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Fixed Point Results for a Family of Interpolative F-Contractions in b-Metric Spaces

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Abstract: In this paper, we introduce a new generalized concept, namely, extended interpolative Cirić–Reich–Rus-type F-contraction in b-metric space. In addition, we put forward the notion of interpolative Kannan-type F-contractions. Fixed point results for these new interpolative contraction mappings are established, and non-trivial examples involving finite and infinite sets are provided to validate the results.

Keywords: Cirić–Reich–Rus type mapping; Kannan-type mapping; fixed point; contraction; b-metric space

MSC: 47H10; 54H25; 54E50

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1. Introduction

Following the most celebrated fixed point results of Banach [1] in 1922, fixed point theory has witnessed breakthrough developments in different directions. One such direction is the replacement of the contraction condition with extended versions; another direction is the development of the metric space itself by inducing advanced properties. In the current research, we follow the first direction, i.e., the enhancement of the contraction condition.

For our work, we consider the F-contraction introduced by Wardowski [2] in 2012. Major improvements to Banach's contraction principle were accomplished by Boyd and Wong [3], Chatterjea [4], Ćirić [5], Kannan [6], and Meir and Keeler [7]. For developments in interpolative contractions, we refer to [8–11].

Bakhtin [12] and Czerwik [13] introduced the b-metric space, which non-trivially extended the class of metric spaces. Subsequently, tremendous improvement in fixed point theory in the framework of b-metric space have taken place [14–23].

In this paper, we put forward the concept of extended interpolative Cirić–Reich–Rus type (CRR-type) F-contraction and interpolative Kannan-type F-contraction in a b-metric space (bMS). These new interpolative results provide a new direction in the area of integral equations to find new solutions. We establish a number of important results while investigating this connection, and provide examples to validate our results. Finally, we present an application of the newly established results towards the solution of a particular type of integral equations.

2. Preliminaries

Several important results in the present context are listed below:

Definition 1 ([12,13])**.** *Consider a mapping* $\Upsilon : M \times M \to [0, \infty)$ *where* $M \neq \phi$ *is a set. When* Υ *satisfies the following conditions:*
(bM1) $\Upsilon(\iota_1, \iota_2) = 0$ *if and only if* $\iota_1 = \iota_2$*;*
(bM2) $\Upsilon(\iota_1, \iota_2) = \Upsilon(\iota_2, \iota_1)$ *for all* $\iota_1, \iota_2 \in X$*;*
(bM3) There exists a real number $s \geq 1$ *such that* $\Upsilon(\iota_1, \iota_3) \leq s[\Upsilon(\iota_1, \iota_2) + \Upsilon(\iota_2, \iota_3)]$ *for all*

$\iota_1, \iota_2, \iota_3 \in X$; then, Υ is known as a b-metric on M and (M, Υ) is b-metric space (bMS) having coefficient s.

Definition 2 ([24]). *Consider a sequence $\{u_n\}$ in M, where (M, Υ) is a b-metric space and $u \in M$. Then:*
(a) $\{u_n\}$ is known as a convergent sequence in (M, Υ), and converges to u if, for every $\varepsilon > 0$ $\exists$, $n_0 \in \mathbb{N}$ such that $\Upsilon(u_n, u) < \varepsilon \; \forall \; n > n_0$, which can be written as $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ as $n \to \infty$.
(b) $\{u_n\}$ is known as a Cauchy sequence in (M, Υ) if, for every $\varepsilon > 0$, $\exists \; n_0 \in \mathbb{N}$ such that $\Upsilon(u_n, u_{n+p}) < \varepsilon \; \forall \; n > n_0, p > 0$.
(c) If every Cauchy sequence in M converges to some $u \in M$, then (M, Υ) is known as a complete b-metric space.

Definition 3 ([2]). *Consider a real mapping $F : (0, \infty) \to (-\infty, +\infty)$ satisfying the following conditions:*
(F1) F follows the strictly increasing property;
(F2) For a sequence $\{t_n\}_{n\in\mathbb{N}} \subset (0, \infty)$, for every $\{t_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} t_n = 0$ iff $\lim_{n\to\infty} F(t_n) = -\infty$;
(F3) $\exists \; s \in (0, 1)$ such that $\lim_{t\to 0} t^s F(t) = 0$.
Suppose $\mathcal{F}$ is the collection of all mappings F. If (M, ρ) is a metric space, then a function $S : M \to M$ is called an F-contraction if $\exists \; \tau > 0$, $F \in \mathcal{F}$ such that $\forall \; p, q \in M$. Then, we have

$$\rho(S(p), S(q)) > 0 \Rightarrow \tau + F(\rho(S(p), S(q))) \leq F(\rho(p, q)).$$

3. Extended Interpolative *F*-Contraction

Here, we present our main results. We first introduce the definition of an extended interpolative CRR-type F-contraction, then establish a fixed point theorem.

Definition 4. *Consider the bMS (M, ξ, s) and a self-map $S : M \to M$. Then, the function S is known as an extended interpolative modified CRR-type F-contraction if, for $\tau > 0$ $\exists \; c_1, c_2 \in [0, 1)$ with $c_1 + c_2 < 1$ and $F \in \mathcal{F}$ such that*

$$\tau + F(\xi(S(\varsigma), S(q))) \leq c_1 F(\xi(\varsigma, q)) + c_2 F(\xi(\varsigma, S(\varsigma))) + (1 - c_1 - c_2) F(\frac{1}{s}\xi(q, S(q))) \quad (1)$$

$\forall \; \varsigma, q \in M \setminus Fix(S)$, where $Fix(S) = \{\varsigma \in M : S(\varsigma) = \varsigma\}$ with $\xi(S(\varsigma), S(q)) > 0$.

Theorem 1. *Consider a complete bMS (M, ξ, s) and a continuous self-map S on M. Then, S has a fixed point in M if S is an extended interpolative CRR-type F-contraction.*

Proof. Consider $\varsigma_0 \in M$. Suppose ς_n is a sequence. Consider $\varsigma_n = S^n(\varsigma_0) \; \forall \; n \in \mathbb{N}$. Then, ς_{n_0} becomes a fixed point of S if $\exists \; n_0 \in \mathbb{N}$ such that $\varsigma_{n_0} = \varsigma_{n_0+1}$. Hence, we assume that $\varsigma_n \neq \varsigma_{n+1} \; \forall \; n \in \mathbb{N}$ with $\varsigma = \varsigma_n$ and $q = \varsigma_{n-1} \; \forall \; n \in \mathbb{N}$. Then, we have the following from the Equation (1):

$$\begin{aligned}
\tau + F(\xi(\varsigma_{n+1}, \varsigma_n)) &= \tau + F(\xi(S(\varsigma_n), S(\varsigma_{n-1}))) \\
&\leq c_1 F(\xi(\varsigma_n, \varsigma_{n-1})) + c_2 F(\xi(\varsigma_n, S(\varsigma_n))) + (1 - c_1 - c_2) F(\frac{1}{s}\xi(\varsigma_{n-1}, S(\varsigma_{n-1}))) \\
&= c_1 F(\xi(\varsigma_n, \varsigma_{n-1})) + c_2 F(\xi(\varsigma_n, \varsigma_{n+1})) + (1 - c_1 - c_2) F(\frac{1}{s}\xi(\varsigma_{n-1}, \varsigma_n)) \\
&\leq c_1 F(\xi(\varsigma_n, \varsigma_{n-1})) + c_2 F(\xi(\varsigma_n, \varsigma_{n+1})) + (1 - c_1 - c_2) F(\xi(\varsigma_{n-1}, \varsigma_n)), \quad (\text{ for } s \geq 1).
\end{aligned} \quad (2)$$

Suppose, $\xi(\varsigma_{n-1}, \varsigma_n) < \xi(\varsigma_n, \varsigma_{n+1})$; then, from (2), we have

$$\begin{aligned}\tau + F(\xi(\varsigma_{n-1},\varsigma_n)) &< c_1F(\xi(\varsigma_n,\varsigma_{n+1})) + c_2F(\xi(\varsigma_n,\varsigma_{n+1})) + (1-c_1-c_2)F(\xi(\varsigma_n,\varsigma_{n+1})) \\ &= F(\xi(\varsigma_n,\varsigma_{n+1})) \\ \implies \tau + F(\xi(\varsigma_{n-1},\varsigma_n)) &< F(\xi(\varsigma_n,\varsigma_{n+1})),\end{aligned}$$

which is a contradiction.

Therefore,

$$\xi(\varsigma_n,\varsigma_{n+1}) < \xi(\varsigma_{n-1},\varsigma_n), \quad \forall\, n \geq 1.$$

Hence,

$$\tau + F(\xi(\varsigma_{n+1},\varsigma_n)) \leq F(\xi(\varsigma_n,\varsigma_{n-1})).$$

Consequently,

$$F(\xi(\varsigma_n,\varsigma_{n+1})) \leq F(\xi(\varsigma_{n-1},\varsigma_n)) - \tau \leq \ldots \leq F(\xi(\varsigma_0,\varsigma_1)) - n\tau, \forall\, n \geq 1.$$

Then, by taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} F(\xi(\varsigma_n,\varsigma_{n+1})) = -\infty.$$

Thus, from condition $F2$ we obtain

$$\lim_{n\to\infty} \xi(\varsigma_n,\varsigma_{n+1}) = 0.$$

Let $\theta_n = \xi(\varsigma_n,\varsigma_{n+1})$.
Hence,

$$\lim_{n\to\infty} \theta_n = 0.$$

Next, from condition $F3$, $\exists\, k \in (0,1)$ $\forall\, n \in \mathbb{N}$, and we have

$$\theta_n^k(F(\theta_n) - F(\theta_0)) \leq -\theta_n^k n\tau < 0. \tag{3}$$

Now, using condition $F3$ and taking the limit as $n \to \infty$, we have

$$\underbrace{\lim}_{n\to\infty} n\theta_n^k = 0.$$

Hence, $\exists\, n_0 \in \mathbb{N}$ such that $n\theta_n^k \leq 1, \forall n \geq n_0$.

$$\theta_n \leq \frac{1}{n^{\frac{1}{k}}}, \quad \forall n \geq n_0.$$

Next, to verify that $\{\varsigma_n\}$ is a Cauchy sequence, we take $m,n \in \mathbb{N}$ such that $m > n \geq n_0$.
From the triangular inequality, we have

$$\begin{aligned}\xi(\varsigma_n,\varsigma_m) &\leq \frac{1}{s^{m-1}}[\xi(\varsigma_n,\varsigma_{n+1}) + \xi(\varsigma_{n+1},\varsigma_{n+2}) + \ldots + \xi(\varsigma_{m-1},\varsigma_m)] \\ &= \frac{1}{s^{m-1}}[\theta_n + \theta_{n+1} + \ldots + \theta_{m-1}] \\ &= \sum_{i=1}^{m-1}\frac{1}{s^i}\sum_{i=n}^{m-1}\theta_i \\ &\leq \sum_{i=1}^{\infty}\frac{1}{s^i}\sum_{i=n}^{\infty}\theta_i \\ &\leq \sum_{i=1}^{\infty}\frac{1}{s^i}\sum_{i=n}^{\infty}\frac{1}{n^{\frac{1}{k}}}.\end{aligned}$$

If we take the limit as $n \to \infty$ for the above inequality, we obtain

$$\lim_{m,n\to\infty} \xi(\varsigma_n,\varsigma_m) = 0.$$

Therefore, $\{\varsigma_n\}$ is a Cauchy sequence in M.
Because (M, ξ, s) is a complete b-MS, $\exists\, \varsigma^* \in M$ such that $\varsigma_n \to \varsigma^* \in M$, as $n \to \infty$.
Next, we have to show that ς^* is a fixed point of S.
Because $\varsigma_n \to \varsigma^* \in M$, as $n \to \infty$,

$$\lim_{n\to\infty} \varsigma_n = \varsigma^*.$$

Therefore, for a subsequence ς_{n_i} of ς_n,

$$\begin{aligned}\varsigma^* &= \lim_{n\to\infty} \varsigma_{n_i+1}\\ &= \lim_{n\to\infty} S(\varsigma_{n_i})\\ &= S(\lim_{n\to\infty} \varsigma_{n_i})\\ &= S(\varsigma^*).\end{aligned}$$

Hence, ς^* is a fixed point in S.
□

Next, we introduce the extended interpolative Kannan-type F-contraction and establish a fixed point result.

Definition 5. *Consider the bMS(M, ξ, s) and a self-map $S : M \to M$. Then, the map S is called an extended interpolative modified Kannan-type F-contraction if for $\tau > 0\ \exists\, c_1, c_2 \in (0, 1)$ with $c_1 + c_2 < 1$ and $F \in \mathcal{F}$ such that*

$$\tau + F(\xi(S(\varsigma), S(q))) \le c_1 F(\xi(\varsigma, S(\varsigma))) + c_2 F(\frac{1}{s}\xi(q, S(q))) \tag{4}$$

$\forall\, \varsigma, q \in M \setminus Fix(S)$ with $S(\varsigma) \neq S(q)$, where $Fix(S)$ denotes the set of all fixed points of S.

Theorem 2. *Consider a complete b-MS (M, ξ, s) and a continuous self-map S on M. Then, S has a unique fixed point in M if S is an extended interpolative Kannan-type F-contraction.*

Proof. Consider $\varsigma_0 \in M$. Construct the sequence ς_n such that $\varsigma_n = S^n(\varsigma_0)\ \forall\, n \ge 0$. If $\exists$ $n_0 \in \mathbb{N}$ such that $\varsigma_{n_0} = \varsigma_{n_0+1}$, then ς_{n_0} is clearly a fixed point of S. Hence, assume that $\varsigma_n \neq \varsigma_{n+1}\ \forall\, n \ge 0$ with $\varsigma = \varsigma_n$ and $q = \varsigma_{n-1}\ \forall\, n \in \mathbb{N}$. Then, we have the following from Equation (4):

$$\begin{aligned}\tau + F(\xi(\varsigma_{n+1}, \varsigma_n)) &= \tau + F(\xi(S(\varsigma_n), S(\varsigma_{n-1})))\\ &\le c_1 F(\xi(\varsigma_n, S(\varsigma_n))) + c_2 F(\frac{1}{s}\xi(\varsigma_{n-1}, S(\varsigma_{n-1})))\\ &< c_1 F(\xi(\varsigma_n, \varsigma_{n+1})) + (1 - c_1) F(\frac{1}{s}\xi(\varsigma_{n-1}, \varsigma_n))\\ &< c_1 F(\xi(\varsigma_n, \varsigma_{n+1})) + (1 - c_1) F(\xi(\varsigma_{n-1}, \varsigma_n)), (\text{ for } s \ge 1).\end{aligned} \tag{5}$$

Suppose $\xi(\varsigma_n, \varsigma_{n-1}) < \xi(\varsigma_{n+1}, \varsigma_n)$; then, from (5), we have

$$\begin{aligned}&\tau + F(\xi(S(\varsigma_n), S(\varsigma_{n-1}))) < c_1 F(\xi(\varsigma_n, \varsigma_{n+1})) + (1 - c_1) F(\xi(\varsigma_{n+1}, \varsigma_n))\\ &\implies \tau + F(\xi(\varsigma_{n+1}, \varsigma_n)) < F(\xi(\varsigma_n, \varsigma_{n+1})).\end{aligned}$$

which is a contradiction.
Therefore,

$$\xi(\varsigma_{n+1}, \varsigma_n) < \xi(\varsigma_n, \varsigma_{n-1}), \quad \forall\, n \ge 1.$$

Hence,
$$\tau + F(\xi(\varsigma_{n+1}, \varsigma_n)) < F(\xi(\varsigma_n, \varsigma_{n-1})).$$
Consequently,
$$F(\xi(\varsigma_n, \varsigma_{n+1})) \leq F(\xi(\varsigma_{n-1}, \varsigma_n)) - \tau \leq \ldots \leq F(\xi(\varsigma_0, \varsigma_1)) - n\tau, \quad \forall\, n \geq 1.$$
Then, taking the limit as $n \to \infty$,
$$\lim_{n\to\infty} F(\xi(\varsigma_n, \varsigma_{n+1})) = -\infty.$$
Thus, from condition $F2$ we obtain
$$\lim_{n\to\infty} \xi(\varsigma_n, \varsigma_{n+1}) = 0.$$
Let $\theta_n = \xi(\varsigma_n, \varsigma_{n+1})$.
Hence,
$$\lim_{n\to\infty} \theta_n = 0.$$
Next, from the condition $F3$, $\exists\, k \in (0,1)\ \forall\, n \in \mathbb{N}$, we have
$$\theta_n^k(F(\theta_n) - F(\theta_0)) \leq -\theta_n^k n\tau < 0. \tag{6}$$
Now, using condition $F3$ and taking the limit as $n \to \infty$,
$$\lim_{n\to\infty} n\theta_n^k = 0.$$
Hence, $n_0 \in \mathbb{N}$ such that $n\theta_n^k \leq 1, \forall n \geq n_0$.
$$\theta_n \leq \frac{1}{n^{\frac{1}{k}}}, \forall n \geq n_0.$$
Next, to verify that $\{\varsigma_n\}$ is a Cauchy sequence, we take $m, n \in \mathbb{N}$ such that $m > n \geq n_0$.
From the triangular inequality, we have
$$\begin{aligned}
\xi(\varsigma_n, \varsigma_m) &\leq \frac{1}{s^{m-1}}[\xi(\varsigma_n, \varsigma_{n+1}) + \xi(\varsigma_{n+1}, \varsigma_{n+2}) + \ldots + \xi(\varsigma_{m-1}, \varsigma_m)] \\
&= \frac{1}{s^{m-1}}[\theta_n + \theta_{n+1} + \ldots + \theta_{m-1}] \\
&= \sum_{i=1}^{m-1} \frac{1}{s^i} \sum_{i=n}^{m-1} \theta_i \\
&\leq \sum_{i=1}^{\infty} \frac{1}{s^i} \sum_{i=n}^{\infty} \theta_i \\
&\leq \sum_{i=1}^{\infty} \frac{1}{s^i} \sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}.
\end{aligned}$$
If we take limit as $n \to \infty$ for the above inequality, we obtain
$$\lim_{m,n\to\infty} \xi(\varsigma_n, \varsigma_m) = 0.$$
Therefore, $\{\varsigma_n\}$ is a Cauchy sequence in M.
Because(M, ξ, s) is a complete bMS,
there exists $\varsigma^* \in M$ such that $\varsigma_n \to \varsigma^* \in M$, as $n \to \infty$.
Next, we have to show that ς^* is a fixed point of S.
Because $\varsigma_n \to \varsigma^* \in M$ and $n \to \infty$,
$$\lim_{n\to\infty} \varsigma_n = \varsigma^*.$$
Therefore, for a subsequence ς_{n_k} of ς_n,
$$\begin{aligned}
\varsigma^* &= \lim_{n\to\infty} \varsigma_{n_k+1} \\
&= \lim_{n\to\infty} S(\varsigma_{n_k}) \\
&= S(\lim_{n\to\infty} \varsigma_{n_k}) \\
&= S(\varsigma^*).
\end{aligned}$$

Hence, ς^* is a fixed point in S.
Finally, we have to show the uniqueness.
Consider two fixed points, ς^* and q^*, such that $\varsigma^* \neq q^*$. Then, $S(\varsigma^*) = \varsigma^* \neq q^* = S(q^*)$.
Thus, we have

$$\begin{aligned}\tau + F(\xi(S(\varsigma^*), S(q^*))) &\leq c_1 F(\xi(\varsigma^*, S(\varsigma^*))) + c_2 F(\frac{1}{s}\xi(q^*, S(q^*))) \\ &= 0,\end{aligned}$$

which is a contradiction.
Therefore, $\varsigma^* = q^*$.
Hence, S has a unique fixed point in bMS.
□

4. Examples

In this section, we provide examples to validate our results.

Example 1. *Consider $M = \{x, y, z\}$. Let $\xi : M \times M \to [0, \infty)$ be defined as $\xi(x, y) = 0$ if $x = y$, $\xi(x, y) = \xi(y, x)\ \forall\ x, y \in M$ and $\xi(x, y) = 1$, $\xi(x, z) = 2.3$, $\xi(y, z) = 1.1$. Then, $(M, \xi, \frac{23}{21})$ is a complete b-MS but not an MS.*

Construct a self-map $S : M \to M$ such that

$$S(x) = \begin{cases} \{x\}, & \text{if } x = x \text{ or } x = y \\ \{y\}, & \text{if } x = z. \end{cases}$$

Next, construct

$$\xi(S(x), S(y)) = \begin{cases} \{\xi(x, x) = 0\}, & \text{if } x \neq z, y \neq z \\ \{\xi(y, x) = 1\}, & \text{if } x = z, y \neq z \\ \{\xi(x, y) = 1\}, & \text{if } x \neq z, y = z \\ \{\xi(y, y) = 0\}, & \text{if } x = z, y = z. \end{cases}$$

Hence, $\xi(S(x), S(y))$ attains its maximum value of 1.
Now, we consider $x, y \in M \setminus Fix(S)$ with $S(x) \neq S(y)$.
Consider $c_1 = \frac{1}{5}$, $c_2 = \frac{1}{2}$, $\tau = \frac{1}{4}\ln(\frac{21}{23})$ and $F(x) = \ln(x)$ for $x > 0$.
For the Cirić–Reich–Rus type inequality, we have

$$\begin{aligned}\tau + F(\xi(S(x), S(y))) &= \frac{1}{4}\ln(\frac{21}{23}) + F(1) \\ &= \frac{1}{4}\ln(\frac{21}{23}) + \ln(1) \\ &= \frac{1}{4}\ln(\frac{21}{23}).\end{aligned}$$

Further,

$$\begin{aligned}c_1[F(\xi(x, y))] + c_2[F(\xi(x, S(x)))] &+ (1 - c_1 - c_2)[F(\frac{1}{s}\xi(y, S(y)))] \\ &= c_1 F(1) + c_2 F(1) + (1 - c_1 - c_2) F(\frac{21}{23}1) \\ &= (1 - c_1 - c_2) F(\frac{21}{23}) \\ &= \frac{3}{10}\ln(\frac{21}{23}).\end{aligned}$$

Therefore,

$$\tau + F(\xi(S(x), S(y))) < c_1[F(\xi(x, y))] + c_2[F(\xi(x, S(x)))] + (1 - c_1 - c_2)[F(\tfrac{1}{2}\xi(y, s(y)))].$$

Hence, the conditions for a CRR-type F-contraction are satisfied.

Next, for a Kannan-type inequality, we have

$$\begin{aligned}\tau + F(\xi(S(x), S(y))) &= \frac{1}{4}\ln(\frac{21}{23}) + F(1)\\ &= \frac{1}{4}\ln(\frac{21}{23}) + \ln(1)\\ &= \frac{1}{4}\ln(\frac{21}{23}),\end{aligned}$$

and

$$\begin{aligned}c_1[F(\xi(x, S(x)))] + c_2[F(\frac{1}{s}\xi(y, S(y)))] &= \frac{1}{5}F(1) + \frac{1}{2}F(\frac{21}{23}1)\\ &= \frac{1}{2}\ln(\frac{21}{23})\end{aligned}$$

Therefore,

$$\tau + F(\xi(S(x), S(y))) < c_1[F(\xi(x, S(x)))] + c_2[F(\tfrac{1}{s}\xi(y, S(y)))].$$

Hence, the conditions for a Kannan-type F-contraction are satisfied.

Example 2. *Consider the space $l_p(0 < p < 1)$, where*

$$l_p = \left\{ (x_n) \subset \mathbb{R} : \overbrace{\sum_{n=1}}^{\infty} |x_n|^p < \infty \right\}.$$

Define a metric on this space such as

$$\xi : l_p \times l_p \to \mathbb{R} \text{ defined by,}$$

$$\xi(x, y) = (\overbrace{\sum_{n=1}}^{\infty} |x_n - y_n|^p)^p, \text{ where } x = x_n, y = y_n \in l_p,$$

having the triangular inequality

$$\xi(x, z) \leq 2^{\frac{1}{p}}[\xi(x, y) + \xi(y, z)].$$

Then, $(l_p, \xi, 2^{\frac{1}{p}})$ is a complete bMS.
Construct a self-map $S : l_p \to l_p$ such that

$$S(x) = \begin{cases} \{0\}, & \text{if } x = (0) \text{ or } x = (x) \text{ where } x \in l_p \\ \{1\}, & \text{otherwise.} \end{cases}$$

Now, for the Ćirić–Reich–Rus type inequality, we have

$$\begin{aligned}\tau + F(\xi(S(x), S(y))) &\leq \tau + F(2^{\frac{1}{p}}(\xi(S(x), x) + \xi(x, S(y))))\\ &\leq \tau + F[2^{\frac{1}{p}}(\xi(S(x), x) + 2^{\frac{1}{p}}(\xi(x, y) + \xi(y, S(y))))]\\ &< \tau + c_1 F(\xi(x, y)) + c_2 F(\xi(x, S(x))) + (1 - c_1 - c_2)F(\frac{1}{2^p}(y, S(y)))\end{aligned}$$

where $c_1, c_2 \in \mathbb{N}$.
Therefore the conditions for a CRR-type F-contraction are satisfied.

Next, for a Kannan-type inequality, we have

$$\begin{aligned}\tau + F(\xi(S(x),S(y))) &\leq \tau + F(2^{\frac{1}{p}}(\xi(S(x),x)+\xi(x,S(y)))) \\ &\leq \tau + F[2^{\frac{1}{p}}(\xi(S(x),x)+2^{\frac{1}{p}}(\xi(x,y)+\xi(y,S(y))))] \\ &< \tau + c_1F(\xi(x,S(x))) + c_2F(\frac{1}{2^p}\xi(y,S(y)))\end{aligned}$$

where $c_1, c_2 \in \mathbb{N}$.

Therefore, the conditions for a Kannan-type F-contraction are satisfied.

5. Application to Integral Equations

Applications of fixed point theorems can be found in various areas of science, such as physics, engineering, etc., in terms of solutions of differential and integral equations. In this section, following the similar lines as in [25,26], we provide an application of Theorem 2 towards the solution of an integral equation.

Let $I = [0,r], r > 0$ and $M = C(I,\mathbb{R})$ be the set of all real-valued continuous functions defined on I. Suppose that

$$\xi(g,h) = \sup_{t\in I}(|g(t)-h(t)|) = \|g-h\|,$$

where ξ is a b-metric on M.

Consider the integral equation

$$g(x) = q(x) + \int_0^r G(x,w)f(w,g(w))dw, x \in [0,r], \tag{7}$$

where

(i) $q : I \to \mathbb{R}$ and $f : I \times \mathbb{R} \to \mathbb{R}$ are continuous;

(ii) for all $x \in I$, $G : I \times I \to \mathbb{R}$ is continuous and measurable at $w \in I$;

(iii) for all $x, w \in I$, $G(x,w) \geq 0$ and for all $x \in I$, $\int_0^r G(x,w)f(w,g(w))dw \leq 1$.

Theorem 3. *Assume that conditions* $(i)-(iii)$ *hold. Let* $\tau > 0$ *and* $c_1, c_2 \in (0,1)$ *with* $c_1 + c_2 < 1$ *exist such that*

$$\begin{aligned}&\Big|f(x,g(x)) - f(x,h(x))\Big| \\ &\leq \frac{1}{s^{c_2}}e^{-\tau}\left(\left\|g - \int_0^r G(x,w)f(w,g(w))dw\right\|\right)^{c_1}. \\ &\left(\left\|h - \int_0^r G(x,w)f(w,h(w))dw\right\|\right)^{c_2},\end{aligned} \tag{8}$$

for each $x \in I$ *and for all* $g, h \in X$ *such that*

$$g(x) \neq \int_0^r G(x,w)f(w,g(w))dw,$$

$$h(x) \neq \int_0^r G(x,w)f(w,h(w))dw,$$

while $g(x) \neq h(x)$ *for all* $x \in I$. *Then, the integral equation (7) has a solution in* M.

Proof. Let us define a self-map $\Gamma : M \to M$ by

$$\Gamma(g(x)) = q(x) + \int_0^r G(x,w)f(w,g(w))dw, x \in [0,r].$$

For each $x \in [0,r]$, we have

$$
\begin{aligned}
\Big|\Gamma(g(x)) - \Gamma(h(x))\Big| &= \Big| \int_0^r G(x,w)\{f(w,g(w)) - f(w,h(w))\}dw\Big| \\
&\leq \int_0^r G(x,w)\Big|f(w,g(w)) - f(w,h(w))\Big|dw \\
&\leq \int_0^r G(x,w) \cdot \frac{1}{s^{c_2}}e^{-\tau}\Big(\|g-\Gamma g\|\Big)^{c_1} \cdot \\
&\Big(\|h-\Gamma h\|\Big)^{c_2}dw \\
&= \frac{1}{s^{c_2}}e^{-\tau} \cdot \Big(\|g-\Gamma g\|\Big)^{c_1} \cdot \Big(\|h-\Gamma h\|\Big)^{c_2} \\
&\int_0^{c_1} G(x,w)dw \\
&\leq \frac{1}{s^{c_2}}e^{-\tau} \cdot \Big(\|g-\Gamma g\|\Big)^{c_1} \cdot \Big(\|h-\Gamma h\|\Big)^{c_2},
\end{aligned} \tag{9}
$$

(because $\int_0^r G(x,w)dw \leq 1$ for all $x \in I$).

Taking the supremum over $x \in I$ on both sides of (9), we have

$$
\begin{aligned}
\xi(\Gamma g, \Gamma h) &= \|\Gamma g - \Gamma h\| \\
&\leq \frac{1}{s^{c_2}}e^{-\tau} \cdot \Big(\|g-\Gamma g\|\Big)^{c_1} \cdot \Big(\|h-\Gamma h\|\Big)^{c_2} \\
&= \frac{1}{s^{c_2}}e^{-\tau} \cdot \Big(\xi(g,\Gamma g)\Big)^{c_1} \cdot \Big(\xi(h,\Gamma h)\Big)^{c_2}
\end{aligned}
$$

$$
\implies \frac{\xi(\Gamma g, \Gamma h)}{\frac{1}{s^{c_2}}\Big(\xi(g,\Gamma g)\Big)^{c_1} \cdot \Big(\xi(h,\Gamma h)\Big)^{c_2}} \leq e^{-\tau}
$$

$$
\begin{aligned}
&\implies \ln\left[\frac{\xi(\Gamma g, \Gamma h)}{\frac{1}{s^{c_2}}\Big(\xi(g,\Gamma g)\Big)^{c_1} \cdot \Big(\xi(h,\Gamma h)\Big)^{c_2}}\right] \leq -\tau \\
&\implies \ln\Big[\xi(\Gamma g, \Gamma h)\Big] - \ln\left[\frac{1}{s^{c_2}}\Big(\xi(g,\Gamma g)\Big)^{c_1} \cdot \Big(\xi(h,\Gamma h)\Big)^{c_2}\right] \leq -\tau \\
&\implies \tau + \ln\Big[\xi(\Gamma g, \Gamma h)\Big] \leq \ln\left[\Big(\xi(g,\Gamma g)\Big)^{c_1} \cdot \Big(\frac{1}{s}\xi(h,\Gamma h)\Big)^{c_2}\right] \\
&\implies \tau + \ln\Big[\xi(\Gamma g, \Gamma h)\Big] \leq c_1 \ln\Big[\Big(\xi(g,\Gamma g)\Big)\Big] + c_2 \ln\left[\frac{1}{s}\Big(\xi(h,\Gamma h)\Big)\right].
\end{aligned} \tag{10}
$$

Taking $F(\delta) = \ln(\delta)$, $\delta > 0$, we have form (10):

$$
\tau + F(\xi(\Gamma g, \Gamma h)) \leq c_1 F(\xi(g,\Gamma g)) + c_2 F(\frac{1}{s}\eta(h,\Gamma h)),
$$

for all $g, h \in M \setminus Fix(\Gamma)$ with $\Gamma g \neq \Gamma h$.

Hence, per Theorem 2, Γ has a unique fixed point; therefore, the integral Equation (7) has a unique solution. $\square$

6. Conclusions

In this paper, we have introduced extended interpolative CRR-type F-contraction and interpolative Kannan-type F-contraction mappings in a bMS. The existence of fixed point results has been established for the new contraction maps. We have provided examples involving both finite and infinite bMSs. This work can be extended in the future to investigate discontinuity results at the fixed points. Applications of these extended results may point out new iterative schemes for the solution of differential and integral equations.

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References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur applications aux tequations integrales. *Fund. Math.* **1922**, *3*, 133–181. [CrossRef]
2. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, 94, 1–6. [CrossRef]
3. Boyd, D.W.; Wong, J.S. On nonlinear contractions. *Proc. Amer. Math. Soc.* **1969**, *20*, 458–464. [CrossRef]
4. Chatterjea, S.K. Fixed point theorems. *C. R. Acad. Bulg. Sci.* **1972**, *25*, 727–730. [CrossRef]
5. Ćirić, L. Fixed point theorems for multivalued contractions in complete metric spaces. *J. Math. Anal. Appl.* **2008**, *348*, 499–507. [CrossRef]
6. Kannan, R. Some results on fixed points. *Bull. Calc. Math. Soc.* **1968**, *60*, 71–77.
7. Meir, A.; Keeler, E. A theorem on contraction mappings. *J. Math. Anal. Appl.* **1969**, *28*, 326–329. [CrossRef]
8. Debnath, P.; Konwar, N.; Radenović, S. *Metric Fixed Point Theory*; Springer: Singapore, 2021.
9. Debnath, P.; Mitrović, Z.; Radenović, S. Interpolative Hardy-Rogers and Reich-Rus-Ćirić type contractions in b-metric spaces and rectangular b-metric spaces. *Math. Vesnik.* **2020**, *72*, 368–374.
10. Karapinar, E.; Agarwal, R.P.; Aydi, H. Interpolative Reich-Rus-Ćirić type contractions on partial metric spaces. *Mathematics* **2018**, *6*, 256. [CrossRef]
11. Karapinar, E.; Alahtani, O.; Aydi, H. On interpolative Hardy-Rogers type contractions. *Symmetry* **2018**, *11*, 8. [CrossRef]
12. Bakhtin, I.A. The contraction mapping principle in almost metric spaces. *Funct. Anal.* **1989**, *30*, 26–37.
13. Czerwik, S. Contraction mappings in b-metric spaces. *Acta Math. Univ. Ostrav.* **1993**, *1*, 5–11.
14. Alharbi, N.; Aydi, H.; Felhi, A.; Ozel, C.; Sahmim, S. α-contractive mappings on rectangular b-metric spaces and an application to integral equations. *J. Math. Anal.* **2018**, *9*, 47–60.
15. Aydi, H.; Chen, C.M.; Karapinar, E. Interpolative Ćirić-Reich-Rus type contractions via the Branciari distance. *Mathematics* **2019**, *7*, 84. [CrossRef]
16. Aydi, H.; Karapinar, E.; Hierro, A.F.R. ω-interpolative Ćirić-Reich-Rus-type contractions. *Mathematics* **2019**, *7*, 57. [CrossRef]
17. Collaco, P.; Silva, J.C. A complete comparison of 25 contraction conditions. *Nonlinear Anal. Theory, Methods Appl.* **1997**, *30*, 471–476. [CrossRef]
18. Ding, H.S.; Imdad, M.; Radenovic, S.; Vujakovic, J. On some fixed point results in b-metric, rectangular and b-rectangular metric spaces. *Arab J. Math. Sci.* **2016**, *22*, 151–164.
19. Ege, O. Complex valued rectangular b-metric spaces and an application to linear equations. *J. Nonlinear Sci. Appl.* **2015**, *8*, 1014–1021. [CrossRef]
20. Gholidahneh, A.; Sedghi, S.; Ege, O.; Mitrovic, Z.D.; de la Sen, M. The Meir-Keeler type contractions in extended modular b-metric spaces with an application. *AIMS Math.* **2021**, *6*, 1781–1799. [CrossRef]
21. Jeong, G.S.; Rhoades, B.E. Maps for which $f(t) = f(t^n)$. *Fixed Point Theory Appl.* **2005**, *6*, 71–105.
22. Kadelburg, Z.; Radenović, S. Pata-type common fixed point results in b-metric and b-rectangular metric spaces. *J. Nonlinear Anal. Appl.* **2015**, *8*, 944–954. [CrossRef]
23. Rhoades, B.E. A comparison of various definitions of contractive mappings. *Trans. Amer. Math. Soc.* **1977**, *226*, 257–290. [CrossRef]
24. George, R.; Radenović, S.; Reshma, K.P.; Shukla, S. Rectangular b-metric space and contraction principles. *J. Nonlinear Sci. Appl.* **2015**, *8*, 1005–1013. [CrossRef]
25. Debnath, P. A new extension of Kannan's fixed point theorem via F-contraction with application to integral equations. *Asian-Eur. J. Math.* **2022**, *15*, 2250123. [CrossRef]
26. Debnath, P.; Srivastava, H. M. New extensions of Kannan's and Reich's fixed point theorems for multivalued maps using Wardowski's technique with application to integral equations. *Symmetry* **2020**, *12*, 1090. [CrossRef]