Article

Certain Geometric Properties of the Fox–Wright Functions

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Abstract: The primary objective of this study is to establish necessary conditions so that the normalized Fox–Wright functions possess certain geometric properties, such as convexity and pre-starlikeness. In addition, we present a linear operator associated with the Fox–Wright functions and discuss its k-uniform convexity and k-uniform starlikeness. Furthermore, some sufficient conditions were obtained so that this function belongs to the Hardy spaces. The results of this work are presumably new and illustrated by several consequences, remarks, and examples.

Keywords: Fox–Wright functions; analytic functions; univalent functions; convex functions; starlike functions; Hardy spaces

MSC: 30D15; 30C45; 30H10

1. Introduction and Motivation

In this article, we consider the Fox–Wright function; we studied several geometric properties in a unit disc. The Fox–Wright function is an important special function that plays a vital role in different branches of science and engineering. Geometric properties, such as starlikeness, pre-starlikeness, convexity, k-uniformly convexity, and k-uniformly starlikeness, are associated with special functions, and have always been focused on by researchers [1–4]. For more descriptions of the geometric properties of special functions, please refer to [1–10] and their references.

The results mentioned above motivated us to discuss the geometric properties related to Fox–Wright functions, such as pre-starlikeness, convex of order δ, k-uniform starlikeness, k-Uniform convexity, and Hardy spaces.

In addition to explaining the definitions of the geometric terms mentioned above, we will now introduce the basic definitions and properties of the Fox–Wright function, which are important in the sequel.

1.1. Fox–Wright Function

The Fox–Wright function $_m\psi_n[.]$ is defined as [11]

$$_m\psi_n\left[\begin{array}{c}(a_1,b_1),\ldots,(a_m,b_m)\\(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\end{array}\Big| z\right] = \sum_{k=0}^{\infty}\frac{\Pi_{i=1}^m\Gamma(a_i+kb_i)z^k}{\Pi_{j=1}^n\Gamma(\alpha_j+k\beta_j)k!},\tag{1}$$

where $a_i, \alpha_j \in \mathbb{C}$, $b_i, \beta_j \in \mathbb{R}^+$ $(i = 1, 2\ldots m, j = 1, 2, \ldots, n)$ and $\Gamma(z)$ denotes Euler's gamma function [12]. The series (1) converges in the entire z-plane, when

$$\Xi = 1 + \sum_{j=1}^{n}\beta_j - \sum_{i=1}^{m}b_j > 0.$$

If $\Xi = -1$, then the series (1) converges for every bounded $|z|$, where $z \in \mathbb{C}$.

The Fox–Wright function $_m\psi_n[.]$ can be represented by the Fox H-function [11] as follows:

$$_m\psi_n \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \end{array} \Big| z \right] = H^{1,m}_{m,n+1} \left[\begin{array}{c} (1 - a_1, b_1), \ldots, (1 - a_m, b_m) \\ (1 - \alpha_1, \beta_1), \ldots, (1 - \alpha_n, \beta_n), (0, 1) \end{array} \Big| - z \right].$$

Many special functions, such as the Wright function, Bessel function, hypergeometric function, and Mittag–Leffler function can be expressed as particular cases of the Fox–Wright function. For example, putting $m = 1, n = 2$ and $a_1 = \alpha_2, b_1 = \beta_2$ in (1), the Wright function $W_{\alpha_1, \beta_1}(z)$ can be obtained. Particularly, when setting $\alpha_1 = 1$ and $\beta_1 = \mu + 1$, we obtain the Bessel function [13], represented as:

$$J_\mu(z) = \left(\frac{z}{2} \right)^\mu W_{1, \mu+1} \left(-\frac{z^2}{4} \right) = \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n+\mu}}{n! \Gamma(n + \mu + 1)}, \quad \mu > -1. \tag{2}$$

In Setting $m = 1 = n = a_1 = b_1$ and $\alpha_1 = \alpha, \beta_1 = \beta$ in (1), we have the Mittag–Leffler function $E_{\alpha, \beta}(z)$, which is given as:

$$E_{\alpha, \beta}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha + \beta n)}, \quad \beta > 0, \; \alpha \in \mathbb{C}. \tag{3}$$

which was introduced by Wiman [14].

1.2. Geometric Functions Theory

Suppose $\mathcal{A}$ is the class of analytic functions of the form

$$g(z) = z + \sum_{k=2}^\infty b_k z^k, \quad z \in \mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

A function $g \in \mathcal{A}$ is called starlike in $\mathbb{D}$ if $g(\mathbb{D})$ is univalent in $\mathbb{D}$ and the starlike domain with respect to the origin in $\mathbb{C}$. The class of starlike functions in $\mathbb{D}$ is denoted by $\mathcal{S}^*$. The analytical descriptions of starlike functions (see [15]) can be described as:

$$g \in \mathcal{S}^* \iff \Re \left(\frac{z g'(z)}{g(z)} \right) > 0 \quad (\forall z \in \mathbb{D}).$$

Let $\mathcal{S}^*(\alpha)$ denote the starlike function of order α. Then analytical characterization of $\mathcal{S}^*(\alpha)$ is given by:

$$\Re \left(\frac{z g'(z)}{g(z)} \right) > \alpha \quad (\forall z \in \mathbb{D}; \; 0 \leq \alpha < 1).$$

A function $g \in \mathcal{A}$ is called convex in a domain $\mathbb{D}$, if g is univalent in $\mathbb{D}$ and $g(\mathbb{D})$ is a convex domain in $\mathbb{C}$. Suppose that $\mathcal{K}$ denotes the class of convex functions. The analytical description of $\mathcal{K}$ is given by

$$g \in \mathcal{K} \iff \Re \left(1 + \frac{z g''(z)}{g'(z)} \right) > 0 \quad (\forall z \in \mathbb{D}).$$

Moreover, assume that $\mathcal{K}(\alpha)$ denotes the class of convex functions of order α. Then the analytical characterization of $\mathcal{K}(\alpha)$ is given as:

$$\Re \left(1 + \frac{z g''(z)}{g'(z)} \right) > \alpha \quad (\forall z \in \mathbb{D}; \; 0 \leq \alpha < 1).$$

A function $g \in \mathcal{A}$ is called close-to-convex in domain $\mathbb{D}$, if there is a starlike function h in $\mathbb{D}$, which satisfy

$$\Re\left(\frac{zg'(z)}{h(z)}\right) > 0 \qquad (\forall z \in \mathbb{D}).$$

It is well known that every close-to-convex in $\mathbb{D}$ is univalent in $\mathbb{D}$.

The convolution (Hadamard product) of two power series is an important tool in the geometric functions theory. To define convolution (Hadamard product), let us consider the following two Taylor series:

$$g(z) = \sum_{n=0}^{\infty} c_n z^n \qquad (|z| < \rho_1) \quad \text{and} \quad h(z) = \sum_{n=0}^{\infty} d_n z^n \qquad (|z| < \rho_2),$$

where ρ_1 and ρ_2 are their radii of convergence, respectively. Then their convolutions (Hadamard product), see, for example, [16], are represented as:

$$(g * h)(z) = \sum_{n=0}^{\infty} c_n d_n z^n \qquad (|z| < \rho_1 \cdot \rho_2).$$

We also require the convolution of two classes of analytic functions. If $\mathcal{M}_1$ and $\mathcal{M}_2$ consist of analytic functions defined on the unit disc, then

$$\mathcal{M}_1 * \mathcal{M}_2 := \{f * g, \; \forall \, f \in \mathcal{M}_1 \quad \text{and} \quad \forall g \in \mathcal{M}_2\}.$$

Pre-starlike functions are introduced by Ruscheweyh in [17]. For $\mu \in (0, 1]$, consider the function

$$g_\mu(z) = \frac{z}{(1-z)^{2-2\mu}}.$$

Then the class of pre-starlike functions of order μ, denoted by $\mathcal{L}_\mu$, is given as

$$\mathcal{L}_\mu := \{g : g \in \mathcal{A} \quad \text{and} \quad g_\mu(z) * g \in \mathcal{S}^*(\mu)\}.$$

In particular, $\mathcal{L}_{\frac{1}{2}} = \mathcal{S}^*\left(\frac{1}{2}\right)$ and $\mathcal{L}_0 = \mathcal{C}$. Sheil-Small et al. [18] generalized the class $\mathcal{L}_\mu$ as $\mathcal{L}[\rho, \mu]$ for $0 \le \rho, \mu < 1$, which is defined as

$$\mathcal{L}[\rho, \mu] := \{g : g \in \mathcal{A} \quad \text{and} \quad g_\rho * g \in \mathcal{S}^*(\mu)\},$$

where $g_\rho(z) = z(1-z)^{-(2-2\rho)}$. Clearly, $\mathcal{L}_\mu = \mathcal{L}[\mu, \mu]$.

If a real function $g \in \mathcal{A}$ on $(-1, 1)$ satisfies the following relation:

$$\Im(z)\Im(g(z)) > 0 \qquad (z \in \mathbb{D}),$$

then it is called a typical real function. In [19], Robertson introduced the class of typically real functions. A function $g \in \mathcal{A}$ is said to be convex in the imaginary axis direction, if the region $g(\mathbb{D})$ is a convex region in the imaginary axis direction, i.e.,

$$\Re\{\nu_1\} = \Re\{\nu_2\} \qquad \left(\forall \, \nu_1, \nu_2 \in g(\mathbb{D})\right).$$

It is well known from [19] that a function $g \in \mathcal{A}$ is convex in the imaginary axis direction containing real coefficients if $zg'(z)$ is a typically real function, which is equivalent to

$$\Re\{(1-z^2)g'(z)\} > 0 \qquad (z \in \mathbb{D}).$$

Two other important classes required in our study are k-ST and k-UCV, introduced by Kanas et al. (see [20] and [21]; see also [22]); they are defined as follows:

$$k\text{-ST} = \left\{ g : g \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zg'(z)}{g(z)}\right) > k\left|\frac{zg'(z)}{g(z)} - 1\right| \quad (z \in \mathbb{D}) \right\}$$

and

$$k\text{-UCV} = \left\{ g : g \in \mathcal{S} \quad \text{and} \quad \Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > k\left|\frac{zg''(z)}{g'(z)}\right| \quad (z \in \mathbb{D}) \right\}.$$

The following two results are significant for k-ST and k-UCV, respectively.

Theorem 1 ([21]). *Assume that $g \in \mathcal{A}$. If*

$$\sum_{k=2}^{\infty} [k + l(k-1)]|b_k| < 1$$

for some l $(0 \le l < \infty)$, then $g \in k$-ST.

Theorem 2 ([20]). *Suppose that $g \in \mathcal{A}$. If*

$$\sum_{k=2}^{\infty} k(k-1)|b_k| \le \frac{1}{l+2}$$

for some l $(0 \le l < \infty)$, then $g \in k$-UCV. The number $\frac{1}{l+2}$ cannot be enlarged.

We also consider the following class:

$$\mathcal{R}_\eta(\zeta) = \left\{ g : g \in \mathcal{A}, \Re\{e^{i\eta}(g'(z) - \zeta)\} > 0 \quad \left(z \in \mathbb{D}; \ \zeta < 1; \ \frac{-\pi}{2} < \eta < \frac{\pi}{2}\right) \right\}.$$

If we take the function $g \in \mathcal{A}$ in the class $\mathcal{R}_\eta(\zeta)$, then we have

$$|b_k| \le \frac{2(1-\zeta)\cos\eta}{k} \quad (k \in \mathbb{N} \setminus \{1\}).$$

1.3. Hardy Space

Let us consider the space of all bounded functions $\mathcal{H}^\infty$ in open unit disk $\mathbb{D}$. We suppose that h is in the class of analytic functions in domain $\mathbb{D}$ and set

$$\mathcal{M}_q(\epsilon, h) = \begin{cases} \left(\frac{1}{2\pi}\int_0^{2\pi} |h(\epsilon e^{(i\theta)}|^q \, d\theta\right)^{\frac{1}{q}} & (0 < q < \infty) \\ \max\{|h(z)| : |z| \le \epsilon\} & (q = \infty). \end{cases}$$

It can be observed from [23] that $h \in \mathcal{H}^q$, if $\mathcal{M}_q(\epsilon, h)$ is bounded for all $\epsilon \in [0, 1)$ and

$$\mathcal{H}^\infty \subset \mathcal{H}^p \subset \mathcal{H}^q \quad (0 < p < q < 1).$$

Let us consider the following results [24] related to the Hardy space $\mathcal{H}^q$:

$$\Re\{h'(z)\} > 0 \implies h' \in \mathcal{H}^q \quad (\forall \, q < 1) \implies h \in \mathcal{H}^{\frac{p}{1-p}} \quad (\forall \, 0 < p < 1). \tag{4}$$

This paper is organized as follows. In Section 2, we present some lemmas that will help derive the main results. Section 3 presents sufficient conditions so that the Fox–Wright function satisfies certain geometric properties, such as pre-starlikeness and convexity of order δ. Furthermore, consequences, important remarks, and examples are shown in this section. In Section 4, we consider the linear operator associated with the Fox–Wright

function for which *k*-uniformly starlikeness and *k*-uniformly convexity are discussed. Furthermore, sufficient conditions are established in Section 5, so that this function belongs to the Hardy space. Consequences and remarks are also presented in this section.

2. Useful Lemmas

Some Lemmas were recalled in this section, which will be helpful to prove the main theorems in this paper.

Lemma 1 ([25]). *For any real number $s > 1$, the digamma function $\Psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ satisfies the following inequality:*

$$\log(s) - \gamma \leq \Psi(s) \leq \log(s),$$

where γ is the Euler–Mascheroni constant.

Lemma 2 ([26]). *If $h \in \mathcal{A}$ and $|(h(z)/z) - 1| < 1$ for all $z \in \mathbb{D}$, then h is starlike and univalent in*

$$\mathbb{D}_{\frac{1}{2}} = \left\{ z : |z| < \frac{1}{2} \quad \text{where} \quad z \in \mathbb{C} \right\}.$$

Lemma 3 ([27]). *If $h \in \mathcal{A}$ and $|h'(z) - 1| < 1$ for each $z \in \mathbb{D}$, then h is convex in*

$$\mathbb{D}_{\frac{1}{2}} = \left\{ z : |z| < \frac{1}{2} \quad \text{where} \quad z \in \mathbb{C} \right\}.$$

Lemma 4 ([28]). *Suppose that the function $h(z) \in \mathcal{A}$ and*

$$|h'(z) - 1| < \frac{2}{\sqrt{5}} \qquad (\forall z \in \mathbb{D}).$$

Then h is starlike in $\mathbb{D}$.

Lemma 5 ([29]). *Assume that $h \in \mathcal{A}$.*

1. *If $\left| \dfrac{zh''(z)}{h'(z)} \right| < \dfrac{1}{2}$, then $h \in$ UCV.*
2. *If $\left| \dfrac{zh'(z)}{h(z)} - 1 \right| < \dfrac{1}{2}$, then $h \in \mathcal{S}_p$.*

Let $\mathcal{M}$ be the class of all analytic functions in $\mathbb{D}$ and $\mu < 1$. In [30], the following classes are introduced:

$$\mathcal{M}_\eta(\mu) = \left\{ p : p \in \mathcal{M}, p(0) = 1, \Re\left\{ e^{i\eta}(p(z) - \mu) \right\} > 0, z \in \mathbb{D}, \eta \in \mathbb{R} \right\}$$

and

$$\mathcal{L}_\eta(\mu) = \left\{ p : p \in \mathcal{M}, \Re\left\{ e^{i\eta}(p'(z) - \mu) \right\} > 0, z \in \mathbb{D}, \eta \in \mathbb{R} \right\}.$$

For $\eta = 0$, we obtain the classes of the analytic function $\mathcal{M}_o(\mu)$ and $L_o(\mu)$, respectively. The following lemmas are required to prove the main results in Section 5.

Lemma 6 ([31]). *$\mathcal{M}_o(\rho) * \mathcal{M}_o(\delta) \subset \mathcal{M}_o(\mu)$, where $\mu = 1 - 2(1-\rho)(1-\delta)$ and $0 \leq \rho, \delta < 1$. The value of μ cannot be improved.*

Lemma 7 ([32]). *If $0 \leq \rho, \delta < 1$ and $\mu = 1 - 2(1-\rho)(1-\delta)$, then*

$$\mathcal{L}_o(\rho) * \mathcal{L}_o(\delta) \subset \mathcal{L}_o(\mu)$$

or, equivalent to,

$$\mathcal{M}_o(\rho) * \mathcal{M}_o(\delta) \subset \mathcal{M}_o(\mu).$$

Lemma 8 ([33]). *If the function g, convex of order λ $(0 \leq \lambda < 1)$, is not of the following type:*

$$g(z) = \begin{cases} n + r \cdot z(1 - ze^{i\zeta})^{2\lambda - 1} & \left(\lambda \neq \frac{1}{2}\right) \\ \\ n + r \cdot \log(1 - ze^{i\zeta}) & \left(\lambda = \frac{1}{2}\right) \end{cases} \tag{5}$$

for $n, r \in \mathbb{C}$ and for $\zeta \in \mathbb{R}$, then each of the following statements holds true:

(i)   *There exists $\rho = \rho(g)$, such that $h' \in \mathcal{H}^{\rho + \frac{1}{2(1-\lambda)}}$.*

(ii) *If $0 \leq \lambda < \frac{1}{2}$, then there exists $\sigma = \sigma(g) > 0$, such that $g \in \mathcal{H}^{\sigma + \frac{1}{1-2\lambda}}$.*

(iii) *If $\lambda \geq \frac{1}{2}$, then $g \in \mathcal{H}^{\infty}$.*

Now we are ready to state and prove the main results in the subsequent sections.

3. Pre-Starlikeness and Convexity

We should note that ${}_m\psi_n \begin{bmatrix} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \end{bmatrix} \notin \mathcal{A}$. We consider the following normalized form of the Fox–Wright function:

$$ {}_m\tilde{\psi}_n \begin{bmatrix} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \end{bmatrix} |z \end{bmatrix} = \frac{\Pi_{j=1}^n \Gamma(\alpha_j)}{\Pi_{i=1}^m \Gamma(a_i)} \sum_{k=0}^{\infty} \frac{\Pi_{i=1}^m \Gamma(a_i + kb_i) z^{k+1}}{\Pi_{j=1}^n \Gamma(\alpha_j + k\beta_j) k!}. $$

Theorem 3. *Assume that $a_i > \frac{1}{2}$, $b_i > \frac{1}{2}$, $\alpha_i > \frac{1}{2}$, $\beta_i > \frac{1}{2}$, and $0 \leq \mu, \rho < 1$. Define $A := \max(a_1, \ldots a_m)$, $a := \min(a_1, \ldots a_m)$, $B := \max(b_1, \ldots b_m)$, $\alpha := \max(\alpha_1, \ldots \alpha_m)$, $\zeta := \min(\alpha_1, \ldots \alpha_m)$, $\beta := \max(\beta_1, \ldots \beta_m)$, $\eta := \min(\beta_1, \ldots \beta_m)$ such that the following conditions hold:*

$$B < \eta, \quad \alpha < a \quad \text{and} \quad \gamma(\beta + 1) + \log(3 - 2\rho) + \eta \log\left(\frac{A + B}{\zeta + \eta}\right) < 0.$$

Further, if

$$\prod_{i=1}^m \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)}(e - 1) < \frac{1 - \mu}{(2 - \mu)(2 - 2\rho)} \prod_{i=1}^m \frac{\Gamma(a_i)}{\Gamma(\alpha_i)},$$

then the Fox–Wright function ${}_m\tilde{\psi}_m[.] \in \mathcal{L}[\rho, \mu]$.

Proof. Let us consider the function $p(z)$ in form of the Hadamard product, defined as

$$p(z) = \left({}_m\tilde{\psi}_m \begin{bmatrix} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m) \end{bmatrix} |.\end{bmatrix} * g_\rho \right)(z) \qquad (0 \leq \rho < 1). \tag{6}$$

To show the results stated by Theorem 3, it suffices to prove that $\Re\left(\frac{zp'(z)}{p(z)}\right) > \mu$ for $z \in \mathbb{D}$. Hence, it is enough to prove that

$$\left| \frac{zp'(z)}{p(z)} - 1 \right| = \frac{\left| [p'(z) - \frac{p(z)}{z}] \right|}{\left| \frac{p(z)}{z} \right|} < 1 - \mu.$$

From (6), we have

$$p(z) = \sum_{k=0}^{\infty} \frac{\Gamma(2 - 2\rho + k)}{\Gamma(2 - 2\rho)} \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)z^{k+1}}{\Gamma(\alpha_i + k\beta_i)(k!)^2}.$$

Now,

$$\left| p'(z) - \frac{p(z)}{z} \right| = \sum_{k=1}^{\infty} k \frac{\Gamma(2 - 2\rho + k)}{\Gamma(2 - 2\rho)} \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)z^{k}}{\Gamma(\alpha_i + k\beta_i)(k!)^2}$$

$$= \sum_{k=1}^{\infty} q_k(a_i, b_i, \alpha_i, \beta_i) \frac{z^k}{k!}, \tag{7}$$

where

$$q_k = q_k(a_i, b_i, \alpha_i, \beta_i) = \frac{\Gamma(2 - 2\rho + k)}{\Gamma(2 - 2\rho)} \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)}{\Gamma(\alpha_i + k\beta_i)\Gamma(k)}, \quad k \geq 1. \tag{8}$$

Let us define the function $l_1(s)$ as:

$$l_1(s) = \frac{\Gamma(2 - 2\rho + s)}{\Gamma(2 - 2\rho)} \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + sb_i)}{\Gamma(\alpha_i + s\beta_i)\Gamma(s)}, \quad s \geq 1. \tag{9}$$

Using the logarithmic derivative on (9), we have

$$l_1'(s) = l_1(s)l_2(s),$$

where

$$l_2(s) = \Psi(2 - 2\rho + s) + \sum_{i=1}^{m} b_i \Psi(a_i + sb_i) - \beta_i \Psi(\alpha_i + s\beta_i) - \Psi(s).$$

Using Lemma 1, we have

$$l_2(s) \leq l_3(s) = \log(2 - 2\rho + s) + \sum_{i=1}^{m} b_i \log(a_i + sb_i) - \beta_i \log(\alpha_i + s\beta_i) - \log(s) + \beta_i \gamma + \gamma$$

which leads to

$$l_3'(s) = \frac{1}{(2 - 2\rho + s)} + \sum_{i=1}^{m} \frac{b_i^2}{a_i + sb_i} - \frac{\beta_i^2}{\alpha_i + s\beta_i} - \frac{1}{s}$$

$$= \frac{1}{(2 - 2\rho + s)} - \frac{1}{s} + \sum_{i=1}^{m} \frac{b_i^2(\alpha_i + s\beta_i) - \beta_i^2(a_i + sb_i)}{(\alpha_i + s\beta_i)(a_i + sb_i)}$$

$$= \frac{1}{(2 - 2\rho + s)} - \frac{1}{s} + \sum_{i=1}^{m} \frac{b_i^2\alpha_i - \beta_i^2 a_i + s(b_i^2\beta_i - \beta_i^2 b_i)}{(\alpha_i + s\beta_i)(a_i + sb_i)}$$

$$\leq \frac{1}{(2 - 2\rho + s)} - \frac{1}{s} + \sum_{i=1}^{m} \frac{B^2\alpha - \eta^2 a + sb_i\beta_i(B - \eta)}{(\alpha_i + s\beta_i)(a_i + sb_i)}$$

$$< \frac{1}{(2 - 2\rho + s)} - \frac{1}{s} + \sum_{i=1}^{m} \frac{\eta^2(\alpha - a) + sb_i\beta_i(B - \eta)}{(\alpha_i + s\beta_i)(a_i + sb_i)}.$$

This shows that $l_3(s)$ is decreasing on $[1, \infty)$ under the given hypothesis. It can also be verified that $l_3(1) < 0$ and $l_2(s) < 0$, for $s \geq 1$. Consequently, $l_1'(s) < 0$ on $[1, \infty)$. Clearly, $(q_k)_{k \geq 1}$ is decreasing. With the help of (7), we have

$$\left| p'(z) - \frac{p(z)}{z} \right| < \sum_{k=1}^{\infty} \frac{q_k(a_i, b_i, \alpha_i, \beta_i)}{k!}$$

$$\leq \sum_{k=1}^{\infty} \frac{q_1(a_i, b_i, \alpha_i, \beta_i)}{k!}$$

$$= q_1(a_i, b_i, \alpha_i, \beta_i)(e - 1). \tag{10}$$

A simple computation leads to

$$\frac{p(z)}{z} > 1 - \sum_{k=1}^{\infty} \frac{r_k(a_i, b_i, \alpha_i, \beta_i)}{k!}, \quad z \in \mathbb{D}, \tag{11}$$

where the sequence $(r_k)_{k \geq 1}$ is given by

$$r_k = r_k(a_i, b_i, \alpha_i, \beta_i) = \frac{\Gamma(2 - 2\rho + k)}{\Gamma(2 - 2\rho)} \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)}{\Gamma(\alpha_i + k\beta_i)\Gamma(k+1)}, \quad k \geq 1.$$

Similarly, it can be proven that (r_k) is decreasing. Now, using (11), we obtain

$$\frac{p(z)}{z} > 1 - r_1(a_i, b_i, \alpha_i, \beta_i)(e - 1). \tag{12}$$

Combining (7) and (11), we have

$$\left| \frac{p'(z) - \frac{p(z)}{z}}{\frac{p(z)}{z}} \right| \leq \frac{q_1(a_i, b_i, \alpha_i, \beta_i)(e - 1)}{1 - r_1(a_i, b_i, \alpha_i, \beta_i)(e - 1)}$$

$$= \frac{(2 - 2\rho) \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)} (e - 1)}{1 - (2 - 2\rho) \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)} (e - 1)} < 1 - \mu, \quad z \in \mathbb{D},$$

which is equivalent to the given condition, i.e.,

$$\prod_{i=1}^{m} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)} (e - 1) < \frac{1 - \mu}{(2 - \mu)(2 - 2\rho)} \prod_{i=1}^{m} \frac{\Gamma(a_i)}{\Gamma(\alpha_i)}.$$

Hence, the theorem is proved. □

Example 1. $_2\tilde{\psi}_2 \left[\begin{array}{c} (1, 0.55), (1, 0.55) \\ (0.55, 3), (0.55, 3) \end{array} \Big| . \right] \in \mathcal{L}[\rho, \mu].$

Example 2. $_1\tilde{\psi}_1 \left[\begin{array}{c} (1, 1) \\ (0.55, 5) \end{array} \Big| . \right] \in \mathcal{L}[\rho, \mu].$

Remark 1. *Setting $\mu = \rho$ in Theorem 3, it can easily be shown that the Fox–Wright function $_m\tilde{\psi}_m[.] \in \mathcal{L}[\mu]$ for $0 \leq \mu < 1$, i.e., the Fox–Wright function $_m\tilde{\psi}_m[.]$ is pre-starlike of the order $\mu \, \forall \, z \in \mathbb{D}$.*

Remark 2. *Putting $\mu = \rho = 0$ in Theorem 3, we have*

$$_m\tilde{\psi}_m[.] * g_o \in \mathcal{S}^*,$$

which is equivalent to $z\left(_m\tilde{\psi}_m[.]\right)'$, belonging to class $\mathcal{S}^*$, which yields $_m\tilde{\psi}_m[.] \in \mathcal{C}$, i.e., $_m\tilde{\psi}_m[.]$ is a convex function.

Remark 3. *It can be verified from Theorem 3 that* $_1\psi_1\left[\begin{array}{c}(1,1)\\(\alpha,\beta)\end{array}\Big|z\right] = \mathbb{E}_{\alpha,\beta}(z)$ *belongs to class* $\mathcal{L}[\rho,\mu]$ *for* $\alpha \in [0.555, 1)$. *In ([3] Theorem 3.1), it is established that* $\mathbb{E}_{\alpha,\beta} \in \mathcal{L}[\rho,\mu]$ *for* $\alpha \geq 1$. *Hence, Theorem 3 improves the existing results in [3].*

Theorem 4. *Suppose that* $a_i, b_i, \alpha_i, \beta_i > \frac{1}{2}$ *and* $0 \leq \delta < 1$. *Define* $A := \max(a_1, \dots a_m)$, $a := \min(a_1, \dots a_m)$, $B := \max(b_1, \dots b_m)$, $\alpha := \max(\alpha_1, \dots \alpha_m)$, $\zeta := \min(\alpha_1, \dots \alpha_m)$, $\beta := \max(\beta_1, \dots \beta_m)$ *and* $\eta := \min(\beta_1, \dots \beta_m)$ *such that* $B < \eta$, $\alpha < a$, $\gamma\beta + \frac{3}{2} + \eta \log \frac{A+B}{\zeta+\eta} < 0$ *and satisfy the following inequality:*

$$\prod_{i=1}^{m} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)}(e - 1) < \frac{1 - \delta}{2(2 - \delta)} \prod_{i=1}^{m} \frac{\Gamma(a_i)}{\Gamma(\alpha_i)}.$$

Then the Fox–Wright function $_m\tilde{\psi}_m[.]$ *is convex of order* δ *in* $\mathbb{D}$.

Proof. Let

$$g(z) = \sum_{k=0}^{\infty} \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)z^{k+1}}{\Gamma(\alpha_i + k\beta_i)k!}.$$

To show the desired results, we have to prove that $h(z) = zg'(z)$ is a starlike function of order δ in $\mathbb{D}$. For this, it is enough to prove that $\Re\left(\frac{zh'(z)}{h(z)}\right) > \delta$, i.e.,

$$\left|\frac{h'(z) - \frac{h(z)}{z}}{\frac{h(z)}{z}}\right| < 1 - \delta.$$

We have

$$\left|h'(z) - \frac{h(z)}{z}\right| = \left|\sum_{k=0}^{\infty}\left((k+1)^2 \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)z^k}{\Gamma(\alpha_i + k\beta_i)k!} - (k+1) \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)z^k}{\Gamma(\alpha_i + k\beta_i)k!}\right)\right|$$

$$= \sum_{k=1}^{\infty} c_k(a_i, b_i, \alpha_i, \beta_i)\frac{z^k}{k!}, \tag{13}$$

where

$$c_k = c_k(a_i, b_i, \alpha_i, \beta_i) = (k^2 + k) \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)}{\Gamma(\alpha_i + k\beta_i)}, \quad k \geq 1. \tag{14}$$

Let us consider the function $h_1(s)$ defined as:

$$h_1(s) = (s^2 + s) \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + sb_i)}{\Gamma(\alpha_i + s\beta_i)}, \quad s \geq 1.$$

Applying the logarithmic derivative on both sides, we have

$$h_1'(s) = h_1(s)h_2(s),$$

where

$$h_2(s) = \frac{2s + 1}{s^2 + s} + \sum_{i=1}^{m} b_i\Psi(a_i + sb_i) - \beta_i\Psi(\alpha_i + s\beta_i).$$

Using Lemma 1, we have

$$h_2(s) \leq h_3(s) = \frac{2s+1}{s^2+s} + \sum_{i=1}^{m}(b_i \log(a_i + sb_i) - (\beta_i \log(\alpha_i + s\beta_i) - \beta_i \gamma)).$$

Therefore,

$$
\begin{aligned}
h_3'(s) &= \frac{-1 - 2s^2 - 2s}{(s^2+s)^2} + \sum_{i=1}^{m} \frac{b_i^2}{a_i + sb_i} - \frac{\beta_i^2}{\alpha_i + s\beta_i} \\
&= \frac{-1 - 2s^2 - 2s}{(s^2+s)^2} + \sum_{i=1}^{m} \frac{b_i^2(\alpha_i + s\beta_i) - \beta_i^2(a_i + sb_i)}{(\alpha_i + s\beta_i)(a_i + sb_i)} \\
&= \frac{-1 - 2s^2 - 2s}{(s^2+s)^2} + \sum_{i=1}^{m} \frac{b_i^2\alpha_i - \beta_i^2 a_i + s(b_i^2\beta_i - \beta_i^2 b_i)}{(\alpha_i + s\beta_i)(a_i + sb_i)} \\
&\leq \frac{-1 - 2s^2 - 2s}{(s^2+s)^2} + \sum_{i=1}^{m} \frac{B^2\alpha - \eta^2 a + sb_i\beta_i(B - \eta)}{(\alpha_i + s\beta_i)(a_i + sb_i)} \\
&< \frac{-1 - 2s^2 - 2s}{(s^2+s)^2} + \sum_{i=1}^{m} \frac{\eta^2(\alpha - a) + sb_i\beta_i(B - \eta)}{(\alpha_i + s\beta_i)(a_i + sb_i)}.
\end{aligned}
$$

This implies that h_3 is the decreasing function on $[1, \infty)$ under the given hypothesis. It can also be observed that $h_3(1) < 0$ and $h_2(s) < 0$, for $s \geq 1$. Consequently, $h_1'(s) < 0$ on $[1, \infty)$. Thus, $(c_k)_{k \geq 1}$ is a decreasing sequence. With the help of (13), we have

$$
\begin{aligned}
\left| h'(z) - \frac{h(z)}{z} \right| &< \sum_{k=1}^{\infty} \frac{c_k(a_i, b_i, \alpha_i, \beta_i)}{k!} \\
&\leq \sum_{k=1}^{\infty} \frac{c_1(a_i, b_i, \alpha_i, \beta_i)}{k!} \\
&= c_1(a_i, b_i, \alpha_i, \beta_i)(e - 1), \quad z \in \mathbb{D}.
\end{aligned}
\tag{15}
$$

By a simple computation, we have

$$\frac{h(z)}{z} > 1 - \sum_{k=1}^{\infty} \frac{d_k(a_i, b_i, \alpha_i, \beta_i)}{k!}, \quad z \in \mathbb{D}. \tag{16}$$

where $(d_k)_{k \geq 1}$ is given by

$$d_k = d_k(a_i, b_i, \alpha_i, \beta_i) = (k+1)\prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)}{\Gamma(\alpha_i + k\beta_i)}, \quad k \geq 1.$$

In view of sequence (c_k), it can be proven that (d_k) is decreasing. Now, using (16), we obtain

$$\frac{h(z)}{z} > 1 - d_1(a_i, b_i, \alpha_i, \beta_i)(e - 1). \tag{17}$$

Combining (15) and (17), we have

$$
\begin{aligned}
\left| \frac{h'(z) - \frac{h(z)}{z}}{\frac{h(z)}{z}} \right| &\leq \frac{c_1(a_i, b_i, \alpha_i, \beta_i)(e - 1)}{1 - d_1(a_i, b_i, \alpha_i, \beta_i)(e - 1)} \\
&= \frac{2\prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)}(e - 1)}{1 - 2\prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)}(e - 1)} < 1 - \delta, \quad z \in \mathbb{D},
\end{aligned}
$$

which is equivalent to the given condition, i.e.,

$$\prod_{i=1}^{m} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)}(e-1) < \frac{1-\delta}{2(2-\delta)} \prod_{i=1}^{m} \frac{\Gamma(a_i)}{\Gamma(\alpha_i)}.$$

Thus, proof of the theorem is completed. □

Remark 4. *Setting* $m = n = a_1 = b_1 = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$ *in Theorem 4 we observed that*
$${}_1\psi_1 \left[\begin{array}{c} (1,1) \\ (\alpha, \beta) \end{array} \Big| z \right] = \mathbb{E}_{\alpha,\beta}^{(1)}(z)$$ *is convex of order* δ *for* $\alpha \in [0.555, 1)$, *but in ([34] corollary 4),*
discussed $\mathbb{E}_{\alpha,\beta}^{(1)}(z)$ *is convex of order* δ *for* $\alpha \geq 1$. *Hence, our Theorem 4 is better than the existing*
results in [34]. This also leads to a generalized form of several results available in [10,34].

4. *k*-Uniformly Starlike and *k*-Uniformly Convex Functions

In this section, we consider a linear operator associated with the Fox–Wright function for which *k*-ST and *k*-UCV are discussed.

Let us define a linear operator using the Hadamard product involving the normalized Fox–Wright function as:

$$I_{m,n} : \mathcal{A} \longrightarrow \mathcal{A} \text{ such that}$$

$$\left[I_{m,n}(g) \right](z) = \left({}_m\tilde{\psi}_n \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \end{array} \Big| . \right] * g \right)(z) \qquad (g \in \mathcal{A}).$$

Or

$$I_{m,n} \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \end{array} \Big| . \right] g(z) = \left({}_m\tilde{\psi}_n \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \end{array} \Big| . \right] * g \right)(z).$$

It can be noted that $\left[I_{m,n}(g) \right](z)$ is a natural extension of the Alexander and Libra operators, denoted by $\mathcal{A}$ and $\mathcal{L}$, respectively, which was first introduced in [35]. It can be defined in terms of the Fox–Wright function as follows:

$$\mathcal{A}(g) = \left({}_2\tilde{\psi}_1 \left[\begin{array}{c} (1,1), (1,1) \\ (2,1) \end{array} \Big| . \right] * g \right)(z)$$

and

$$\mathcal{L}(g) = \left({}_2\tilde{\psi}_1 \left[\begin{array}{c} (1,1), (2,1) \\ (3,1) \end{array} \Big| . \right] * g \right)(z).$$

These operators are very useful in fractional calculus. Applications of fractional derivatives involving the Alexander integral operator were discussed in [36]. In [22], relations between *k*-UCV, *k*-ST, and $\mathcal{R}_\eta(\zeta)$ were studied for the above-mentioned similar type of linear operator associated with the hypergeometric function.

Now, we establish some important theorems associated with the classes *k*-ST and *k*-UCV for the linear operator $\left[I_{m,n}(g) \right](z)$.

Theorem 5. *Let the assertion of Theorem 4 hold and* $l \in [0, \infty)$ *be such that* $g(z) \in \mathcal{R}_\eta(\zeta)$ *and*

$$2(1-\zeta)\cos\eta \prod_{i=1}^{m} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)}(e-1) \leq \prod_{i=1}^{m} \frac{\Gamma(a_i)}{\Gamma(\alpha_i)} \frac{1}{l+2}. \tag{18}$$

Then $\left[I_{m,m}(g) \right] \in k$-*UCV.*

Proof. To prove this theorem, it is enough to prove that

$$\sum_{n=2}^{\infty} n(n-1) \left| \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + nb_i - b_i)}{\Gamma(\alpha_i + n\beta_i - \beta_i)(n-1)!} b_n \right| \leq \frac{1}{l+2}.$$

Since $g \in \mathcal{R}_\eta(\zeta)$; therefore,

$$|b_n| \leq \frac{2(1-\zeta)\cos\eta}{n}.$$

Now,

$$\sum_{n=2}^{\infty} n(n-1)\left|\prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + nb_i - b_i)}{\Gamma(\alpha_i + n\beta_i - \beta_i)(n-1)!} b_n\right|$$

$$\leq \sum_{n=2}^{\infty} n(n-1)\left|\prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + nb_i - b_i)2(1-\zeta)\cos\eta}{\Gamma(\alpha_i + n\beta_i - \beta_i)n(n-1)!}\right|$$

$$= 2(1-\zeta)\cos\eta \sum_{n=1}^{\infty} \frac{b_n}{(n)!}, \tag{19}$$

where

$$b_n = n\prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + nb_i)}{\Gamma(\alpha_i + n\beta_i)}, \quad n \geq 1. \tag{20}$$

Let us consider the function $h_1(s)$ defined as:

$$h_1(s) = s\prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + sb_i)}{\Gamma(\alpha_i + s\beta_i)}, \quad s \geq 1,$$

which yields

$$h_1'(s) = h_1(s)h_2(s),$$

where

$$h_2(s) = \frac{1}{s} + \sum_{i=1}^{m} b_i\Psi(a_i + sb_i) - \beta_i\Psi(\alpha_i + s\beta_i).$$

Using Lemma 1, we have

$$h_2(s) \leq h_3(s) = \frac{1}{s} + \sum_{i=1}^{m} b_i\log(a_i + sb_i) - \beta_i\log(\alpha_i + s\beta_i) + \beta_i\gamma.$$

Which leads to

$$h_3'(s) = \frac{-1}{s^2} + \sum_{i=1}^{m} \frac{b_i^2}{a_i + sb_i} - \frac{\beta_i^2}{\alpha_i + s\beta_i}.$$

It can be easily shown that $h_3(s)$ is a decreasing function on $[1,\infty)$ under the given hypothesis. It can also noted that $h_3(1) < 0$ and further $h_2(s) < 0$, for $s \geq 1$. Consequently, $h_1'(s) < 0$ on $[1,\infty)$. Thus, $(b_n)_{n\geq 1}$ is a decreasing sequence. Therefore, by (19), we obtain

$$2(1-\zeta)\cos\eta \sum_{n=1}^{\infty} \frac{b_n}{(n)!} \leq 2(1-\zeta)\cos\eta \sum_{n=1}^{\infty} \frac{b_1}{(n)!}$$

$$= 2(1-\zeta)\cos\eta \prod_{i=1}^{m} \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)}(e-1).$$

In view of condition (18), proof of this theorem is completed. $\quad\square$

Theorem 6. *Let the given supposition of Theorem 4 hold with $l \in [0, \infty)$ and $f(z) \in \mathcal{R}_\eta(\zeta)$. Moreover, assume that the following inequality holds:*

$$2(1-\zeta)\cos\eta(e-1)\prod_{i=1}^{m}\frac{\Gamma(a_i+b_i)}{\Gamma(\alpha_i+\beta_i)} < \frac{1}{l+2}\prod_{i=1}^{m}\frac{\Gamma(a_i)}{\Gamma(\alpha_i)}.$$

Then $\left[I_{m,m}(f)\right] \in k$-ST.

Proof. To find the required result, we have to show that

$$\sum_{n=2}^{\infty}[n+l(n-1)]\left|\prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+nb_i-b_i)}{\Gamma(\alpha_i+n\beta_i-\beta_i)(n-1)!}b_n\right| < 1. \tag{21}$$

Since $g \in \mathcal{R}_\eta(\zeta)$, then

$$|b_n| \leq \frac{2(1-\zeta)\cos\eta}{n}.$$

A simple computation leads to

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n+l(n-1)]\left|\prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+nb_i-b_i)}{\Gamma(\alpha_i+n\beta_i-\beta_i)(n-1)!}b_n\right| \\
&\leq \sum_{n=2}^{\infty}[n+l(n-1)]\left|\prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+nb_i-b_i)}{\Gamma(\alpha_i+n\beta_i-\beta_i)}\frac{2(1-\zeta)\cos\eta}{n!}\right| \\
&= 2(1-\zeta)\cos\eta\sum_{n=1}^{\infty}[n+1+nl]\prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+nb_i)}{\Gamma(\alpha_i+n\beta_i)(n+1)!} \\
&< 2(1-\zeta)\cos\eta(1+l)\sum_{n=1}^{\infty}\frac{c_n}{n!}, \tag{22}
\end{aligned}$$

where

$$c_n = \prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+nb_i)}{\Gamma(\alpha_i+n\beta_i)}, \quad n \geq 1. \tag{23}$$

Using a similar technique proof of Theorem 4 and applying the assumption of this theorem, it can be observed that c_n is a decreasing sequence. Given (22), we obtain

$$\begin{aligned}
2(1-\zeta)\cos\eta(1+l)\sum_{n=1}^{\infty}\frac{c_n}{n!} &\leq 2(1-\zeta)\cos\eta(1+l)\sum_{n=1}^{\infty}\frac{c_1}{n!} \\
&= 2(1-\zeta)\cos\eta(1+l)\prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+b_i)}{\Gamma(\alpha_i+\beta_i)}(e-1).
\end{aligned}$$

Finally, using the given hypothesis, the desired result can be established. $\square$

5. Hardy Space of the Fox–Wright Function

In this section, we will study the inclusion properties of the Fox–Wright function in the Hardy space.

Theorem 7. *Under the same supposition and statement of Theorem 4, the following relation holds:*

$$
{}_m\psi_m \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m) \end{array} \Big| . \right] \in \begin{cases} \mathcal{H}^{\frac{1}{1-2\lambda}} & \left(0 \leq \lambda < \frac{1}{2} \right) \\[2mm] \mathcal{H}^\infty & \left(\lambda \geq \frac{1}{2} \right). \end{cases}
\tag{24}
$$

Proof. Using the definition of the hypergeometric function ${}_2\phi_1(p, q; r; z)$ [13], we have

$$
n + \frac{r \cdot z}{(1 - ze^{i\xi})^{1-2\lambda}} = n + r \cdot z\, {}_2\phi_1\left(1, 1 - 2\lambda; 1; ze^{i\xi}\right) \qquad \left(\lambda \neq \frac{1}{2} \right)
$$

and

$$
n + r \cdot \log(1 - ze^{i\xi}) = n + r \cdot z\, {}_2\phi_1\left(1, 1; 2; ze^{i\xi}\right) \qquad \left(\lambda = \frac{1}{2} \right).
$$

It can be easily seen that the normalized Fox–Wright function ${}_m\tilde{\psi}_m \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n) \end{array} \Big| z \right]$ is not of the given types:

$$
n + r \cdot z(1 - ze^{i\xi})^{2\lambda - 1} \qquad \left(\lambda \neq \frac{1}{2} \right)
$$

and

$$
n + r \cdot \log(1 - ze^{i\xi}) \qquad \left(\lambda = \frac{1}{2} \right).
$$

Hence, by applying Theorem 4, we observe that ${}_m\tilde{\psi}_m \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m) \end{array} \Big| z \right]$ is convex of order λ in $\mathbb{D}$. Finally, with the help of Lemma 8, the required result would readily follow. $\square$

Remark 5. *It can be noted from Theorem 7 that* ${}_1\psi_1 \left[\begin{array}{c} (1, 1) \\ (\alpha, \beta) \end{array} \Big| z \right] = \mathbb{E}_{\alpha, \beta}^{(1)}(z)$ *belongs to the Hardy space for* $\alpha \in [0.555, 1)$, *but in ([34] Theorem 7), the authors note that* $\mathbb{E}_{\alpha, \beta}^{(1)}(z)$ *is in the Hardy space for* $\alpha \geq 1$. *Therefore, our Theorem 3 improves the existing result in [3] and in the generalized form.*

Theorem 8. *Let the assertion of Theorem 4 hold along with the condition*

$$
\prod_{i=1}^m \frac{\Gamma(a_i + b_i)}{\Gamma(\alpha_i + \beta_i)} (e - 1) \leq (1 - \mu) \prod_{i=1}^m \frac{\Gamma(a_i)}{\Gamma(\alpha_i)}.
\tag{25}
$$

Then $\frac{1}{z}{}_m\tilde{\psi}_m \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m) \end{array} \Big| z \right] \in \mathcal{L}(\mu)$.

Proof. To prove this theorem, it is enough to show that $|q(z) - 1| < 1$, where

$$
q(z) = \frac{1}{1 - \mu} \left(\frac{1}{z}{}_m\tilde{\psi}_m \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m) \end{array} \Big| z \right] - \mu \right).
$$

Now by using Lemma 1, we have

$$
\left| \frac{1}{1 - \mu} \left(\frac{1}{z}{}_m\tilde{\psi}_m \left[\begin{array}{c} (a_1, b_1), \ldots, (a_m, b_m) \\ (\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m) \end{array} \Big| z \right] - \mu \right) - 1 \right| = \frac{1}{1 - \mu} \sum_{k=1}^\infty \prod_{i=1}^m \frac{\Gamma(\alpha_i)}{\Gamma(a_i)} \frac{\Gamma(a_i + kb_i)z^k}{\Gamma(\alpha_i + k\beta_i)k!}.
$$

By applying the similar way proof of Theorem 4, with the assumption of this Theorem, we have

$$\frac{1}{1-\mu}\sum_{k=1}^{\infty}\prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+kb_i)z^k}{\Gamma(\alpha_i+k\beta_i)k!} < \frac{1}{1-\mu}\prod_{i=1}^{m}\frac{\Gamma(\alpha_i)}{\Gamma(a_i)}\frac{\Gamma(a_i+b_i)}{\Gamma(\alpha_i+\beta_i)}(e-1).$$

Hence, followed by the given condition, the proof of this theorem is complete. □

Theorem 9. *Suppose that the same assumptions of Theorem 4 hold and the following inequality is satisfied:*

$$\prod_{i=1}^{m}\frac{\Gamma(a_i+b_i)}{\Gamma(\alpha_i+\beta_i)}(e-1) \le (1-\rho)\prod_{i=1}^{m}\frac{\Gamma(a_i)}{\Gamma(\alpha_i)}. \tag{26}$$

If $g \in \mathcal{L}(\delta)(\delta < 1)$, then $\left({}_m\tilde{\psi}_m\left[\begin{array}{c}(a_1,b_1),\ldots,(a_m,b_m)\\(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)\end{array}\Big| \cdot\right] * g\right)(z) \in \mathcal{L}(\mu)$, where $\mu = 1 - 2(1-\rho)(1-\delta)$.

Proof. If $g \in \mathcal{L}(\delta)$ $(\delta < 1)$, then by using Lemma 7 $f' \in \mathcal{M}(\delta)$, suppose that

$$u(z) = {}_m\tilde{\psi}_m\left[\begin{array}{c}(a_1,b_1),\ldots,(a_m,b_m)\\(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)\end{array}\Big| z\right] * g(z).$$

We have

$$u'(z) = \frac{1}{z}\,{}_m\tilde{\psi}_m\left[\begin{array}{c}(a_1,b_1),\ldots,(a_m,b_m)\\(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)\end{array}\Big| z\right] * g'(z).$$

Now, with the help of Theorem 8, it can be easily observed that the normalized Fox–Wright function is:

$$\frac{1}{z}\,{}_m\tilde{\psi}_m\left[\begin{array}{c}(a_1,b_1),\ldots,(a_m,b_m)\\(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)\end{array}\Big| z\right] \in \mathcal{M}(\rho),$$

using the given conditions. Hence, by using Lemma 7, it can be noted that $u' \in \mathcal{M}(\rho)$, and $u \in \mathcal{L}(\mu)$. Thus, the proof of Theorem 9 would follow readily. □

6. Conclusions

In our present article, we investigated geometric properties, such as convexity of order δ and pre-starlikeness for the Fox–Wright function ${}_m\tilde{\psi}_m[.]$. It can be observed from Remark 3 that some of the results obtained in this manuscript improved and generalized certain results established in [34]. Moreover, sufficient conditions were derived, such as k-uniformly starlike and k-uniformly convex associated with this function. Many other conditions were also provided for the Fox–Wright function belonging to the Hardy space. From Remark 4 and Remark 5, we can observe that the results derived in this paper improved and generalized several results available in the literature [3,34]. Interesting examples and consequences were provided to support the desired results obtained in this investigation. Further research directions on the subjects of the present considerations were discussed analogously, i.e., for the Fox–Wright type functions [37], hypergeometric function [13], and Srivastava's unification [38] $\mathcal{E}_{\alpha,\beta}(\varphi;z,s,\kappa)$ of the Mittag–Leffler type functions.

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