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Article

An Application of Rabotnov Functions on Certain Subclasses of Bi-Univalent Functions

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Abstract: In this study, a new class $\mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,\delta,\beta)$ of bi-univalent functions studied by means of Gegenbauer polynomials (GP) with Rabotnov functions is introduced. The coefficient of the Taylor coefficients $|a_2|$and $|a_3|$ and Fekete-Szegö problems for functions belonging to $\mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,\delta,\beta)$ have been derived as well. Furthermore, a variety of new results will appear by considering the parameters in the main results.

Keywords: Rabotnov functions; bi-univalent; Taylor series; Fekete-Szegö; Gegenbauer polynomials

MSC: 30C45

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1. Introduction

In 1948, Rabotnov [1] introduced a special function applied to viscoelasticity. This function, known today as the Rabotnov fractional exponential function or briefly Rabotnov function, is given by:

$$\Phi_{\alpha,\lambda}(\xi) = \xi^{\alpha} \sum_{n=0}^{\infty} \frac{(\lambda)^n \xi^{n(1+\alpha)}}{\Gamma((n+1)(1+\alpha))}, \quad (\alpha, \lambda, \xi \in \mathbb{C}\ ). \tag{1}$$

The Rabotnov function is a particular case of the familiar Mittag-Leffler widely used in the solution of fractional order integral equations or fractional order differential equations. The relation between the Rabotnov and Mittag-Leffler functions [2] can be written as follows

$$\Phi_{\alpha,\lambda}(\xi) = \xi^{\alpha} E_{1+\alpha,1+\alpha}(\lambda \xi^{1+\alpha}),$$

where E is Mittag-Leffler and $\alpha, \lambda, z \in \mathbb{C}$. Various properties of the generalized Mittag-Leffler function can be found in [3–6].

Let $\mathcal{A}$ denote the class of analytic functions f defined in $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ and normalized by $f'(0) - 1 = 0 = f(0)$. Thus $f \in \mathcal{A}$ has a Taylor series expansion

$$f(\xi) = \xi + a_2\xi^2 + a_3\xi^3 + \cdots = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \ \ (\xi \in \mathbb{U}). \tag{2}$$

Let $\mathcal{S}$ denote the class of all $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

Let the function f and function g be analytic in $\mathbb{U}$. We say that f is subordinate to g, written as $f \prec g$, if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$|\omega(\xi)| < 1 \text{ and } \omega(0) = 0$$

such that

$$g(\omega(\xi)) = f(\xi).$$

Moreover, if g is univalent in $\mathbb{U}$, then the equivalence holds

$$f(\xi) \prec g(\xi) \quad \text{iff} \quad f(\mathbb{U}) \subset g(\mathbb{U}) \quad \text{and} \quad f(0) = g(0).$$

It is well known that all functions $f \in \mathcal{S}$ have an inverse f^{-1}, defined by

$$\xi = f^{-1}(f(\xi)) \qquad (\xi \in \mathbb{U})$$

and

$$w = f^{-1}(f(w)) \qquad (|w| < r_0(f);\ r_0(f) \geq \frac{1}{4})$$

where

$$f^{-1}(w) = g(w) = w - a_2 w^2 + w^3(2a_2^2 - a_3) - w^4(5a_2^3 - 5a_2 a_3 + a_4) + \cdots. \tag{3}$$

$f(\xi)$ is said to be bi-univalent in $\mathbb{U}$ if both $f(\xi)$ and $f^{-1}(\xi)$ are univalent in $\mathbb{U}$.

Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (2). For interesting subclasses of functions in Σ, see ([7–16]).

In 1784, Legendre [17] discovered orthogonal polynomials, which have been studied extensively. The importance of orthogonal polynomials and their applications is manifested in various disciplines involving contemporary mathematics, physics and engineering. These polynomials play an essential role in problems of the approximation theory [18,19].

Ala Amourah et al. [20], in 2020, first studied the function of Gegenbauer polynomials (GP) in the following

$$H_\mu(x,\xi) = \frac{1}{(1 - 2x\xi + \xi^2)^\mu}, \tag{4}$$

where $\xi \in \mathbb{U}$ and $x \in [-1,1]$. For fixed x, the function H_μ is analytic in $\mathbb{U}$, so it can be expanded in a Taylor series as

$$H_\mu(x,\xi) = \sum_{n=0}^{\infty} C_n^\mu(x)\xi^n, \tag{5}$$

where $C_n^\mu(x)$ is the (GP) of degree n.

Obviously, H_μ generates nothing when $\mu = 0$. Therefore, the generating of the (GP) is set to be

$$H_0(x,\xi) = 1 - \log\left(1 - 2x\xi + \xi^2\right) = \sum_{n=0}^{\infty} C_n^0(x)\xi^n \tag{6}$$

for $\mu = 0$. Moreover, it is worth mentioning that normalization of μ greater than $-1/2$ is desirable [19]. (GP) can be defined

$$C_n^\mu(x) = \frac{1}{n}\left[2x(n+\mu-1)C_{n-1}^\mu(x) - (n+2\mu-2)C_{n-2}^\mu(x)\right], \tag{7}$$

with the initial values

$$C_0^\mu(x) = 1,\ C_1^\mu(x) = 2\mu x \text{ and } C_2^\mu(x) = 2\mu(1+\mu)x^2 - \mu. \tag{8}$$

We note that for $\mu = 1$, we get the Chebyshev polynomials $C_n^1(x) = C_n(x)$ and for $\mu = \frac{1}{2}$, we get the Legendre polynomials $C_n^{0.5}(x)$.

It is clear that the Rabotnov function $\Phi_{\alpha,\lambda}(\xi)$ does not belong to $\mathcal{A}$. Thus, it is natural to consider the normalization of the Rabotnov function for $\alpha \geq 0$ and $\lambda > 0$ defined by

$$\begin{aligned}\mathbb{R}_{\alpha,\lambda}(\xi) &= \xi^{\frac{1}{1+\alpha}}\Gamma(1+\alpha)\Phi_{\alpha,\lambda}(\xi^{\frac{1}{1+\alpha}})\\ &= \xi + \sum_{n=2}^{\infty}\frac{\lambda^{n-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)n)}\xi^n,\ \xi \in \mathbb{U}.\end{aligned}$$

Geometric properties, including convexity, close-to-convexity and starlikeness, for the normalized Rabotnov function $\mathbb{R}_{\alpha,\lambda}(\xi)$ were recently investigated by Eker and Ece in [21].

We now consider the linear operator $V_{\alpha,\lambda} : \mathcal{A} \to \mathcal{A}$ defined by

$$V_{\alpha,\lambda}f(\xi) = \mathbb{R}_{\alpha,\lambda}(\xi) * f(\xi) = \xi + \sum_{n=2}^{\infty}\frac{\lambda^{n-1}\Gamma(1+\alpha)}{\Gamma((1+\alpha)n)}a_n\xi^n,\ \xi \in \mathbb{U}, \tag{9}$$

where $\alpha \geq 0$ and $\lambda > 0$.

Numerous scholars have recently been investigating bi-univalent functions related to orthogonal polynomials [22–26]. As far as we are aware, there are few works papers on bi-univalent functions for the Gegenbauer polynomial.

Mainly motivated by the work of Ala Amourah [20], a subclass of Σ involving Rabotnov function associated with (GP) is introduced, and additionally, the bounds for the Taylor coefficients $|a_2|$ and $|a_3|$ and Fekete-Szegö problems are obtained.

2. Coefficient Bounds of the Class $\mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,\delta,\lambda)$

In this section, we begin defining associated Rabotnov functions, the new subclass $\mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,\lambda)$.

Definition 1. *Let $f \in \Sigma$ given by (2) be said to be in the class $\mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,\delta,\lambda)$ if the following subordinations:*

$$(1-\gamma)\frac{V_{\alpha,\lambda}f(\xi)}{\xi} + \gamma(V_{\alpha,\lambda}f(\xi))' + \delta\xi(V_{\alpha,\lambda}f(\xi))'' \prec H_{\mu}(x,\xi) \tag{10}$$

and

$$(1-\gamma)\frac{V_{\alpha,\lambda}g(w)}{w} + \gamma(V_{\alpha,\lambda}g(w))' + \delta w(V_{\alpha,\lambda}g(w))'' \prec H_{\mu}(x,w), \tag{11}$$

where $\mu > 0$, $\alpha \geq 0$, $x \in (\frac{1}{2},1]$, $\lambda > 0$, $\gamma \geq 0$, , and function $g = f^{-1}$ is given by (3) and H_{μ} is the generating function of the (GP) given by (4).

Example 1. *For $\delta = 0$, we have, $\mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,0,\lambda) = \mathfrak{H}^{\mu}_{\Sigma}(x,\gamma,\alpha,\lambda)$, in which $\mathfrak{H}^{\mu}_{\Sigma}(x,\gamma,\alpha,\lambda)$ denotes the class of $f \in \Sigma$ given by (2) and satisfying the condition*

$$(1-\gamma)\frac{V_{\alpha,\lambda}f(\xi)}{\xi} + \gamma(V_{\alpha,\lambda}f(\xi))' \prec H_{\mu}(x,\xi) \tag{12}$$

and

$$(1-\gamma)\frac{V_{\alpha,\lambda}g(w)}{w} + \gamma(V_{\alpha,\lambda}g(w))' \prec H_{\mu}(x,\xi), \tag{13}$$

where $\mu > 0$, $x \in (\frac{1}{2},1]$, $\gamma \geq 0$, and function $g = f^{-1}$ are given by (3), and H_{μ} is the generating function of the (GP) given by (4).

Example 2. *For $\gamma = 1$ and $\delta = 0$, we have, $\mathfrak{R}^{\mu}_{\Sigma}(x,1,\alpha,0,\lambda) = \mathfrak{R}^{\mu}_{\Sigma}(x,\alpha,\lambda)$, in which $\mathfrak{R}^{\mu}_{\Sigma}(x,\alpha,\lambda)$ denotes the class of $f \in \Sigma$ given by (2) and satisfying the following condition*

$$(V_{\alpha,\lambda}f(\xi))' \prec H_{\mu}(x,\xi) \tag{14}$$

and

$$(V_{\alpha,\lambda}g(w))' \prec H_\mu(x,\xi), \tag{15}$$

where $x \in (\frac{1}{2},1]$, $\mu > 0$, *and function* $g = f^{-1}$ *is given by (3) and* H_μ *is the generating function of the (GP) given by (4).*

Unless otherwise mentioned, we shall assume in this paper that $\lambda, \mu > 0$, $\gamma, \alpha \geq 0$ *and* $x \in (\frac{1}{2}, 1]$.

First, we give the maximum value for $|a_2|$ and $|a_3|$ given in Definition 1.

Theorem 1. *Let* $f \in \Sigma$ *given by (2) belong to the class* $\mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,\delta,\lambda)$. *Then*

$$|a_2| \leq \frac{2|\mu|\Gamma(2(1+\alpha))x\sqrt{2\Gamma(3(1+\alpha))x}}{\lambda\sqrt{\left|\Big(\digamma(\mu,\gamma,\alpha,\delta,\lambda)x^2 + (1+2\gamma+2\delta)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha))\Big)\Gamma(1+\alpha)\right|}},$$

and

$$|a_3| \leq \frac{4\mu^2x^2[\Gamma(2(1+\alpha))]^2}{\lambda^2(1+\gamma+2\delta)^2[\Gamma(1+\alpha)]^2} + \frac{2|\mu|x\Gamma(3(1+\alpha))}{(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)},$$

where

$$\digamma(\mu,\gamma,\alpha,\delta) = 4\mu(1+2\gamma+6\delta)[\Gamma(2(1+\alpha))]^2 - 2(1+\mu)(1+2\gamma+2\delta)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha)).$$

Proof. From Definition 1 and $f \in \mathfrak{R}^{\mu}_{\Sigma}(x,\gamma,\alpha,\delta,\lambda)$, for some regular functions φ, v such that $\varphi(0) = 0 = v(0)$ and $|\varphi(\xi)| < 1$, $|v(w)| < 1$ for all $\xi, w \in \mathbb{U}$, then

$$(1-\gamma)\frac{V_{\alpha,\lambda}f(\xi)}{\xi} + \gamma(V_{\alpha,\lambda}f(\xi))' + \delta\xi(V_{\alpha,\lambda}f(\xi))'' = H_\mu(x,\varphi(\xi)) \tag{16}$$

and

$$(1-\gamma)\frac{V_{\alpha,\lambda}g(w)}{w} + \gamma(V_{\alpha,\lambda}g(w))' + \delta w(V_{\alpha,\lambda}g(w))'' = H_\mu(x,v(w)), \tag{17}$$

From (16) and (17), we obtain

$$\begin{aligned}(1-\gamma)\frac{V_{\alpha,\lambda}f(\xi)}{\xi} &+ \gamma(V_{\alpha,\lambda}f(\xi))' + \delta\xi(V_{\alpha,\lambda}f(\xi))'' \\ &= 1 + C^{\mu}_1(x)c_1\xi + \left[C^{\mu}_1(x)c_2 + C^{\mu}_2(x)c_1^2\right]\xi^2 + \cdots\end{aligned} \tag{18}$$

and

$$\begin{aligned}(1-\gamma)\frac{V_{\alpha,\lambda}g(w)}{w} &+ \gamma(V_{\alpha,\lambda}g(w))' + \delta w(V_{\alpha,\lambda}g(w))'' \\ &= 1 + C^{\mu}_1(x)d_1w + \left[C^{\mu}_1(x)d_2 + C^{\mu}_2(x)d_1^2\right])w^2 + \cdots.\end{aligned} \tag{19}$$

It is well known

$$|\varphi(\xi)| = \left|c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots\right| < 1, \ \ (\xi \in \mathbb{U})$$

and

$$|v(w)| = \left|d_1w + d_2w^2 + d_3w^3 + \cdots\right| < 1, \ \ (w \in \mathbb{U}),$$

then

$$|c_j| \leq 1 \text{ and } |d_j| \leq 1 \text{ for all } j \in \mathbb{N}. \tag{20}$$

Upon comparing the coefficients in (18) and (19), we have

$$(1+\gamma+2\delta)\frac{\lambda\Gamma(1+\alpha)}{\Gamma(2(1+\alpha))}a_2 = C_1^{\mu}(x)c_1, \tag{21}$$

$$(1+2\gamma+6\delta)\frac{\lambda^2\Gamma(1+\alpha)}{\Gamma(3(1+\alpha))}a_3 = C_1^{\mu}(x)c_2 + C_2^{\mu}(x)c_1^2, \tag{22}$$

$$-(1+\gamma+2\delta)\frac{\lambda\Gamma(1+\alpha)}{\Gamma(2(1+\alpha))}a_2 = C_1^{\mu}(x)d_1, \tag{23}$$

and

$$(1+2\gamma+6\delta)\frac{\lambda^2\Gamma(1+\alpha)}{\Gamma(3(1+\alpha))}\left[2a_2^2 - a_3\right] = C_1^{\mu}(x)d_2 + C_2^{\mu}(x)d_1^2. \tag{24}$$

It follows from (21) and (23) that

$$c_1 = -d_1 \tag{25}$$

and

$$2(1+\gamma+2\delta)^2\frac{\lambda^2[\Gamma(1+\alpha)]^2}{[\Gamma(2(1+\alpha))]^2}a_2^2 = \left[C_1^{\mu}(x)\right]^2\left(c_1^2+d_1^2\right). \tag{26}$$

If we add (22) and (24), we get

$$2(1+2\gamma+6\delta)\frac{\lambda^2\Gamma(1+\alpha)}{\Gamma(3(1+\alpha))}a_2^2 = C_1^{\mu}(x)(c_2+d_2) + C_2^{\mu}(x)\left(c_1^2+d_1^2\right). \tag{27}$$

Substituting the value of $\left(c_1^2+d_1^2\right)$ from (26) into (27), we deduce that

$$\begin{aligned} &2\lambda^2\Gamma(1+\alpha)\left[\frac{(1+2\gamma+6\delta)}{\Gamma(3(1+\alpha))} - \frac{(1+\gamma+2\delta)^2\Gamma(1+\alpha)}{[\Gamma(2(1+\alpha))]^2}\,\frac{C_2^{\mu}(x)}{\left[C_1^{\mu}(x)\right]^2}\right]a_2^2 \\ &= C_1^{\mu}(x)(c_2+d_2). \end{aligned} \tag{28}$$

Moreover, using computations (19), (20) and (28), we find that

$$|a_2| \leq \frac{2|\mu|\Gamma(2(1+\alpha))x\sqrt{\Gamma(3(1+\alpha))x}}{\lambda\sqrt{\left|\left(\digamma(\mu,\gamma,\alpha,\delta)x^2 + (1+2\gamma+2\delta)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha))\right)\Gamma(1+\alpha)\right|}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (24) from (22), we obtain

$$2(1+2\gamma+6\delta)\frac{\lambda^2\Gamma(1+\alpha)}{\Gamma(3(1+\alpha))}\left(a_3 - a_2^2\right) = C_1^{\mu}(x)(c_2-d_2) + C_2^{\mu}(x)\left(c_1^2-d_1^2\right). \tag{29}$$

Then, in view of (26) and (29), it becomes

$$\begin{aligned} a_3 = &\frac{[\Gamma(2(1+\alpha))]^2\left[C_1^{\mu}(x)\right]^2}{2\lambda^2(1+\gamma+2\delta)^2[\Gamma(1+\alpha)]^2}\left(c_1^2+d_1^2\right) \\ &+ \frac{C_1^{\mu}(x)\Gamma(3(1+\alpha))}{2(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)}(c_2-d_2). \end{aligned}$$

Thus applying (8), we conclude that

$$|a_3| \leq \frac{4\mu^2 x^2 [\Gamma(2(1+\alpha))]^2}{\lambda^2 (1+\gamma+2\delta)^2 [\Gamma(1+\alpha)]^2} + \frac{2|\mu| x \Gamma(3(1+\alpha))}{(1+2\gamma+6\delta)\lambda^2 \Gamma(1+\alpha)}.$$

The proof is complete. □

3. Fekete–Szegö Inequality

In this section, we prove a sharp bound of the Fekete and Szegö functional $\eta a_2^2 - a_3$ [27], where function f belongs to the class $\mathfrak{R}_\Sigma^\mu(x,\gamma,\alpha,\delta,\lambda)$.

Theorem 2. *Let $f \in \Sigma$ given by (2) belong to class $\mathfrak{R}_\Sigma^\mu(x,\gamma,\alpha,\delta,\lambda)$. Then*

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2\Gamma(3(1+\alpha))|\mu| x}{(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)}, & |\eta - 1| \leq Q(\mu,\gamma,\alpha,\delta) \\ \frac{8\mu^2 x^3 \Gamma(3(1+\alpha))[\Gamma(2(1+\alpha))]^2 |\eta-1|}{\lambda^2 \Gamma(1+\alpha) |L(\mu,\gamma,\alpha,\delta,\lambda)|}, & |\eta - 1| \geq Q(\mu,\gamma,\alpha,\delta), \end{cases}$$

where

$$\begin{aligned} L(\mu,\gamma,\alpha,\delta) &= 4\mu x^2 (1+2\gamma+6\delta)[\Gamma(2(1+\alpha))]^2 \\ &\quad - (1+\gamma+2\delta)^2 \Gamma(1+\alpha)\Gamma(3(1+\alpha))\Big(2(1+\mu)x^2 - 1\Big) \end{aligned}$$

and

$$Q(\mu,\gamma,\alpha,\delta) = \left|1 - \frac{(1+\gamma+2\delta)^2 (2(1+\mu)x^2 - 1)\Gamma(3(1+\alpha))\Gamma(1+\alpha)}{4\mu x^2 (1+2\gamma+6\delta)[\Gamma(2(1+\alpha))]^2}\right|.$$

Proof. From (28) and (29) given by

$$\begin{aligned} & a_3 - \eta a_2^2 \\ &= \frac{(1-\eta)\left[C_1^\mu(x)\right]^3 (c_2+d_2)\Gamma(3(1+\alpha))[\Gamma(2(1+\alpha))]^2}{2\lambda^2\Gamma(1+\alpha)\left[(1+2\gamma+6\delta)[\Gamma(2(1+\alpha))]^2\left[C_1^\mu(x)\right]^2 - (1+\gamma+2\delta)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha))C_2^\mu(x)\right]} \\ &+ \frac{C_1^\mu(x)\Gamma(3(1+\alpha))}{2(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)}(c_2 - d_2) \\ &= C_1^\mu(x)\left[h(\eta) + \frac{\Gamma(3(1+\alpha))}{2(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)}\right]c_2 + C_1^\mu(x)\left[h(\eta) - \frac{\Gamma(3(1+\alpha))}{2(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)}\right]d_2, \end{aligned}$$

where

$$h(\eta) = \frac{\left[C_1^\alpha(x)\right]^2 \Gamma(3(1+\alpha))[\Gamma(2(1+\alpha))]^2 (1-\eta)}{2\lambda^2\Gamma(1+\alpha)\left[(1+2\gamma+6\delta)[\Gamma(2(1+\alpha))]^2\left[C_1^\mu(x)\right]^2 - (1+\gamma+2\delta)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha))C_2^\mu(x)\right]}.$$

Then, in view of (8), we conclude that

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{\Gamma(3(1+\alpha))\left|C_1^\alpha(x)\right|}{(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)} & 0 \leq |h(\eta)| \leq \frac{\Gamma(3(1+\alpha))}{2(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)}, \\ 2\left|C_1^\mu(x)\right||h(\eta)| & |h(\eta)| \geq \frac{\Gamma(3(1+\alpha))}{2(1+2\gamma+6\delta)\lambda^2\Gamma(1+\alpha)}, \end{cases}$$

the proof is complete 2. □

4. Corollaries and Consequences

Conformable essentially to Examples 1 and 2, Theorems 1 and 2 yield the following corollaries.

Corollary 1. *Let $f \in \Sigma$ given by (2) belong to class $\mathfrak{H}_\Sigma^\mu(x,\gamma,\alpha,\lambda)$. Then*

$$|a_2| \leq \frac{2|\mu|\Gamma(2(1+\alpha))x\sqrt{2\Gamma(3(1+\alpha))x}}{\lambda\sqrt{\left|\Big(\digamma(\mu,\gamma,\alpha)x^2 + (1+2\gamma)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha))\Big)\Gamma(1+\alpha)\right|}}$$

$$|a_3| \leq \frac{4\mu^2 x^2 [\Gamma(2(1+\alpha))]^2}{\lambda^2(1+\gamma)^2[\Gamma(1+\alpha)]^2} + \frac{2|\mu| x \Gamma(3(1+\alpha))}{(1+2\gamma)\lambda^2\Gamma(1+\alpha)},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2\Gamma(3(1+\alpha))|\mu|x}{(1+2\gamma)\lambda^2\Gamma(1+\alpha)}, & |\eta - 1| \leq Q(\mu,\gamma,\alpha) \\ \frac{8\mu^2 x^3 \Gamma(3(1+\alpha))[\Gamma(2(1+\alpha))]^2|\eta-1|}{\lambda^2\Gamma(1+\alpha)|L(\mu,\gamma,\alpha,0,\lambda)|}, & |\eta - 1| \geq Q(\mu,\gamma,\alpha), \end{cases}$$

where

$$\digamma(\mu,\gamma,\alpha) = 4\mu(1+2\gamma)[\Gamma(2(1+\alpha))]^2 - 2(1+\mu)(1+2\gamma)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha)),$$

$$\begin{aligned} L(\mu,\gamma,\alpha) = {} & 4\mu x^2(1+2\gamma)[\Gamma(2(1+\alpha))]^2 \\ & - (1+\gamma)^2\Gamma(1+\alpha)\Gamma(3(1+\alpha))\Big(2(1+\mu)x^2 - 1\Big), \end{aligned}$$

and

$$Q(\mu,\gamma,\alpha) = \left|1 - \frac{(1+\gamma)^2\big(2(1+\mu)x^2 - 1\big)\Gamma(3(1+\alpha))\Gamma(1+\alpha)}{4\mu x^2(1+2\gamma)[\Gamma(2(1+\alpha))]^2}\right|.$$

Corollary 2. *Let $f \in \Sigma$ given by (2) belong to class $\mathfrak{G}_\Sigma^\mu(x,\alpha,\lambda)$. Then*

$$|a_2| \leq \frac{2|\mu|\Gamma(2(1+\alpha))x\sqrt{2\Gamma(3(1+\alpha))x}}{\lambda\sqrt{|(\digamma(\mu,1,\alpha)x^2 + 9\Gamma(1+\alpha)\Gamma(3(1+\alpha)))\Gamma(1+\alpha)|}}$$

$$|a_3| \leq \frac{\mu^2 x^2[\Gamma(2(1+\alpha))]^2}{\lambda^2[\Gamma(1+\alpha)]^2} + \frac{2|\mu| x\Gamma(3(1+\alpha))}{3\lambda^2\Gamma(1+\alpha)},$$

and

$$\left|a_3 - \eta a_2^2\right| \leq \begin{cases} \frac{2\Gamma(3(1+\alpha))|\mu|x}{3\lambda^2\Gamma(1+\alpha)}, & |\eta - 1| \leq Q(\mu,1,\alpha) \\ \frac{8\mu^2 x^3\Gamma(3(1+\alpha))[\Gamma(2(1+\alpha))]^2|\eta-1|}{\lambda^2\Gamma(1+\alpha)|L(\mu,1,\alpha,0)|}, & |\eta - 1| \geq Q(\mu,1,\alpha), \end{cases}$$

where

$$\digamma(\mu,1,\alpha) = 12\mu[\Gamma(2(1+\alpha))]^2 - 18(1+\mu)\Gamma(1+\alpha)\Gamma(3(1+\alpha)),$$

$$L(\mu,1,\alpha) = 12\mu x^2[\Gamma(2(1+\alpha))]^2 - 4\Gamma(1+\alpha)\Gamma(3(1+\alpha))\Big(2(1+\mu)x^2 - 1\Big),$$

and

$$Q(\mu,1,\alpha) = \left|1 - \frac{(2(1+\mu)x^2 - 1)\Gamma(3(1+\alpha))\Gamma(1+\alpha)}{3\mu x^2[\Gamma(2(1+\alpha))]^2}\right|.$$

Remark 1. *The results presented in this article would lead to various other new results for the classes $\mathfrak{R}_\Sigma^{0.5}(x,\gamma,\alpha,\delta,\lambda)$ for Legendre polynomials, and $\mathfrak{R}_\Sigma^{1}(x,\gamma,\alpha,\delta,\lambda)$ for Chebyshev polynomials.*

5. Conclusions

In our study, a new class $\mathfrak{R}_\Sigma^\mu(x,\gamma,\alpha,\delta,\lambda)$ of normalized analytic functions and bi-univalent functions associated with the normalized Rabotnov function series was introduced. For functions belonging to this class, the estimates of the Taylor coefficients $|a_2|$ and $|a_3|$ and Fekete-Szegö functional problems were derived. This study could inspire researchers to introduce new classes of analytic and bi-univalent functions associated with the normalized Rabotnov function series.

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