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Representations, Translations and Reductions for Ternary Semihypergroups

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Abstract: The concept of ternary semihypergroups can be considered as a natural generalization of arbitrary ternary semigroups. In fact, each ternary semigroup can be constructed to a ternary semihypergroup. In this article, we investigate some interesting algebraic properties of ternary semihypergroups induced by semihypergroups. Then, we extend the well-known result on group theory and semigroup theory, the so-called Cayley's theorem, to study on ternary semihypergroups. This leads us to construct the ternary semihypergroups of all multivalued full binary functions. In particular, we investigate that each element of a ternary semihypergroup induced by a semihypergroup can be represented by a multivalued full binary function. Moreover, we introduce the concept of translations for ternary semihypergroups which can be considered as a generalization of translations on ternary semigrgoups. Then, we construct ternary semihypergroups of all multivalued full functions and ternary semihypergroups via translations. So, some interesting algebraic properties are investigated. At the last section, we discover that there are ternary semihypergroups satisfying some significant conditions which can be reduced to semihypergroups. Furthermore, ternary semihypergroups with another one condition can be reduced to idempotent semihypergroups.

Keywords: ternary semihypergroups; semihypergroups; multivalued full binary functions; Cayley's theorem; idempotent semihypergroups; translations

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1. Introduction

The concept of ternary semigroups was initiated by Banach, S. (cf. [1]) who gave an example of a ternary semigroup which does not necessarily be reduced to a (binary) semigroup. In fact, every ternary semigroup can be constructed as a semigroup. As a result, there are many mathematicians who study the algebraic properties and applications of ternary semigroups in primary and advance ways. In 1932, Lehmer, D.H. [2] introduced the definition and theory of ternary semigroups, and also investigated some of its algebraic properties. Previously, in 1904, algebraic ternary structures were studied by Kanser, E. [3] who presented the notion of n-ary algebras, where n is a fixed natural number. In 1963, Sioson, F.M. [4] introduced the concept of regular ternary semigroups and characterized the structures by ideals. Then, Santiago, M.L. [5] investigated some interesting properties of regular ternary semigroups. Moreover, Santiago, M.L. and Sri Bala, S. [6] characterized regular ternary semigroups via idempotent pairs. In 1997, Dixit, V.N. and Dewan, S. [7] studied equivalence relations on ternary semigroups. In particular, they gave some significant conditions on ternary semigroups which implies that ternary semigroups can be reduced to semigroups.

Based on the ternary algebraic structures and its generalization, which were called n-ary structures, there are some applications in Physics. In 1937, Nambu, Y. [8] introduced the idea of the novel dynamical system based on the canonical triplex, which was called

Nambu mechanics (cf. [9]). Furthermore, the theory of functional equations and the stability of functional equations on ternary (n-ary) algebraic structures have been investigated by many researchers, see [10–13]. For more applications related to Physics, see [9,14–16].

A ternary semigroup $(T, \cdot)$ is a pair of a nonempty set T under one ternary operation $\cdot : T^3 \longrightarrow T$ which satisfies the ternary associative law as follows: for each $a, b, c, d, e \in T$

$$\cdot(\cdot(a, b, c), d, e) = \cdot(a, \cdot(b, c, d), e) = \cdot(a, b, \cdot(c, d, e)).$$

Moreover, ternary semigroups can be considered as a special case of the so-called n-ary semigroups, which are regarded as generalizations of semigroups and ternary semigroups, where $n = 2$ and $n = 3$, respectively. Nowadays, there is another one generalization of ternary semigroups, the so-called ternary Menger algebras of rank n, which was firstly established by the authors in 2021, see [17]. In fact, an algebraic structure of ternary Menger algebras of rank n can immediately be reduced to an algebraic structure of ternary semigroups when setting the fixed natural number n is equal to 1. For more information of n-ary semigroups and ternary Menger algebras, see [18,19].

The concept of hyperstructures was firstly introduced by Marty, F. [20] in 1934, when he defined hypergroups based on the notion of hyperoperations. The hyperstructures can be regarded as a generalization of classical algebraic structures, i.e., based on classical structures, the composition of two elements (or n elements) is an element; while in algebraic hyperstructures, the composition of two elements (or n elements) is a set. Since then, there are many mathematicians who have studied algebraic properties on this topic in various aspects. Moreover, there are several books which were written on this topic. A book on hyperstructures [21] have pointed out on some applications of hyperstructures in graphs, hypergraphs cryptography, fuzzy set theory, rough set theory, codes, automata, geometry and lattices. For another related book, see [22,23]. The hyperstucture of ternary semihypergroups is a suitable generalization of the structure of ternary semigroups which is different from n-ary semigroups and Menger algebras of rank n. As a result, many authors investigated algebraic properties of semihypergroups in different aspects. In 2010, Davvaz, B. and Leoreanu-Fotea, V. [24] studied binary relations on ternary semihypergroups. Up to 2014, Hila, K., Davvaz, B. and Naka, K. [25] extended the concept of ideals to study on ternary semihypergroups. For more related articles, see [25–27].

Let G be a nonempty set. The set of all permutations on G ($\sigma : G \longrightarrow G$ is bijective) was denoted by $Sym(G)$. It is well-known that the set $Sym(G)$ with the composition $\circ$ of functions forms a group. The group $(Sym(G), \circ)$ is called a *symmetric group*. Every subgroup of $(Sym(G), \circ)$ is called a *permutation group*. Based on this structure, the well-known result *Cayley's theorem for groups*, which was introduced by Cayley, A., is raised. The theorem stated that every abstract group can be embedded into some permutation groups of symmetric groups on some sets. For each element x of a group $(G, *)$, a function L_x was defined by $L_x(g) = *(x, g)$ for all $g \in G$. So, the set $G' = \{L_x \mid x \in G\}$ under the composition of functions is a permutation group on G. Hence, there is an isomophism from $(G, *)$ onto $(G', \circ)$. It means that every element x of a group $(G, *)$ can be represented by L_x.

Similar to group theory, the *Cayley's theorem for semigroups* was also investigated. The analogous set of all permutations on G is the set $\mathcal{T}(G)$ of all full transformations on G. The structure $(\mathcal{T}(G), \circ)$ under the composition $\circ$ of functions forms a semigroup, which is called a *semigroup of all full transformations (or a transformation semigroup)*. In fact, every semigroup is isomorphic to a transformation semigroup.

In recent year, the authors investigated the *Cayley's theorem for ternary semigroups induced by semigroups*, see [28]. For every ternary semigroup $(T, \cdot)$ induced by a semigroup $(T, \bullet)$, we can define a function $\alpha : T^1 \times T^1 \longrightarrow T^1$ called a *full binary transformation*, where T^1 is a ternary semigroup with adjoin an identity element. We conclude that every element x of a ternary semigroup $(T, \cdot)$ can be represented by a full binary transformation λ_x, where $\lambda_x(a, b) = \star(x, a, b)$ for all $a, b \in T$ and $\star$ is a ternary operation on a ternary semihypergroup $(T^1, \star)$.

Motivated by the previously results and the recent studies, this article is constructed as follows: First, we provide some significant notions on ternary semihypergroups in Section 2.

Second, some interesting algebraic properties of ternary semihypergroups induced by semihypergroups are investigated in Section 3. In particular, we obtain that a scalar identity element can be added into ternary semihypergroups which is induced by a semihypergroup. We continue in Section 4 by introducing a new concept of ternary semihypergroups of all multivalued full binary transformations. As a result, the important and the well-known theorem called *Cayley's theorem for ternary semihypergroups induced by semihypergroups* is investigated. In Section 5, we start with a construction of ternary semihypergroups of all multivalued full functions. Then, we define the concept of left (resp. right, lateral) translations on ternary semihypergroups, and investigate some of its algebraic properties. We complete the section by showing that ternary semihypergroups can be constructed via left (resp. right) translations. In Section 6, we prove that ternary semihypergroups satisfying some significant conditions can be reduced to semihypergroups. Finally, the summary of this article and the interesting questions to do research in the future are provided in Section 7.

2. Basic Notions and Concepts

In order to get the main results of this article, we start by recalling some significant definitions and results in the theory of ternary semihypergroups.

Let T be a nonempty set. A mapping $\circ : T \times T \times T \longrightarrow P^*(T)$, where $P^*(T)$ denotes the family of all nonempty subset of T, is called a *ternary hyperoperation* on T. The algebraic hyperstructure $(T, \circ)$ is called a *ternary hypergroupoid*. For any nonempty subsets X, Y and Z of T, we define

$$\circ(X, Y, Z) = \bigcup_{x \in X,\ y \in Y,\ z \in Z} \circ(x, y, z).$$

A ternary hypergroupoid $(T, \circ)$ is said to be a *ternary semihypergroup* if the ternary hyperoperation $\circ$ satisfies the *ternary associative law*, i.e., for every $a, b, c, d, e \in T$

$$\circ(\circ(a, b, c), d, e) = \circ(a, \circ(b, c, d), e) = \circ(a, b, \circ(c, d, e)).$$

The above assertion means that

$$\bigcup_{x \in \circ(a,b,c)} \circ(x, d, e) = \bigcup_{y \in \circ(b,c,d)} \circ(a, y, e) = \bigcup_{z \in \circ(c,d,e)} \circ(a, b, z).$$

For each nonempty subset S of T, a hyperstructure $(S, \circ)$ is called a *ternary subsemihypergroup* of $(T, \circ)$, if $\circ(S, S, S) \subseteq S$, it means that

$$\bigcup_{x,y,z \in S} \circ(x, y, z) \subseteq S.$$

A ternary semihypergroup $(T, \circ)$ is said to be *commutative* if for each $x_1, x_2, x_3 \in T$,

$$\circ(x_1, x_2, x_3) = \circ(x_{\mu(1)}, x_{\mu(2)}, x_{\mu(3)})$$

for every permutation μ of $\{1, 2, 3\}$.

There is an important relationship between semihypergroups and ternary semihypergroups which is shown as follows.

Remark 1. *Let $(T, \bullet)$ be a semihypergroup. Define a ternary hyperoperation $\circ$ on T by*

$$\circ\,(x, y, z) = \bullet(\bullet(x, y), z) \quad \textit{for all } x, y, \in T. \tag{1}$$

By the associativity of the binary hyperoperation $\bullet$ on T, the ternary hyperoperation $\circ$ on T satisfies the ternary associative law. Then $(T, \circ)$ forms a ternary semihypergroup. Furthermore, it is obvious that $(T, \circ)$ and $(T, \bullet)$ are isomorphic via an identity mapping.

The main point of Remark 1 is to show that every semihypergroup can be constructed to a ternary semihypergroup. However, there are some ternary semihypergroups which do not necessarily be reduced to semihypergroups. Moreover, the ternary semihypergroup $(T, \circ)$ under the ternary hyperoperation $\circ$ defined as in (1) is called a *ternary semihypergroup* $(T, \circ)$ *induced by a semihypergroup* $(T, \bullet)$.

Next, the following result plays an important role in the studying the algebraic properties in this article.

Remark 2 ([29]). *Let $(T, \bullet)$ be a semihypergroup. Generally, $(T, \bullet)$ does not necessarily contain a scalar identity element e (i.e., $\bullet(x, e) = \{x\} = \bullet(e, x)$ for all $x \in T$). Then, we can add the element $e \notin T$ into the semihypergroup $(T, \bullet)$. Thus, the hyperstructure $(T \cup \{e\}, \diamond)$ forms a semihypergroup, i.e., a binary hyperoperation $\diamond$ on T is given by*

$$\begin{aligned} \diamond(x, y) &= \bullet(x, y) \quad \text{for all } x, y \in T, \\ \diamond(x, e) &= \{x\} = \diamond(e, x) \quad \text{for all } x \in T, \\ \diamond(e, e) &= \{e\}. \end{aligned}$$

Based on the algebraic hyperstructure of ternary semihypergroups, the following remarkable fact is given to show that the concept of ternary semihypergroups can be considered as a generalization of ternary semigroups.

Remark 3. *Every ternary semigroup $(T, \cdot)$ can be constructed to a ternary semihypergroup $(T, \circ)$ where $\circ$ is defined by*

$$\circ(x, y, z) = \{\cdot(x, y, z)\} \quad \text{for all } x, y, z \in T.$$

The ternary semihypergroup $(T, \circ)$, which is defined as in Remark 3, is called a *trivial ternary semihypergroup*. Analogous to the semihypergroup theory, some special elements on ternary semihypergroups are defined.

Definition 1. *Let $(T, \circ)$ be a ternary semihypergroup. An element $e \in T$ is said to be*

(i) a left scalar identity element if $\{x\} = \circ(e, e, x)$ for all $x \in T$,
(ii) a right scalar identity element if $\{x\} = \circ(x, e, e)$ for all $x \in T$,
(iii) a lateral scalar identity element if $\{x\} = \circ(e, x, e)$ for all $x \in T$,
(iv) a scalar identity element if $\{x\} = \circ(e, e, x) = \circ(x, e, e) = \circ(e, x, e)$ for all $x \in T$.

Definition 2. *Let $(T, \circ)$ be a ternary semihypergroup. An element $0 \in T$ is called*

(i) a left scalar zero element if $\{0\} = \circ(0, x, y)$ for all $x, y \in T$,
(ii) a right scalar zero element if $\{0\} = \circ(x, y, 0)$ for all $x, y \in T$,
(iii) a lateral scalar zero element if $\{0\} = \circ(x, 0, y)$ for all $x, y \in T$,
(iv) a scalar zero element if $\{0\} = \circ(0, x, y) = \circ(x, y, 0) = \circ(x, 0, y)$ for all $x, y \in T$.

Furthermore, a ternary semihypergroup such that each element is a (resp. left, right, lateral) scalar zero element is called a *(resp. left, right, lateral) scalar zero ternary semihypergroup*. Here are some examples of ternary semihypergroups.

Example 1. *(i) Let $\mathbb{Z}$ be the set of all integers. Define a ternary hyoperation $\circ$ on $\mathbb{Z}$ by*

$$\circ(x, y, z) = \{x \times y \times z\} \quad \text{for all } x, y, z \in \mathbb{Z}.$$

where $\times$ is the usual multiplication on $\mathbb{Z}$. Then, $(\mathbb{Z}, \circ)$ is a ternary semihypergroup with scalar identity elements 1 and -1. Moreover, $0 \in \mathbb{Z}$ is a scalar zero element.

(ii) Consider the set $\mathbb{R}$ of all real numbers with respect to a ternary hyperoperation $\circ$ which is defined as follows

$$\circ(x, y, z) = \{m \in \mathbb{R} \mid m \geq \max\{x, y, z\}\} \quad \text{for all } x, y, z \in \mathbb{R}.$$

Therefore, $(\mathbb{R}, \circ)$ is a ternary semihypergroup.

(iii) *Let T be the unit interval $[0, 1]$. For each x, y, z, we define*

$$\circ(x, y, z) = \left[0, \frac{x \times y \times z}{3}\right] \quad \textit{for all } x, y, z \in T.$$

where $\times$ is the usual multiplication on $\mathbb{R}$. Hence, $(T, \circ)$ forms a ternary semihypergroup.

3. Some Results on Ternary Semihypergroups Induced by Semihypergroups

In this section, we investigate some interesting algebraic properties of ternary semihypergroups which are induced by semihypergroups. Moreover, we give a characterization that a scalar identity element can be added into a ternary semihypergroup if and only if the ternary semihypergroup is induced by a semihypergroup.

Lemma 1. *Let $(T, \circ)$ be a ternary semihypergroup containing a scalar identity element. Then $(T, \circ)$ can be induced by a semihypergroup.*

Proof. Assume that $(T, \circ)$ is a ternary semihypergroup with a scalar identity element e. Define a binary hyperoperation $\bullet$ on T by

$$\bullet\,(x, y) = \circ(x, e, y) \quad \text{for all } x, y \in T. \tag{2}$$

Indeed, for each $x, y, z \in T$, we have

$$\begin{aligned}
\bullet(\bullet(x, y), z) &= \circ(\circ(x, e, y), e, z) \\
&= \circ(x, e, \circ(y, e, z)) \\
&= \bullet(x, \bullet(y, z)).
\end{aligned}$$

Hence, the binary hyperoperation $\bullet$ is associative. Therefore, the hyperstructure $(T, \bullet)$ forms a binary semihypergroup.

Next, suppose that $(T, \circ')$ is a ternary semihypergroup which is induced by the semihypergroup $(T, \bullet)$ where the ternary hyperoperation $\circ'$ is defined by

$$\circ'(x, y, z) = \bullet(\bullet(x, y), z) \quad \text{for all } x, y, z \in T.$$

By using the definition of the binary hyperoperation $\bullet$ on T define as in (2), we have

$$\begin{aligned}
\circ'(x, y, z) &= \bullet(\bullet(x, y), z) \\
&= \bullet(\circ(x, e, y), z) \\
&= \bigcup_{a \in \circ(x, e, y)} \bullet(a, z) \\
&= \bigcup_{a \in \circ(x, e, y)} \circ(a, e, z) \\
&= \circ(\circ(x, e, y), e, z) \\
&= \circ(x, \circ(e, y, e), z) \\
&= \circ(x, \{y\}, z) \\
&= \circ(x, y, z),
\end{aligned}$$

for all $x, y, z \in T$.

Therefore, the ternary semihypergroups $(T, \circ')$ and $(T, \circ)$ are the same. It means that the ternary semihypergroup $(T, \circ)$ is induced from the semihypergroup $(T, \bullet)$. $\square$

In Section 4, a scalar identity element plays important role to construct a ternary hyperoperation of a ternary semihypergroup of all multivalued full binary functions. So, the following property need to be shown.

Proposition 1. *A scalar identity element can be added into each ternary semihypergroup which is induced by a semihypergroup.*

Proof. Assume that $(T, \circ)$ is a ternary semihypergroup which is induced by a semihypergroup $(T, \bullet)$, i.e., a ternary hyperoperation $\circ$ on T is defined by

$$\circ(x, y, z) = \bullet(\bullet(x, y), z) \quad \text{for all } x, y, z \in T.$$

Now, let $e \notin T$. By using Remark 2, we can add the element $e \notin T$ into the semihypergroup $(T, \bullet)$. Then, the hyperstructure $(T \cup \{e\}, \diamond)$ forms a semihypergroup respect to a binary hyperoperation $\diamond$ on T which is defined as the same way in Remark 2. It is clear that the element e is a scalar identity element in $(T \cup \{e\}, \diamond)$.

Next, we obtain that a hyperstructure $(T \cup \{e\}, \star)$ forms a ternary semihypergroup together with a ternary hyperoperation $\star$ defined by

$$\star (x, y, z) = \diamond(\diamond(x, y), z) \quad \text{for all } x, y, z \in T \cup \{e\}. \tag{3}$$

Indeed, for every $x \in T$, we have

$$\star(x, e, e) = \diamond(\diamond(x, e), e) = \diamond(x, e) = \{x\}$$

Analogously, we can prove that $\star(e, x, e) = \{x\} = \star(e, e, x)$. It follows that the element $e \notin T$ is a scalar identity element in $(T \cup \{e\}, \star)$.

Furthermore, for each $x, y, z \in T$, we get

$$\circ(x, y, z) = \bullet(\bullet(x, y), z) = \diamond(\diamond(x, y), z) = \star(x, y, z).$$

It means that $\circ(x, y, z) = \star(x, y, z)$ for each $x, y, z \in T$. Therefore, the element $e \notin T$ can be added into $(T, \circ)$ as a scalar identity element. In particular, the element e is a scalar identity element in $(T \cup \{e\}, \star)$. $\square$

4. Representations for Ternary Semihypergroups Induced by Semihypergroups

In this section, we investigate the well-known theorem on semigroup theory, the so-called Cayley's theorem. We extend the theorem to study on ternary semihypergroups. So, the main aim of this section is to establish the representation of ternary semihypergroups which are induced by semihypergroups. In order to get the result, we introduce the concept of ternary semihypergroups of all multivalued full binary functions, and then we prove that ternary semihypergroups induced by semihypergroups can be represented by multivalued full binary functions.

Firstly, throughout in this section, we assume that $(T, \circ)$ is a ternary semihypergroup induced by a semihypergroup $(T, \bullet)$, i.e., a ternary hyperoperation $\circ$ is given by

$$\circ(x, y, z) = \bullet(\bullet(x, y), z) \quad \text{for all } x, y, z \in T.$$

By Remark 2, a scalar identity element $e \notin T$ can be added into the ternary semihypergroup $(T, \circ)$. First, we get a semihypergroup $(T \cup \{e\}, \diamond)$ with a scalar identity element e. Second, we obtain that $(T \cup \{e\}, \star)$ forms a ternary semihypergroup induced by the semihypergroup $(T \cup \{e\}, \diamond)$ together with a scalar identity element e, where a ternary hyperoperation $\star$ is defined by

$$\star (x, y, z) = \diamond(\diamond(x, y), z) \quad \text{for all } x, y, z \in T \cup \{e\}. \tag{4}$$

Furthermore, we also have $\circ(x, y, z) = \star(x, y, z)$ for all $x, y, z \in T$.

So, for convenience, for every ternary semihypergroup $(T, \circ)$ induced by a semihypergroup $(T, \bullet)$, we denote

$$T^e = \begin{cases} T & \text{if } T \text{ has scalar identity element,} \\ T \cup \{e\} & \text{if } T \text{ has no scalar identity elements.} \end{cases}$$

Consequently, if the ternary semihypergroup $(T, \circ)$ has a scalar identity element, then ternary semihypergroups $(T, \circ)$ and $(T^e, \star)$ are the same. In addition, $(T, \circ)$ is a ternary subsemihypergroup of $(T^e, \star)$.

Now, let $T^e \times T^e$ be the Cartesian product of the base set of the ternary semihypergroup $(T^e, \star)$. On the set $\mathcal{T}(T^e \times T^e, P^*(T^e))$ of all *multivalued full binary functions* (or *binary hyperoperations*) $\alpha : T^e \times T^e \longrightarrow P^*(T^e)$, we define a ternary operation $\circ^e : \mathcal{T}(T^e \times T^e, P^*(T^e))^3 \longrightarrow \mathcal{T}(T^e \times T^e, P^*(T^e))$ as follows

$$\circ^e(\alpha, \beta, \gamma)(x, y) = \alpha(e, \beta(e, \gamma(x, y))), \tag{5}$$

for all $\alpha, \beta, \gamma \in \mathcal{T}(T^e \times T^e, P^*(T^e)), x, y \in T^e$. In particular, the right-hand side of the definition of the ternary operation $\circ^e$ is meant that

$$\alpha(e, \beta(e, \gamma(x, y))) = \bigcup_{z \in \beta(e, \gamma(x,y))} \alpha(e, z) \quad \left(= \bigcup_{q \in \beta(e,p),\ p \in \gamma(x,y)} \alpha(e, q) \right).$$

So, we call the ternary operation $\circ^e$ defined as in (5) , a *composition of multivalued full functions via a scalar identity element e*. As a result, the primary result is investigated as follows.

Lemma 2. *Let $(T, \circ)$ be a ternary semihypergroup induced by a semihypergroup. Then the composition $\circ^e$ of multivalued full functions via a scalar identity element e is ternary associative, i.e., if $\alpha_i \in \mathcal{T}(T^e \times T^e, P^*(T^e)), i = 1, 2, \ldots, 5$ and $x, y \in T^e$, then*

$$\circ^e(\circ^e(\alpha_1, \alpha_2, \alpha_3), \alpha_4, \alpha_5) = \circ^e(\alpha_1, \circ^e(\alpha_2, \alpha_3, \alpha_4), \alpha_5) = \circ^e(\alpha_1, \alpha_2, \circ^e(\alpha_3, \alpha_4, \alpha_5)).$$

Proof. It is evident that $\circ^e$ is a ternary operation on $\mathcal{T}(T^e \times T^e, P^*(T^e))$. Now, let $\alpha_i \in \mathcal{T}(T^e \times T^e, P^*(T^e)), i = 1, 2, \ldots, 5$ and $x, y \in T^e$.

Consider,

$$\begin{aligned}
\circ^e(\circ^e(\alpha_1, \alpha_2, \alpha_3), \alpha_4, \alpha_5)(x, y) &= \circ^e(\alpha_1, \alpha_2, \alpha_3)(e, \alpha_4(e, \alpha_5(x, y))) \\
&= \bigcup_{p \in \alpha_4(e, \alpha_5(x,y))} \circ^e(\alpha_1, \alpha_2, \alpha_3)(e, p) \\
&= \bigcup_{p \in \alpha_4(e, \alpha_5(x,y))} \alpha_1(e, \alpha_2(e, \alpha_3(e, p))) \\
&= \bigcup_{q \in \alpha_2(e, \alpha_3(e,p)),\ p \in \alpha_4(e, \alpha_5(x,y))} \alpha_1(e, q) \\
&= \bigcup_{q \in \alpha_2(e, \alpha_3(e, \alpha_4(e, \alpha_5(x,y))))} \alpha_1(e, q) \\
&= \bigcup_{q \in \circ^e(\alpha_2, \alpha_3, \alpha_4)(e, \alpha_5(x,y))} \alpha_1(e, q) \\
&= \alpha_1(e, \circ^e(\alpha_2, \alpha_3, \alpha_4)(e, \alpha_5(x, y))) \\
&= \circ^e(\alpha_1, \circ^e(\alpha_2, \alpha_3, \alpha_4), \alpha_5)(x, y).
\end{aligned}$$

On another hand, we have

$$\begin{aligned}
\circ^e(\circ^e(\alpha_1,\alpha_2,\alpha_3),\alpha_4,\alpha_5)(x,y) &= \circ^e(\alpha_1,\alpha_2,\alpha_3)(e,\alpha_4(e,\alpha_5(x,y))) \\
&= \bigcup_{p\in\alpha_4(e,\alpha_5(x,y))} \circ^e(\alpha_1,\alpha_2,\alpha_3)(e,p) \\
&= \bigcup_{p\in\alpha_4(e,\alpha_5(x,y))} \alpha_1(e,\alpha_2(e,\alpha_3(e,p))) \\
&= \bigcup_{q\in\alpha_2(e,\alpha_3(e,\alpha_4(e,\alpha_5(x,y))))} \alpha_1(e,q) \\
&= \bigcup_{q\in\alpha_2(e,\circ^e(\alpha_3,\alpha_4,\alpha_5)(x,y))} \alpha_1(e,q) \\
&= \alpha_1(e,\alpha_2(e,\circ^e(\alpha_3,\alpha_4,\alpha_5)(x,y))) \\
&= \circ^e(\alpha_1,\alpha_2,\circ^e(\alpha_3,\alpha_4,\alpha_5))(x,y).
\end{aligned}$$

It follows that the ternary operation $\circ^e$ is ternary associative on $\mathcal{T}(T^e\times T^e,P^*(T^e))$. □

Theorem 1. *Let $(T,\circ)$ be a ternary semihypergroup induced by a semihypergroup. Then the set $\mathcal{T}(T^e\times T^e,P^*(T^e))$ under the composition $\circ^e$ of multivalued full functions via a scalar identity element e forms a ternary semihypergroup.*

Proof. The proof follows from Lemma 2. □

The set $\mathcal{T}(T^e\times T^e,P^*(T^e))$ of all multivalued full binary functions, which is defined via a ternary semihypergroup $(T,\circ)$ induced by a semihypergroup $(T,\bullet)$, is closed with respect to the composition $\circ^e$ of multivalued full binary functions via a scalar identity element e. Consequently, the algebraic structure forms a ternary semihypergroup $(\mathcal{T}(T^e\times T^e,P^*(T^e)),\circ^e)$. We call such a ternary semihypergroup, a *ternary semihypergroup of all multivalued full binary functions*. For convenience, a *ternary semihypergroup of multivalued full binary functions* is referred to each ternary subsemihypergroup of $(\mathcal{T}(T^e\times T^e,P^*(T^e)),\circ^e)$.

Lemma 3. *Each ternary semigroup induced by a semigroup can be constructed as a ternary semihypergroup induced by a semihypergroup.*

Proof. Let $(T,\diamond)$ be a ternary semigroup induced by a semigroup $(T,\cdot)$. On the structure $(T,\cdot)$, we immediately obtain a semihypergroup $(T,\bullet)$ where a binary hyperoperation $\bullet$ is defined by

$$\bullet(x,y)=\{\cdot(x,y)\}\quad \text{for all } x,y\in T.$$

By Remark 3, on the structure $(T,\diamond)$, we obtain a ternary semihypergroup $(T,\circ)$ under a ternary hyperoperation $\circ$ defined as follows

$$\circ(x,y,z)=\{\diamond(x,y,z)\}\quad \text{for all } x,y,z\in T.$$

Indeed, for every $x,y,z\in T$, we have

$$\begin{aligned}
\circ(x,y,z) &= \{\diamond(x,y,z)\} \\
&= \{\cdot(\cdot(x,y),z)\} \\
&= \bigcup_{a\in\{\cdot(x,y)\}} \cdot(a,z) \\
&= \bigcup_{a\in\bullet(x,y)} \bullet(a,z) \\
&= \bullet(\bullet(x,y),z).
\end{aligned}$$

It means that the ternary semihypergroup $(T, \circ)$ is induced by the semihypergroup $(T, \bullet)$. $\square$

Now, let $(T, \diamond)$ be a ternary semigroup induced by a semigroup $(T, \cdot)$. By using the well-know result on semigroups, an identity element $e \notin T$ can be added into the semigroup $(T, \cdot)$. Then, we obtain that $(T^e = T \cup \{e\}, *)$ forms a semigroup with an identity element e under a binary operation $*$ is given by

$$* (x, y) = \cdot(x, y) \text{ for all } x, y \in T, *(x, e) = x = *(e, x) \text{ for all } x \in T \text{ and } * (e, e) = e. \quad (6)$$

Furthermore, we also obtain that $(T^e, \circledast)$, where $\circledast$ is defined by $\circledast(x, y, z) = *(*(x, y), z)$ for all $x, y, z \in T^e$, forms a ternary semigroup with an identity element e.

Next, in order to get the next main result, we firstly introduce another one ternary semihypergroup. Let $(T, \diamond)$ be a ternary semigroup induced by a semigroup $(T, \cdot)$. The set $\mathcal{T}(T^e \times T^e, T^e)$ of all full binary transformations (or binary operations), $\alpha : T^e \times T^e \longrightarrow T^e$, is a ternary semigroup under a ternary composition $\diamond^e$ via an identity element e defined by

$$\diamond^e(\alpha, \beta, \gamma)(x, y) = \alpha(e, \beta(e, \gamma(x, y))) \quad \text{for all } \alpha, \beta, \gamma \in \mathcal{T}(T^e \times T^e, T^e), x, y \in T^e$$

For more information of the ternary semigroup $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e)$ of all full binary transformations, see [28].

Then, the following theorem is given to show that the ternary semihypergroup $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e)$ can be considered as a contraction of a ternary semihypergroup $((\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$ of all mutivalued full binary functions in sense of injective homomorphisms.

Theorem 2. *Let $(T, \diamond)$ be a ternary semigroup induced by a semigroup $(T, \cdot)$. A ternary semigroup $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e)$ can be embedded into a ternary semihypergroup $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$.*

Proof. Firstly, let $(T, \diamond)$ be a ternary semigroup induced by a semigroup $(T, \cdot)$. By the proof of Lemma 3, we obtain that $(T, \circ)$ forms a ternary semihypergroup induced by a semihypergroup $(T, \bullet)$ with a scalar identity element e.

Based on the algebraic structures of $(T, \diamond)$ and $(T, \circ)$, we obtain that a ternary semigroup $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e))$ of full binary transformations and a ternary semihypergroup $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$ of multivalued full functions exist.

Secondly, we assume that $\alpha : T^e \times T^e \longrightarrow T^e$ be a full binary transformation, i.e., $\alpha \in \mathcal{T}(T^e \times T^e, T^e)$. Let $\bar{\alpha} : T^e \times T^e \longrightarrow P^*(T^e)$ be a multivalued full function which is defined by

$$\bar{\alpha}(x, y) = \{\alpha(x, y)\} \quad \text{for all } x, y \in T^e,$$

where α is a full binary transformation on T^e.

Finally, we define a mapping $\pi : \mathcal{T}(T^e \times T^e, T^e) \longrightarrow \mathcal{T}(T^e \times T^e, P^*(T^e))$ by

$$\pi(\alpha) = \bar{\alpha} \quad \text{for all } \alpha \in \mathcal{T}(T^e \times T^e, T^e).$$

Indeed, for each $\alpha, \beta, \gamma \in \mathcal{T}(T^e \times T^e, T^e)$ and $x, y \in T^e$, we have

$$\begin{aligned}
\pi(\diamond^e(\alpha,\beta,\gamma))(x,y) &= \overline{\diamond^e(\alpha,\beta,\gamma)}(x,y) \\
&= \{\diamond^e(\alpha,\beta,\gamma)(x,y)\} \\
&= \{\alpha(e,\beta(e,\gamma(x,y)))\} \\
&= \bar{\alpha}(e,\beta(e,\gamma(x,y))) \\
&= \bigcup_{p\in\{\beta(e,q)\},\ q\in\{\gamma(x,y)\}} \bar{\alpha}(e,p) \\
&= \bigcup_{p\in\bar{\beta}(e,q),\ q\in\bar{\gamma}(x,y)} \bar{\alpha}(e,p) \\
&= \bar{\alpha}(e,\bar{\beta}(e,\bar{\gamma}(x,y))) \\
&= \circ^e(\bar{\alpha},\bar{\beta},\bar{\gamma})(x,y) \\
&= \circ^e(\pi(\alpha),\pi(\beta),\pi(\gamma))(x,y),
\end{aligned}$$

which implies that π is a homomorphism. Let $\alpha, \beta \in \mathcal{T}(T^e \times T^e, T^e)$ and $x, y \in T^e$ be such that $\bar{\alpha}(x,y) = \bar{\beta}(x,y)$. Then $\{\alpha(x,y)\} = \{\beta(x,y)\}$, and hence $\bar{\alpha} = \bar{\beta}$. Therefore, π is an injective homomorphim.

Consequently, the ternary semigroup $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e))$ can be embedded into the ternary semihypergroup $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$ via the injective homomorphim π. □

To illustrate the previous theorem, the following example is presented.

Example 2. *Let $(T = \{a,b,0\}, \diamond)$ be a ternary semihypergroup induced by a semigroup $(T, \cdot)$ where a binary operation $\cdot$ is given as follows.*

$\cdot$	a	b	0
a	a	a	0
b	b	b	0
0	0	0	0

Obviously, the semigroup $(T, \cdot)$ has no an identity element. So, we can extend the semigroup $(T, \cdot)$ to a semigroup $(T^e = \{a,b,0,e\}, *)$ with an identity element e under a binary operation $*$ is defined as in (6). Then, we obtain that $(T^e, \circledast)$ forms a ternary semigroup induced by the semigroup $(T^e, *)$ with an identity element e where a ternary operation $\circledast$ is defined by $\circledast(x,y,z) = *(*(x,y),z)$ for all $x,y,z \in T^e$. Hence, a ternary semigroup $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e)$ exists.

On the semigroup $(T^e, *)$, we can construct a semihypergroup $(T^e, \bullet)$ with a scalar identity element e under a binary hyperoperation $\bullet$ is defined by $\bullet(x,y) = \{*(x,y)\}$ for all $x,y \in T^e$. By Remark 1, we get that the algebraic hyperstructure $(T^e, \circ)$ forms a ternary semihypergroup with a scalar identity element e which is induced by the semihypergroup $(T^e, \bullet)$ where $\circ(x,y,z) = \bullet(\bullet(x,y),z)$ for all $x,y,z \in T^e$. Thus, a ternary semihypergroup $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$ exists.

Now, we will show that $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e)$ can be embedded into $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$. Define $\alpha : T^e \times T^e \longrightarrow T^e$ by $\alpha(x,y) = 0$ for all $x,y \in T^e$, where 0 is a fixed element of T^e. Next, we define $\bar{\alpha} : T^e \times T^e \longrightarrow P^*(T^e)$ by $\bar{\alpha} = \{\alpha(x,y)\} = \{0\}$ for all $x,y \in T^e$. By Theorem 2, there is an injective homomorphism π from $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e))$ into $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$ defined by $\pi(\beta) = \bar{\beta}$ for all $\beta \in (\mathcal{T}(T^e \times T^e, T^e), \diamond^e))$. Hence, $\bar{\alpha}$ is an element of $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$ corresponding to α in $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e))$. Therefore, $(\mathcal{T}(T^e \times T^e, T^e), \diamond^e)$ is embedded into $(\mathcal{T}(T^e \times T^e, P^*(T^e), \circ^e)$.

Next, by using the algebraic hyperstructure of the ternary semihypergroups of all multivalued full binary functions, we can extend the Cayley's Theorem of semigroup theory to study on ternary semihypergroups which are induced by semihypergroups.

Let $(T, \circ)$ be a ternary semihypergroup induced by a semihypergroup $(T, \bullet)$. By the proof of Proposition 1, a ternary semihypergroup $(T^e, \star)$ exists where $\star$ is defined on T^e as in (3). For each $a \in T^e$, we define a mapping $\lambda_a : T^e \times T^e \longrightarrow P^*(T^e)$ as follows

$$\lambda_a(x, y) = \star(a, x, y) \quad \text{for all } x, y \in T^e. \tag{7}$$

Moreover, for every nonempty set A of T^e, $\lambda_A : T^e \times T^e \longrightarrow P^*(T^e)$ is defined by

$$\lambda_A(x, y) = \star(A, x, y) \quad \text{for all } x, y \in T^e,$$

which means that $\lambda_A(x, y) = \bigcup_{a \in A} \star(a, x, y)$. It is evident that $\lambda_A = \bigcup_{a \in A} \lambda_a$.

Thus, the multivaled full binary function λ_a defined as in (7) can be considered as an element of $\mathcal{T}(T^e \times T^e, P^*(T^e))$ which is called an *inner left translation* of $T^e \times T^e$ corresponding to an element a of T^e. Similar to the definition of the inner left translations, we can define others mappings which are elements of $\mathcal{T}(T^e \times T^e, P^*(T^e))$. For each $a \in T^e$, we define others mappings $\rho_a, \sigma_a : T^e \times T^e \longrightarrow P^*(T^e)$ by

$$\rho_a(x, y) = \star(x, y, a) \text{ and } \sigma_a(x, y) = \star(x, a, y) \quad \text{for all } x, y \in T^e.$$

Hence, ρ_a and σ_a are elements of $\mathcal{T}(T^e \times T^e, P^*(T^e))$. We call ρ_a (σ_a) an inner right (lateral) translation of $T^e \times T^e$ corresponding to an element a of T^e. Obviously, if ternary semihypergroup $(T^e, \star)$ is commutative, then λ_a, ρ_a and σ_a are the same for all $a \in T^e$.

Lemma 4. *Let $(T, \circ)$ be a ternary semihypergroup induced by a semihypergroup. Then*

$$\lambda_{\star(a,b,c)} = \circ^e(\lambda_a, \lambda_b, \lambda_c) \quad \textit{for all } a, b, c \in T^e.$$

Proof. By using the ternary associativity of a ternary hyperoperation $\star$ on T^e, we get

$$\begin{aligned}
\lambda_{\star(a,b,c)}(x, y) &= \star(\star(a, b, c), x, y) \\
&= \star(\star(a, \{b\}, c), x, y) \\
&= \star(\star(a, \star(e, b, e), c), x, y) \\
&= \star(a, e, \star(b, e, \star(c, x, y))) \\
&= \star(a, e, \star(b, e, \lambda_c(x, y))) \\
&= \bigcup_{p \in \lambda_c(x,y)} \star(a, e, \star(b, e, p)) \\
&= \bigcup_{q \in \lambda_b(e,p),\ p \in \lambda_c(x,y)} \lambda_a(e, q) \\
&= \lambda_a(e, \lambda_b(e, \lambda_c(x, y))) \\
&= \circ^e(\lambda_a, \lambda_b, \lambda_c(x, y)),
\end{aligned}$$

for all $a, b, c \in T^e$. □

Lemma 5. *Let $(T, \circ)$ be a ternary semihypergroup induced by a semihypergroup. Then*

$$\rho_{\star(a,b,c)} = \circ^e(\rho_c, \rho_b, \rho_a) \quad \textit{for all } a, b, c \in T^e.$$

Proof. The proof is similar to Lemma 4. □

For each ternary semihypergroup $(T, \circ)$ induced by a semihypergroup, let $T' = \{\lambda_a \mid a \in T\}$ and $T'' = \{\rho_a \mid a \in T\}$. In order to get the next main aims, we first show that the hyperstructures $(T', \circ^e)$ and $(T'', \circ^e)$ are ternary subsemihypergroups of $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$.

Lemma 6. *Let $(T, \circ)$ be a ternary semihypergroup induced by a semihypergroup. Then $(T', \circ^e)$ forms a ternary subsemihypergroup of $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$, and it is a ternary semihypergroup of multivalued full binary functions.*

Proof. By the definition of T', it is obvious that $\emptyset \neq T' \subseteq \mathcal{T}(T^e \times T^e, P^*(T^e))$. By Lemma 4, $\lambda_{\star(a,b,c)} = \circ^e(\lambda_a, \lambda_b, \lambda_c)$ for all $a, b, c \in T^e$. Thus, $(T', \circ^e)$ forms a ternary subsemihypergroup of $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$. It means that $(T', \circ^e)$ is a ternary semihypergroup of multivalued full binary functions. $\square$

Lemma 7. *Let $(T, \circ)$ be a ternary semihypergroup induced by a semihypergroup. Then $(T'', \circ^e)$ forms a ternary subsemihypergroup of $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$, and it is a ternary semihypergroup of multivalued full binary functions.*

Proof. The proof is similar to Lemma 3, and it follows from Lemma 5. $\square$

Theorem 3 (Cayley's theorem for ternary semihypergroups induced by semihypergroups)**.** *Every ternary semihypergroup induced by a semihypergroup is isomorphic to some ternary semihypergroups of multivalued full binary functions.*

Proof. Let $(T, \circ)$ be a ternary semihypergroup induced by a semihypergroup $(T, \bullet)$. We will show that the ternary semihypergroups $(T, \circ)$ and $(T', \circ^e)$, where $T' = \{\lambda_a \mid a \in T\}$, are isomorphic.

Since $(T, \bullet)$ is a semihypergroup, we obtain that $(T^e, \diamond)$ forms a semihypergroup which is constructed as in the proof of Proposition 1. Then, a hyperstructure $(T^e, \star)$ forms a ternary semihypergroup induced by a semihypergroup $(T^e, \diamond)$ with a scalar identity element e, where a ternary hyperoperation $\star$ is defined by

$$\star(x, y, z) = \diamond(\diamond(x, y), z) \quad \text{for all } x, y, z \in T^e.$$

Indeed, for each $a \in T$, we define a mapping $\pi : T \longrightarrow T'$ by

$$\pi(a) = \lambda_a \quad \text{for all } a \in T. \tag{8}$$

By the definition of the ternary hyperoperation $\star$ on T^e, we get

$$\star(x, y, z) = \diamond(\diamond(x, y), z) = \bullet(\bullet(x, y), z) = \circ(x, y, z),$$

for all $x, y, z \in T$.

By Lemma 4, we have

$$\begin{aligned}
\pi(\circ(x, y, z)) &= \bigcup_{p \in \circ(x,y,z)} \pi(p) \\
&= \bigcup_{p \in \circ(x,y,z)} \lambda_p \\
&= \lambda_{\circ(x,y,z)} \\
&= \lambda_{\star(x,y,z)} \\
&= \circ^e(\lambda_x, \lambda_y, \lambda_z) \\
&= \circ^e(\pi(x), \pi(y), \pi(z)),
\end{aligned}$$

for all $x, y, z \in T$. Therefore, π is a homomorphism.

If $\lambda_x = \lambda_y$, then $\lambda_x(a, b) = \lambda_y(a, b)$ for all $a, b \in T^e$. It follows that $\star(x, e, e) = \star(y, e, e)$, and hence $\{x\} = \{y\}$. So, $x = y$, and hence π is injective.

Finally, let $\lambda_a \in T'$. There exists an element a in T such that $\pi(a) = \lambda_a$, which implies that π is surjective. Consequently, π is an isomorphism from the ternary semihypergroup $(T, \circ)$ onto the ternary semihypergroup $(T', \circ^e)$ of multivalued full binary functions. $\square$

By Theorem 3, we notice that each element a of a ternary semihypergroup $(T, \circ)$ induced by a semihypergroup can be represented by a multivalued full binary function λ_a of a ternary semihypergroup $(T', \circ^e)$ of multivalued full binary functions. In addition, we call the representation π, which is defined as in (8), an *extended regular representation*. In particular, since the ternary semihypergroup $(T', \circ^e)$ is a ternary subsemihypergroup of the ternary semihypergroup $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$, the ternary semihypergroup $(T, \circ)$ is embedded into $(\mathcal{T}(T^e \times T^e, P^*(T^e)), \circ^e)$.

Furthermore, we can define another one mapping ϕ from $(T, \bullet)$ into $(T'', \circ^e)$ by $\phi(x) = \rho_x$ for all $x \in T$. Then the mapping ϕ is an anti-homomorphism which follows from Lemma 5, and we also obtain that ϕ is both injective and surjective. Thus $(T, \circ)$ is isomorphic with $(T'', \circ^e)$. We call the presentation ϕ, an *extended regular anti-representation*.

In order to illustrate the Cayley's theorem for ternary semihypergroups which are induced by semihypergroups, the following example must be shown.

Example 3. *Let $(T = \{a, b, c\}, \circ)$ be a ternary semihypergroup induced by a semihypergroup $(T, \bullet)$ where the ternary hyperoperation $\circ$ is defined by $\circ(x, y, z) = \bullet(\bullet(x, y), z)$ for all $x, y, z \in T$ and the binary hyperoperation $\bullet$ is given by the following table.*

$\bullet$	a	b	c
a	$\{a\}$	$\{a, b\}$	T
b	$\{a\}$	$\{a, b\}$	T
c	$\{a\}$	$\{a, b\}$	$\{c\}$

Obviously, the semihypergroup $(T, \bullet)$ does not contain a scalar identity element. So, we extend $(T, \bullet)$ to construct a semihypergroup $(T^e, \diamond)$, where $T^e = \{a, b, c, e\}$, with a scalar identity element e and the binary hyperoperation $\diamond$ is defined as the same way in Remark 2, i.e.,

$$\diamond(x, y) = \bullet(x, y) \text{ for all } x, y \in T, \diamond(x, e) = \{x\} = \diamond(e, x) \text{ for all } x \in T, \text{ and } \diamond(e, e) = \{e\}.$$

On the algebraic hyperstructure of $(T^e, \diamond)$, we can construct a ternary semihypergroup $(T^e, \star)$, which is induced by the semihypergroup $(T^e, \diamond)$, and the ternary hyperoperation $\star$ is defined by $\star(x, y, z) = \diamond(\diamond(x, y), z)$ for all $x, y, z \in T^e$.

Next, we define an inner left translation λ_x for all $x \in T$ as follows.

λ_a	a	b	c	e
a	$\{a\}$	$\{a, b\}$	T	$\{a\}$
b	$\{a\}$	$\{a, b\}$	T	$\{a, b\}$
c	$\{a\}$	$\{a, b\}$	T	T
e	$\{a\}$	$\{a, b\}$	T	$\{a\}$

λ_b	a	b	c	e
a	$\{a\}$	$\{a, b\}$	T	$\{a\}$
b	$\{a\}$	$\{a, b\}$	T	$\{a, b\}$
c	$\{a\}$	$\{a, b\}$	T	T
e	$\{a\}$	$\{a, b\}$	T	$\{b\}$

λ_c	a	b	c	e
a	$\{a\}$	$\{a, b\}$	T	$\{a\}$
b	$\{a\}$	$\{a, b\}$	T	$\{a, b\}$
c	$\{a\}$	$\{a, b\}$	T	T
e	$\{a\}$	$\{a, b\}$	$\{c\}$	$\{c\}$

By Lemma 6, the set $T' = \{\lambda_a, \lambda_b, \lambda_c\}$ under the ternary hyperoperation $\circ^e$ defined as in (5) forms a ternary semihypergroup. Finally, we define a mapping $\pi : T \longrightarrow T'$ by $a \mapsto \lambda_a, b \mapsto \lambda_b$, and $c \mapsto \lambda_c$. By Theorem 3, π is an isomorphism. Consequently, the ternary semihypergroup $(T, \circ)$ induced by the semihypergroup $(T, \bullet)$ is isomorphic to the ternary semihypergroup $(T', \circ^e)$ of multivalued full binary functions.

5. Translations for Ternary Semihypergroups

In this section, we introduce the concept of translations for ternary semihypergroups which can be considered as a generalization of translations on ternary semigrgoups. Moreover, based on the concept of the ternary semihypergroup of all multivalued full binary functions which are define in the previous section, we can construct *ternary semihypergroups of all multivalued full functions*. Then, some interesting results related the ternary semihypergroups are investigated.

Let T be a nonempty set. On the set $\mathcal{T}(T, P^*(T))$ of all *multivalued full functions* or *unary hyperoperations* $\alpha : T \longrightarrow P^*(T)$, we define a ternary hyperoperation $\oplus : \mathcal{T}(T, P^*(T))^3 \longrightarrow \mathcal{T}(T, P^*(T))$ by

$$\oplus(\alpha, \beta, \gamma)(x) = \alpha(\beta(\gamma(x))) \tag{9}$$

for all $\alpha, \beta, \gamma \in \mathcal{T}(T, P^*(T)), x \in T$. In particular, the right-hand side is meant that

$$\alpha(\beta(\gamma(x))) = \bigcup_{a\in\beta(b), b\in\gamma(x)} \alpha(a).$$

By using the definition of the ternary hyperoperation $\oplus$, the following important result is investigated.

Theorem 4. *Let T be a nonempty set. Then, the ternary hyperoperation $\oplus$ defined as in (9) is ternary associative on the set $\mathcal{T}(T, P^*(T))$, i.e.,*

$$\oplus(\oplus(\alpha, \beta, \gamma), \delta, \mu)(x) = \oplus(\alpha, \oplus(\beta, \gamma, \delta), \mu)(x) = \oplus(\alpha, \beta, \oplus(\gamma, \delta, \mu))(x),$$

for all $\alpha, \beta, \gamma, \delta, \mu \in \mathcal{T}(T, P^*(T)), x \in T$.

Proof. For each $\alpha, \beta, \gamma, \delta, \mu \in \mathcal{T}(T, P^*(T)), x \in T$, we have

$$\begin{aligned}
\oplus(\oplus(\alpha, \beta, \gamma), \delta, \mu)(x) &= \bigcup_{a\in\delta(b),\ b\in\mu(x)} \oplus(\alpha, \beta, \gamma)(a) \\
&= \bigcup_{m\in\beta(n),\ n\in\gamma(a),\ a\in\delta(b),\ b\in\mu(x)} \alpha(m) \\
&= \bigcup_{m\in\bigcup_{n\in\gamma(a),\ a\in\delta(b)} \beta(n),\ b\in\mu(x)} \alpha(m) \\
&= \bigcup_{m\in\oplus(\beta,\gamma,\delta)(b),\ b\in\mu(x)} \alpha(m) \\
&= \oplus(\alpha, \oplus(\beta, \gamma, \delta), \mu)(x).
\end{aligned}$$

Similar to the above argument, we can show that

$$\oplus(\oplus(\alpha, \beta, \gamma), \delta, \mu)(x) = \oplus(\alpha, \beta, \oplus(\gamma, \delta, \mu))(x) \quad \text{for all } x \in T.$$

Therefore, the ternary hyperoperation $\oplus$ is ternary associative on the set $\mathcal{T}(T, P^*(T))$. □

Now, we extend the previous concept into consideration on the base set of ternary semihypergroup $(T, \circ)$ and then we obtain the following result.

Corollary 1. *Let $(T, \circ)$ be a ternary semihypergroup. Then, $(\mathcal{T}(T, P^*(T)), \oplus)$ forms a ternary semihypergroup under the ternary hyperoperation $\oplus$ defined as in (9).*

Definition 3. *Let $(T, \circ)$ be a ternary semihypergroup. A multivalued full function $\lambda : T \longrightarrow P^*(T)$ is said to be*

(i) a left translation if $\lambda(\circ(x, y, z)) = \bigcup_{a\in\lambda(x)} \circ(a, y, z)$ for all $x, y, z \in T$,

(ii) a right translation if $\lambda(\circ(x, y, z)) = \bigcup_{a\in\lambda(z)} \circ(x, y, a)$ for all $x, y, z \in T$,

(iii) a lateral translation if $\lambda(\circ(x, y, z)) = \bigcup_{a\in\lambda(y)} \circ(x, a, z)$ for all $x, y, z \in T$.

Example 4. *Let $(T, \circ)$ be a ternary semihypergroup. A multivalued full function $\pi : T \longrightarrow P^*(T)$ is defined by $\pi(x) = \{x\}$ for all $x \in T$. Indeed, for each $x, y, z \in T$, we have*

$$\pi(\circ(x,y,z)) = \circ(x,y,z) = \bigcup_{a \in \{x\}} \circ(a,y,z) = \bigcup_{a \in \pi(x)} \circ(a,y,z).$$

Hence, π is a left translation. Furthermore, π is both a right translation and a lateral translation.

Proposition 2. *Let $(T, \circ)$ be a ternary semihypergroup. A left (resp. right) translation λ maps a right (resp. left) scalar zero element 0 (if it exists) to itself, i.e., $\lambda(0) = \{0\}$.*

Proof. Let λ be a left translation on T and 0 be a right scalar zero element. Hence, $\circ(x, y, 0) = \{0\}$ for all $x, y \in T$. It follows that

$$\lambda(0) = \lambda(\{0\}) = \lambda(\circ(x,y,0)) = \bigcup_{a \in \lambda(x)} \circ(a,y,0) = \{0\}.$$

Thus, $\lambda(0) = \{0\}$. Similarly, we can show the rest. □

Proposition 3. *Let $(T, \circ)$ be a ternary semihypergroup. A lateral translation π maps a right (resp. left) scalar zero element 0 (if it exists) to itself, i.e., $\pi(0) = \{0\}$.*

Proof. The proof is similar to the proof of Proposition 2. □

Theorem 5. *If $(T, \circ)$ is a ternary semihypergroup under a ternary hyperoperation $\circ$ defined by*

$$\circ(x,y,z) = \{c\} \quad \text{for all } x,y,z \in T,$$

where $c \in T$ is a fixed element, then each multivalued full function λ satisfying $\lambda(c) = \{c\}$ is a left (resp. right, lateral) translation. Furthermore, the converse is true.

Proof. Assume that the hypothesis holds. So, $\circ(x, y, z) = \{c\}$ for all $x, y, z \in T$. Now, let λ be a multivalued full function on T satisfying $\lambda(c) = \{c\}$. Indeed, for each $x, y, z \in T$, we obtain that

$$\lambda(\circ(x,y,z)) = \lambda(\{c\}) = \lambda(c) = \{c\} = \bigcup_{a \in \lambda(x)} \circ(a,y,z).$$

It means that $\lambda(\circ(x, y, z)) = \bigcup_{a \in \lambda(c)} \circ(x, y, a)$, and hence λ is a left translation on T.

Next, we will show the converse. Assume that each multivalued full function λ satisfying $\lambda(c) = \{c\}$ is a left translation. Suppose that there are $a, x, y, z \in T$ such that $a \neq c$ and $\circ(x, y, z) = \{a\}$. We choose a multivalued full function ϕ on T with the following conditions:

$$\phi(c) = \{c\},\ \phi(a) = \{c\} \text{ and } \phi(x) = \{x\}.$$

Since ϕ is a left translation, we immediately obtain that

$$\{c\} = \phi(a) = \phi(\{a\}) = \phi(\circ(x,y,z)) = \bigcup_{b \in \phi(x)} \circ(b,y,z) = \circ(x,y,z).$$

It implies that $\{c\} = \circ(x, y, z)$, which is a contradiction with $\circ(x, y, z) = \{a\} \neq \{c\}$.

Consequently, $(T, \circ)$ forms a ternary semihypergroup with respect to a ternary hyperoperation $\circ$ defined by

$$\circ(x,y,z) = \{c\} \quad \text{for all } x,y,z \in T.$$

Similar to the above argument, we can show the rest. Thus, the proof of the theorem has been completed. □

On a algebraic hyperstructure of a ternary semihypergroup $(T, \circ)$, we construct a set of all left (resp. right, lateral) translations defined on the base set T and denote by $\Lambda(T)$ (resp. $\Phi(T), \Pi(T)$), i.e.,

$$\begin{aligned}
\Lambda(T) &= \{\lambda \in \mathcal{T}(T, P^*(T)) \mid \lambda \text{ is a left translation on } T\},\\
\Phi(T) &= \{\phi \in \mathcal{T}(T, P^*(T)) \mid \lambda \text{ is a right translation on } T\},\\
\Pi(T) &= \{\pi \in \mathcal{T}(T, P^*(T)) \mid \lambda \text{ is a lateral translation on } T\}.
\end{aligned}$$

Next, we obtain the following result that show a relationship between $\Lambda(T)$ (resp. $\Phi(T), \Pi(T)$) and $\mathcal{T}(T, P^*(T))$.

Theorem 6. *Let $(T, \circ)$ be a ternary semihypergroup. Then*

(i) $\Lambda(T) = \mathcal{T}(T, P^*(T))$ *if and only if $(T, \circ)$ is a left scalar zero ternary semihypergroup.*
(ii) $\Phi(T) = \mathcal{T}(T, P^*(T))$ *if and only if $(T, \circ)$ is a right scalar zero ternary semihypergroup.*
(iii) $\Pi(T) = \mathcal{T}(T, P^*(T)))$ *if and only if $(T, \circ)$ is a lateral scalar zero ternary semihypergroup.*

Proof. (i) $(\Longrightarrow)$ Firstly, assume that $\Lambda(T) = \mathcal{T}(T, P^*(T))$. Now, we suppose that $(T, \circ)$ is not a left scalar zero ternary semihypergroup. It means that there are $x, y, z \in T$ such that $\circ(x, y, z) \neq \{x\}$.

Finally, we choose a multivalued full function $\lambda \in \mathcal{T}(T, P^*(T))$ such that $\lambda(x) = \{x\}$ and $\lambda(a) = \{x\}$ for all $a \in \circ(x, y, z)$. By the assumption, λ is also a left translation on T. Hence, we obtain that

$$\{x\} = \bigcup_{a \in \circ(x,y,z)} \lambda(a) = \lambda(\circ(x, y, z)) = \bigcup_{b \in \lambda(x)} \circ(b, y, z) = \bigcup_{b \in \{x\}} \circ(b, y, z) = \circ(x, y, z).$$

It implies that $\circ(x, y, z) = \{x\}$, which contradicts with $\circ(x, y, z) \neq \{x\}$. Consequently, $(T, \circ)$ forms a left scalar zero ternary semihypergroup.

$(\Longleftarrow)$ Assume that $(T, \circ)$ is a left scalar zero ternary semihypergroup. Hence, $\circ(x, y, z) = \{x\}$ for all $x, y, z \in T$. By the definition of $\Lambda(T)$, we immediately $\Lambda(T) \subseteq \mathcal{T}(T, P^*(T))$. So, we only show that $\mathcal{T}(T, P^*(T)) \subseteq \Lambda(T)$. Let $\lambda \in \mathcal{T}(T, P^*(T))$ and $x, y, z \in T$. Then,

$$\lambda(\circ(x, y, z)) = \lambda(\{x\}) = \lambda(x) = \bigcup_{a \in \lambda(x)} \{a\} = \bigcup_{a \in \lambda(x)} \circ(a, y, z).$$

It yields that λ is a left translation on T. Hence, $\lambda \in \Lambda(T)$, which implies that $\mathcal{T}(T, P^*(T)) \subseteq \Lambda(T)$. Therefore, $\Lambda(T) = \mathcal{T}(T, P^*(T))$.

(ii) and (iii) can be proved similar to (i). $\square$

Finally, we complete this section by introducing others ternary semihypergroups which can be constructed by some left (resp. right) translations on the base set of ternary semihypergroups.

Let $(T, \circ)$ be a ternary semihypergroup and λ be a left translation on T. Now, let T_λ be the set of all elements of T, i.e., $T_\lambda = T$. Define a ternary hyperoperation $\odot$ on T_λ by

$$\odot(x, y, z) = \circ(x, \lambda(y), \lambda(z)) \quad \text{for all } x, y, z \in T_\lambda, \tag{10}$$

In addition, the right-hand side is meant that

$$\circ(\lambda(x), \lambda(y), z), = \bigcup_{a \in \lambda(y),\ b \in \lambda(z)} \circ(x, a, b).$$

Analogously, we can define another one base set T_ϕ via a right translation ϕ on T such that $T_\phi = T$. Then, we define a ternary hyperoperation $\ominus$ on T_ϕ by

$$\ominus(x, y, z) = \circ(\phi(x), \phi(y), z) \quad \text{for all } x, y, z \in T_\phi. \tag{11}$$

In order to get others main results, the following lemmas must be shown.

Lemma 8. *Let $(T, \circ)$ be a ternary semihypergroup and let λ and ϕ be a left translation and a right translation on T, respectively. Then,*

(i) $\lambda(\circ(x, \lambda(y), \lambda(z))) = \circ(\lambda(x), \lambda(y), \lambda(z))$ *for all $x, y, z \in T$.*
(ii) $\phi(\circ(\phi(x), \phi(y), z)) = \circ(\phi(x), \phi(y), \phi(z))$ *for all $x, y, z \in T$.*

Proof. (i) Indeed, for every $x, y, z \in T$, we have

$$\begin{aligned}
\lambda(\circ(x,\lambda(y),\lambda(z))) &= \lambda\left(\bigcup_{b\in\lambda(y),\ c\in\lambda(z)} \circ(x,b,c)\right) \\
&= \bigcup_{b\in\lambda(y),\ c\in\lambda(z)} \lambda(\circ(x,b,c)) \\
&= \bigcup_{b\in\lambda(y),\ c\in\lambda(z)} \left(\bigcup_{a\in\lambda(x)} \circ(a,b,c)\right) \\
&= \bigcup_{a\in\lambda(x),\ b\in\lambda(y),\ c\in\lambda(z)} \circ(a,b,c) \\
&= \circ(\lambda(x),\lambda(y),\lambda(z)).
\end{aligned}$$

Similar to the above argument, we can show that (ii) is true. □

Theorem 7. *Let $(T, \circ)$ be a ternary semihypergroup and let λ and ϕ be a left translation and a right translation on T, respectively. Then, the algebraic hyperstructures $(T_\lambda, \odot)$ and $(T_\phi, \ominus)$ form ternary semihypergroups.*

Proof. It is clear that the set T_λ is close under a ternary hyperoperation $\odot$ which is defined as in (10).

By using Lemma 8, we obtain that

$$\begin{aligned}
\odot(\odot(a,b,c),d,e) &= \circ(\circ(a,\lambda(b),\lambda(c)),\lambda(d),\lambda(e)) \\
&= \circ(a,\circ(\lambda(b),\lambda(c),\lambda(d)),\lambda(e)) \\
&= \circ(a,\lambda(\circ(b,\lambda(c),\lambda(d))),\lambda(e)) \\
&= \odot(a,\odot(b,c,d),e),
\end{aligned}$$

for every $a, b, c, d, e \in T_\lambda$. Similarly, we also obtain that

$$\odot(\odot(a,b,c),d,e) = \odot(a,b,\odot(c,d,e)).$$

Then, $\odot$ is ternary associative, which implies that $(T_\lambda, \odot)$ forms a ternary semihypergroup.

Analogously, we can show that $(T_\phi, \ominus)$ is a ternary semihypergroup. Therefore, the proof has been completed. □

6. Reductions for Ternary Semihypergroups into Semihypergroups

Based on the well-known remark on algebraic hyperstructures of semihypergroups and ternary semihypergroups (also in semigroups and ternary semigroups), some ternary semihypergroups do not necessarily be reduced to semihypergroups. So, in this section, we have found that ternary semihypergroups satisfying sufficient and necessary conditions can be reduced to semihypergroups. Furthermore, ternary semihypergroups with a special condition can be reduced to idempotent semihypergroups.

Lemma 9. *Let $(T, \circ)$ be a ternary semihypergroup which admits a scalar idempotent element e satisfying the properties (α) and (β) which are given below. Then (β) implies (α).*

$$\begin{array}{lll}
(\alpha) & (i) & \circ(e,e,x) = \circ(e,x,e) = \circ(x,e,e) = \{x\} \text{ for all } x \in T \\
 & (ii) & \circ(e,x,y) \subseteq T \text{ for all } x,y \in T \text{ such that } x \neq y \text{ and } x,y \neq e \\
(\beta) & & \circ(x,x,e) = \circ(x,e,x) = \circ(e,x,x) = \{x\} \text{ for all } x \in T.
\end{array}$$

Proof. For each $x \in T$, we get

$$\begin{aligned}
\circ(e,e,x) &= \circ(e,e,\circ(e,x,x)) = \circ(\circ(e,e,e),a,a) = \circ(e,a,a) = \{x\}, \\
\circ(e,x,e) &= \circ(e,\circ(e,x,x),e) = \circ(\circ(e,e,x),x,e) = \circ(x,x,e) = \{x\},
\end{aligned}$$

$$\circ(x,e,e) = \circ(\circ(x,x,e),e,e) = \circ(x,x,\circ(e,e,e)) = \circ(x,x,e) = \{x\}.$$

For each $x, y \in T$, we have

$$\circ(e,x,y) = \circ(e,\circ(e,x,x),y) = \circ(\circ(e,e,x),x,y) = \circ(x,x,y) \subseteq T.$$

□

By Lemma 9, we note that, in general, (α) does not imply (β). In fact, (α) implies $\circ(x,x,e) = \circ(x,e,x) = \circ(e,x,x)$ for all $x \in T$ only, which is consequent from

$$\circ(e,x,x) = \circ(e,\circ(e,x,e),\circ(e,e,x)) = \circ(\circ(e,e,x),\circ(e,e,e),x) = \circ(x,e,x), \text{and}$$
$$\circ(x,e,x) = \circ(x,e,\circ(e,x,e)) = \circ(x,\circ(e,e,x),e) = \circ(x,x,e).$$

Thus, $\circ(x,x,e) = \circ(x,e,x) = \circ(e,x,x)$ for all $x \in T$, but it is not equal to $\{x\}$. The counterexample can be found in Example 4.16 [7] by considering that the ternary semigroup in the example is a trivial ternary semihypergroup.

Theorem 8. *Let $(T,\circ)$ be a ternary semihypergroup which admits a scalar idempotent element e satisfying the property (α) in Lemma 9. Then there exists a binary hyperoperation $\bullet$ on the same base set T such that $(T,\bullet)$ forms a semihypergroup and $\circ(x,y,z) = \bullet(\bullet(x,y),z)$ for all $x,y,z \in T$.*

Proof. Firstly, we define a binary hyperoperation $\bullet : T \times T \longrightarrow P^*(T)$ as follows:

$$\bullet\,(x,y) = \circ(e,x,y) \quad \text{for all } x,y \in T. \tag{12}$$

Since the property (ii) of (α) holds, the base set T is closed under the binary hyperoperation $\bullet$ defined as in (12).

Finally, by using the fact that $(T,\circ)$ is a ternary semihypergroup and the property (ii) of (α), we have

$$\begin{aligned}
\circ(x,y,z) &= \circ(\circ(e,e,x),y,z) \\
&= \circ(e,\circ(e,x,y),z) \\
&= \bigcup_{a\in\circ(e,x,y)} \circ(e,a,z) \\
&= \bigcup_{a\in\circ(e,x,y)} \bullet(a,z) \\
&= \bullet(\circ(e,x,y),z) \\
&= \bullet(\bullet(x,y),z),
\end{aligned}$$

for all $x,y,z \in T$.

Similarly, we also get $\circ(x,y,z) = \bullet(x,\bullet(y,z))$. It immediately implies that the binary hyperoperation $\bullet$ is associative.

Consequently, the hyperstructure $(T,\bullet)$ forms a semihypergroup. □

Theorem 9. *Let $(T,\circ)$ be a ternary semihypergroup which admits a scalar idempotent element e satisfying the property (β) in Lemma 9. Then there exists a binary hyperoperation $\bullet$ on the same base set T such that $(T,\bullet)$ forms an idempotent semihypergroup (a semihypergroup with $x \in \bullet(x,x)$ for all $x \in T$, see [30]) and $\circ(x,y,z) = \bullet(\bullet(x,y),z)$ for all $x,y,z \in T$.*

Proof. Firstly, by Lemma 9, we obtain that (β) implies (α). Then we can define a binary hyperoperation $\bullet$ on T as in (12), i.e.,

$$\bullet(x,y) = \circ(e,x,y) \quad \text{for all } x,y \in T.$$

By the same argument of Lemma 8, the binary hyperoperation $\bullet$ on T is associative, and hence $(T,\bullet)$ forms a semihypergroup.

Finally. we shall show that each element of T is idempotent. Indeed, for each $x \in T$, we have

$$\bullet(x, x) = \circ(e, x, x) = \{x\},$$

which implies from the property (β) in Lemma 9. It means that $x \in \bullet(x, x)$ for all $x \in T$, and hence x is idempotent. Therefore, $(T, \bullet)$ forms an idempotent semihypergroup. $\square$

In order to illustrate the results on this section, the following examples are prepared.

Example 5. *Let $\mathcal{I} = [0, 1]$ be a unit interval. An algebraic hyperstructure $(\mathcal{I}, \circ)$ under a ternary hyperoperation $\circ$ defined by*

$$\circ(x, y, z) = \{m \in \mathbb{R} \mid m = \max\{x, y, z\}\} \quad \text{for all } x, y, z \in \mathcal{I}$$

forms a ternary semihypergroup with an element 0 is a scalar idempotent element. Furthermore, the element 0 satisfies the properties (α) and (β) in Lemma 9. By using Theorem 9, $(\mathcal{I}, \circ)$ forms a ternary semihypergroup such that it can be reduced to an idempotent semihypergroup $(\mathcal{I}, \bullet)$ under a binary hyperoperation $\bullet$ defined as in (12).

Example 6. *Let $T = \{a, b, c, e\}$. Define a ternary hyperoperation $\circ$ on T by the following table.*

$\circ$	a	b	c	e
a,a	$\{a\}$	T	T	$\{a\}$
a,b	T	T	T	T
a,c	T	T	T	T
a,e	$\{a\}$	T	T	$\{a\}$

$\circ$	a	b	c	e
b,a	T	T	T	T
b,b	T	$\{b\}$	$\{b\}$	$\{b\}$
b,c	T	$\{b\}$	$\{b\}$	$\{b\}$
b,e	T	$\{b\}$	$\{b\}$	$\{b\}$

$\circ$	a	b	c	e
c,a	T	T	T	T
c,b	T	$\{b\}$	$\{b\}$	$\{b\}$
c,c	T	$\{b\}$	$\{c\}$	$\{c\}$
c,e	T	$\{b\}$	$\{c\}$	$\{c\}$

$\circ$	a	b	c	e
e,a	$\{a\}$	T	T	$\{a\}$
e,b	T	$\{b\}$	$\{b\}$	$\{b\}$
e,c	T	$\{b\}$	$\{c\}$	$\{c\}$
e,e	$\{a\}$	$\{b\}$	$\{c\}$	$\{e\}$

Then, $(T, \circ)$ is a ternary semihypergroup with an idempotent element e satisfying the properties (α) and (β) in Lemma 9. By using Theorem 9, $(T, \circ)$ can be reduced to an idempotent semihypergroup $(T, \bullet)$ where a binary hyperoperation $\bullet$ is defined as in (12).

7. Conclusions and Future Works

The main objectives of this article are to construct the important and well-known theorem called Cayley's theorem for ternary semihypergroups induced by semihypergroups, to define translations on ternary semihypergroups and to give some reductions of ternary semihypergroups to semihypergroups. In ordered to have all main results, we started by investigating some algebraic properties of ternary semihypergroups induced by semihypergroups. In particular, we attempted to show that a scalar identity element can be added into ternary semihypergroups which are induced by semihypergroups. Based on the previous results, we constructed a new ternary semihypergroup called a ternary semihypergroup of all multivalued full binary functions under a ternary hyperoperation defined via a scalar identity element. These lead us to get the Cayley's theorem for ternary semihypergroups induced by semihypergroups. Furthermore, the first main objective can be considered as an extension of Cayley's theorem for ternary semigroups induced by semigroups. In fact, it can be seen as a generalization of the work which was studied in [28].

Then, we introduced the concept of left (resp. right, lateral) translations on ternary semihypergroups. Hence, by using the idea of the ternary semihypergroups of all multivalued full binary functions, we constructed the ternary semihypergroups of all multivalued full functions and investigated some algebraic properties. In addition, we have shown that ternary semihypergroups are constructed by using left (resp. right, lateral) translations defined on its base sets.

Based on the well-known fact, every semihypergroup can be constructed to a ternary semihypergroup, while some ternary semihypergroups do not necessarily be reduced to semihypergroups, we discovered that there are some ternary semihypergroups with some significant and necessary conditions which can be reduced to semihypergroups.

Based on the structure of n-ary semihypergroups, which can be regarded as a natural generalization of ternary semihypergroups (including semihypergroups), we immediately have interesting research questions as follows.

(i) Can we extend Cayley's theorem on the certain results in this article to study on n-ary semihypergroups?
(ii) What are significant and necessary conditions on n-ary semihypergroups such that the hyperstructures can be reduced to $(n-1)$-ary semihypergroups?

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