c}{e^d - e^c} + 1 \leq \frac{d+c}{2}e^{(d+c)}\left(\frac{d-c}{e^d - e^c}\right)^2.$$

For the rest, we only need to prove that

$$e^{(c+d)}\left(\frac{d-c}{e^d-e^c}\right)^2 \leq 1. \tag{13}$$

We write $e^d = u$ and $e^c = v$; then, the above inequality is equivalent to the well-known log-geometric mean inequality.

$$L(v,u) = \frac{v-u}{\log v - \log u} \geq \sqrt{vu} = B(v,u).$$

□

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